CORRELATION FUNCTIONS AND RENORMALIZATION IN A SCALAR FIELD THEORY ON THE FUZZY SPHERE

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K.H. and A.T., PTEP 2017, 063B01 (2017) [arXiv:1704.01698]

+ work in progress

Discrete Approaches to the Dynamics of Fields and Space-time @APCTP, 19th-23rd Sept.

Contents

Introduction

Noncommutative space-time

- Noncommutative space-time is considered as playing a crucial role in quantum gravity
- Indeed, it appears in various contexts of string theory
- Noncommutative Yang-Mills is a low energy effective theory for D-branes in a constant B-field   Seiberg-Witten ('98)
- Noncommutative gauge theories are naturally realized in matrix models which are expected to give nonperturbative formulation of string theory
- String field theories seem to be formulated as Chern-Simons-like gauge theory on a certain huge noncommutative space-time

Field theory on noncommutative space

- It is important to elucidate the difference b/w field theories on ordinary space and noncommutative space

- It is well-known that there are the IR divergences that originate from UV divergences in field theories on noncommutative space ~ UV/IR mixing

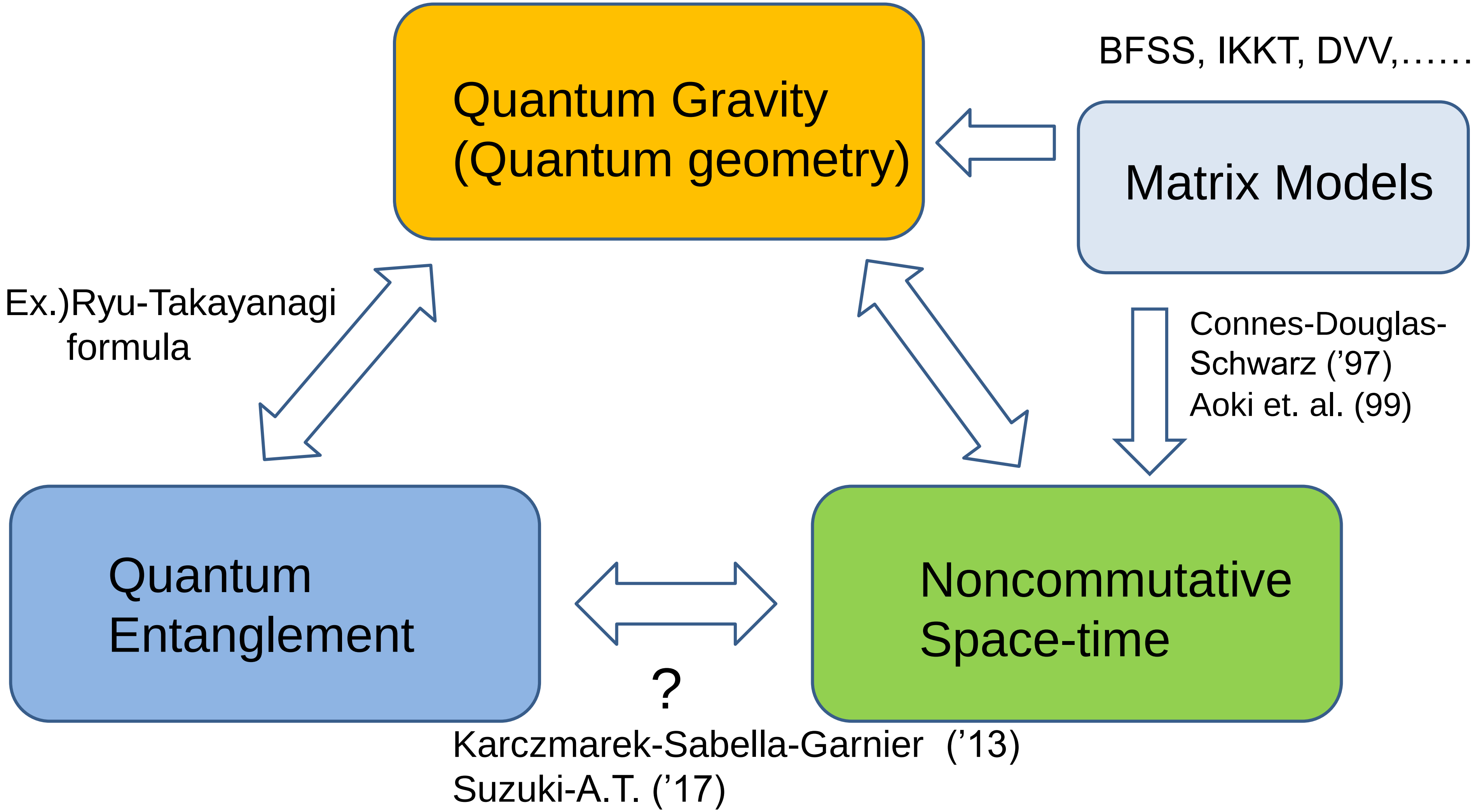
  Minwalla-Raamsdonk-Seiberg ('99)

- The UV/IR mixing prevents a theory from being perturbatively renormalizable

What we will do

- Here we study whether a scalar field theory on the fuzzy sphere, which is a typical compact noncommutative space, is nonperturbatively renormalizable
- We want to see how meaningful the local quantities are on noncommutative space
- We define multi-point local correlation functions on the fuzzy sphere using the Berezin symbol and calculate them by Monte Carlo simulation
- Cf.) Bietenholz-Hofheinz-Nishimura ('04)
  calculate 2-point function in a scalar field theory on noncommutative torus and take the double scaling (continuum and thermodynamic) limit

Motivation


Quantum Gravity
(Quantum geometry)
BFSS, IKKT, DVV,……
Matrix Models
Ex.)Ryu-Takayanagi
formula
Connes-Douglas-
Schwarz ('97)
Aoki et. al. (99)
Quantum
Entanglement
Noncommutative
Space-time
?
Karczmarek-Sabella-Garnier ('13)
Suzuki-A.T. ('17)

Noncommutative plane

Noncommutative plane

- Noncommutative plane (Moyal plane)

Cf.) talks of Kaneko, Muraki and Yang

simplest example of noncommutative space

$$[\hat{x}_1, \hat{x}_2] = i\theta$$

θ : real constant

- Conjugate momenta

$$\hat{p}_1 = \theta^{-1}\hat{x}_2, \;\; \hat{p}_2 = -\theta^{-1}\hat{x}_1$$

$$\Rightarrow \begin{cases} [\hat{p}_1, \hat{p}_2] = i\theta^{-1} \\ [\hat{x}_i, \hat{p}_j] = i\delta_{ij} \;\; (i, j = 1, 2) \end{cases}$$

coordinates and momenta are not independent

$\rightarrow$ Hilbert space for a particle in one dimension

Matrix vs field

$\hat{f}$: operator (matrix with infinite matrix size)

$f(x)$: field in two dimensions (Weyl-ordered symbol)

$$\hat{f} = \int \frac{d^2k}{(2\pi)^2} \int d^2x f(x) e^{-ik\cdot x} e^{ik\cdot\hat{x}}$$

$$f(x) = \int \frac{d^2k}{(2\pi)^2} e^{ik\cdot x} \mathrm{tr}(\hat{f} e^{-ik\cdot\hat{x}})$$

Correspondence

$$[\hat{p}_i, \hat{f}] \Longleftrightarrow -i\partial_i f(x) \longleftarrow [\hat{x}_i, \hat{p}_j] = i\delta_{ij}$$

$$\mathrm{tr}(\hat{f}) \Longleftrightarrow \int \frac{d^2x}{2\pi\theta} f(x)$$

$$\hat{f}\hat{g} \Longleftrightarrow f(x) \star g(x)$$

$$= e^{\frac{i\theta}{2}(\partial_{x_1}\partial_{y_2} - \partial_{x_2}\partial_{y_1})} f(x)g(y)|_{x=y}$$

Moyal product (star product)

Noncommutative & nonlocal

Field theory on noncommutative plane

- a matrix model with infinite matrix size

$$S = 2\pi\theta \mathrm{tr}\left(-\frac{1}{2}[\hat{p}_i, \hat{\phi}]^2 + \frac{m^2}{2}\hat{\phi}^2 + \frac{\lambda}{4}\hat{\phi}^4\right)$$

- correspondence between matrix and field

$$\hat{\phi} = \int \frac{d^2k}{(2\pi)^2} \int d^2x \phi(x) e^{-ik\cdot x} e^{ik\cdot\hat{x}}$$

- scalar field theory on noncommutative plane

$$S = \int d^2x \left(\frac{1}{2}(\partial_{x_i}\phi(x))^2 + \frac{m^2}{2}\phi(x)^2 + \frac{\lambda}{4}\phi(x)\star\phi(x)\star\phi(x)\star\phi(x)\right)$$

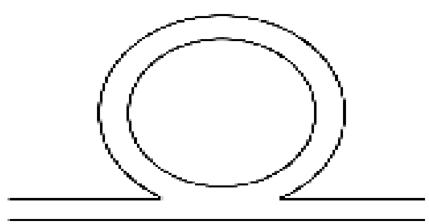
effect of noncommutativity appears only in interaction term

UV/IR mixing

Minwalla-Raamsdonk-Seiberg ('99)

Ex.) 1-loop correction to propagator

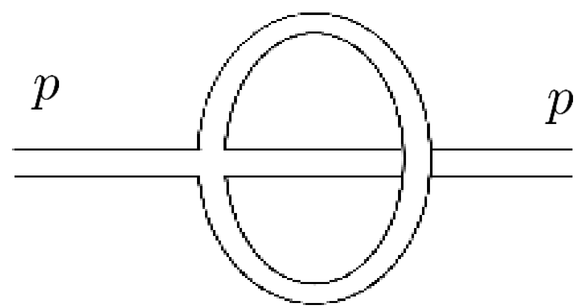
planar



$$-2\lambda \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2+m^2}$$

same as the one
in ordinary field theory

non-planar



$$-\lambda \int \frac{d^2q}{(2\pi)^2} \frac{\boxed{e^{-i\theta(p_1q_2-p_2q_1)}}}{q^2+m^2}$$

suppress UV divergence

different form the one
in ordinary field theory

UV/IR mixing (cont'd)

$$-\lambda \int \frac{d^2q}{(2\pi)^2} \frac{e^{-i\theta(p_1q_2-p_2q_1)}}{q^2+m^2} = \frac{\lambda}{2\pi}\left(\gamma + \log\left(\frac{m}{2}\sqrt{\theta^2p^2+\frac{1}{\Lambda^2}}\right)\right)$$

Λ : UV cutoff

$\theta = 0$ $\Lambda \to \infty$ ⇨ $\sim \log \Lambda$ ~ UV div. in ordinary field theory

$\theta \neq 0$ $\Lambda \to \infty$ ⇨ $\sim \log\left(\frac{m}{2}\sqrt{\theta^2p^2}\right)$ $p \to 0$ IR div.

UV/IR mixing

Scalar field theory on the fuzzy sphere

Fuzzy sphere

- Definition of fuzzy sphere

Cf.) Ishiki' s talk

$$\begin{cases} \hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 = R^2 \\ \hat{x}_i = \dfrac{R}{\sqrt{j(j+1)}} L_i \\ L_i \ \text{: generators of spin } j \text{ rep. of SU(2)} \end{cases}$$

$$[L_i, L_j] = i\epsilon_{ijk} L_k$$

- Noncommutativity

$$[\hat{x}_i, \hat{x}_j] = i\frac{R}{\sqrt{j(j+1)}}\epsilon_{ijk}\hat{x}_k \xrightarrow[\substack{j \to \infty \\ R\,:\,\text{fixed}}]{} [\hat{x}_i, \hat{x}_j] = 0$$

commutative limit

Fuzzy sphere (cont'd)

- Derivative

$$[L_i, \hat{x}_j] = i\epsilon_{ijk}\hat{x}_k$$

→ $[L_i, *]$ ⟺ $\mathcal{L}_i * = -i\epsilon_{ijk}x_j\partial_k *$

- Spin l for adjoint operation $[L_i, *] = L_i * - * L_i$
  tensor product of two spin j

