# Criticality at absolute zero from Ising model on 2d dynamical triangulations

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Sep. 23rd, 2017

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@ Discrete Approaches to the Dynamics of Fields and Space-Time, APCTP

**Ising model on 2d dynamical triangulations (DT)** [Kazakov, 1986]

(1) Continuous phase transition at non-zero temperature $T_c$.

(2) Physics around the critical point is described by 2d gravity coupled to Majorana fermion

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Our work:

**Reconsider criticality of Ising model on 2d DT**. [YS, Tanaka 2017]

(1) Introduce a "loop-counting" parameter $\theta$.

(2) Tuning $\theta$, one can reduce $T_c(\theta)$ to absolute zero.

(3) Continuum theory around absolute zero is NOT 2d gravity coupled to Majorana fermions

Ising model on honeycomb lattice (T):

$$Z_T(\beta) = \sum_{\sigma(T)} \prod_{<i,j>} e^{\beta \sigma_i \sigma_j}$$

Ising spin:

$$\sigma_i = \pm 1$$

Exactly solved in the thermodynamic limit:

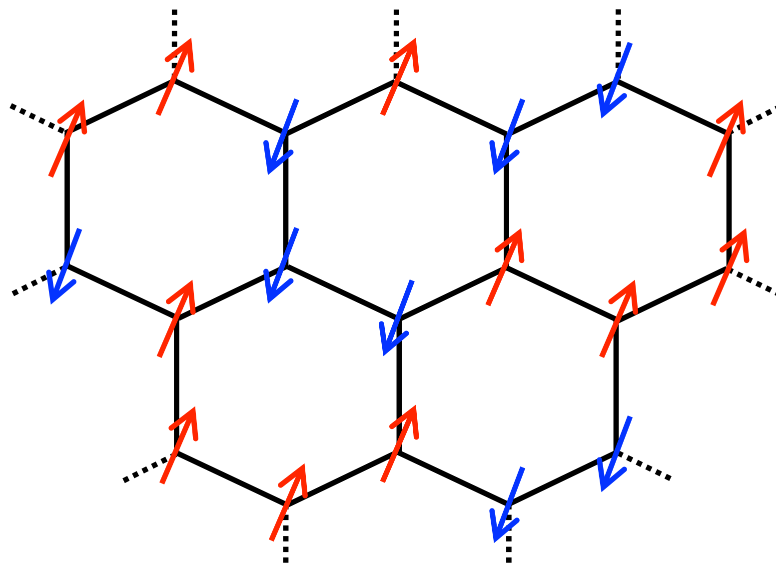
[Weiner, 1950, Houtappel, 1950]

(1) $2^{nd}$ order phase transition at

$$\beta = \beta_c \neq 0$$

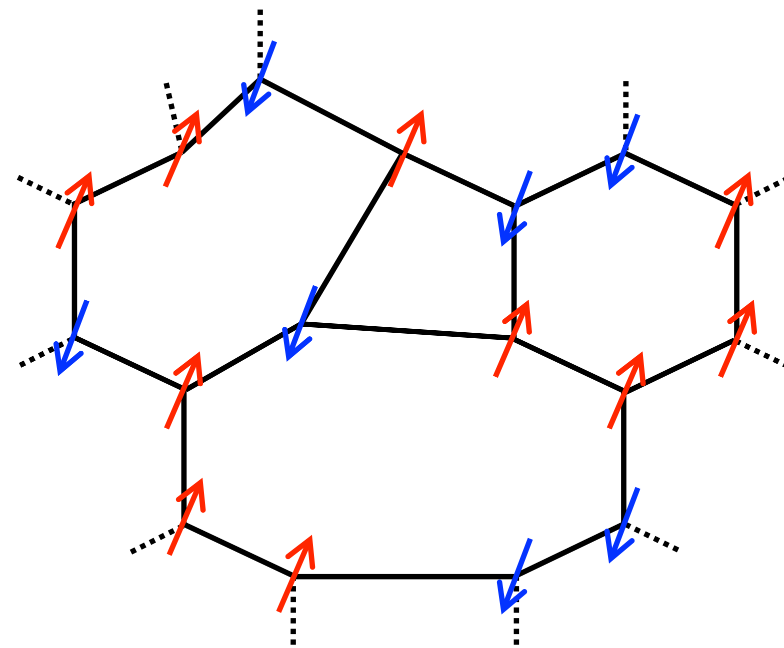
(2) Physics around $\beta_c$ described by

2d Majorana fermion



Ising model on planar lattice (w/ coordination number = 3) (T'):

$$Z_{T'}(\beta) = \sum_{\sigma(T')} \prod_{<i,j>} e^{\beta \sigma_i \sigma_j}$$



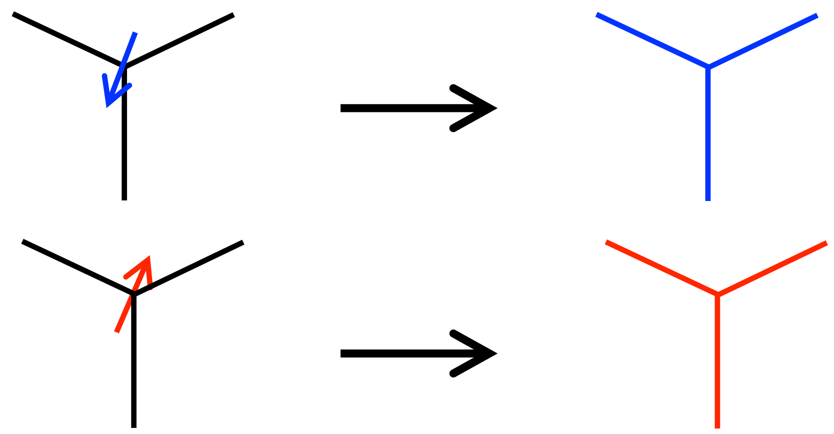
Summing all planar lattices (w/ coordination number = 3),

