Emergent Symmetries in the Canonical Tensor Model

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Discrete Approaches to the Dynamics of Fields and Space-Time
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Outline

Highlighting of symmetries

Symmetries in physics

- Underlying symmetry of the system
- Symmetries of specific configurations
- Underlying symmetry determines the relevant configurations?

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Highlighting of symmetries

Physical quantities with some underlying symmetries may have a highlighting mechanism for partial symmetries.

$$\Psi(P) = \int d\phi e^{iS(\phi,P)}, \quad d\phi = \prod_{a=1}^{N} d\phi_a \tag{1}$$

Consider a symmetry operation:

$$S(\phi^{(g)}, P^{(g)}) = S(\phi, P)$$
$$d\phi^{(g)} = d\phi$$

Where

$$\phi_a^{(g)} = R(g)_a{}^b \phi_b, \quad P_i^{(g)} = \tilde{R}_i{}^j(g) P_j \tag{2}$$

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Approximation by stationary phase approximation:

$$\Psi(P) \approx \sum_{\sigma} a_{\sigma} e^{iS(\phi^{\sigma}, P)} \approx 0 \quad (3)$$

ϕ^{σ}: critical points $\left.\frac{\partial S}{\partial \phi_a}\right|_{\phi=\phi^{\sigma}} = 0$

a_{σ}: pre-factor due to Gaussian integration

Highlighting of symmetries

Symmetric configurations: $R(h)_i{}^j P_j = P_i$

$$S(\phi, P) = S(\phi^{(h)}, P^{(h)}) = S(\phi^{(h)}, P) \tag{4}$$

Orbit of $\phi^{(h)}$ with the same phase!

$$\Psi(P) \approx \sum_{\sigma} Vol(H) \left| \frac{\partial \phi}{\partial h} \right| a_{\sigma} e^{iS(\phi^{\sigma}, P)} + \sum_{\sigma'} a_{\sigma'} e^{iS(\phi^{\sigma'}, P)} \tag{5}$$

Highlighting of symmetries

Example, ϕ_a real N-dimensional vector and P_{abc} real symmetric N-dimensional 3-tensor:

$$\Psi(P) = \int \prod_a d\phi_a e^{iS(\phi,P)-\epsilon\phi^2}$$
$$S = \phi^2 + P_{abc}\phi_a\phi_b\phi_c$$

Underlying $O(N)$ symmetry:

$$\phi_a \to T_{aa'}\phi_{a'} \quad (6)$$
$$P_{abc} \to T_{aa'}T_{bb'}T_{cc'}P_{a'b'c'} \quad (7)$$

Highlighting of symmetries

$N = 4$: $G = O(4)$, $H = O(3), O(2)$

$$S = \phi^2 + (x + z_1)\phi_1^2\phi_4 + (x + z_2)\phi_2^2\phi_4 + x\phi_3^2\phi_4 + y\phi_4^2 \qquad (8)$$

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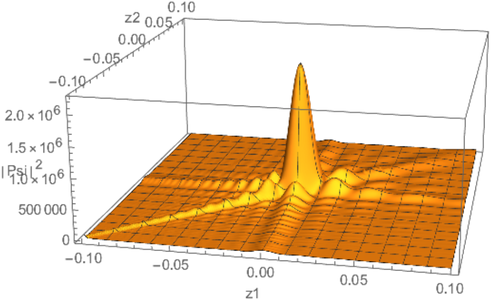


Canonical Tensor Model

- Model for QG in the Canonical framework
- Classically: $\{P, Q\}$ invariant under $O(N)$ transformations
- Invariance remains in Quantum model
- Underlying O(N) symmetry can be used for the mechanism!

CTM wave function

$$\Psi(P) = \int d\phi d\tilde{\phi}\; e^{i(P\phi^3+\phi^2\tilde{\phi}-\frac{4}{27\lambda}\tilde{\phi}^3)} \tag{9}$$

Similar to general setup for the mechanism with

$$S(\phi, P) = P\phi^3 + \phi^2\tilde{\phi} - \frac{4}{27\lambda}\tilde{\phi}^3. \tag{10}$$

$O(N)$ symmetry $\rightarrow$ investigate $O(n)$ subgroups

CTM wave function

Stationary phase analysis

$$\left.\frac{\partial S}{\partial \phi_a}\right|_{\phi=\phi^\sigma} = 0 \Rightarrow 3(P\phi^2)_a + 2\phi_a\tilde{\phi} = 0$$

$$\phi^2 - \frac{4}{9\lambda}\tilde{\phi}^2 = 0$$

$$\Rightarrow \lambda = \left.\frac{(P\phi^3)^2}{(\phi^2)^3}\right|_{\phi=\phi^\sigma} \qquad (11)$$

So $\lambda > 0$ for interesting behaviour!

