Emergent symmetries in the Canonical Tensor Model

Naoki Sasakura

Yukawa Institute for Theoretical Physics, Kyoto University

Based on collaborations with Dennis Obster

arXiv:1704.02113 and a paper coming soon

"Discrete Approaches to the Dynamics of Fields and Space-Time"

September 19-23, 2017 APCTP, Pohang, Korea

Contents (My part)

1. Introduction

   Tensor model

   Canonical Tensor Model (CTM)

2. Formalism of CTM

3. A wave function

The latter (main) part will be done by Dennis.

1. Introduction

Discrete model of quantum gravity — An approach to QG

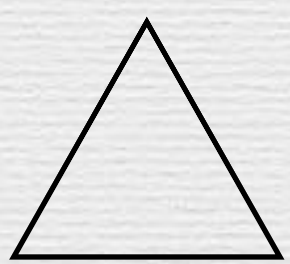
Spacetime (and other fundamental degrees of freedom) are described by discrete variables.

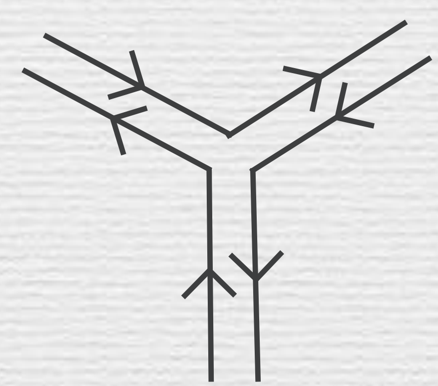
⋆ The most successful case — 2 dim.

Dual

Random surface ⟷ Matrix model

Triangulated surfaces ⟷ Feynman diagrams

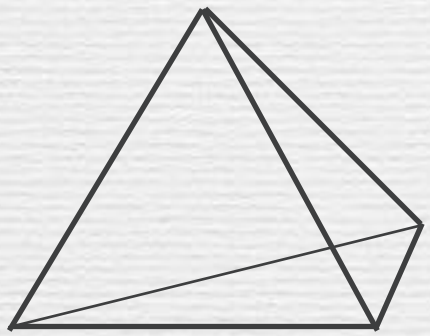




⋆ **D>2 dim.**

Random volume

Dynamical triangulation



Dual ⟷

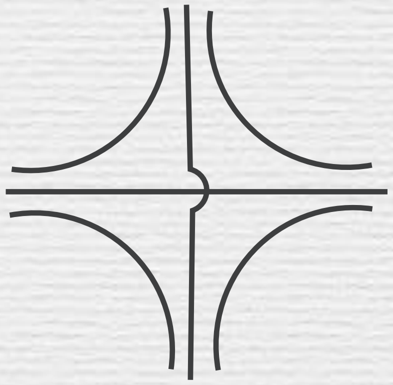
Tensor models

Original models '91

Ambjorn et al, NS, Gross et al

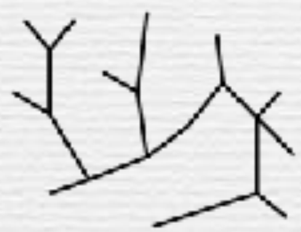
Colored tensor model '09

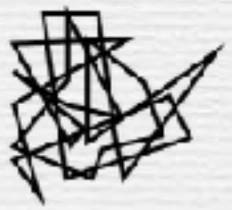
Gurau



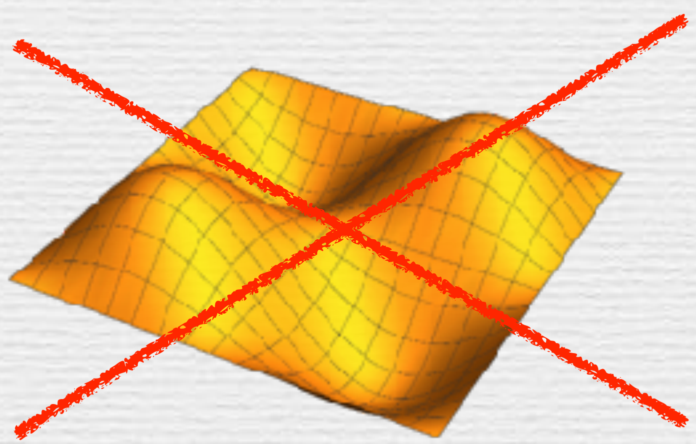
However, these Euclidean gravity discrete models in D>2 suffer from generation of singular spaces.