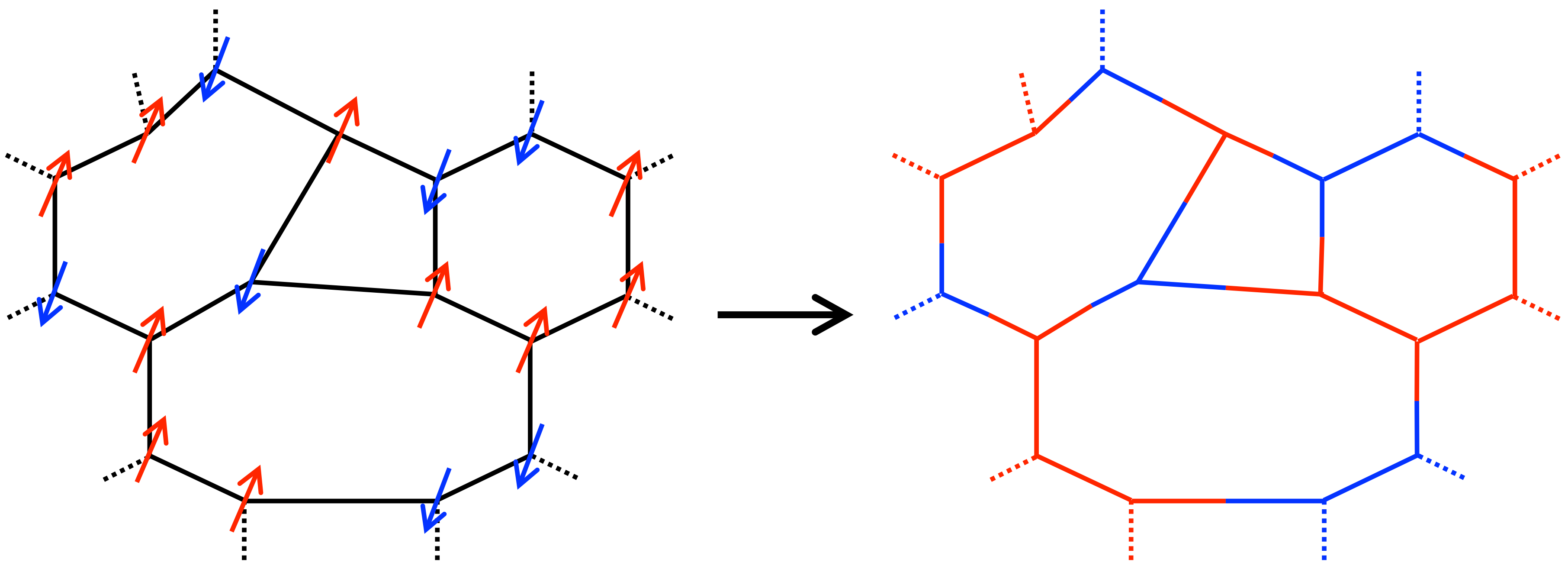
$$\{T, T', T'', T''', \cdots\}$$

one can construct a solvable model (Ising model on 2d DT).

[Kazakov, 1986]

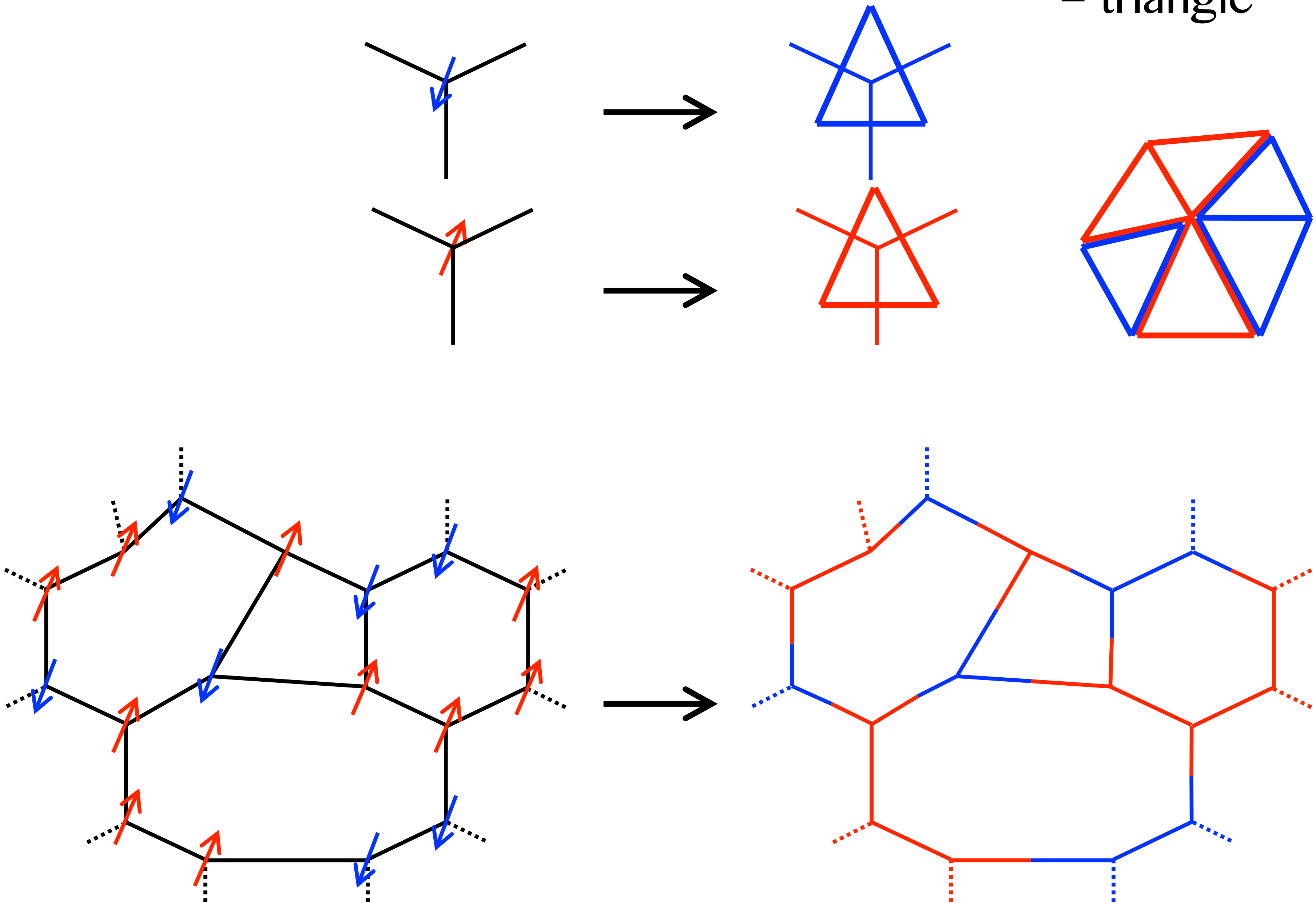
Change notation:





Change notation:

dual graph = triangle



# Ising model on 2d dynamical triangulations

Ising model on 2d dynamical triangulations (DT): [Kazakov, 1986]

$$Z(\beta, \tilde{g}) = \sum_T \frac{1}{|\mathrm{Aut}(T)|} \tilde{g}^{v(T)} Z_T(\beta)$$

$\sum_T$: Sum over all connected planar lattices (w/ coordination number = 3).

$|\mathrm{Aut}(T)|$: Order of automorphism group of T.