Evaluation of the wave function

$$(\phi, \tilde{\phi}) \to \phi, \quad P\phi^3 + \phi^2\tilde{\phi} - \frac{4}{27\lambda}\tilde{\phi}^3 \to \tilde{P}\phi^3$$

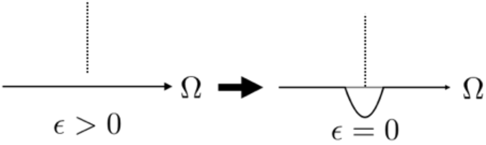
$$\Psi(P) = \int d\phi d\tilde{\phi}\; e^{i(P\phi^3 + \phi^2\tilde{\phi} - \frac{4}{27\lambda}\tilde{\phi}^3)} \to \int d\phi\; e^{i\tilde{P}\phi^3}$$

Hyperspherical coordinates $\phi \to \phi_\Omega r$

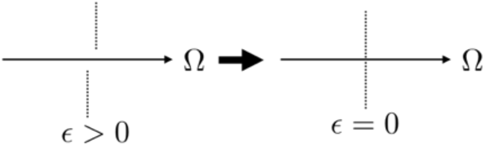
$$\begin{aligned}\Psi(P) &= \int_{S^N} d\Omega \int_0^\infty dr\; r^N e^{i\tilde{P}\phi_\Omega^3 r^3 - \epsilon r^3} \\ &= \frac{1}{3}\Gamma\left(\frac{1+N}{3}\right)\int_{S^N} d\Omega(\epsilon - i\tilde{P}\phi_\Omega^3)^{-\frac{N+1}{3}}\end{aligned}$$

$\epsilon \to 0_+$

Generic P_{abc} will have a well defined $\epsilon \to 0_+$ limit



However, sometimes this doesn't work:



Deforming contour

Generic case: deform contour.

$$\Omega'_j = \Omega_j + i\delta \frac{\partial(P\phi_\Omega^3)}{\partial\Omega_j} \tag{12}$$

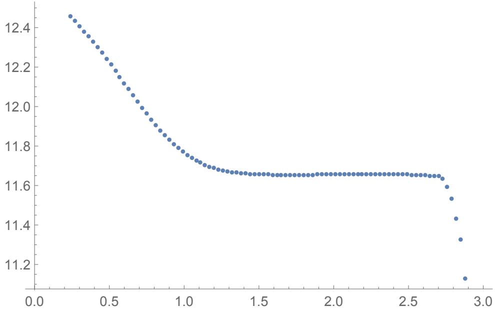
For small δ this is like ϵ:

$$\tilde{P}\phi_{\Omega'} = \tilde{P}\phi_\Omega^3 + i\delta \sum_{j=1}^{N} \left(\frac{\partial(\tilde{P}_\Omega^3)}{\partial\Omega_j} \right)^2 + \mathcal{O}(\delta^2) \tag{13}$$

Safe to take $\epsilon \to 0_+$ limit:

$$\Psi(P) = \frac{1}{3}\Gamma\left(\frac{N+1}{3}\right) \int_{S^N} d\Omega \left|\frac{\partial\Omega'}{\partial\Omega}\right| (-i\tilde{P}\phi_{\Omega'}^3)^{-\frac{N+1}{3}} \tag{14}$$

Deforming contour: check



Here: $\delta = 10^{-\Delta}$

$N = 3$

$$S = x(\phi_1^2 + \phi_2^2)\phi_3 + y\phi_3^3 + z\phi_1\phi_3^2 + \phi^2\tilde{\phi} - \frac{4}{27\lambda}\tilde{\phi}^3$$

$O(2)$ symmetry for $z = 0$.

$$\phi = \tilde{\phi} = 0$$

$$\phi_1 = \phi_2 = 0, \ \phi_3 = -\frac{2}{3y}\tilde{\phi}, \quad \lambda = y^2$$

$$\phi_1^2 + \phi_2^2 = \frac{2x - 3y}{x^3}\tilde{\phi}^2, \ \phi_3 = -\frac{1}{x}\tilde{\phi}, \ \lambda = \frac{4x^3}{27(x-y)}$$

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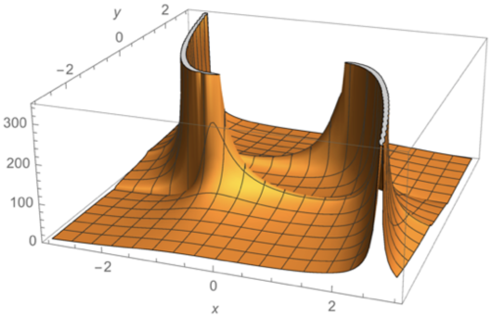
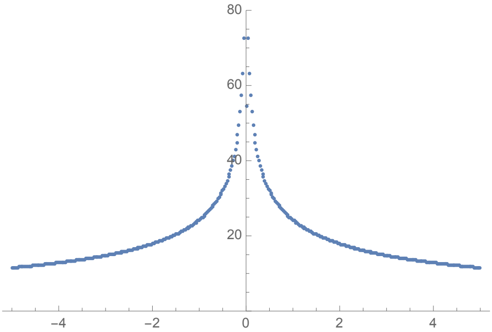