Branched polymer phase or Crumpled phase





No emergence of macroscopic spaces

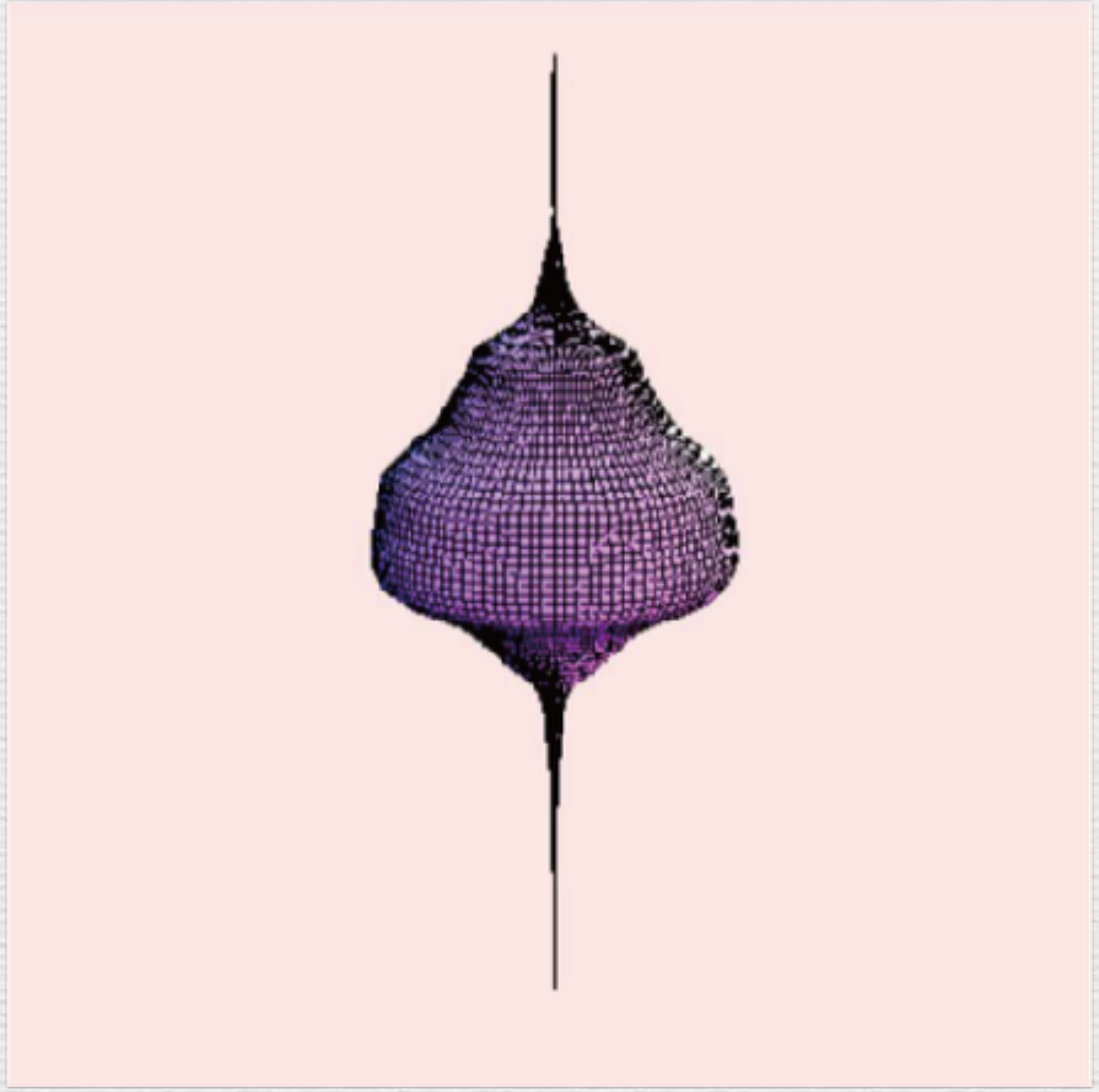


cf. But, recently, relation to SYK model was pointed out (Witten '16). Related to gravity by holography ? This is still an open problem.

On the other hand,

Causal dynamical triangulation — Dynamical triangulation with **time** direction

Emergence of macroscopic spacetime in 3+1 dim (and 2+1)



Time is important !

Seems essential for emergence of macroscopic spacetime.

J. Ambjorn, J. Jurkiewicz, R. Loll, Phys.Rev.Lett. 93 (2004) 131301

Proposal of Canonical Tensor Model (CTM)

NS, Int.J.Mod.Phys. A27 (2012) 1250020

Motivation : Formulate a tensor model with **time**

Strategy:

- Use canonical formalism
- Time must be a gauge direction (not a physical observable)

$\downarrow$

Hamiltonian is a constraint (first-class)

$$H = N_a \mathcal{H}_a + N_{ab} \mathcal{J}_{ab}$$

$$\{\mathcal{H}, \mathcal{H}\} \sim \mathcal{J}$$

$$\{\mathcal{J}, \mathcal{H}\} \sim \mathcal{H}$$

$$\{\mathcal{J}, \mathcal{J}\} \sim \mathcal{J}$$

$\mathcal{H}, \mathcal{J}$: **First-class constraints** (Poisson algebra of constraints is closed)