$Z_T(\beta)$: Partition fun of Ising model on T:

$$Z_T(\beta) = \sum_{\sigma(T)} \prod_{<i,j>} e^{\beta \sigma_i \sigma_j}$$

$\tilde{g}$ : weight for vertices

$v(T)$ : #(vertices) in T

Ising model on 2d dynamical triangulations (DT): [Kazakov, 1986]

Definition via hermitian NxN two-matrix model:

$$Z_N(c,g) = \int D\psi_+ D\psi_- \; e^{-N\mathrm{tr}U(\psi_+,\psi_-)}$$

where

$$U(\psi_+,\psi_-) = \frac{1}{2}\left(\psi_+^2 + \psi_-^2 - 2c\psi_+\psi_-\right) - \frac{g}{3}\left(\psi_+^3 + \psi_-^3\right)$$

Propagators:

$$\langle\psi_+\psi_+\rangle_0 \sim \qquad\qquad \langle\psi_-\psi_+\rangle_0 \sim$$

$$\langle\psi_-\psi_-\rangle_0 \sim \qquad\qquad \langle\psi_+\psi_-\rangle_0 \sim$$

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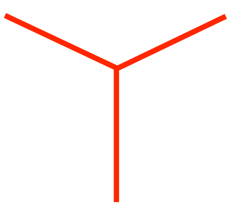
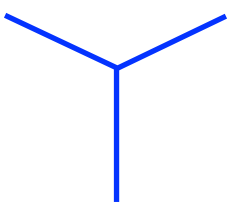
Vertices:

$$\mathrm{tr}(\psi_+^3) \sim$$



$$\mathrm{tr}(\psi_-^3) \sim$$



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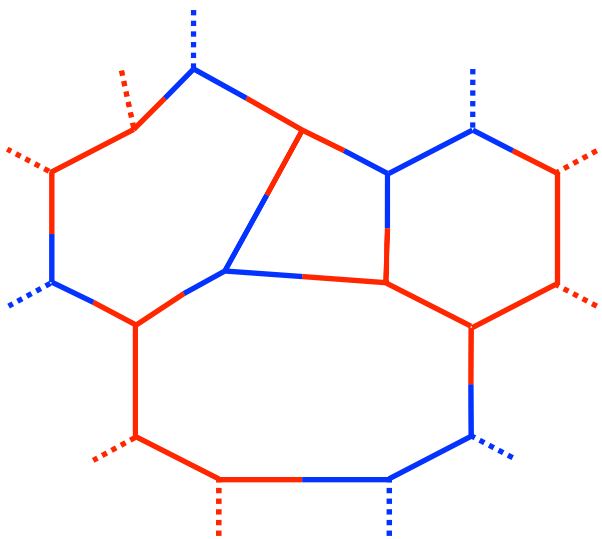

$\sim N^{2-2h}$

h=0 for planar
h>0 for non-planar

In the large-N limit, planar graphs become dominant.

Ising model on 2d dynamical triangulations (DT): [Kazakov, 1986]

$$Z(\beta,\tilde{g}) = \sum_T \frac{1}{|\mathrm{Aut}(T)|}\tilde{g}^{v(T)} \sum_{\sigma(T)} \prod_{<i,j>} e^{\beta\sigma_i\sigma_j}$$ DT

$$= \lim_{N\to\infty} \frac{1}{N^2} \log\left(\frac{Z_N(c,g)}{Z_N(c,0)}\right)$$ Matrix model

where

$$g = (2e^{-\beta}\sinh(2\beta))^3\tilde{g} \qquad c = e^{-2\beta}$$

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Exactly solved in the large-N limit:

(1) Continuous phase transition at non-zero temperature.

(2) Physics around the critical point is described by

2d gravity coupled to Majorana fermions

Ising model on 2d dynamical triangulations (DT): [Kazakov, 1986]

$$Z(\beta,\tilde{g}) = \sum_T \frac{1}{|\mathrm{Aut}(T)|}\tilde{g}^{v(T)} \sum_{\sigma(T)} \prod_{<i,j>} e^{\beta\sigma_i\sigma_j}$$

DT

$$= \lim_{N\to\infty}\frac{1}{N^2}\log\left(\frac{Z_N(c,g)}{Z_N(c,0)}\right)$$

Matrix model

where

$$g = (2e^{-\beta}\sinh(2\beta))^3\tilde{g} \qquad c = e^{-2\beta}$$

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The same critical behavior is obtained by generic potential,

$$U(\psi_+,\psi_-) = \frac{1}{2}(\psi_+^2+\psi_-^2-2c\psi_+\psi_-) - \sum_{i=1}^{m}\frac{g}{i}t_i\left(\psi_+^i+\psi_-^i\right)$$

$$(t_i \geq 0)$$

Universality

# Model

Deform the potential,

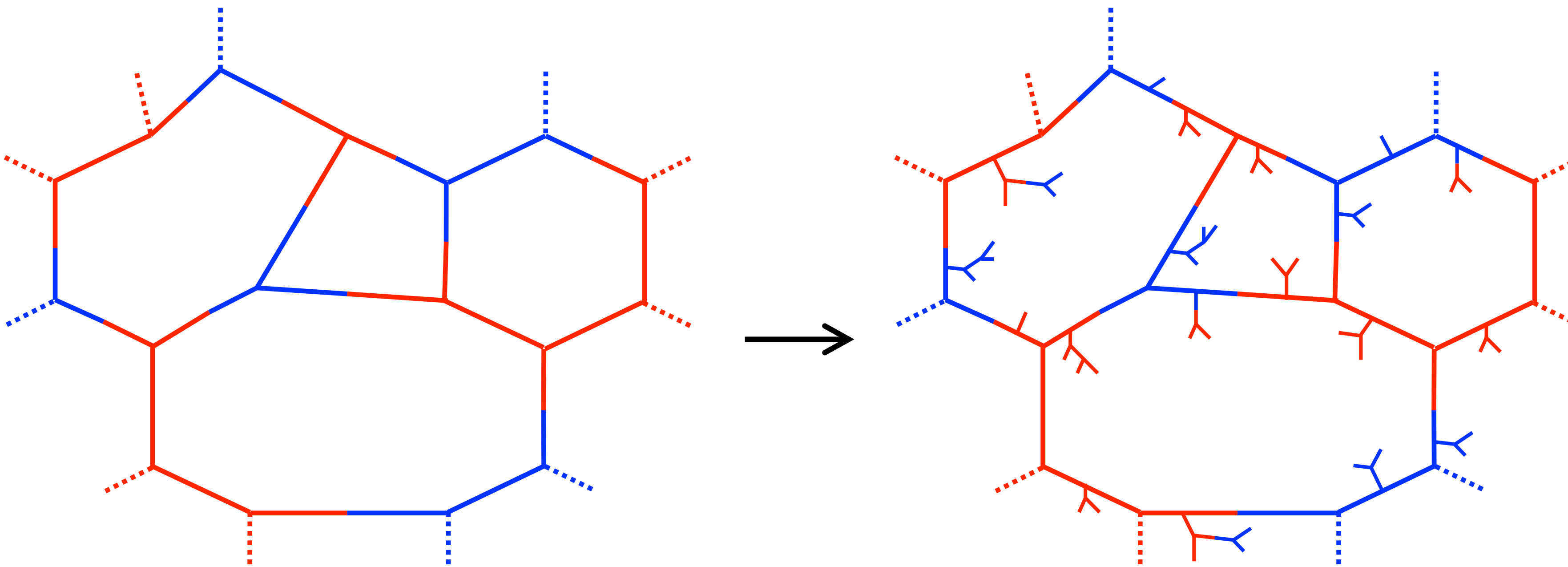
$$U(\psi_+,\psi_-) = \frac{1}{2}\left(\psi_+^2 + \psi_-^2 - 2c\psi_+\psi_-\right) - \frac{g}{3}\left(\psi_+^3 + \psi_-^3\right)$$ [Kazakov, 1986]