⇒ $l = 0, 1, \cdots, 2j$

⇒ $2j$: UV cutoff

Standard basis for SU(2) algebra

L_i : generators of spin j rep. of SU(2)

$$L_\pm = L_1 \pm i L_2$$

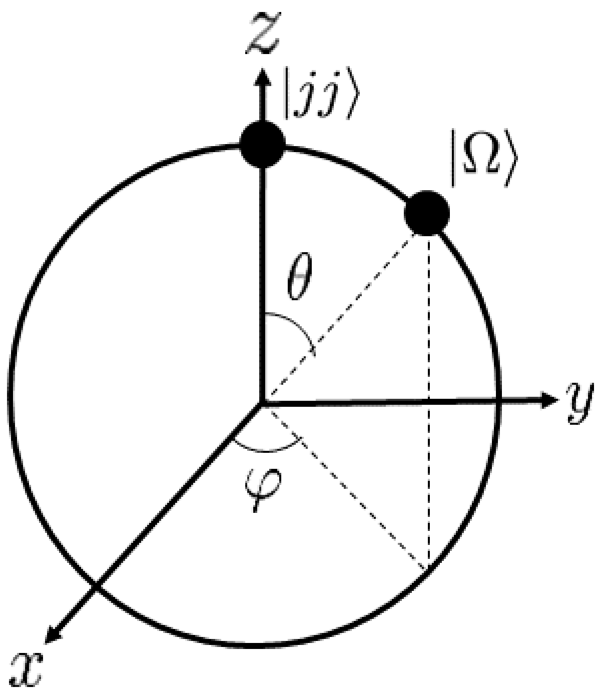
$$L_\pm |jm\rangle = \sqrt{(j \mp m)(j \pm m + 1)}|jm \pm 1\rangle$$

$$m = -j, -j+1, \cdots, j$$

$$L_3 |jm\rangle = m|jm\rangle$$

Bloch coherent states

Gazeau et al.(09)



$$\Omega = (\theta, \varphi)$$

$$|\Omega\rangle = e^{i\theta(\sin\varphi L_1 - \cos\varphi L_2)}|jj\rangle$$

rotation operator

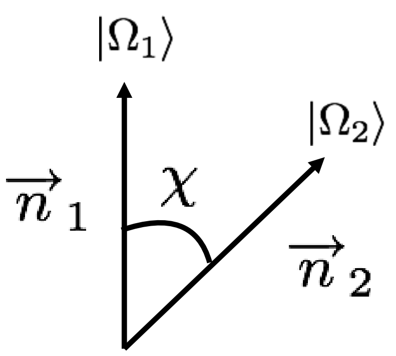
$$\vec{n}\cdot\vec{L}|\Omega\rangle = j|\Omega\rangle$$

Localized around $\Omega = (\theta, \varphi)$
with width $1/\sqrt{j}$

Bloch coherent states (cont'd)

- $\sum_i (\Delta L_i)^2$ is minimum
- completeness

$$\frac{2j+1}{4\pi}\int d\Omega|\Omega\rangle\langle\Omega| = \sum_{m=-j}^{j}|jm\rangle\langle jm| = 1$$



- inner product

$$|\langle\Omega_1|\Omega_2\rangle| = \left(\cos\frac{\chi}{2}\right)^{2j}$$

$$\chi = \frac{2}{\sqrt{j}} \Rightarrow \left(\cos\frac{\chi}{2}\right)^{2j} \approx \left(1-\frac{1}{2j}\right)^{2j} \approx e^{-1}$$

width of wave packet ~ $1/\sqrt{j}$

Berezin symbol

- Berezin symbol

$$f_A(\Omega) = \langle \Omega | A | \Omega \rangle$$

- Derivative

$$f_{[L_i, A]}(\Omega) = \mathcal{L}_i f_A(\Omega)$$

$\mathcal{L}_i$: angular momentum operators

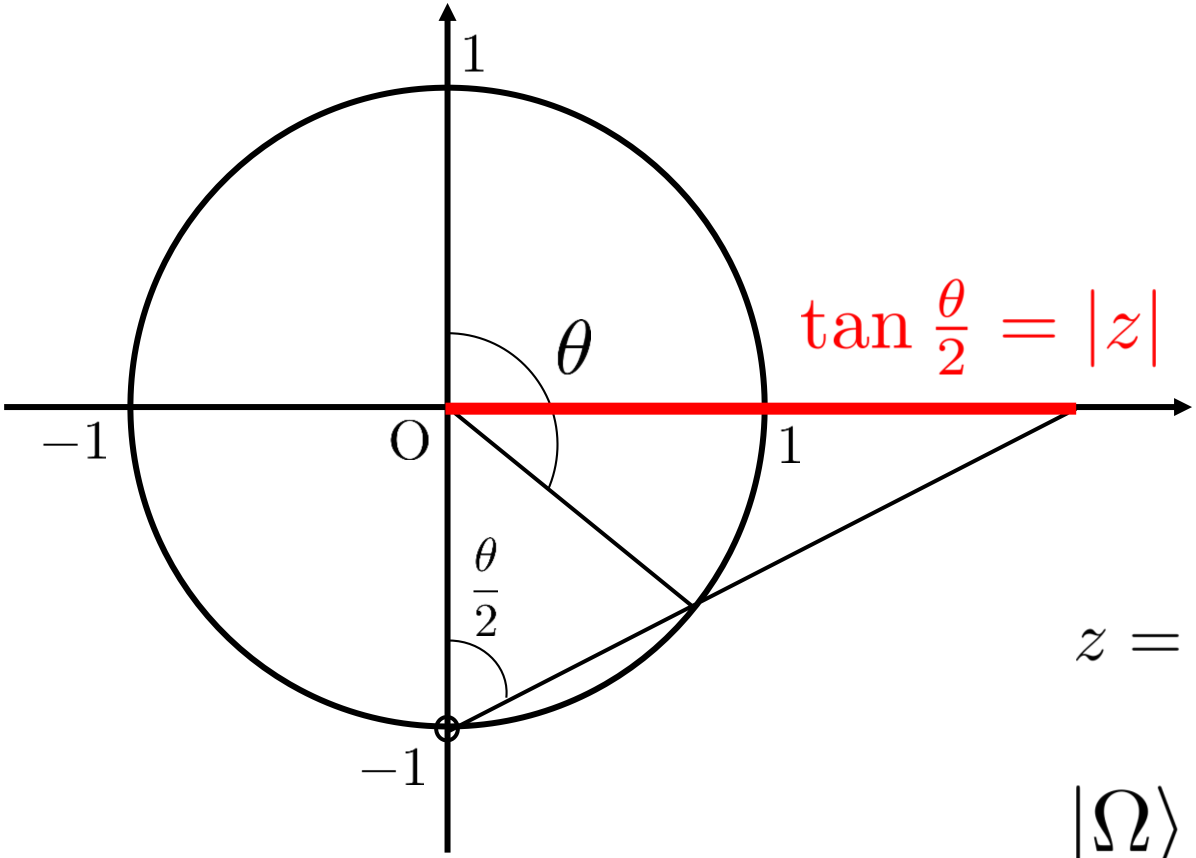
- Star product

$$f_A(\Omega) \star f_B(\Omega) \equiv f_{AB}(\Omega)$$

- Trace

$$\frac{1}{2j+1} \mathrm{Tr} A = \int \frac{d\Omega}{4\pi} f_A(\Omega)$$

Stereographic projection



$$z = \tan\frac{\theta}{2}\, e^{i\varphi}$$

$$|\Omega\rangle \equiv |z\rangle$$

$$\langle z|A|z\rangle \equiv f_A(z, \bar{z})$$

Star product

$$\frac{\langle\beta|A|\alpha\rangle}{\langle\beta|\alpha\rangle} = e^{-\beta\frac{\partial}{\partial\alpha}}\frac{\langle\beta|A|\alpha+\beta\rangle}{\langle\beta|\alpha+\beta\rangle} = e^{-\beta\frac{\partial}{\partial\alpha}}e^{\alpha\frac{\partial}{\partial\beta}}\frac{\langle\beta|A|\beta\rangle}{\langle\beta|\beta\rangle}$$

$$= e^{-\beta\frac{\partial}{\partial\alpha}}e^{\alpha\frac{\partial}{\partial\beta}}f_A(\beta,\bar\beta)$$

$$f_A\star f_B(\beta,\bar\beta) = \langle\beta|AB|\beta\rangle$$

$$= \frac{2j+1}{4\pi}\,4\int\frac{d^2\alpha}{\left(1+|\alpha|^2\right)^2}e^{-\beta\frac{\partial}{\partial\alpha}}e^{\alpha\frac{\partial}{\partial\beta}}f_A(\beta,\bar\beta)e^{-\bar\beta\frac{\partial}{\partial\bar\alpha}}e^{\bar\alpha\frac{\partial}{\partial\bar\beta}}f_B(\beta,\bar\beta)|\langle\beta|\alpha\rangle|^2$$

$$\Rightarrow\quad f_A\star f_B(0,0)\approx e^{\frac{1}{2j}\frac{\partial}{\partial z}\frac{\partial}{\partial w}}f_A(z,\bar z)f_B(w,\bar w)|_{z=w=0}$$

$$\theta\sim 1/(2j)$$

$$\xrightarrow[j\to\infty]{}\quad f_A(\beta,\bar\beta)f_B(\beta,\bar\beta)$$

Matrix vs Field

➢ correspondence between matrix and field

$$\varphi(\Omega) = f_\Phi = \langle \Omega | \Phi | \Omega \rangle$$

$$S_{NC} = \frac{R^2}{2j+1} \mathrm{tr} \left(-\frac{1}{2R^2} [L_i, \Phi]^2 + \frac{\mu^2}{2} \Phi^2 + \frac{\lambda}{4} \Phi^4 \right)$$

Φ : $(2j+1) \times (2j+1)$ Hermitian matrix

classically

$j \to \infty$

$$\varphi(\Omega) = \phi(\Omega)$$

at quantum level ⇒ UV/IR anomaly

Chu-Madore-Steinacker ('01)

$$S_C = \frac{R^2}{4\pi} \int d\Omega \left(-\frac{1}{2R^2} (\mathcal{L}_i \phi)^2 + \frac{\mu^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4 \right)$$

What we will calculate by Monte Carlo simulation

- n-point correlation functions in matrix model

$$\langle \varphi(\Omega_1)\varphi(\Omega_2)\cdots\varphi(\Omega_n)\rangle = \frac{\int d\Phi \varphi(\Omega_1)\varphi(\Omega_2)\cdots\varphi(\Omega_n)e^{-S_{NC}}}{\int d\Phi e^{-S_{NC}}}$$

$$S_{NC} = \frac{1}{2j+1}\mathrm{tr}\left(-\frac{1}{2}[L_i,\Phi]^2 + \frac{\mu^2}{2}\Phi^2 + \frac{\lambda}{4}\Phi^4\right)$$

We put $R=1$ without loss of generality

$$d\Phi = \prod_{i=1}^{N} d\Phi_{ii} \prod_{1\leq j<k\leq N} d\,\mathrm{Re}\,\Phi_{jk} d\,\mathrm{Im}\,\Phi_{jk}$$

$$\langle \varphi(\Omega_1)\varphi(\Omega_2)\cdots\varphi(\Omega_n)\rangle \longleftrightarrow \langle \phi(\Omega_1)\phi(\Omega_2)\cdots\phi(\Omega_n)\rangle$$

We put $N = 2j+1$: UV cutoff

What we will show

- We will show that 2-pt and 4-pt functions of the Berezin symbol are independent of the UV cutoff N by tuning one parameter and making wave function renormalization

  $\rightarrow$ Hatakeyama's part

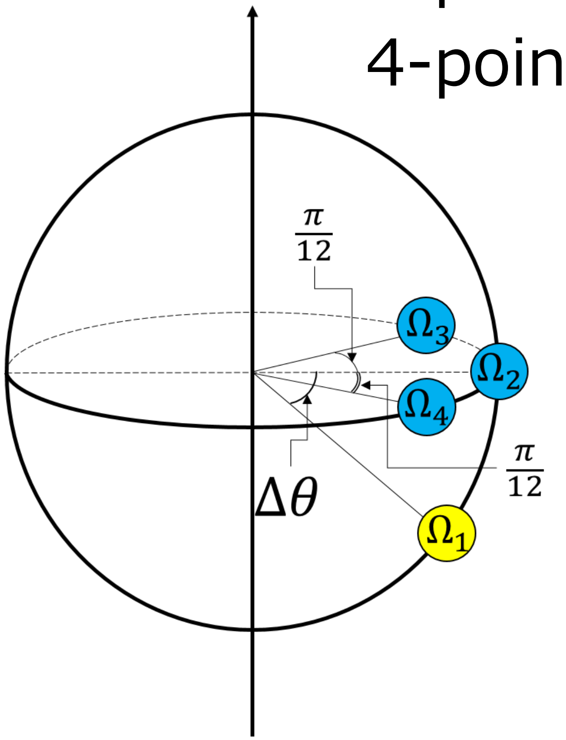
4. Calculation of correlation functions and renormalization

◆Correlation functions calculated by Monte Carlo simulation

1-point function: $\langle \varphi(\Omega_i) \rangle \quad (1 \leq i \leq 4),$

2-point function: $\langle \varphi(\Omega_i)\varphi(\Omega_j) \rangle \quad (1 \leq i < j \leq 4),$

4-point function: $\langle \varphi(\Omega_1)\varphi(\Omega_2)\varphi(\Omega_3)\varphi(\Omega_4) \rangle$



$$\Omega_1 = \left(\frac{\pi}{2} + \Delta\theta, 0\right)$$

$\Delta\theta$ is taken in steps of 0.1 in the range $0 \leq \Delta\theta \leq 1.5$.

$$\begin{aligned} \Omega_2 &= \left(\frac{\pi}{2}, 0\right) \\ \Omega_3 &= \left(\frac{\pi}{2}, \frac{\pi}{12}\right) \\ \Omega_4 &= \left(\frac{\pi}{2}, -\frac{\pi}{12}\right) \end{aligned}$$

Fixed on the equator

Renormalization

◆Renormalization

$\Phi = \sqrt{Z}\Phi_r$ (Z: the factor of the wave function renormalization)