N_a, N_{ab}: Non-dynamical multipliers

Very similar structure as ADM formalism of GR

ADM formalism of GR

$$H_{ADM} = \int d^D x \left(N(x)\mathcal{H}(x) + N_i(x)\mathcal{J}^i(x) \right)$$

$$\{\mathcal{H}, \mathcal{H}\} \sim \mathcal{J}$$

$$\{\mathcal{J}, \mathcal{H}\} \sim \mathcal{H}$$

$$\{\mathcal{J}, \mathcal{J}\} \sim \mathcal{J}$$

First class

$\mathcal{H}(x)$: Hamiltonian constraint

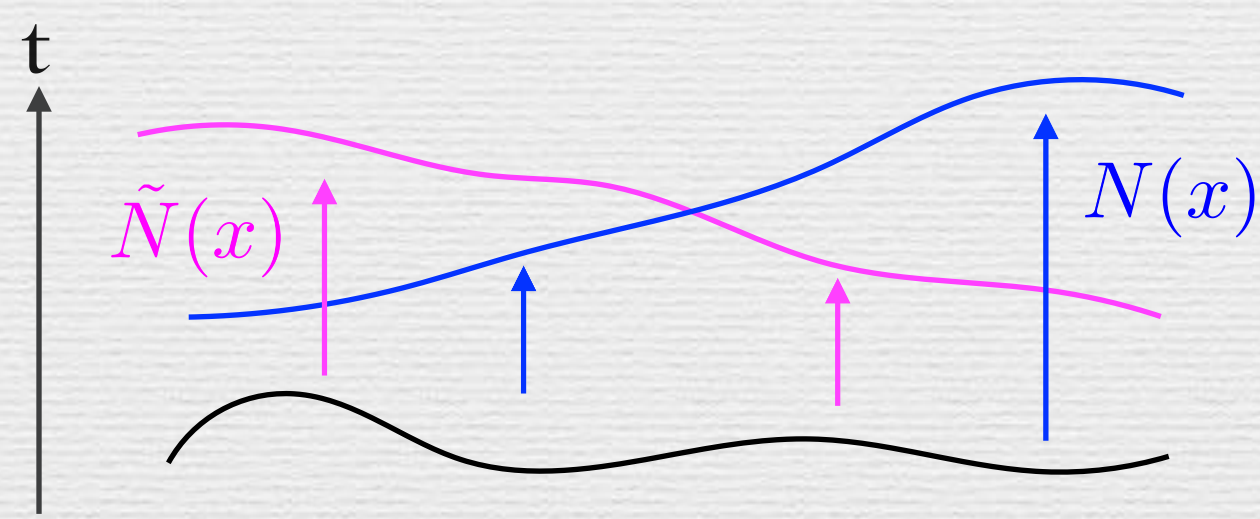
$\mathcal{J}^i(x)$: Momentum constraint (Spatial diffeo)

$N(x)$: Lapse

$N_i(x)$: Shift

Guarantees mutual consistencies of locally defined time-evolutions

$\rightarrow$ **General covariance**



CTM is unique under some reasonable assumptions.

(Details later)

NS, Int.J.Mod.Phys. A27 (2012) 1250096

So, we just need to study one such model.

If CTM does not work at some point, then it should be discarded right away.

Very nice. Life is short.

Classical properties of CTM work very nicely so far.

- ★ **N=1 classical CTM = Mini-superspace treatment of GR (FRW)** NS, Y.Sato, Phys.Lett. B732 (2014) 32-35

- ★ **Classical CTM in a formal continuum limit→GR system**

Constraint algebra→Algebra of ADM constraints

Hypersurface deformation algebra

(Spacetime diffeomorphism symmetry)

NS, Y.Sato, JHEP 1510 (2015) 109

EOM → GR+scalar+higher spins in Hamil.-Jacobi formalism

$$S = \int d^{D+1}x\sqrt{g}\left(2R - \frac{1}{2}(\partial\phi)^2 - e^{\sqrt{\frac{6-D}{8(D-1)}}\phi} + \cdots\right)$$

H.Chen, NS, Y.Sato, Phys.Rev. D95 (2017) no.6, 066008

A formal continuum limit

- **Continuous indices**

$$P_{abc} \to P_{xyz} \qquad x, y, z \in \mathbb{R}^D \qquad \text{Implicitly N} \to \infty$$

- **Locality**

$$P_{xyz} \neq 0 \text{ only at } x \sim y \sim z$$

More specifically, a derivative expansion is assumed.