$$\downarrow$$

$$U^{(0)}(\varphi_+,\varphi_-) = \frac{1}{\theta}\left(\frac{1}{2}\left(\varphi_+^2 + \varphi_-^2 - 2c\varphi_+\varphi_-\right) - g(\varphi_- + \varphi_-) - \frac{g}{3}\left(\varphi_+^3 + \varphi_-^3\right)\right)$$

[YS, Tanaka 2017]

Graphs can terminate to form trees.



$\theta$ was introduced in one-matrix model [Ambjorn, et. al, 2008]

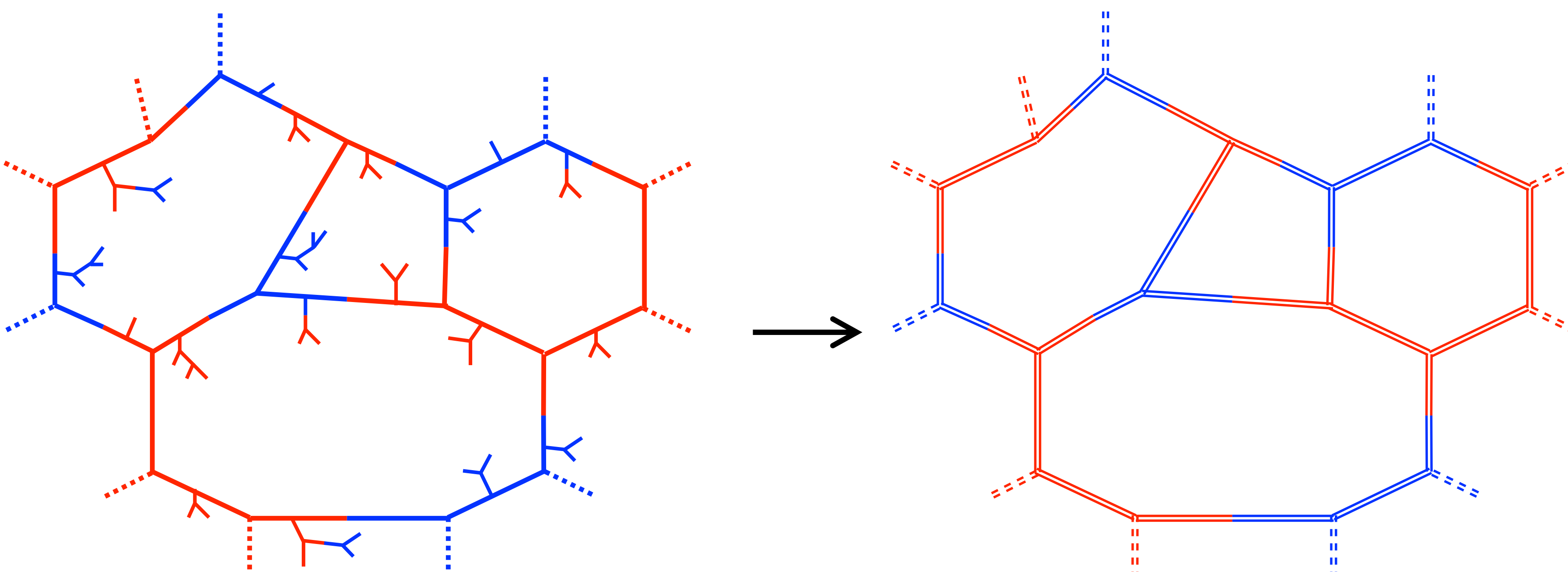
Remove the linear terms,

$$U^{(0)}(\varphi_+,\varphi_-)=\frac{1}{\theta}\left(\frac{1}{2}\left(\varphi_+^2+\varphi_-^2-2c\varphi_+\varphi_-\right)-g(\varphi_-+\varphi_-)-\frac{g}{3}\left(\varphi_+^3+\varphi_-^3\right)\right)$$

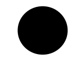
$$\varphi_\pm=\tilde{\varphi}_\pm+Z_{\rm tree}(g,c)\qquad Z_{\rm tree}=\frac{1-c-\sqrt{(1-c)^2-4g^2}}{2g}$$

$$U^{(1)}(\tilde{\varphi}_+,\tilde{\varphi}_-)=\frac{1}{\theta}\left(\frac{1-2gZ_{\rm tree}}{2}(\tilde{\varphi}_+^2+\tilde{\varphi}_-^2)-c\tilde{\varphi}_+\tilde{\varphi}_--\frac{g}{3}(\tilde{\varphi}_+^3+\tilde{\varphi}_-^3)\right)$$

Trees are integrated out:

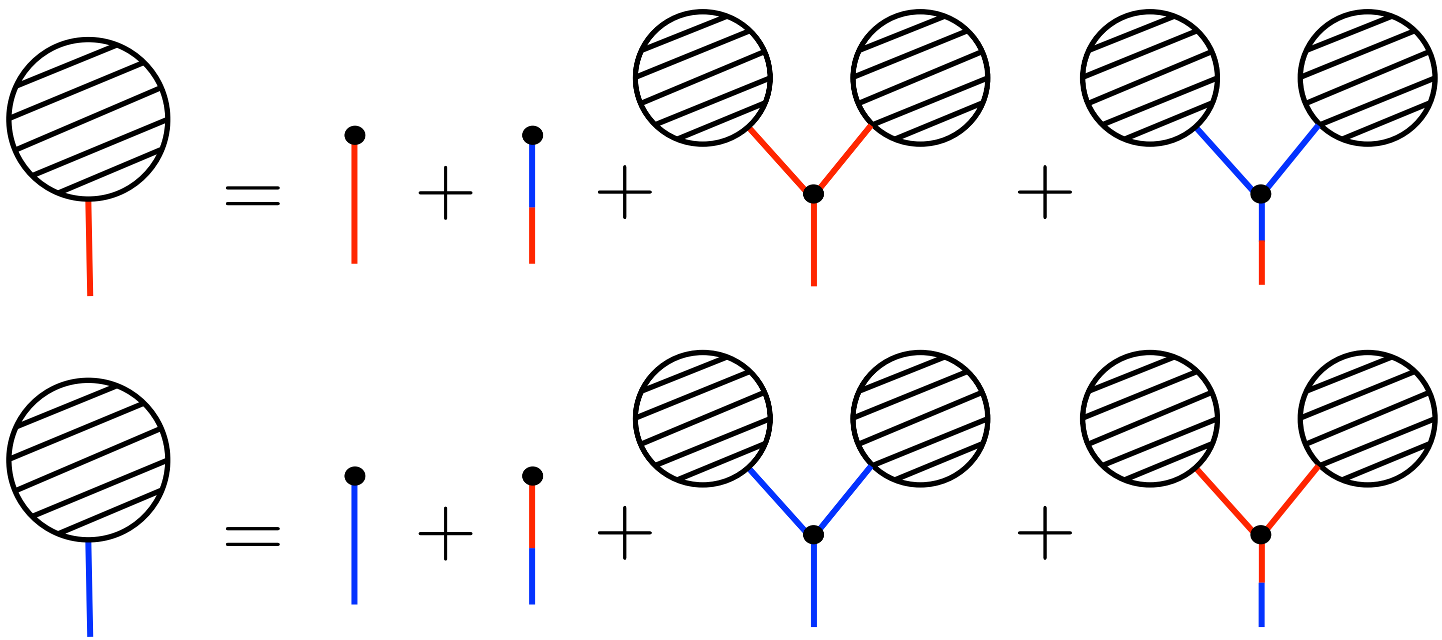


## Planar tree generated by $U^{(0)}$:

 $\sim \dfrac{\theta}{N(1-c^2)}$  $\sim \dfrac{\theta c}{N(1-c^2)}$  $\sim \dfrac{gN}{\theta}$

 $\sim \dfrac{\theta}{N(1-c^2)}$  $\sim \dfrac{\theta c}{N(1-c^2)}$

## Partition functions of planar tree :



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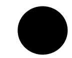
 $\sim \dfrac{\theta}{N(1-c^2)}$  $\sim \dfrac{\theta c}{N(1-c^2)}$  $\sim \dfrac{gN}{\theta}$

 $\sim \dfrac{\theta}{N(1-c^2)}$  $\sim \dfrac{\theta c}{N(1-c^2)}$

## Partition functions of planar tree :

$$Z_+ = \frac{\theta}{N(1-c^2)}\frac{gN}{\theta} + \frac{\theta c}{N(1-c^2)}\frac{gN}{\theta} + \frac{\theta}{N(1-c^2)}\frac{gN}{\theta}Z_+^2 + \frac{\theta c}{N(1-c^2)}\frac{gN}{\theta}Z_-^2$$

$$Z_- = \frac{\theta}{N(1-c^2)}\frac{gN}{\theta} + \frac{\theta c}{N(1-c^2)}\frac{gN}{\theta} + \frac{\theta}{N(1-c^2)}\frac{gN}{\theta}Z_-^2 + \frac{\theta c}{N(1-c^2)}\frac{gN}{\theta}Z_+^2$$

$$\longrightarrow \quad Z_+ = Z_- = Z_{\rm tree} = \frac{1-c-\sqrt{(1-c)^2-4g^2}}{2g}$$

Normalize quadratic terms,

$$U^{(0)}(\varphi_+,\varphi_-) = \frac{1}{\theta}\left(\frac{1}{2}\left(\varphi_+^2+\varphi_-^2-2c\varphi_+\varphi_-\right) - g(\varphi_- + \varphi_-) - \frac{g}{3}\left(\varphi_+^3+\varphi_-^3\right)\right)$$

$$\downarrow \quad \varphi_\pm = \tilde{\varphi}_\pm + Z_{\text{tree}}(g,c)$$

$$U^{(1)}(\tilde{\varphi}_+,\tilde{\varphi}_-) = \frac{1}{\theta}\left(\frac{1-2gZ_{\text{tree}}}{2}(\tilde{\varphi}_+^2+\tilde{\varphi}_-^2) - c\tilde{\varphi}_+\tilde{\varphi}_- - \frac{g}{3}(\tilde{\varphi}_+^3+\tilde{\varphi}_-^3)\right)$$

$$\downarrow \quad \tilde{\varphi}_\pm = \sqrt{\frac{\theta}{1-2gZ_{\text{tree}}}}\psi_\pm$$

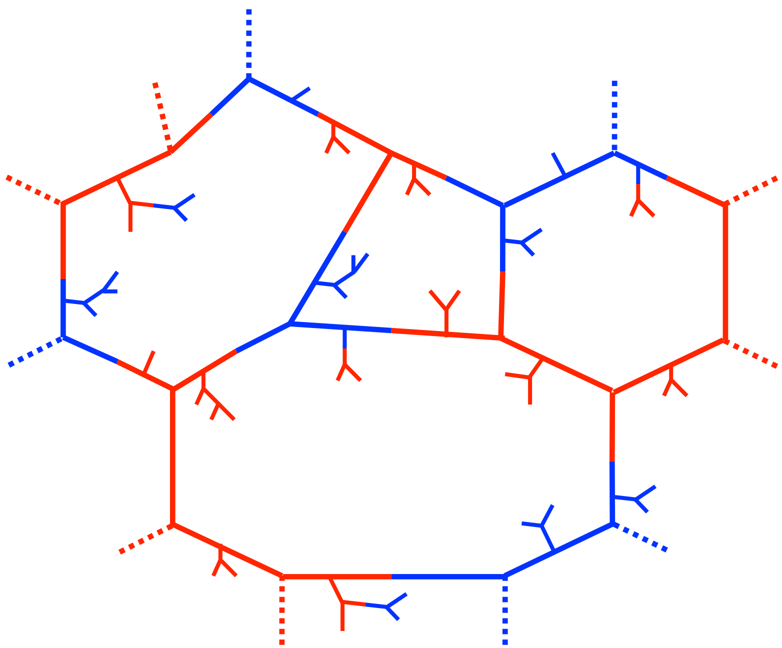
$$U^{(2)}(\psi_+,\psi_-) = \frac{1}{2}(\psi_+^2+\psi_-^2-2c_{\text{dt}}\psi_+\psi_-) - \frac{g_{\text{dt}}}{3}(\psi_+^3+\psi_-^3)$$

where $c_{\text{dt}} = \dfrac{g}{1-2gZ_{\text{tree}}}$ $\qquad g_{\text{dt}} = \dfrac{\theta g}{(1-2gZ_{\text{tree}})^{3/2}}$

The linear terms are not important when $\theta \sim O(1)$

$\theta$ -dependence of planar graphs generated by $U^{(0)}$,

$$U^{(0)}(\varphi_+,\varphi_-) = \frac{1}{\theta}\left(\frac{1}{2}\left(\varphi_+^2+\varphi_-^2-2c\varphi_+\varphi_-\right) - g(\varphi_- + \varphi_-) - \frac{g}{3}\left(\varphi_+^3+\varphi_-^3\right)\right)$$



$$\sim \theta^{\#(loops)-2}$$

When $\theta << 1$, loops are suppressed and trees become dominant.

clarified first in one-matrix model
[Ambjorn-Butt, 2013, Ambjorn-Butt-Watabiki, 2014]

For a fixed $\theta$, obtain critical point (CP) of Ising on 2d DT,

$$(c_c(\theta), g_c(\theta)) \quad \text{[CP of Ising on 2d DT when } \theta \sim O(1)]$$

$$\Big\downarrow \; \theta \to 0$$

$$(c_c(0) = c_*, g_c(0) = g_*) \quad \text{[CP of Ising on tree when } \theta = 0)]$$

Ising model on tree can be critical only at the zero temperature.