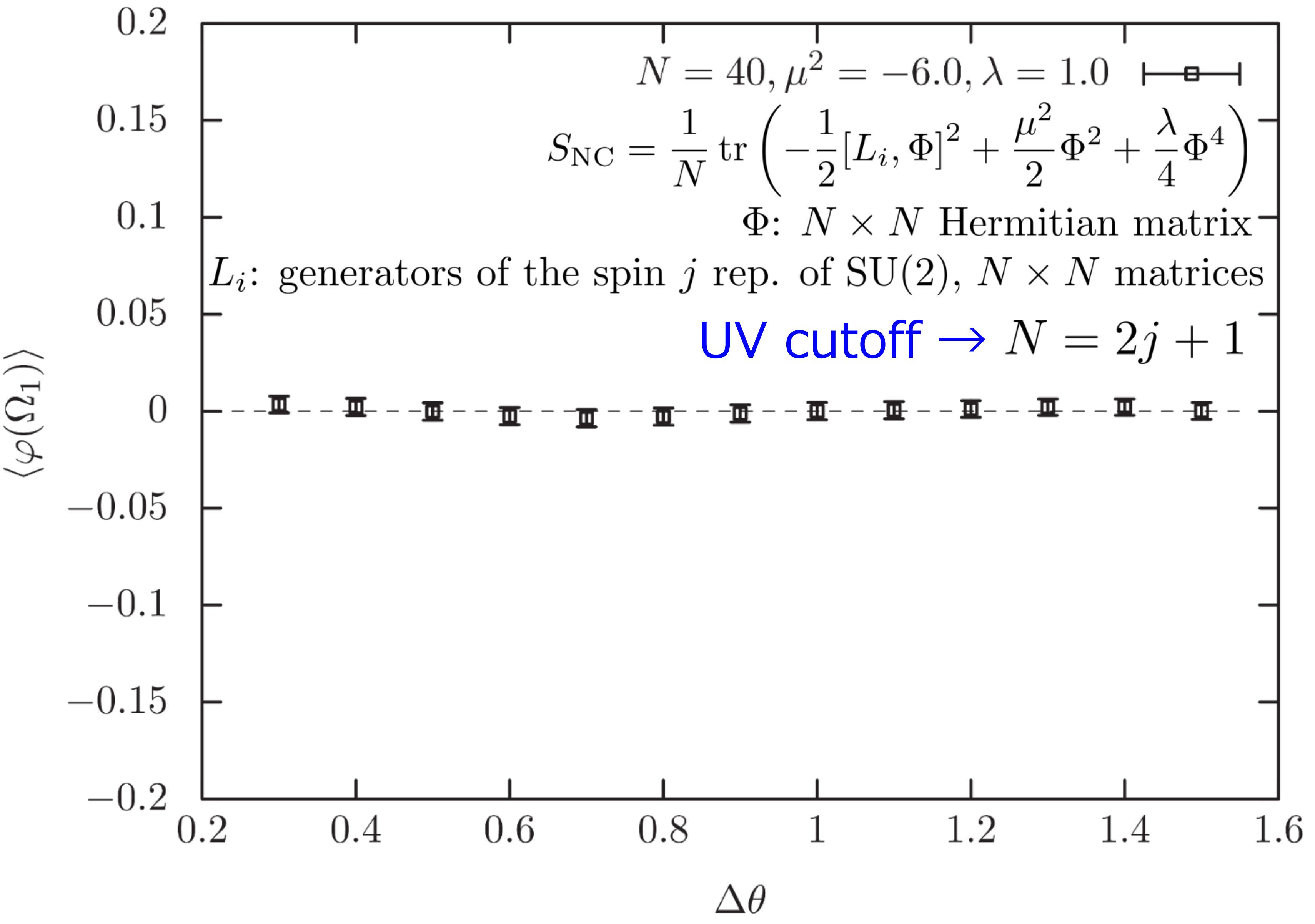
renormalized matrix

$\varphi_r(\Omega) = \langle\Omega|\Phi_r|\Omega\rangle$: the renormalized Berezin symbol

$$\begin{aligned}\langle\varphi(\Omega_i)\rangle &= \sqrt{Z}\,\langle\varphi_r(\Omega_i)\rangle\,,\\ \langle\varphi(\Omega_i)\varphi(\Omega_j)\rangle_c &= Z\,\langle\varphi_r(\Omega_i)\varphi_r(\Omega_j)\rangle_c\,,\\ \langle\varphi(\Omega_1)\varphi(\Omega_2)\varphi(\Omega_3)\varphi(\Omega_4)\rangle_c &= Z^2\,\langle\varphi_r(\Omega_1)\varphi_r(\Omega_2)\varphi_r(\Omega_3)\varphi_r(\Omega_4)\rangle_c\end{aligned}$$

In the following,
we show that renormalized correlation functions are independent of the matrix size N which is the UV cutoff by tuning 1-parameter.

1-point function



◆Correlation functions calculated by Monte Carlo simulation

1-point function: $\langle\varphi(\Omega_1)\rangle = 0$,

2-point function: $\langle\varphi(\Omega_i)\varphi(\Omega_j)\rangle = \langle\varphi(\Omega_i)\varphi(\Omega_j)\rangle_c$,

4-point function: $\langle\varphi(\Omega_1)\varphi(\Omega_2)\varphi(\Omega_3)\varphi(\Omega_4)\rangle$

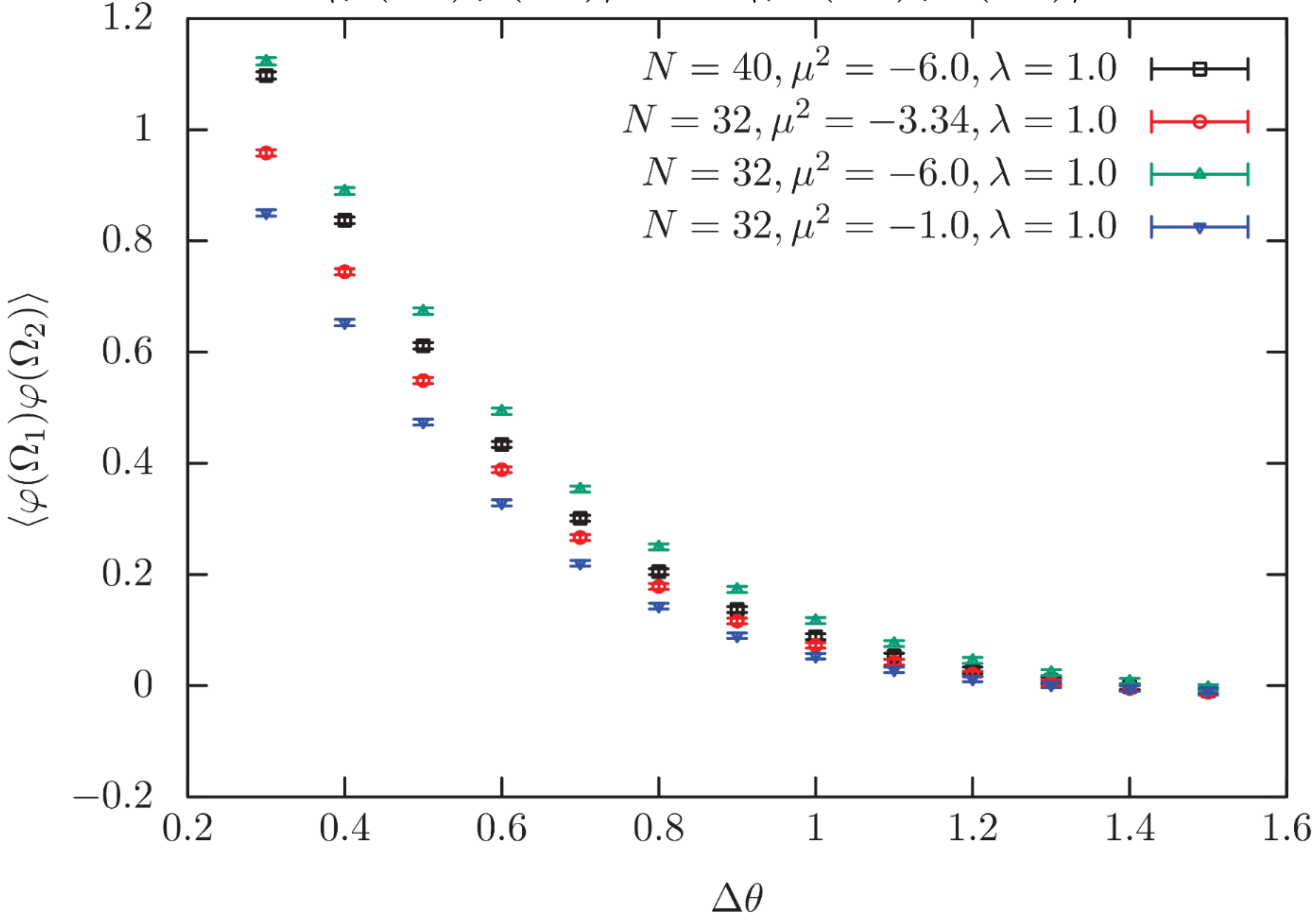
◆Connected 4-point function

$$\begin{aligned}
&\big\langle\varphi(\Omega_1)\varphi(\Omega_2)\varphi(\Omega_3)\varphi(\Omega_4)\big\rangle_c \\
= \;& \big\langle\varphi(\Omega_1)\varphi(\Omega_2)\varphi(\Omega_3)\varphi(\Omega_4)\big\rangle - \big\langle\varphi(\Omega_1)\varphi(\Omega_2)\big\rangle\big\langle\varphi(\Omega_3)\varphi(\Omega_4)\big\rangle \\
& -\big\langle\varphi(\Omega_1)\varphi(\Omega_3)\big\rangle\big\langle\varphi(\Omega_2)\varphi(\Omega_4)\big\rangle - \big\langle\varphi(\Omega_1)\varphi(\Omega_4)\big\rangle\big\langle\varphi(\Omega_2)\varphi(\Omega_3)\big\rangle
\end{aligned}$$

Renormalization with λ fixed (λ=1.0)

2-point function（N=40 and 32, λ=1.0）

$$\langle \varphi(\Omega_1)\varphi(\Omega_2)\rangle = Z \langle \varphi_r(\Omega_1)\varphi_r(\Omega_2)\rangle$$



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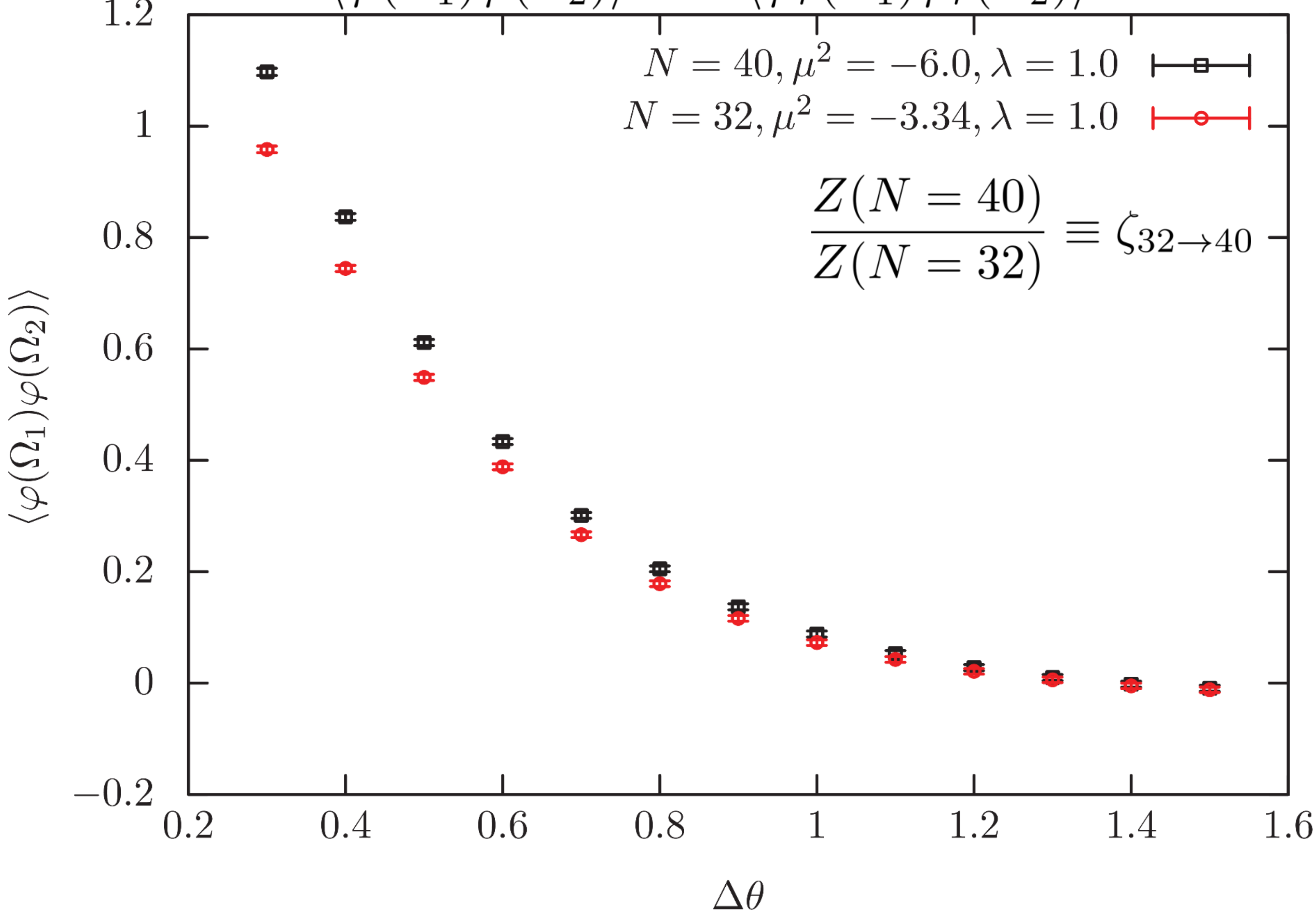