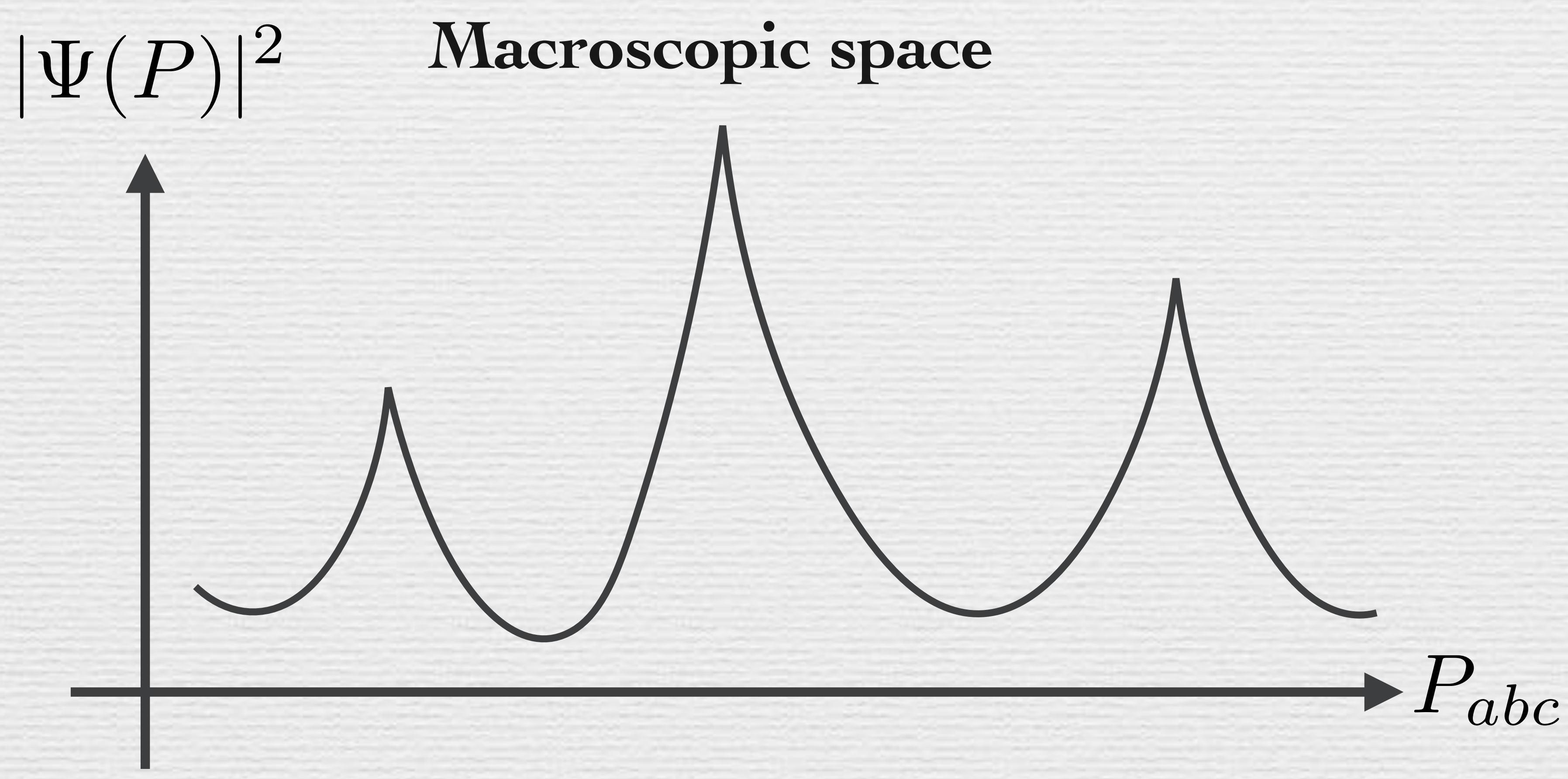
$$\begin{aligned} P_{xyz} = \beta(x)\delta^D(x-y)\delta^D(x-z) \\ +\beta^\mu(x)\partial_\mu\delta^D(x-y)\delta^D(x-z) + \cdots \\ +\beta^{\mu\nu}(x)\partial_\mu\delta^D(x-y)\partial_\nu\delta^D(x-z) + \cdots \\ +\cdots \end{aligned}$$

Formal continuum limit $\approx$ **Presetting a macroscopic space with continuity and locality**

Classically, it can be regarded as a sort of an initial condition, and is therefore a consistent treatment.