[Ambjorn-Durhuus-Jonsson-Thorleifsson, 1993]

In fact, CP of Ising model on tree computed by $Z_{tree}(c,g)$ is

$$c_* = 0, \quad g_* = 1/2$$

Therefore, one expects that

tuning $\theta \to 0$, critical temperature approaches absolute zero.

CP where fluctuations of spins and graphs are divergent:

$$g_c^2(\theta) = \left( -\frac{9}{4\times 10^{3/2}}\theta^{2/3} + \frac{3^{1/3}\theta^{1/3}(243\theta - 80) + H^2}{4\times 30^{2/3} H} \right)^3$$

$$c_c(\theta) = \frac{2}{5^{1/3}}\theta^{1/3} \left( -\frac{9}{4\times 10^{3/2}}\theta^{2/3} + \frac{3^{1/3}\theta^{1/3}(243\theta - 80) + H^2}{4\times 30^{2/3} H} \right)$$

where $(0 \leq \theta \leq 1.01378\cdots)$

$$H = \left[ 81(40-81\theta)\theta + 80(90 + \sqrt{8100 + 3(2510 - 5103\theta)\theta}) \right]^{1/3}$$

Small-$\theta$ expansion gives,

$$g_c^2(\theta) \cong \frac{1}{4} - \frac{1}{4\times 5^{1/3}}\theta^{1/3} + \cdots$$

CP of Ising on tree, $g^2{}_*$

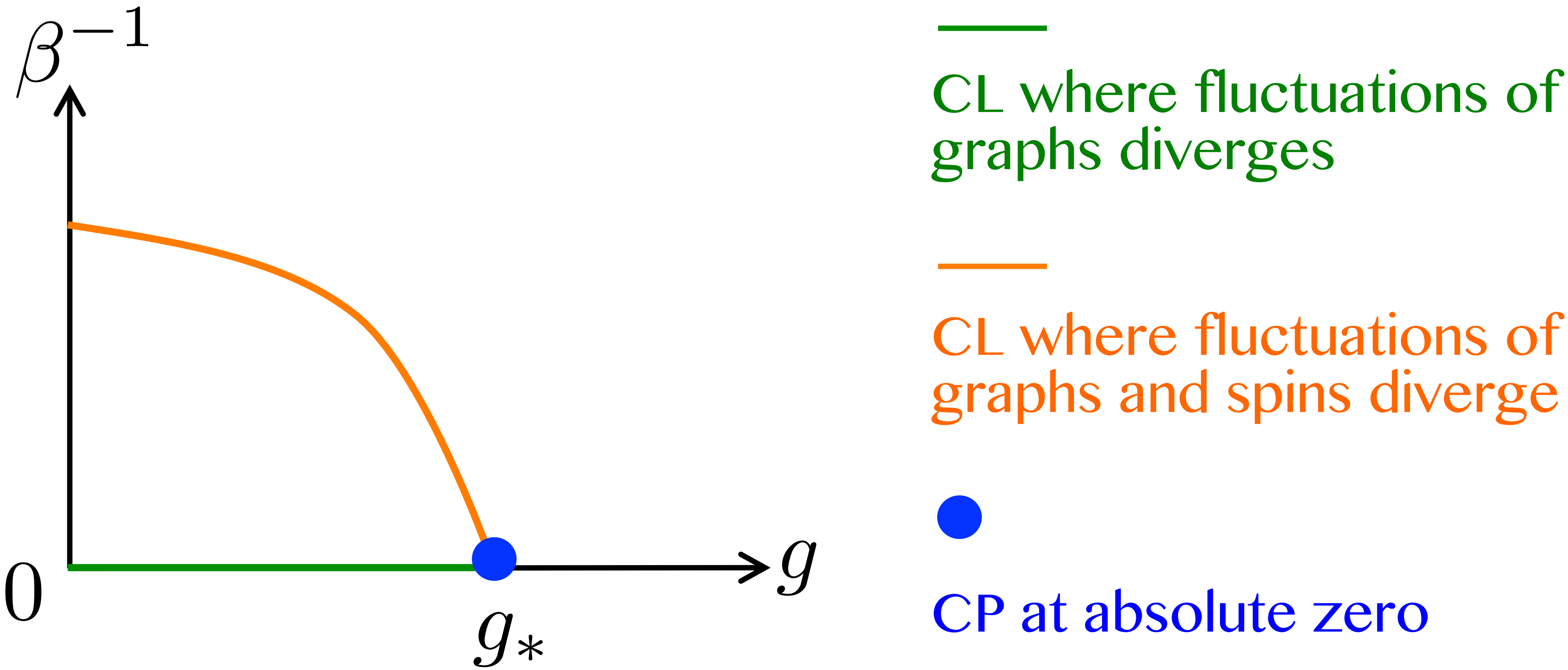
In the small-$\theta$ limit,

$$\lim_{\theta\to 0} \beta_c^{-1}(\theta) = 0$$

Criticality at absolute zero

$$(c_c(\theta) =: e^{-2\beta_c(\theta)})$$

Critical lines (CL):