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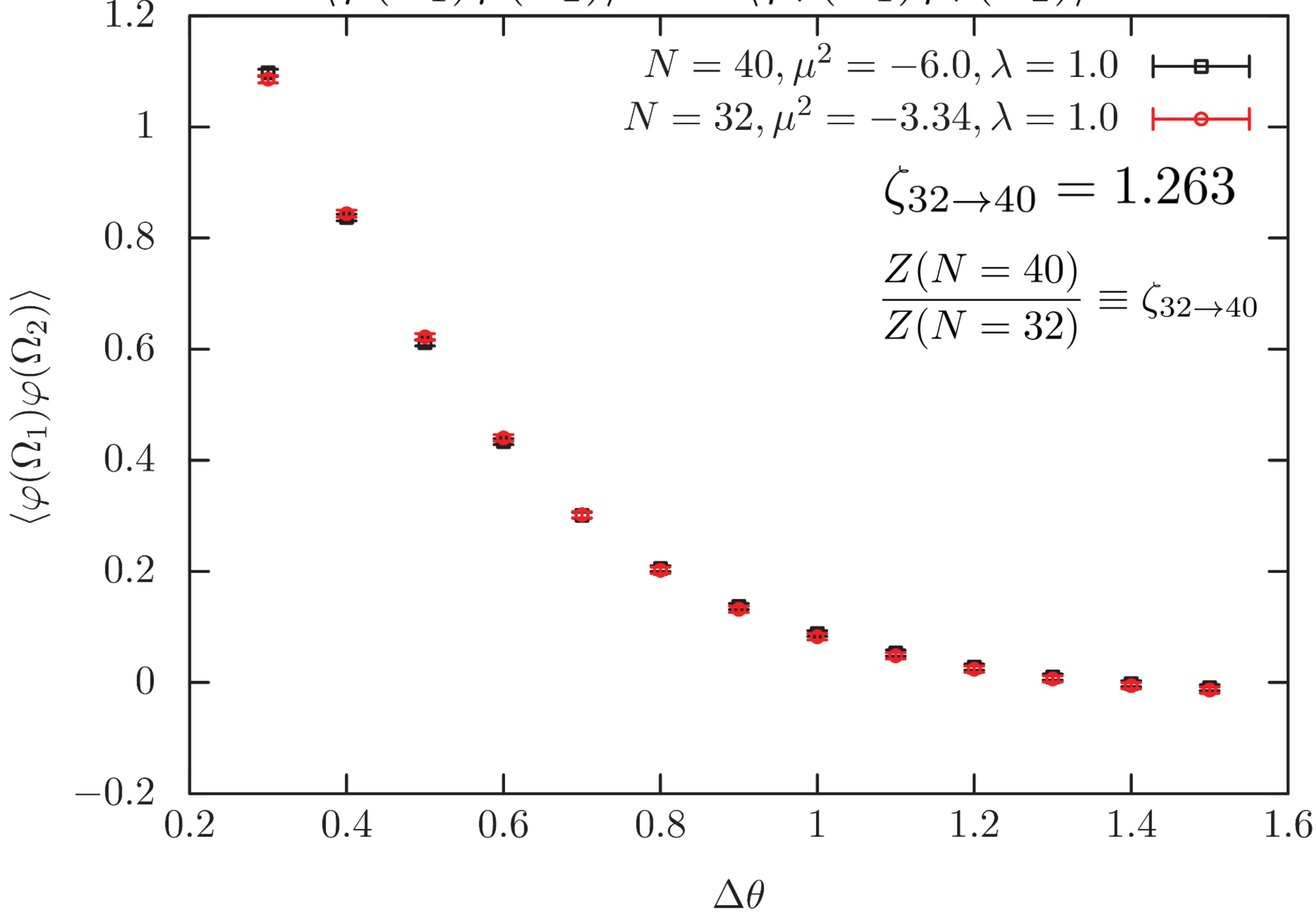

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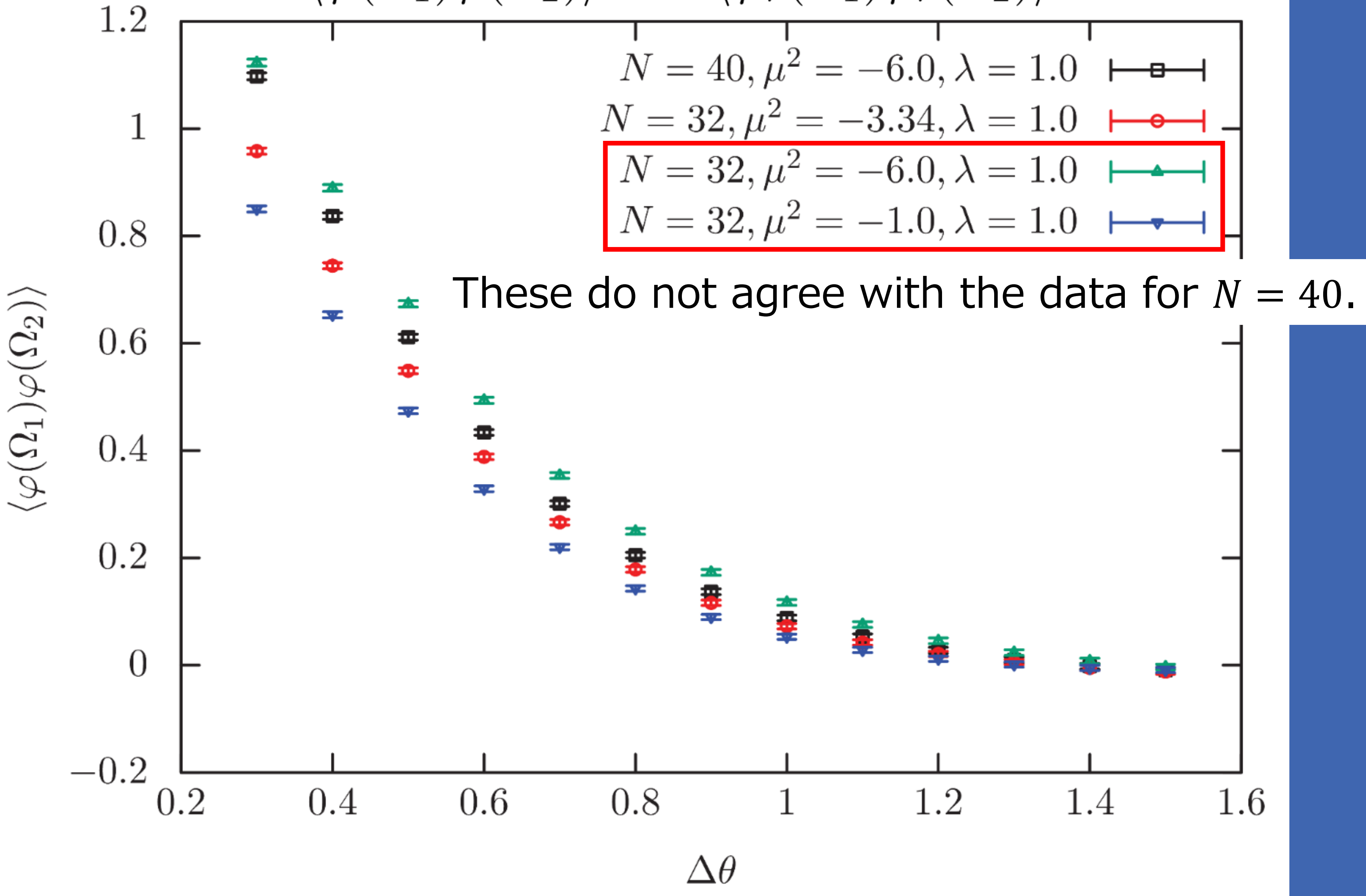
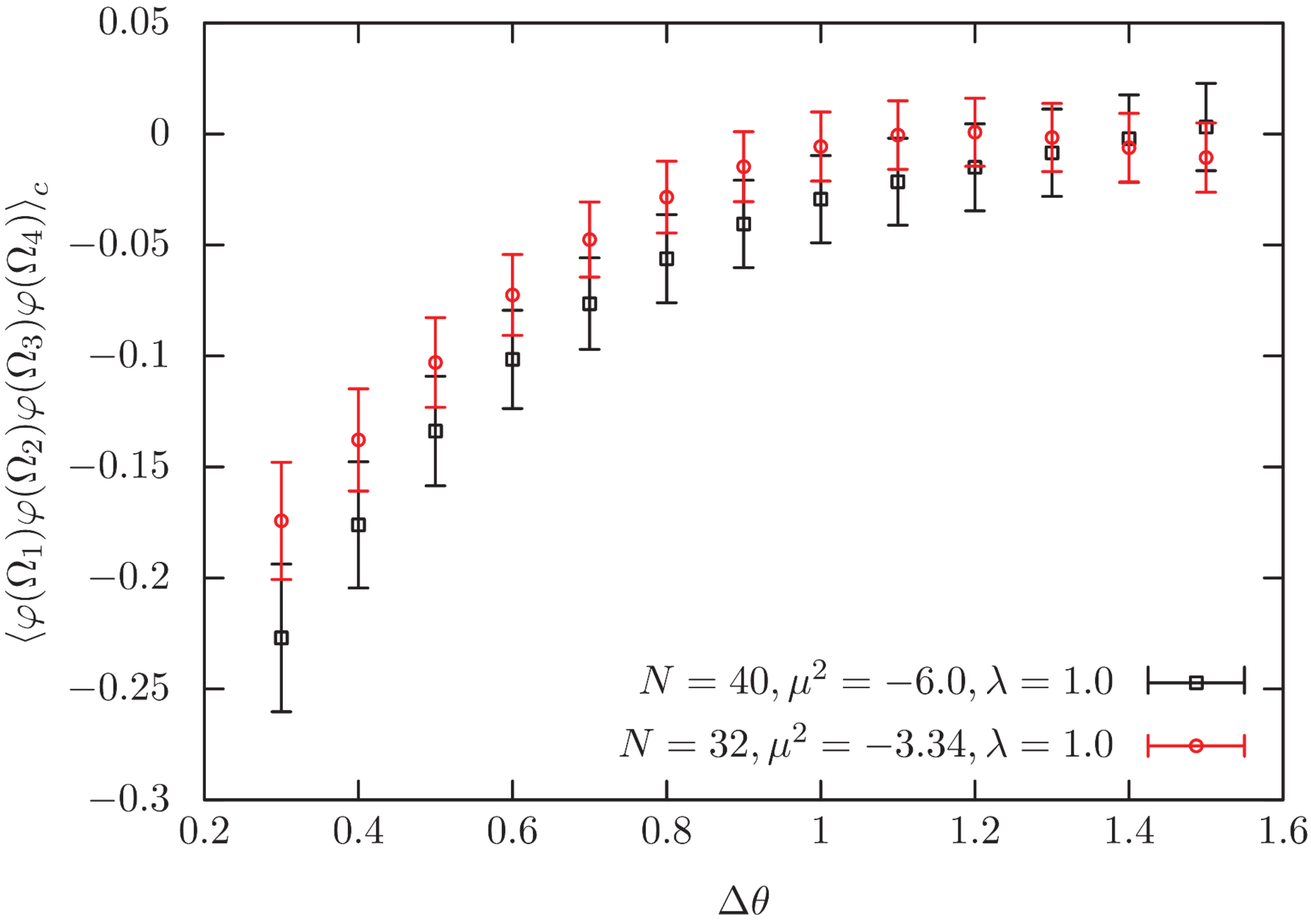


Connected 4-point function（N=40 and 32, λ=1.0）

$$\langle\varphi(\Omega_1)\varphi(\Omega_2)\varphi(\Omega_3)\varphi(\Omega_4)\rangle_c = Z^2 \langle\varphi_r(\Omega_1)\varphi_r(\Omega_2)\varphi_r(\Omega_3)\varphi_r(\Omega_4)\rangle_c$$