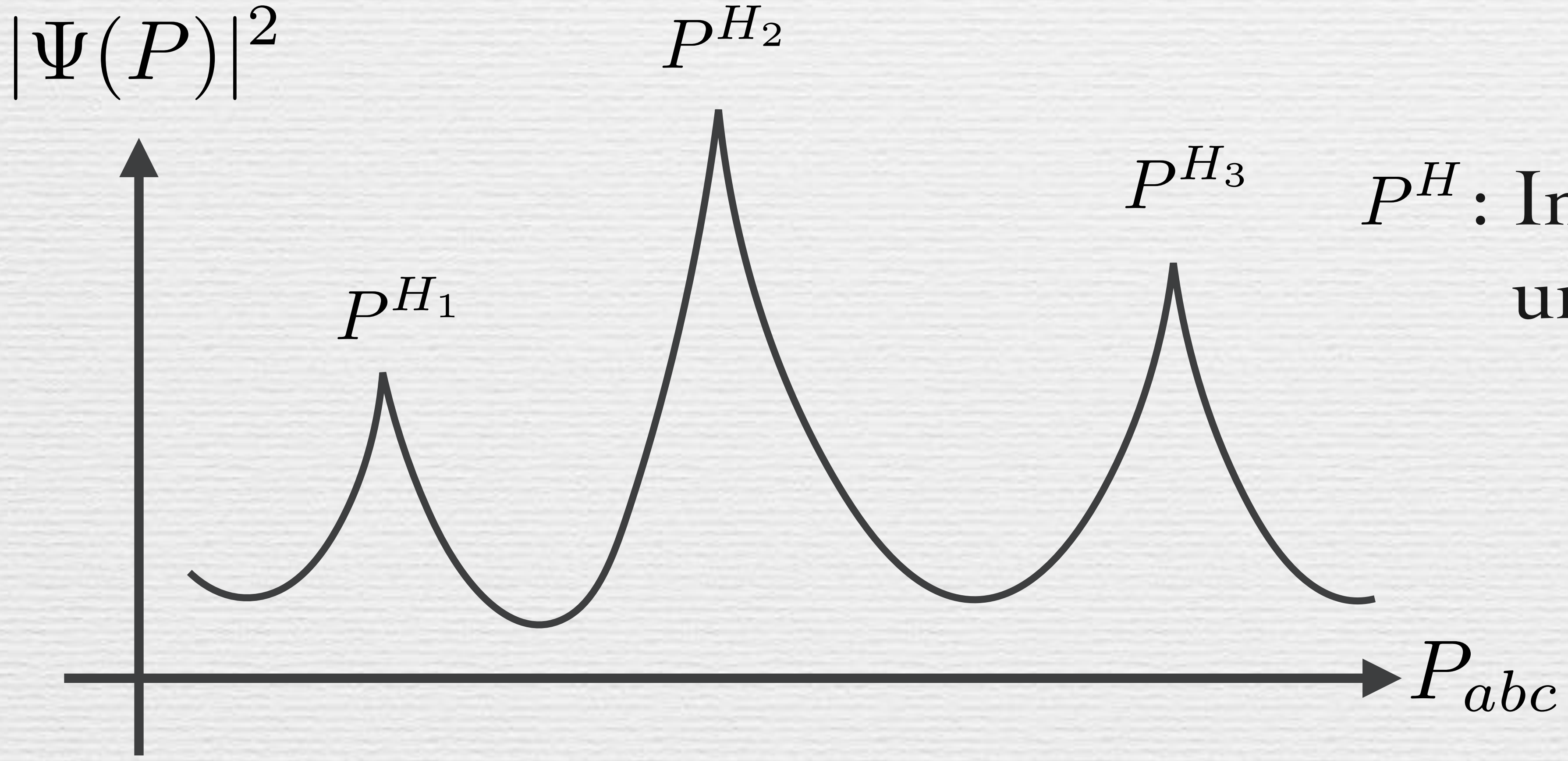
Rather, we want to derive it as a peak of a wave function of quantum CTM. Emergence of macroscopic spaces.

Ideally,



Instead, in this talk, we will show

A wave function of quantum CTM has the preference of Lie-group symmetric configurations.



P^H: Invariant configuration under Lie group H

Encouraging, because Lie group symmetries determine global structures of spacetimes. Eg. translation, rotation, etc.

2. Formalism of CTM

NS, Int.J.Mod.Phys. A27 (2012) 1250020
NS, Int.J.Mod.Phys. A27 (2012) 1250096

⋆ **Dynamical variables**

A conjugate pair of **real symmetric rank-3** tensors

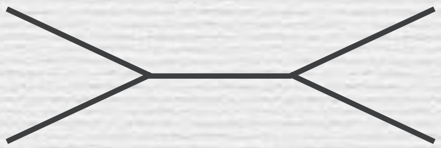
$$\{Q_{abc}, P_{def}\} = \sum_{\sigma} \delta_{a\sigma_d}\delta_{b\sigma_e}\delta_{c\sigma_f} \qquad a, b, \ldots = 1, 2, \ldots, N$$

Permutations

$$\{Q_{abc}, Q_{def}\} = \{P_{abc}, P_{def}\} = 0$$

Rank-3, because

- The simplest. Try the simplest first !
- Higher tensors may be composites of rank-3.



Rank-4, $\cdots$

⋆ Hamiltonian

$$H = N_a \mathcal{H}_a + N_{ab} \mathcal{J}_{ab}$$

Following the names in ADM.

N_a : Lapse
N_{ab} : Shift

Non-dynamical multipliers

$\mathcal{H}_a$: Hamiltonian constraint
$\mathcal{J}_{ab}$: Momentum constraint

First-class constraints

$\mathcal{J}_{ab}$ **: so(N) generators — Rotation of indices**
Analogous to spatial diffeomorphism

⋆ Constraints and Poisson algebra

$$\mathcal{H}_a = \frac{1}{2}\left(P_{abc}P_{bde}Q_{cde} - \lambda\, Q_{abb}\right) \qquad \lambda = 0, \pm 1$$

$$\mathcal{J}_{ab} = -\mathcal{J}_{ba} = \frac{1}{4}(Q_{acd}P_{bcd} - Q_{bcd}P_{acd}) \qquad \left(N = 1 = \begin{array}{l}\text{FRW} \\ \text{with}\,\Lambda = \lambda\end{array} \right)$$

Form a first-class constraint Poisson algebra (closed)

$$\{\mathcal{H}(\xi^1), \mathcal{H}(\xi^2)\} = \mathcal{J}\left([\tilde{\xi}^1, \tilde{\xi}^2] + 2\lambda\, \xi^1 \wedge \xi^2\right)$$

$$\{\mathcal{J}(\eta), \mathcal{H}(\xi)\} = \mathcal{H}(\eta\xi)$$

Non-linearity exists similarly to ADM

$$\{\mathcal{J}(\eta^1), \mathcal{J}(\eta^2)\} = \mathcal{J}\left([\eta^1, \eta^2]\right)$$

so(N)