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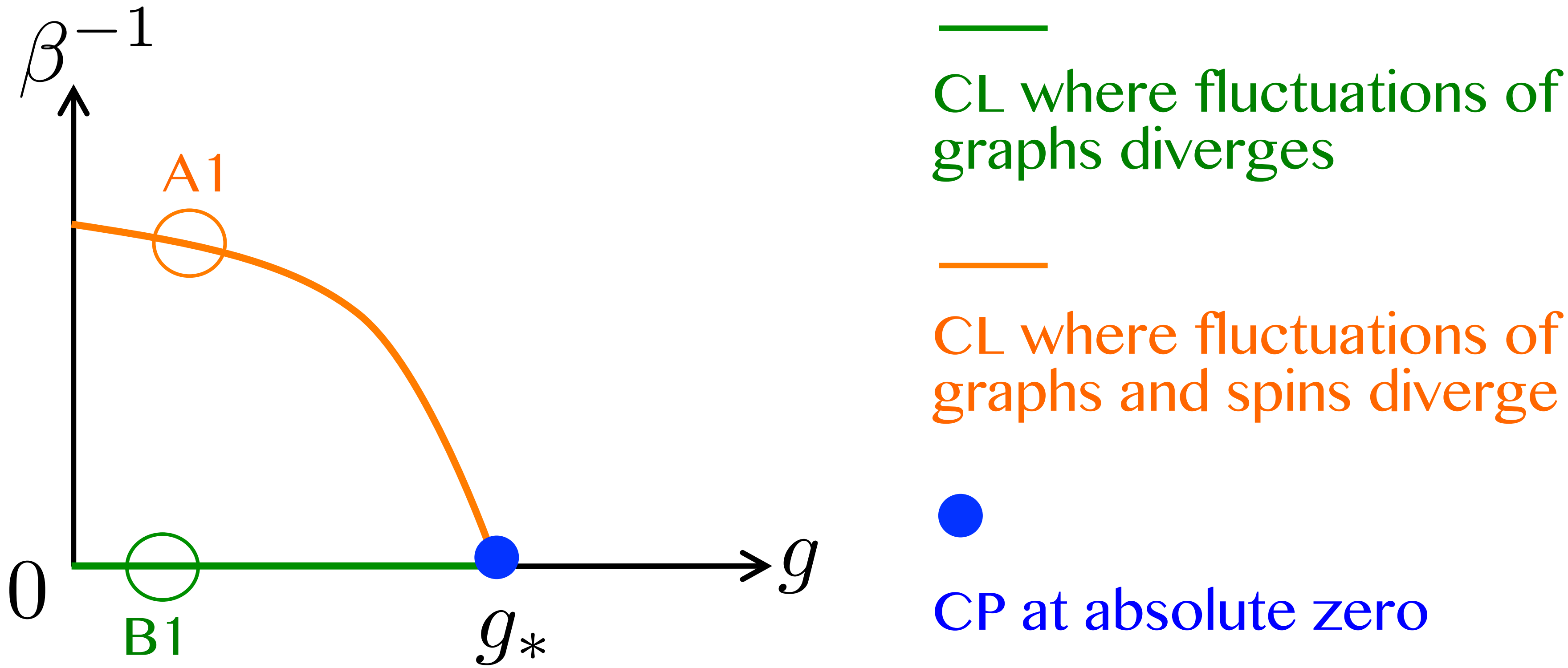
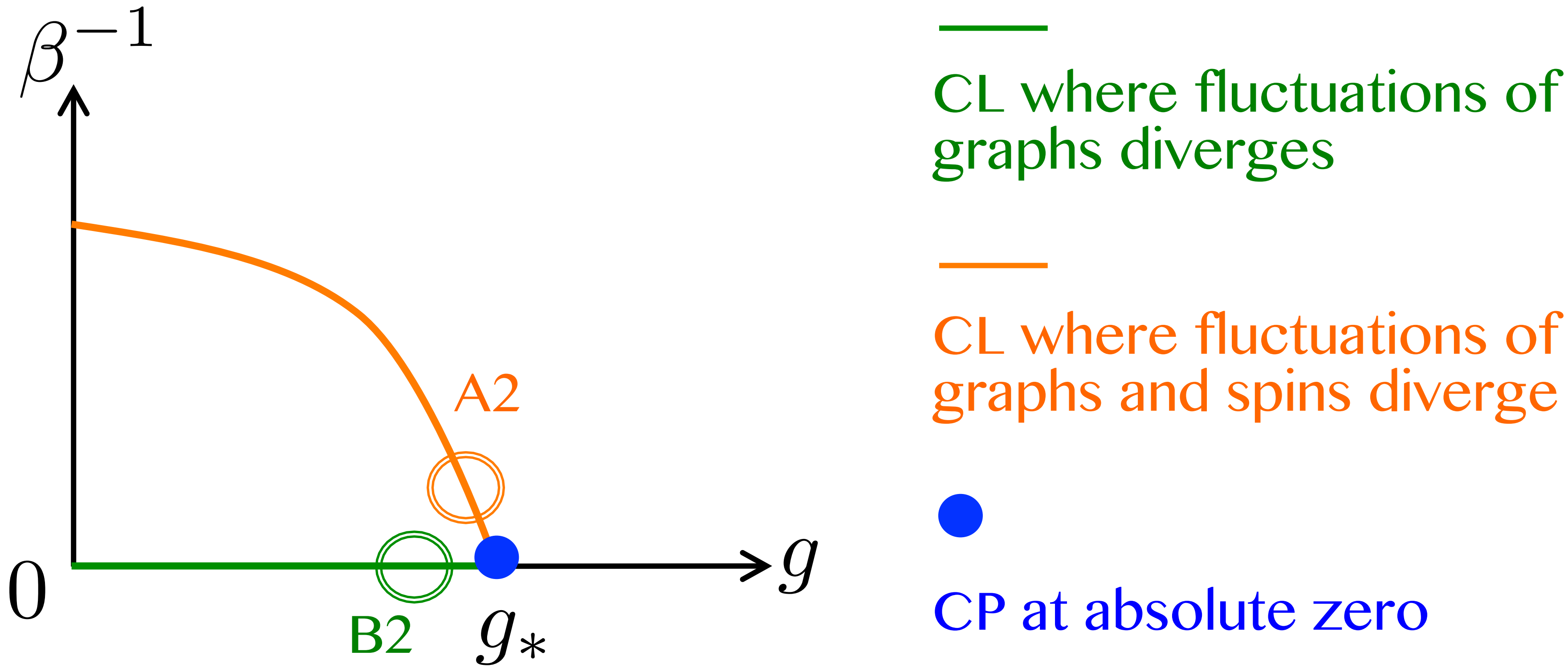



Continuum limits for $\theta \sim O(1)$ :

A1: Gravity coupled to fermions [Kazakov, 1986]

B1: Pure gravity

Critical lines (CL):