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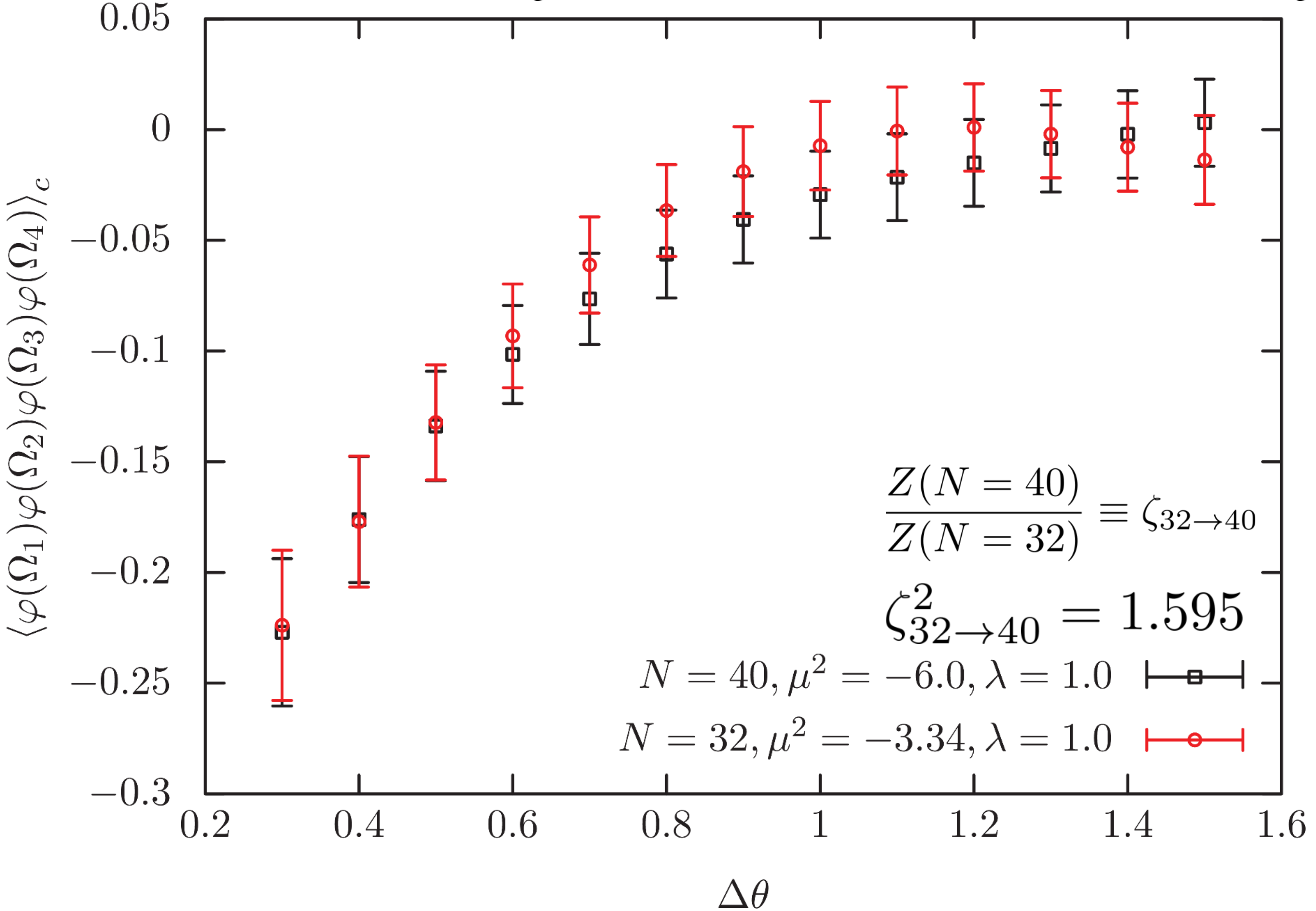
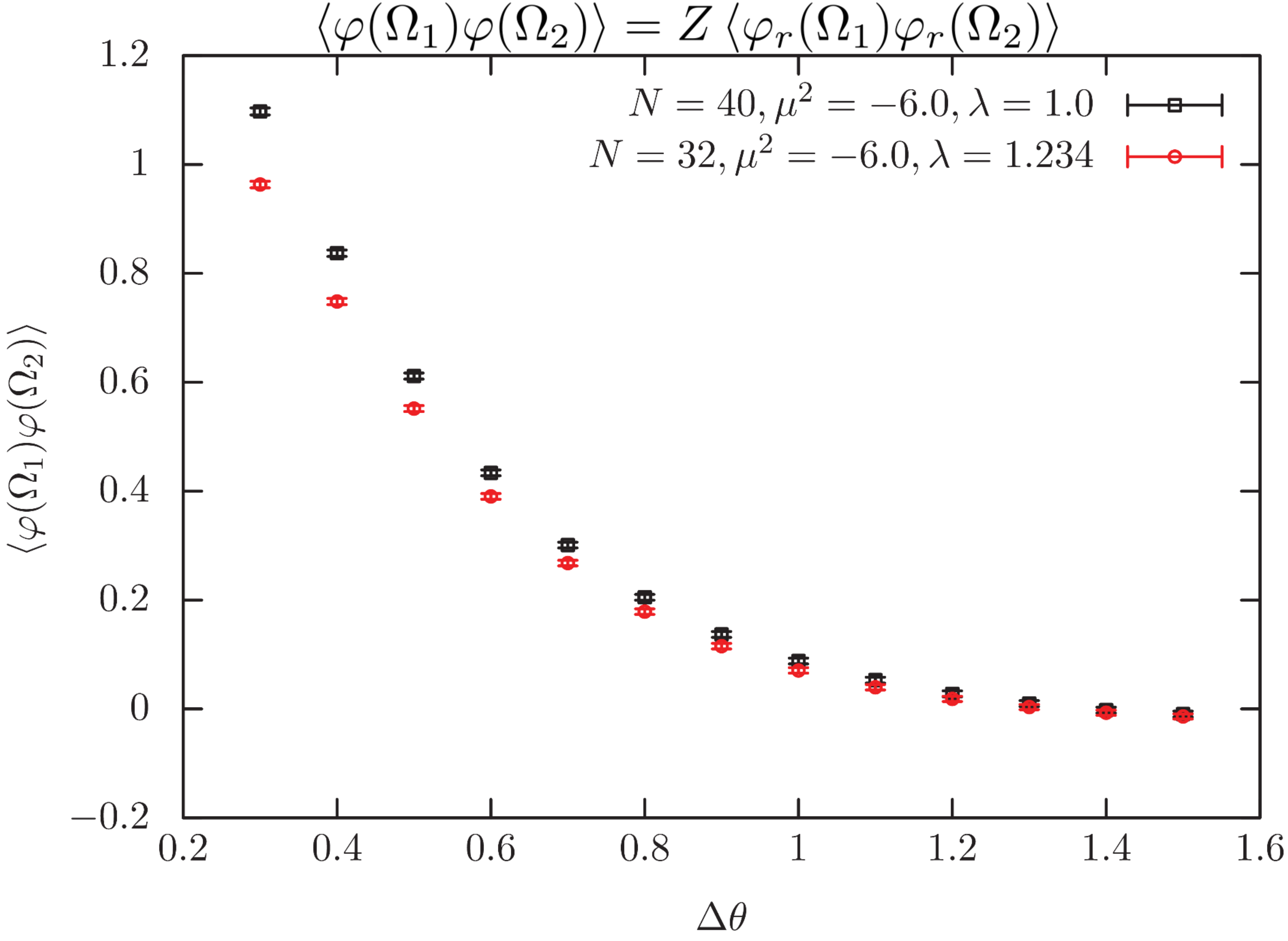
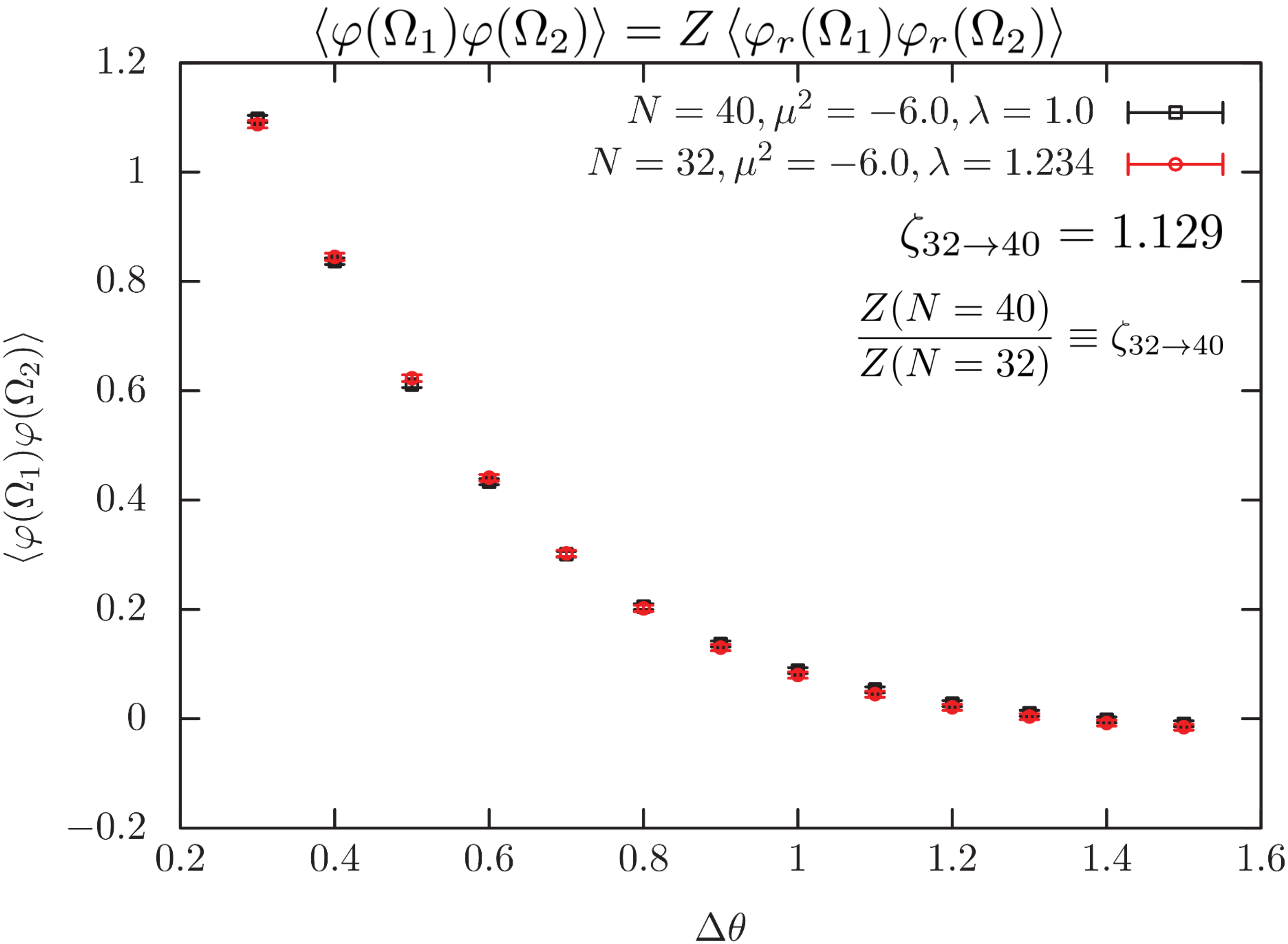


Renormalization with μ^2 fixed (μ^2=-6.0)

2-point function（N=40 and 32, μ^2=-6.0）

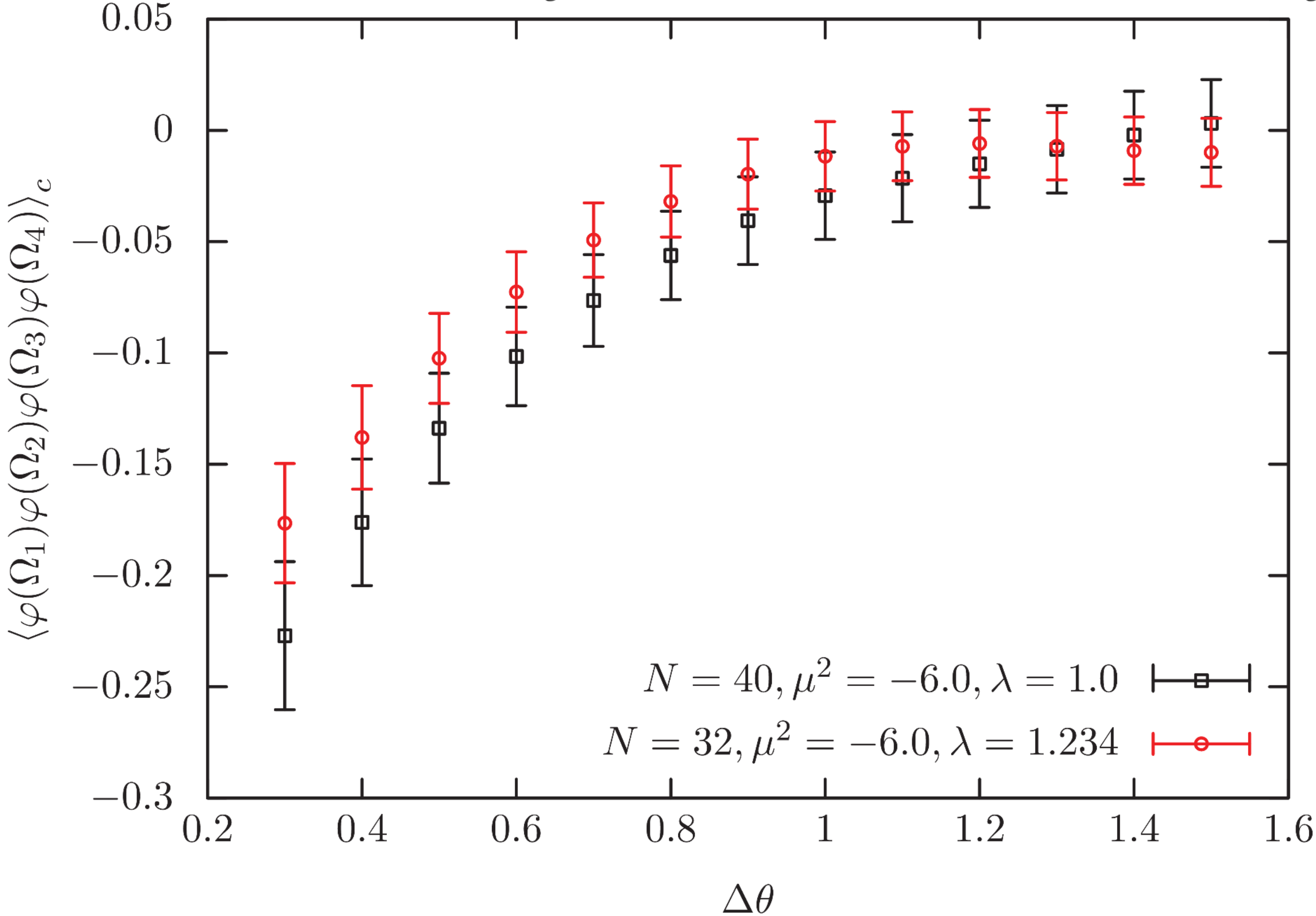


2-point function（N=40 and 32, μ^2=-6.0）



Connected 4-point function（N=40 and 32, μ^2=-6.0 ）

$$\langle \varphi(\Omega_1)\varphi(\Omega_2)\varphi(\Omega_3)\varphi(\Omega_4)\rangle_c = Z^2 \langle \varphi_r(\Omega_1)\varphi_r(\Omega_2)\varphi_r(\Omega_3)\varphi_r(\Omega_4)\rangle_c$$