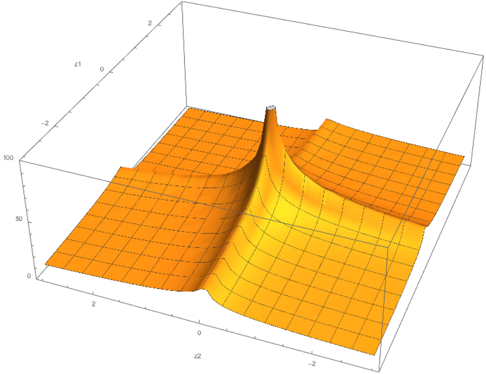
$y = x - \frac{4}{27\lambda}x^3$ and $\lambda = y^2$

$N = 3$



$z = 0$ clearly the preferred configuration: symmetric configuration!

$N = 4$



$z_i = 0$ clearly the preferred configuration: symmetric configuration!

Simple case with general N

$$S = \sum_{i}^{N-1} x_i \phi_i^2 \phi_N + y\phi_N^3 + \phi^2 \tilde{\phi} - \frac{4}{27\lambda}\tilde{\phi}^3$$

For $x_i = x$ $SO(N-1)$ symmetric. Reducible to a single integration:

$$\Psi(x_i, y) = \frac{2\pi^{\frac{N-1}{2}}}{3}\Gamma\left(\frac{5-N}{6}\right) \mathrm{Re}\left[\int d\tilde{\phi} \prod_{i=1}^{N-1} \frac{1}{\sqrt{-i(x_i+\tilde{\phi})}} (-ih(\tilde{\phi}))^{\frac{N-5}{6}}\right]$$

$$h(\tilde{\phi}) = y + \tilde{\phi} - \frac{4}{27\lambda}\tilde{\phi}^3$$

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$$h(\tilde{\phi}) = y + \tilde{\phi} - \frac{4}{27\lambda} \tilde{\phi}^3$$

Gives a kind of toy model to test some large N behaviour.

Simple case with general N

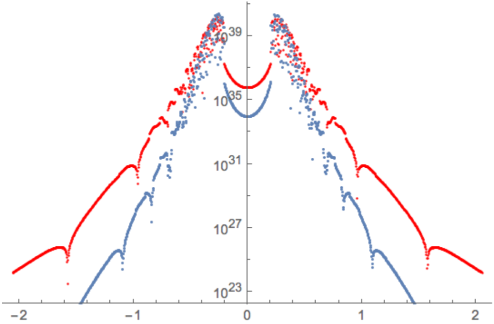
Possible to create representations of $SO(n_1)$ subgroups

$$\vec{x} = (x_1, x_1, x_2, x_2, x_2, x_3, \ldots) \tag{15}$$

However; just two subgroups can have singular behaviour at the same time due to the cubic form of $h(\tilde{\phi})$:

Simple case with general N

Large N behaviour nontrivial



Simple case with general N

Product groups: $N = 7$, $H = SO(6)$, $SO(3)SO(3)$

$SO(6)$

$$\phi_1^2 + \phi_2^2 + \ldots + \phi_6^2 = R^2\tilde{\phi}^2, \qquad \phi_7 = -\frac{1}{x}\tilde{\phi}$$

$SO(3)$x$SO(3)$

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = R_1^2\tilde{\phi}^2, \qquad \phi_7 = -\frac{1}{x_1}\tilde{\phi}$$

and

$$\phi_4^2 + \phi_5^2 + \phi_6^2 = R_2^2\tilde{\phi}^2, \qquad \phi_7 = -\frac{1}{x_2}\tilde{\phi}$$

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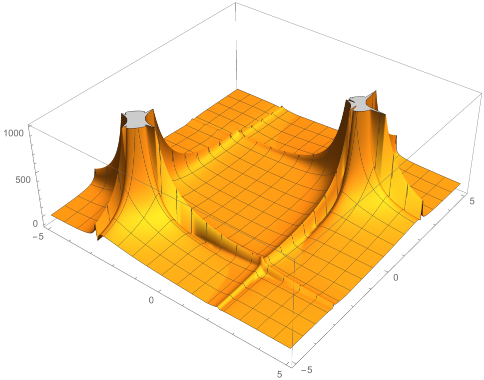

Simpler symmetry preferred in the simplified model

Summary

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- We can understand some of the symmetry properties by a simple mechanism: symmetric configurations preferred
- Known solutions have very rich structure

Future research

- Investigate Hilbert space:
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  - General properties of functions
  - Timeflow
- Other wave functions
- Emergent spaces
- Many more...

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