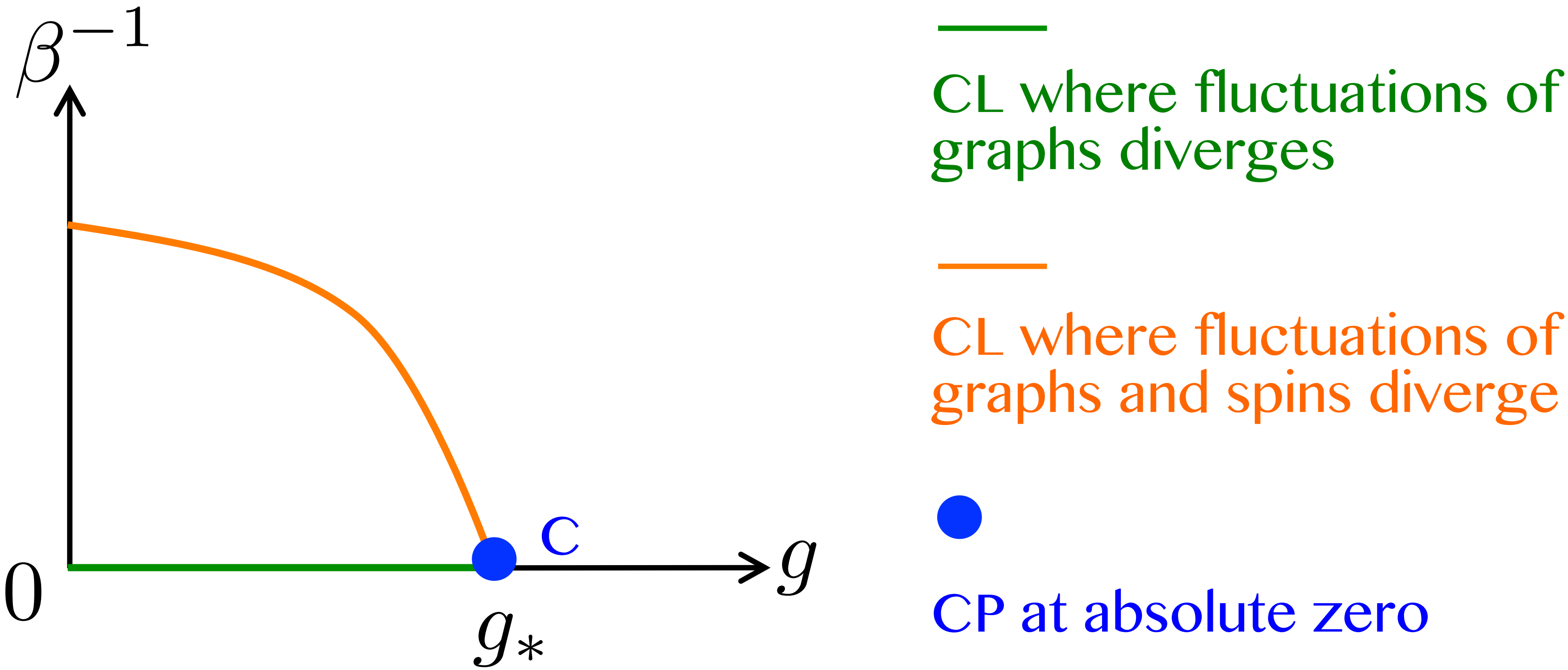



Continuum limits for $\theta << 1$ :

A2: A continuum matrix model [YS-Tanaka, 2017]

B2: Generalized CDT [Ambjorn-Loll-Watabiki-Westra-Zohren, 2008]

Critical lines (CL):




Continuum limits for $\theta = 0$ :

C: 2d projectable Horava-Lifshiz gravity

[Ambjorn-Loll-Watabiki-Westra-Zohren, 2008]

[Ambjorn-Glaser-YS-Watabiki, 2013]

## A2: A continuum matrix model [YS-Tanaka, 2017]

Tuning coupling constants,

$$\theta = \Theta \varepsilon^3 \qquad (\,\varepsilon : \text{lattice spacing})$$

$$g^2 = g_c^2(\theta)(1 - \Lambda\varepsilon^2) = \frac{1}{4}\left(1 - 5^{-1/3}\Theta^{1/3}\varepsilon - \Lambda\varepsilon^2\right) + \cdots$$

$$c = c_c(\theta) = 2^{-1} \times 5^{-1/3}\Theta^{1/3}\varepsilon + \cdots$$

obtain the continuum theory described by the matrix integral:

$$I_N(\Lambda, \Theta) = \int D\Phi_+ D\Phi_- \; e^{-N\mathrm{tr}V(\Phi_+, \Phi_-)}$$

where

$$V = \frac{1}{\Theta}\left(\frac{\Lambda}{2}(\Phi_+ + \Phi_-) - \frac{1}{6}(\Phi_+^3 + \Phi_-^3) - \frac{\Theta^{1/3}}{2 \times 5^{1/3}}\Phi_+\Phi_-\right)$$

# Summary

# Summary (1/2)

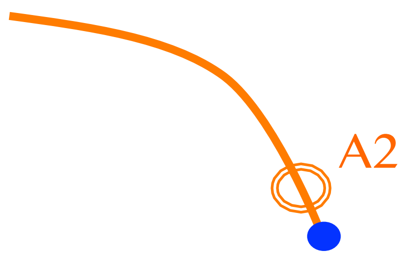
Considered two-matrix model w/ the potential,

$$U^{(0)}(\varphi_+, \varphi_-) = \frac{1}{\theta}\left(\frac{1}{2}\left(\varphi_+^2 + \varphi_-^2 - 2c\varphi_+\varphi_-\right) - g(\varphi_- + \varphi_-) - \frac{g}{3}\left(\varphi_+^3 + \varphi_-^3\right)\right)$$

Reduced critical temperature to absolute zero by $\theta \to 0$:

$$(c_c(\theta), g_c(\theta)) \longrightarrow (c_*, g_*) = (0, 1/2)$$

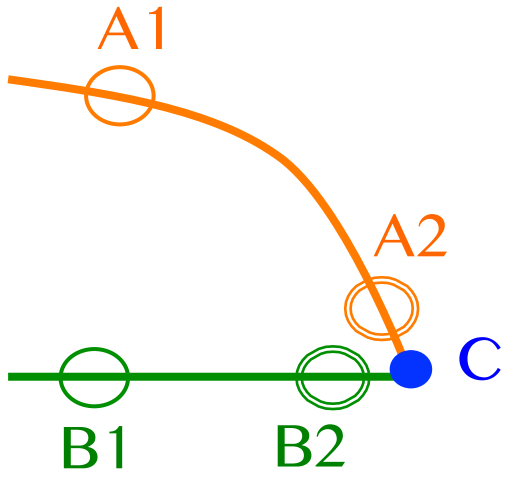
CP of Ising on 2d DT → CP of Ising on tree




Continuum theory defined around A2
is a two-matrix model w/ the potential,

$$V = \frac{1}{\Theta}\left(\frac{\Lambda}{2}(\Phi_+ + \Phi_-) - \frac{1}{6}(\Phi_+^3 + \Phi_-^3) - \frac{\Theta^{1/3}}{2 \times 5^{1/3}}\Phi_+\Phi_-\right)$$

## Summary (2/2)




A critical exponent (string susceptibility):

$$Z \sim (g_c(\theta) - g)^{2-\gamma_{str}}$$

| $\gamma_{str} = -1/2$ | $\gamma_{str} = -1/3$ | $\gamma_{str} = 1/2$ |
|---|---|---|
| B1 | A1 | A2 B2 C |

Future work:

Introduce magnetic fields
to see back-reactions of graphs on Ising spins. [YS, Fraser]