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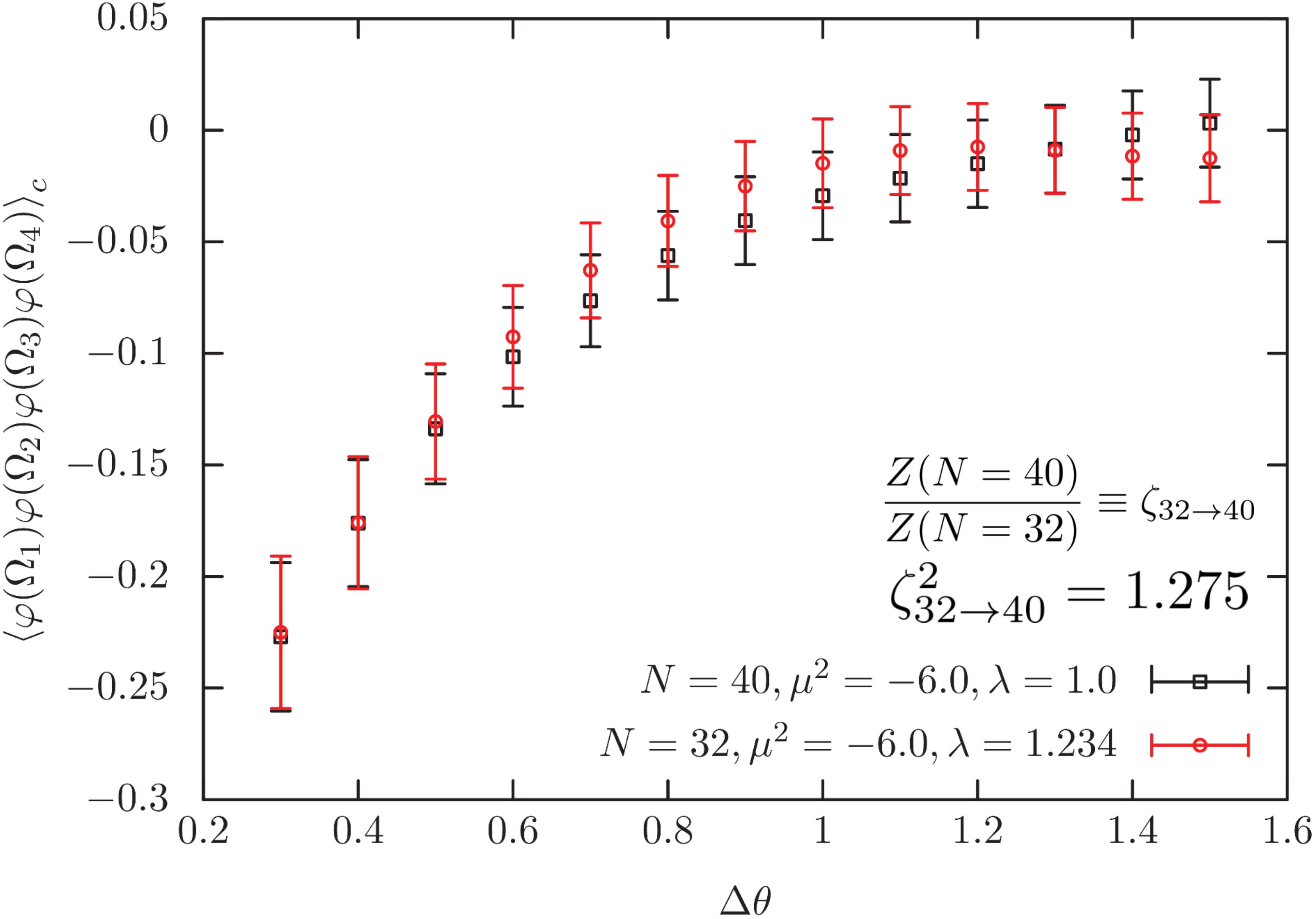


5. Conclusion and outlook

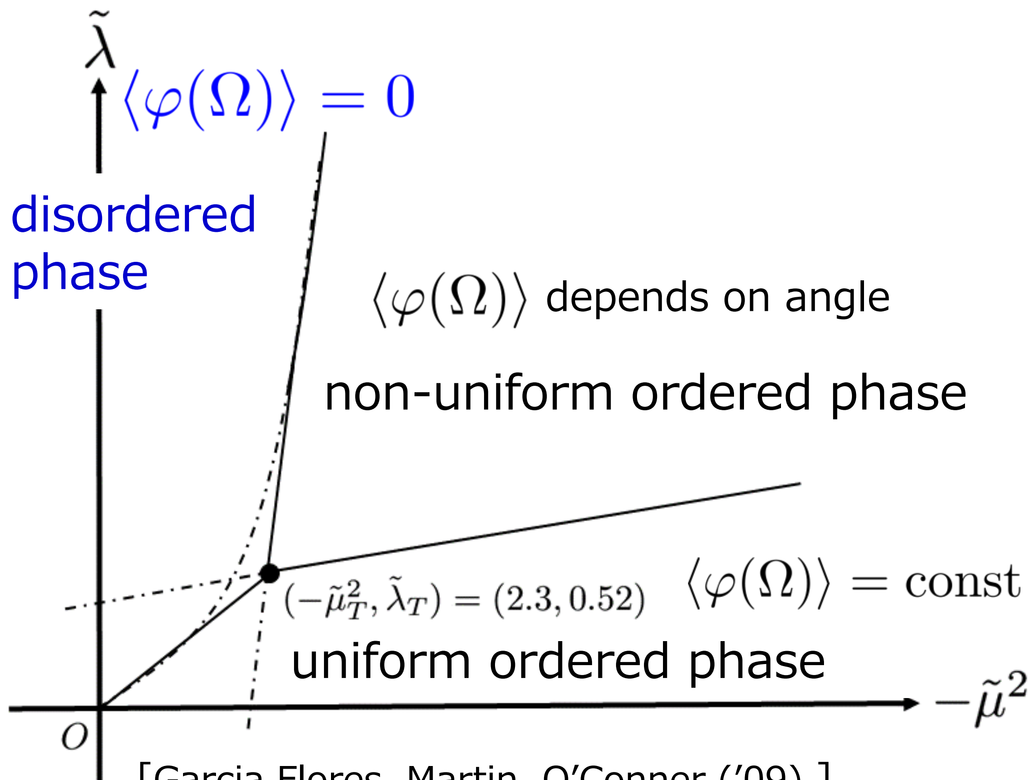
Conclusion

- We constructed the correlation functions in a scalar field theory on the fuzzy sphere by using the Berezin symbol.
  We calculated them by Monte Carlo simulation.

- We found that
  the non-trivial agreement of correlation functions at different N after tuning one parameter (μ^2 or λ) and performing the wave function renormalization,
  which strongly suggests that
  correlation functions are independent of the cutoff N,
  namely,
  the theory on the fuzzy sphere is renormalizable.

Outlook

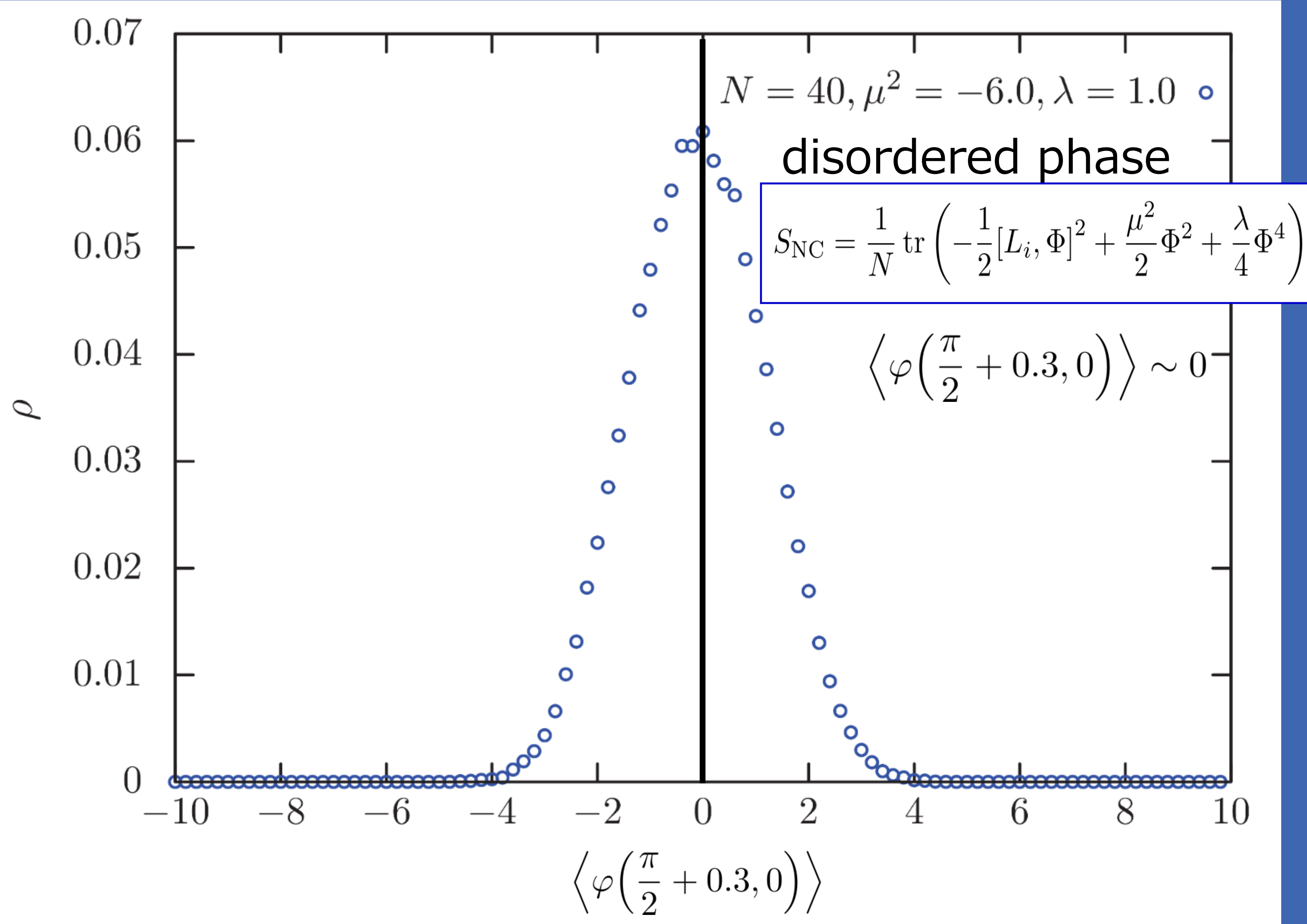
- Renormalization in different phases
  - uniform ordered phase : in progress
  - non-uniform ordered phase