$\mathcal{H}(\xi) = \mathcal{H}_a\xi_a \qquad \mathcal{J}(\eta) = \mathcal{J}_{ab}\eta_{ab} \qquad \tilde{\xi}_{ab} = P_{abc}\xi_c$

$(\xi^1 \wedge \xi^2)_{ab} = \xi_a\xi_b - \xi_b\xi_a \qquad [\ ,\ ]$: Matrix commutator

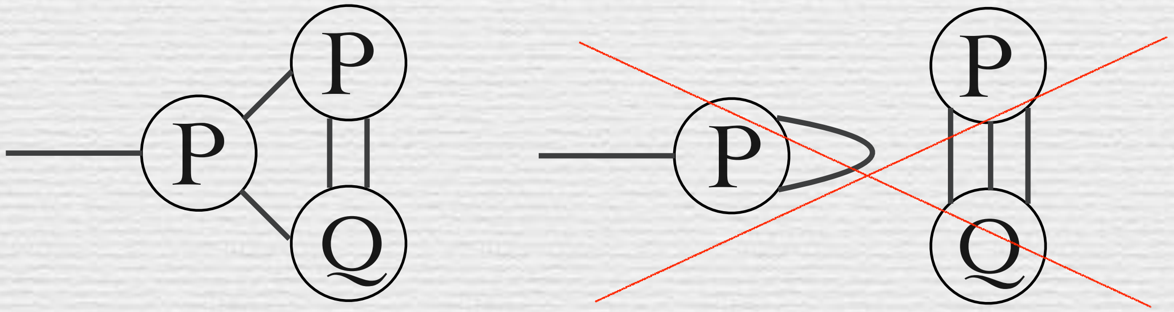
ξ_a, η_{ab} : c-auxiliary variables

⋆ **Uniqueness**

NS, Int.J.Mod.Phys. A27 (2012) 1250096

Assumptions for the proof of the uniqueness

- Dynamical variables : A conjugate pair of rank-3 real symmetric tensors
- SO(N) kinematical symmetry, $\mathcal{J}_{ab}$
- There is a constraint with one index $\mathcal{H}_a$
- $\mathcal{H}_a$ is at most cubic.
- The constraints form a closed Poisson algebra (First-class)
- Time reversal symmetry $P \to -P,\ Q \to Q$
- Connected (Locality)



⋆ Quantization

NS, Int.J.Mod.Phys. A28 (2013) 1350111

Straightforward by the canonical way

$$Q_{abc}, P_{abc} \to \hat{Q}_{abc}, \hat{P}_{abc} \qquad \{\ ,\ \} \to \frac{1}{i}[\ ,\ ]$$

$$\hat{\mathcal{H}}_a = \frac{1}{2}\left(\hat{P}_{abc}\hat{P}_{bde}\hat{Q}_{cde} + i\,\lambda_H\,\hat{P}_{abb} - \lambda\,\hat{Q}_{abb}\right)$$

Normal ordering term

$$\lambda_H = \frac{1}{2}(N+2)(N+3)$$

Determined by hermiticity

$$\hat{\mathcal{J}}_{ab} = \frac{1}{4}\left(\hat{Q}_{acd}\hat{P}_{bcd} - \hat{Q}_{bcd}\hat{P}_{acd}\right)$$

No anomalies !

Quantum constraint algebra is the same as classical.

In particular, it is closed — Consistency guaranteed.

⋆ Physical states

$$\hat{\mathcal{H}}_a|\Psi\rangle = \hat{\mathcal{J}}_{ab}|\Psi\rangle = 0$$

⋆ Wave functions

In P-rep, $Q_{abc} \to iD_{abc}$ $\quad D_{abc}P_{def} = \sum_{\sigma} \delta_{a\sigma_d}\delta_{b\sigma_e}\delta_{c\sigma_f}$

A set of linear 1st order partial differential eqs.

$$(P_{abc}P_{bde}D_{cde} + \lambda_H P_{abb} - \lambda\, D_{abb})\,\Psi_{phys}(P) = 0$$

Analogue to Wheeler-DeWitt eq.

$$(P_{acd}D_{bcd} - P_{bcd}D_{acd})\,\Psi_{phys}(P) = 0$$

In Q-rep., a set of linear 2nd order partial differential eqs.

3. A wave function

Looks very difficult to solve the equations. But, **one can find an exact solution for general N**, which looks very interesting.

G.Narain, NS, Y.Sato, JHEP 1501 (2015) 010

$$\Psi_{phys}(P) = \Psi(P)^{\frac{\lambda_H}{2}}$$ ← Normal ordering constant

$$\Psi(P) = \int_{\mathbb{R}^{N+1}} d\phi d\tilde{\phi}\; e^{i\left(P\phi^3 + \phi^2\tilde{\phi} - \frac{4}{27\lambda}\tilde{\phi}^3\right)}$$

$$P\phi^3 = P_{abc}\phi_a\phi_b\phi_c \qquad \phi^2 = \phi_a\phi_a \qquad d\phi = \prod_{a=1}^{N} d\phi_a$$

Found by being inspired by an intimate connection between CTM and **randomly connected tensor networks**.

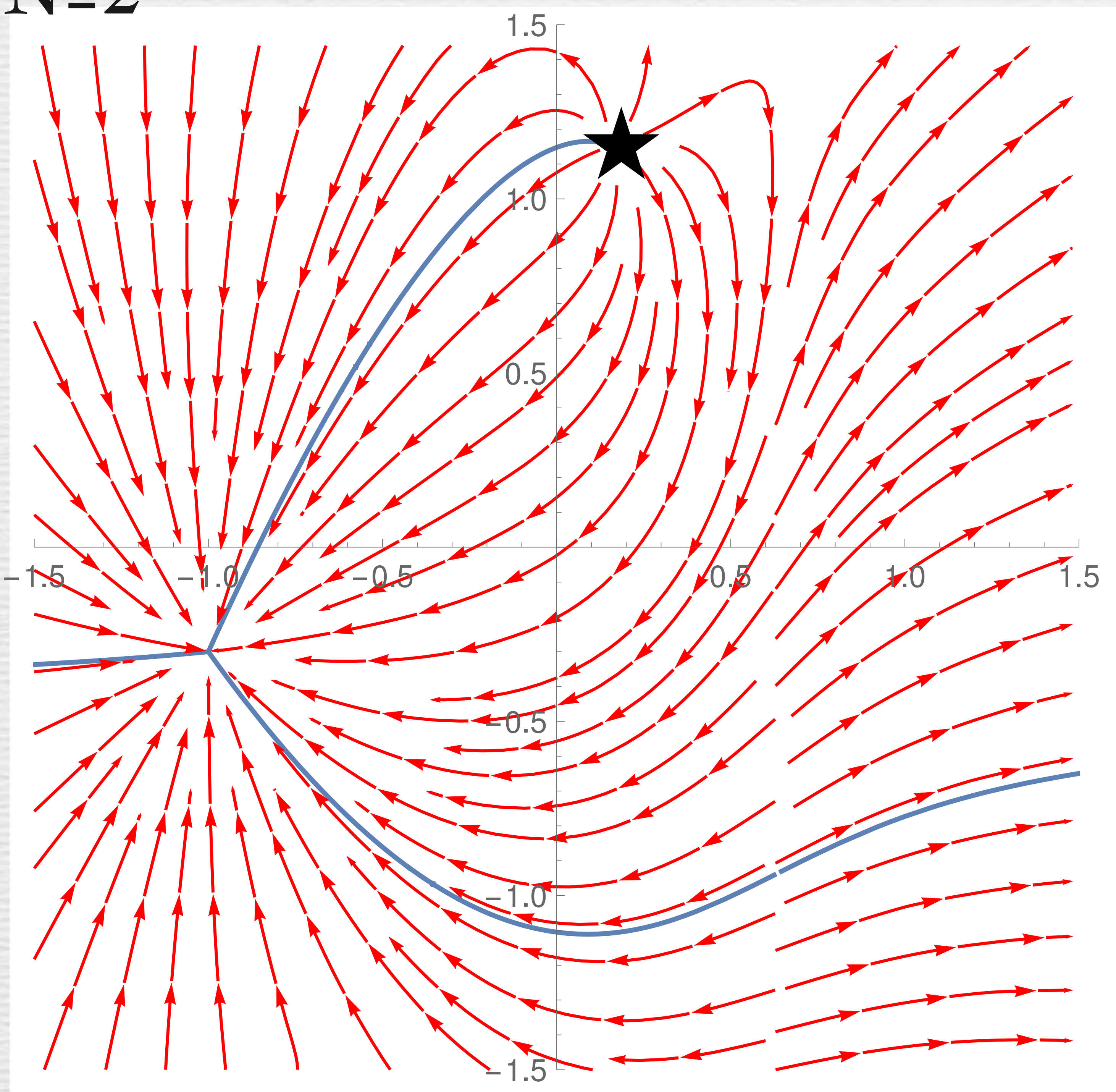
Cf.

Connection between CTM and randomly connected tensor networks

NS, Y.Sato, PTEP 2014 (2014) no.5, 053B03; PTEP 2015 (2015) no.4, 043B09

+ yet to appear

N=2



1st order phase-trans. line

Hamilton vector flow of CTM

★ 2nd order phase-trans. point

Looks very much like an RG flow. (Not yet proven)

- **Proof of the solution**

What is necessary is essentially the validity of the **partial integration** over $\phi, \tilde\phi$.

eg: The first term of $\hat{\mathcal{H}}_a$

$$P_{abc}P_{bde}D_{cde}\int_{\mathcal{C}} d\phi d\tilde\phi\ e^{i\left(P\phi^3+\phi^2\tilde\phi-\frac{4}{27\lambda}\tilde\phi^3\right)}$$

$$=6i\int_{\mathcal{C}} d\phi d\tilde\phi\ P_{abc}P_{\underline{bde}}\phi_c\underline{\phi_d\phi_e}e^{i\left(P\phi^3+\phi^2\tilde\phi-\frac{4}{27\lambda}\tilde\phi^3\right)}$$

$$=2\int_{\mathcal{C}} d\phi d\tilde\phi\ P_{abc}\phi_c\left(\partial_b^\phi e^{iP\phi^3}\right)e^{i\left(\phi^2\tilde\phi-\frac{4}{27\lambda}\tilde\phi^3\right)}$$

Partial integration

$$=-2P_{abb}\Psi(P)-4i\int_{\mathcal{C}} d\phi d\tilde\phi\ P_{abc}\phi_b\phi_c\tilde\phi\, e^{i\left(P\phi^3+\phi^2\tilde\phi-\frac{4}{27\lambda}\tilde\phi^3\right)}$$

$$=\ldots$$

Contributions from the boundaries have been ignored.