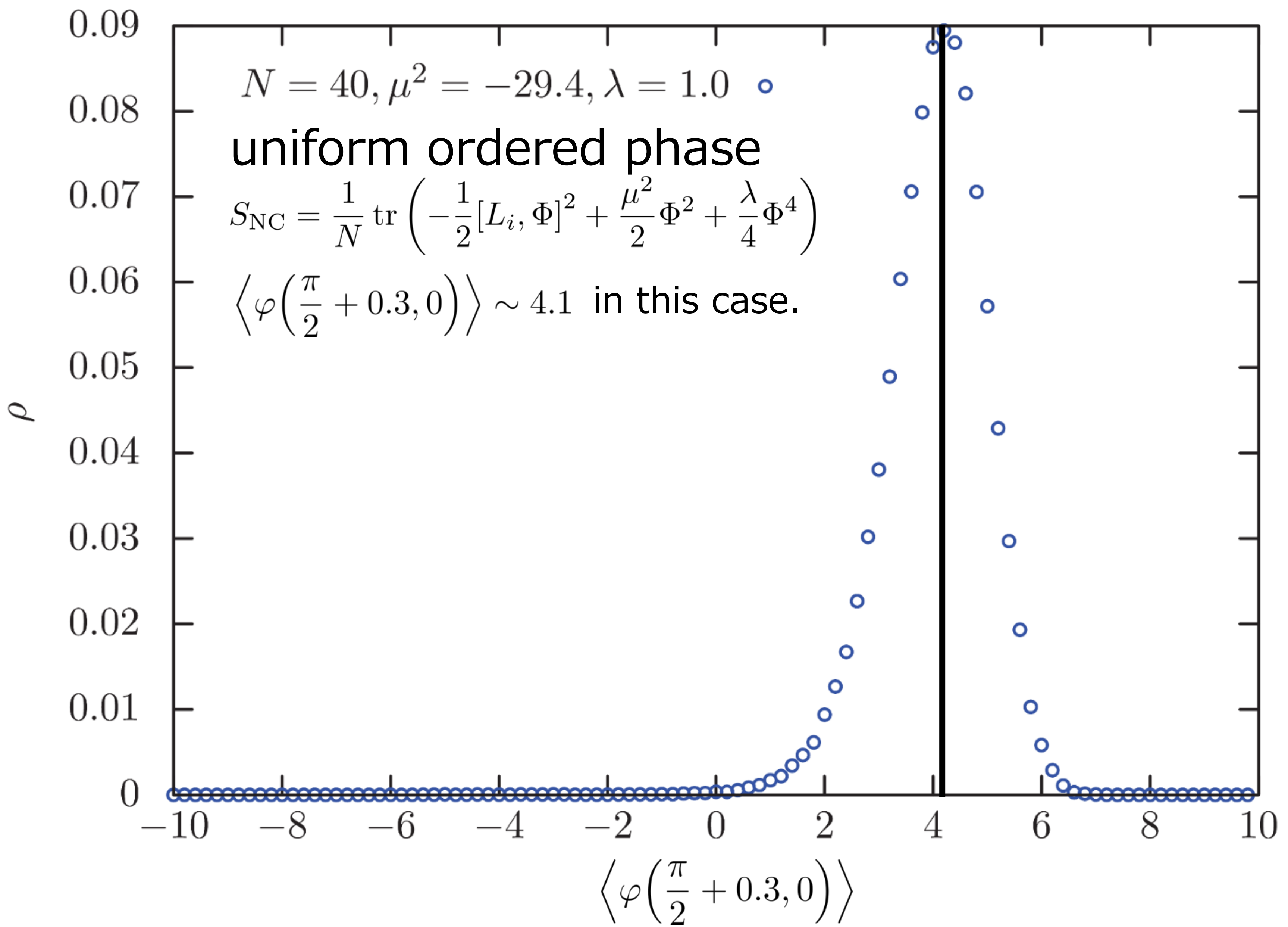
[Garcia Flores, Martin, O'Conner ('09)]

$$\tilde{\lambda} \equiv \lambda/N, \ \tilde{\mu}^2 \equiv \mu^2/N^{3/2}$$

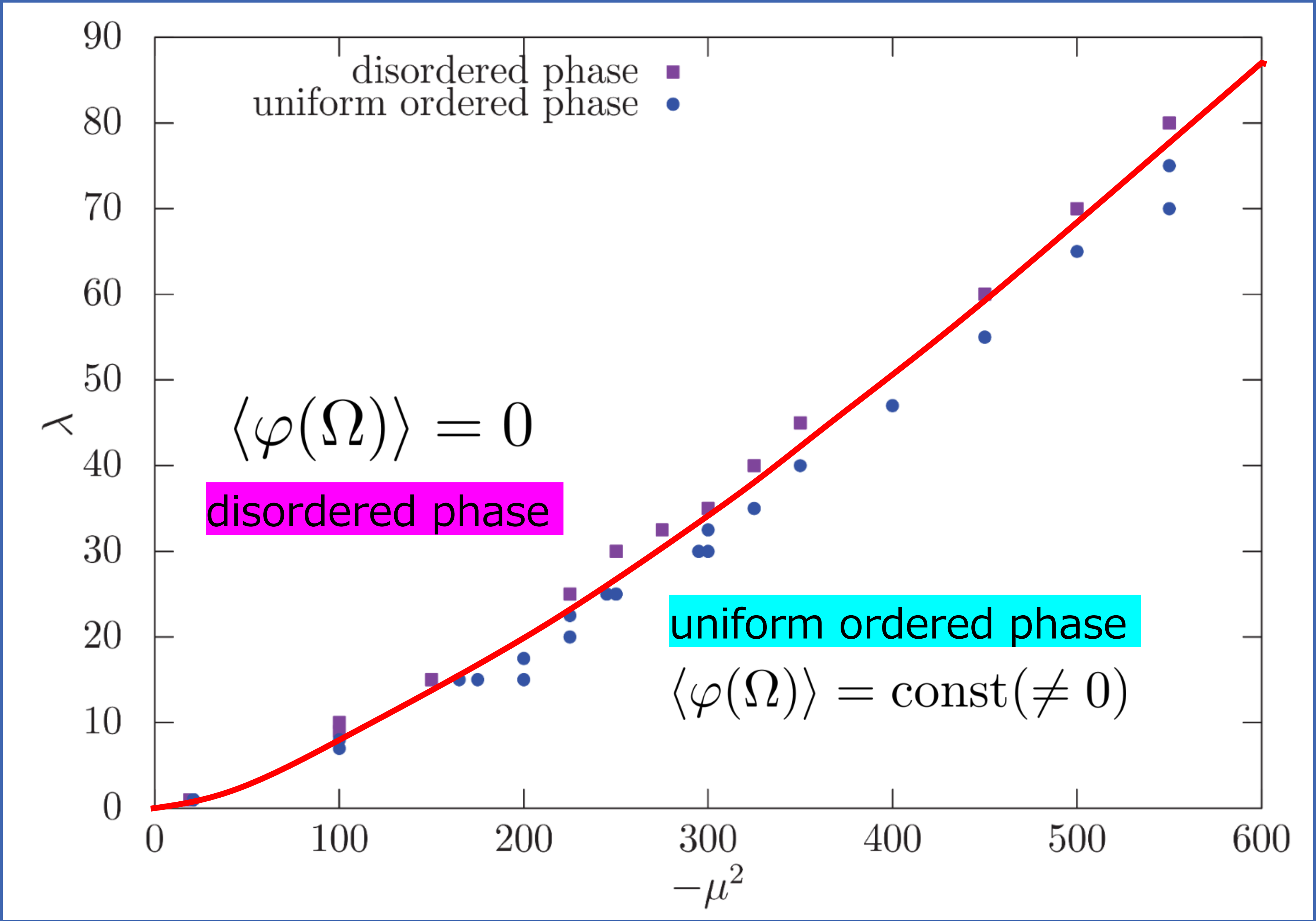
Distribution of 1-point function



Distribution of 1-point function (preliminary)



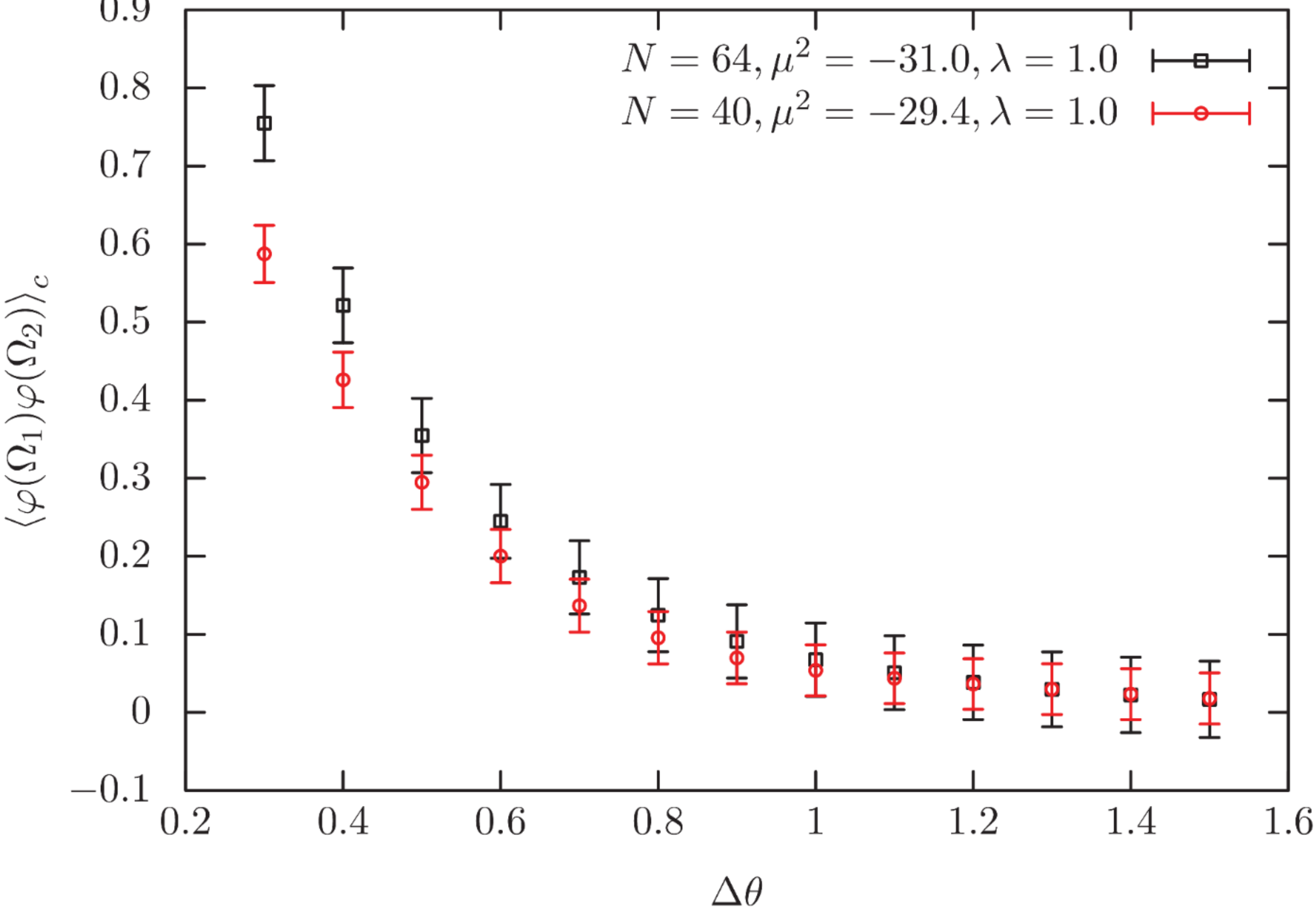
Phase diagram at $N = 40$ (preliminary)



Renormalization with λ fixed (λ=1.0) in the uniform ordered phase (preliminary)

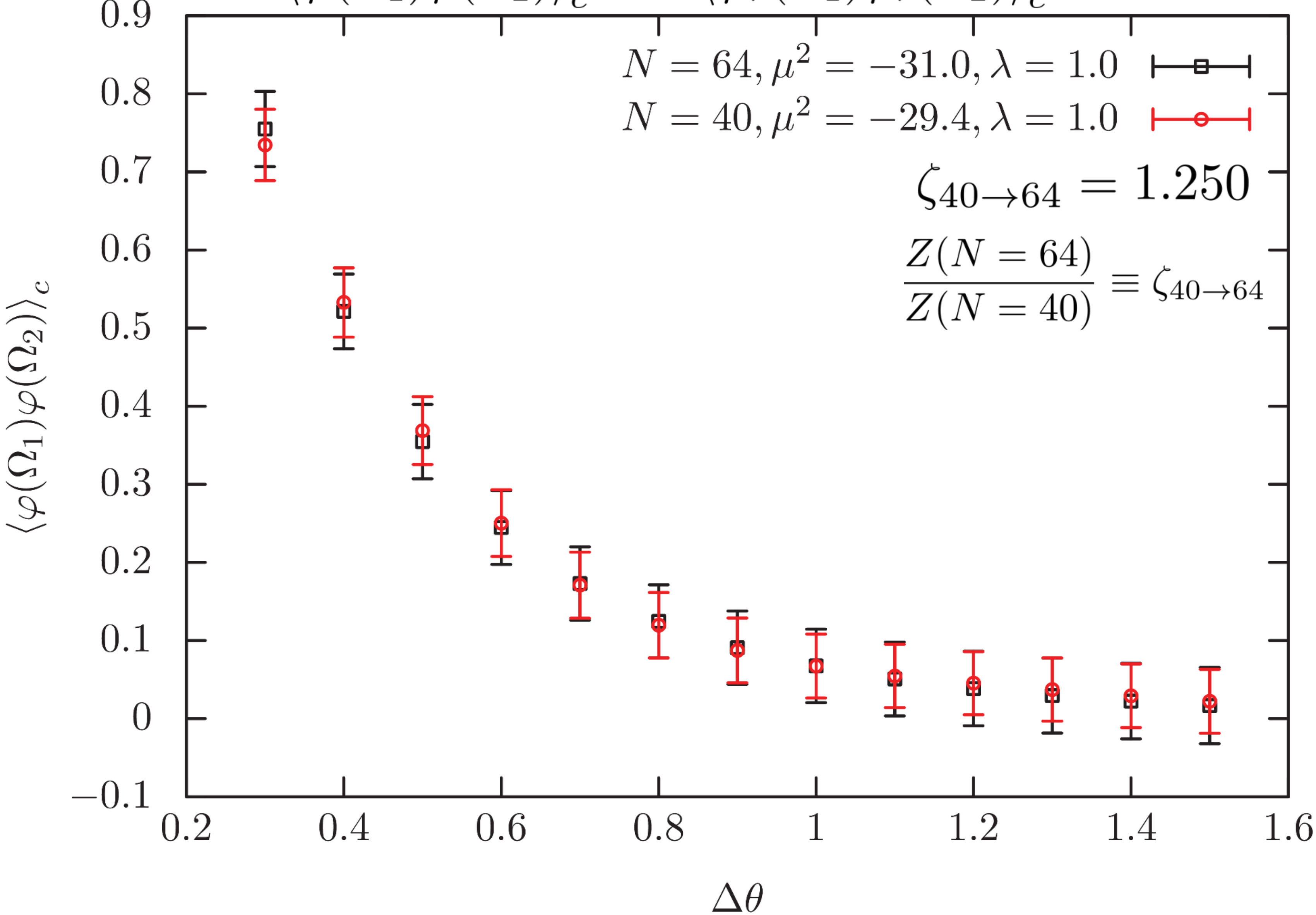
Connected 2-point function (N=40 and 64, λ=1.0) (preliminary)