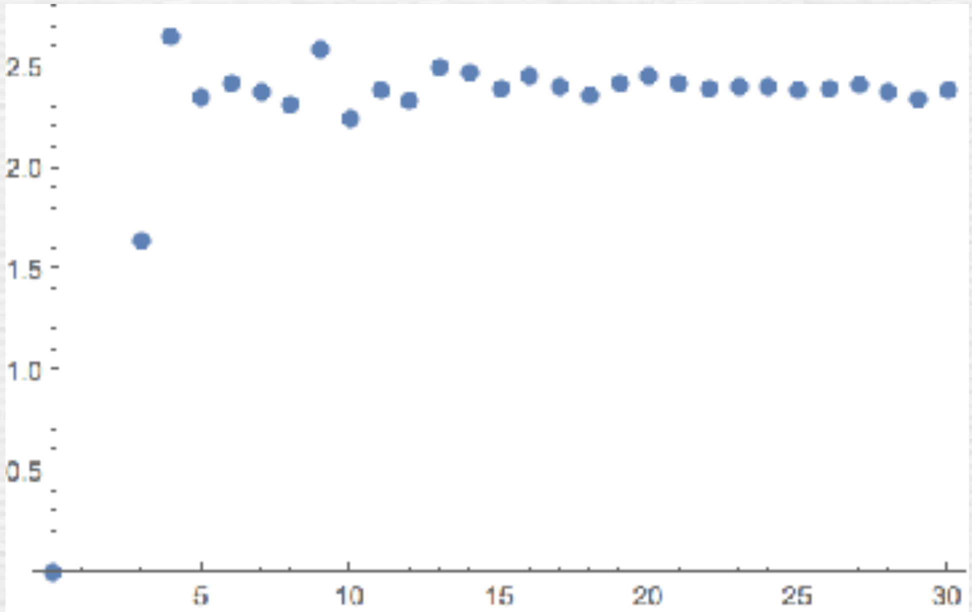
(Discussed shortly)

- **Convergence of the expression and regularization**

Conditionally convergent for generic P_{abc} except for **special** P_{abc} discussed later. The convergence is due to the **infinitely fast oscillation of integrand at infinity**.

eg. $$\int_{x^2+y^2\le R} dxdy\ e^{i(x^3-y^3)}$$

Well-known eg.:
Airy function, Fresnel integral



R

We rather use **ε-prescription** with a similar role as R :

Eg: $$\Psi(P)=\lim_{\epsilon\to+0}\int_{\mathbb{R}^{N+1}} d\phi d\tilde{\phi}\ e^{i\left(P\phi^3+\phi^2\tilde{\phi}-\frac{4}{27\lambda}\tilde{\phi}^3\right)-\epsilon(\phi^2+\tilde{\phi}^2)}$$

- **The ε-prescription**

The violation of the constraints

$$\epsilon \int_{R^{N+1}} d\phi d\tilde{\phi} \left(c_1 P_{abc}\phi_b\phi_c + c_2\, \phi_a \tilde{\phi} \right) e^{i\left(P\phi^3+\phi^2\tilde{\phi}-\frac{4}{27\lambda}\tilde{\phi}^3\right)-\epsilon\phi^2-\epsilon\tilde{\phi}^2}$$

c_1, c_2: Numerical constants

→ 0 for ε→+0

because the integral converges.

Numerical studies (later) also support the result.

$\mathcal{H}_a \Psi(P)$ ~ 10^{-8} for N=2

~ 10^{-2} for N=3

- **λ>0 is required for the physical sensibility of the wave function**

For λ=0 $\Psi(P) = \int_{R^N} d\phi \; e^{P\phi^3} \sim |P|^{-\frac{N}{3}} f(\Omega_P)$

$P_{abc} \to 0$ is favored. **Physically, nothing is favored**.

For λ<0

No critical (stationary) points exist, as shown later.

→ **No apparent peaks of the wave function**

For λ>0, various peaks and structures exist.

λ=Λ: Positive cosmological constant (N=1 CTM ⟷ FRW)

Summary of my part

- Time is introduced to tensor model.
  Time is gauged : CTM $\sim$ ADM

- Classical CTM → GR system in H-J formalism
  In a formal continuum limit $\sim$ presetting a macroscopic space

- Want to find the formal continuum limit as a peak of wave function

$$\Psi(P) = \int_{\mathbb{R}^{N+1}} d\phi d\tilde{\phi}\ e^{i\left(P\phi^3 + \phi^2\tilde{\phi} - \frac{4}{27\lambda}\tilde{\phi}^3\right)}$$

- λ>0 required for the sensibility