$$\langle \varphi(\Omega_1)\varphi(\Omega_2)\rangle_c = Z\,\langle \varphi_r(\Omega_1)\varphi_r(\Omega_2)\rangle_c$$



Connected 2-point function（N=40 and 64, λ=1.0）(preliminary)

$$\langle \varphi(\Omega_1)\varphi(\Omega_2)\rangle_c = Z\,\langle \varphi_r(\Omega_1)\varphi_r(\Omega_2)\rangle_c$$



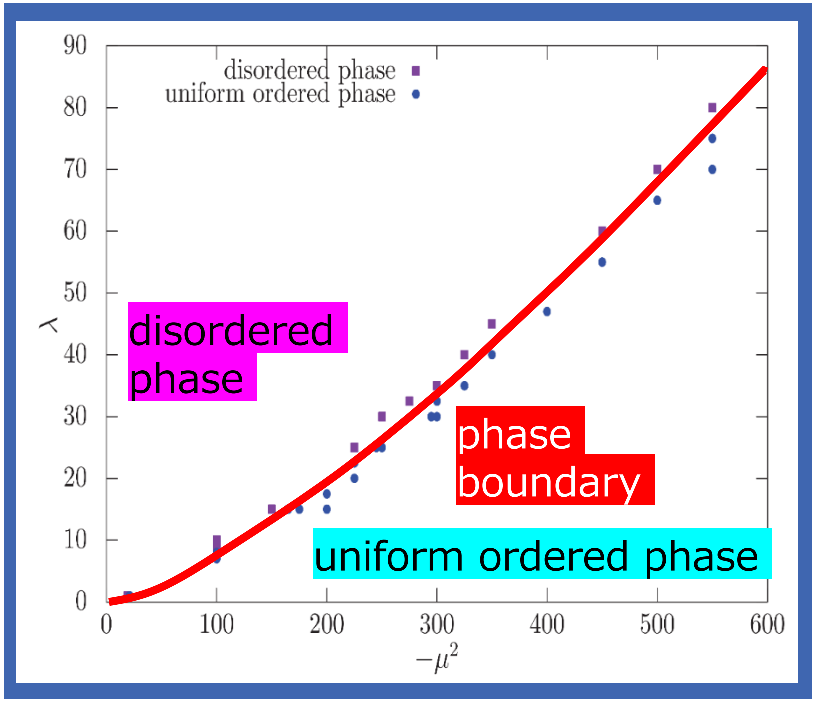
Outlook

- **Behavior on the phase boundary**

  In the ordinary theory,
  the theory behaves as CFT on the phase boundary,
  which implies 2-point function behaves as power of distance.

- **Effect of the noncommutativity**

  By examining the behavior of 2-point function on the boundary,
  we expect to understand the difference between
  the theory on the fuzzy sphere
  and that on the ordinary sphere,
  that is, we expect to understand
  the effect of the noncommutativity.



Outlook

- Renormalization in different phases
- Behavior on the phase boundary
- Effect of the noncommutativity
- Renormalization in different limits

  Ex) fuzzy sphere limit: $N, R \to \infty$

  [Kawamoto-Kuroki ('15)]
- Generalization to higher dimensions

  $R \times \text{fuzzy } S^2$ $\text{fuzzy } CP^2$ $\text{fuzzy } S^2 \times \text{fuzzy } S^2$ etc.
- Quantum entanglement in noncommutative space

  Karczmarek-Sabella-Garnier ('13)
  Sabella-Garnier ('14)
  Suzuki-A.T. ('17)

Backup

Fuzzy spherical harmonics

- As a basis of operators on the Hilbert space, we define fuzzy spherical harmonics.

$$\hat{Y}_{lm} = \sqrt{2j+1} \sum_{r,r'} (-1)^{-j+r'} C^{lm}_{jr\ j-r'} |jr\rangle\langle jr'| \longleftrightarrow Y(\Omega)$$

: spherical harmonics

This corresponds to composition of two angular momenta j.

$C^{lm}_{jr\ j-r'}$ is a Clebsch-Gordan coefficient and $0 \leq l \leq 2j,\ -l \leq m \leq l$.

Arbitrary operator $\hat{f}$ is expanded in terms of $\hat{Y}_{lm}$.

$$\hat{f} = \sum_{l=0}^{2j} \sum_{m=-l}^{l} f_{lm} \hat{Y}_{lm}$$

The maximum of the angular momentum l corresponds to the UV cutoff.

Correspondence between $\hat{Y}_{lm}$ and Y_{lm}

Fuzzy spherical harmonics $\hat{Y}_{lm}$

- Commutation relation

$$L_\pm \equiv L_1 \pm i L_2$$

$$\left[L_\pm, \hat{Y}_{lm}\right] = \sqrt{(l \mp m)(l \pm m + 1)}\hat{Y}_{lm\pm1},$$

$$\left[L_3, \hat{Y}_{lm}\right] = m\hat{Y}_{lm}$$

- Hermitian conjugate

$$\hat{Y}_{lm}^\dagger = (-1)^m \hat{Y}_{l-m}$$

- Orthonormal relation

$$\frac{1}{N}\mathrm{tr}\left(\hat{Y}_{lm}^\dagger \hat{Y}_{l'm'}\right) = \delta_{ll'}\delta_{mm'}$$

- Product

$$\hat{Y}_{lm}^\dagger \hat{Y}_{l'm'}$$

$j \to \infty$

Spherical harmonics Y_{lm}

- Acting angular momentum

$$\mathcal{L}_\pm \equiv \mathcal{L}_1 \pm i\mathcal{L}_2 = e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right), \quad \mathcal{L}_3 = -i\frac{\partial}{\partial\varphi}$$

$$\mathcal{L}_\pm Y_{lm}(\Omega) = \sqrt{(l \mp m)(l \pm m + 1)}Y_{lm\pm1}(\Omega),$$

$$\mathcal{L}_3 Y_{lm}(\Omega) = m Y_{lm}(\Omega)$$

- Complex conjugate

$$Y_{lm}^*(\Omega) = (-1)^m Y_{l-m}(\Omega)$$

- Orthonormal relation

$$\int \frac{d\Omega}{4\pi} Y_{lm}^*(\Omega) Y_{l'm'}(\Omega) = \delta_{ll'}\delta_{mm'}$$

- Product

$$Y_{lm}^*(\Omega) Y_{l'm'}(\Omega)$$

UV/IR anomaly ①

- UV/IR anomaly is a quantum effect which is caused by the noncommutativity between quantization and taking commutative limit.

Consider 1-loop correction.

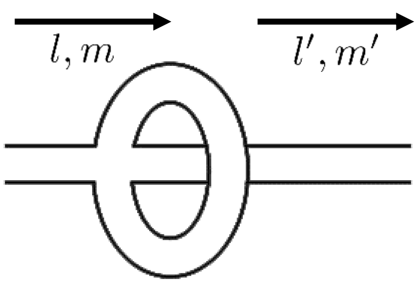
Planar diagram

$$\left(\Gamma^{(2)}_{\text{planar}}\right)^{ll'}_{mm'} = 2\lambda\delta_{ll'}\delta_{m-m'}(-1)^m I^P$$

Log div. in $j \to \infty$

$$I^P \equiv \sum_{K=0}^{2j} \frac{2K+1}{K(K+1)+\mu^2} \overset{j\to\infty}{\sim} \log j + O(1)$$

Non-Planar diagram



$$\left(\Gamma^{(2)}_{\text{nonplanar}}\right)^{ll'}_{mm'} = \lambda\delta_{ll'}\delta_{m-m'}(-1)^m I^{NP}$$

$6j$ symbol

$$I^{NP} \equiv \sum_{K=0}^{2j} (-1)^{l+K+2j} \frac{(2K+1)(2j+1)}{K(K+1)+\mu^2} \begin{Bmatrix} j & j & l \\ j & j & K \end{Bmatrix}$$

l, m and l', m' are momenta of external lines.

UV/IR anomaly ②

◆Calculation of $I^{NP} - I^{P}$

$$I^{NP} - I^{P} \approx \int_{-1}^{1} dt \frac{P_l(t) - 1}{1 - t} + O\left(\frac{1}{j}\right)$$

Legendre polynomial

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l$$

‖

- For large l, log l
- For small l, finite → non IR div.

Therefore, this term causes a nonlocal difference in the 1-loop effective action.

⇒Renormalization is nontrivial.

(UV/IR anomaly ~ a finite analog of UV/IR mixing)

UV/IR anomaly ③

- Using approximate formula of $6j$ symbol

$$\begin{Bmatrix} j & j & l \\ j & j & K \end{Bmatrix} \approx \frac{(-1)^{l+2j+K}}{2j} P_l\left(1-\frac{K^2}{2j^2}\right),\ l \ll j,\ 0 \le K \le 2j$$

$$\begin{aligned} I^{NP} - I^{P} &= \sum_{K=0}^{2j} \frac{2K+1}{K(K+1)+\mu^2}\left[(-1)^{l+K+2j}(2j+1)\begin{Bmatrix} j & j & l \\ j & j & K \end{Bmatrix} - 1\right] \\ &\approx \sum_{K=0}^{2j} \frac{2K+1}{K(K+1)+\mu^2}\left[P_l\left(1-\frac{K^2}{2j^2}\right) - 1\right] \\ &\approx \int_0^2 du \frac{2u+\frac{1}{j}}{u^2+\frac{u}{j}+\left(\frac{\mu}{j}\right)^2}\left[P_l\left(1-\frac{u^2}{2}\right)-1\right] \\ &= \int_{-1}^{1} dt \frac{P_l(t)-1}{1-t} + O\left(\frac{1}{j}\right) \end{aligned}$$

Quadratic terms of action

For the Berezin symbol $\langle\Omega|\Phi|\Omega\rangle = \varphi(\Omega)$,
we obtain the following relations.

$$\langle\Omega|[L_i,\Phi]|\Omega\rangle = \mathcal{L}_i\varphi(\Omega)\ ,\ \frac{1}{N}\mathrm{tr}\longleftrightarrow\int\frac{d\Omega}{4\pi}$$ and

$$\langle\Omega|\Phi_1|\Omega\rangle\star\langle\Omega|\Phi_2|\Omega\rangle\xrightarrow{N\to\infty}\langle\Omega|\Phi_1|\Omega\rangle\ \langle\Omega|\Phi_2|\Omega\rangle$$ ← ordinary product

If we set $\varphi(\Omega)=\phi(\Omega)$, the quadratic terms of

$$S_{\mathrm{NC}} = \frac{1}{N}\mathrm{tr}\left(-\frac{1}{2}[L_i,\Phi]^2+\frac{\mu^2}{2}\Phi^2+\frac{\lambda}{4}\Phi^4\right)$$ and $$S_{\mathrm{C}} = \int\frac{d\Omega}{4\pi}\left(-\frac{1}{2}[\mathcal{L}_i\phi(\Omega)]^2+\frac{\mu^2}{2}\phi(\Omega)^2+\frac{\lambda}{4}\phi(\Omega)^4\right)$$

agree with each other. The quartic terms agree at tree level,
but including the quantum correction, they do not agree.

◆ $\mu^2 < 0$

$\mu_{\text{bare}}^2 = \mu_{\text{phys}}^2 - (\text{quantum correction})$ and our μ^2 are μ_{bare}^2.

Classically, $\mu_c^2 = 0$.

However, at quantum theory,

$\mu_c^2 < 0$ due to the quantum correction.