Theory of metallic transport in strongly coupled matter

3. Hydrodynamics and conductivity bounds

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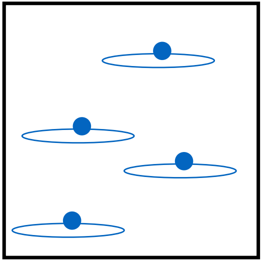
collisionless:

$\omega t_{\mathrm{ee}} \gg 1$



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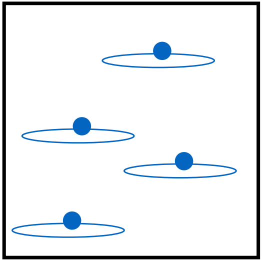

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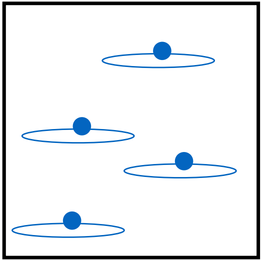

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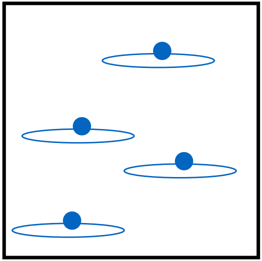

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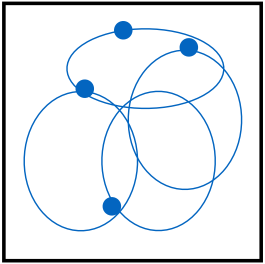

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Collisionless-to-Hydrodynamic Crossover

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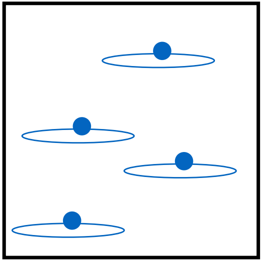

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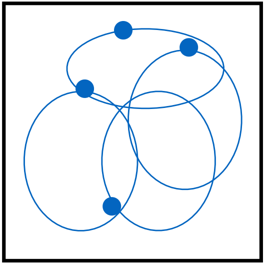

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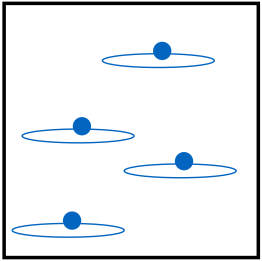

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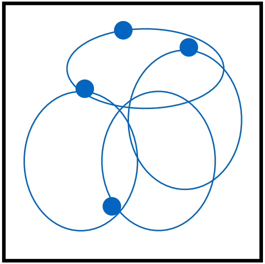
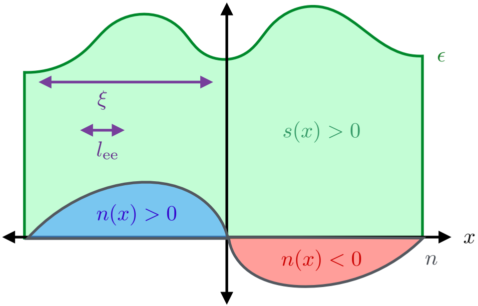


- universal *classical* effective field theory
- slow modes: conserved quantities



- (relatively) universal features of transport in the hydrodynamic regime: smooth potentials on the scale ℓ_{ee}

[Andreev, Kivelson, Spivak; 1011.3068], [Lucas; 1506.02662], [Lucas *et al*; 1510.01738], [Lucas, Hartnoll; 1704.07384] we now turn to a relativistic fluid with conserved U(1) charge, energy and momentum:

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$$\partial_\mu J^\mu = 0, \qquad \partial_\mu T^{\mu\nu} = 0.$$

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- thermodynamics controlled by single function P:

$$\mathrm{d}P = \rho \mathrm{d}\mu + s\mathrm{d}T, \quad \epsilon + P = \mu\rho + Ts$$

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- zeroth order hydrodynamic equations of motion, together with $\mathrm{d}P = n\mathrm{d}\mu + s\mathrm{d}T$, $\epsilon + P = \mu\rho + Ts$:

$$\partial_\mu \left(Tsu^\mu\right) = 0.$$

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- look for J^μ, $T^{\mu\nu}$ as an expansion in *derivatives*:

$$J^\mu = \rho u^\mu + a_1(\tau_{\mathrm{ee}}\partial^\mu)\mu + a_2(\tau_{\mathrm{ee}}\partial^\mu)T + \cdots$$

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- final answer at first order:

$$J^\mu = \rho u^\mu - T\sigma_{\mathrm{Q}} \mathcal{P}^{\mu\nu}\partial_\nu \frac{\mu}{T},$$

$$T^{\mu\nu} = (\epsilon + P)u^\mu u^\nu + P\eta^{\mu\nu} - \eta\mathcal{P}^{\mu\rho}\mathcal{P}^{\nu\sigma}\partial_{(\rho}u_{\sigma)} - \left(\zeta - \frac{2\eta}{d}\right)\mathcal{P}^{\mu\nu}\partial_\rho u^\rho$$

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- the entropy current is the heat current:

$$TJ^\mu_{\mathrm{s}} = (\epsilon + P)u^\mu - \mu J^\mu$$

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- memory matrix formalism $\implies$

$$\sigma \approx \frac{\rho^2 \tau}{\epsilon + P}$$
$$\frac{1}{\tau} \approx \varepsilon^2 \left(\frac{\partial \rho}{\partial \mu}\right)^2 \left(\frac{1}{\sigma_{\mathrm{Q}}(\epsilon + P)} + \frac{4\eta\mu^2}{\xi^2(\epsilon+P)^3}\right).$$

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- this formula looks complicated...we will discover an easier derivation later need to know behavior of η, σ_{Q} to make transport predictions:

$$\eta, \sigma_{\mathrm{Q}} \sim \tau_{\mathrm{ee}}.$$

- Fermi liquid (doped graphene):

$$\tau_{\mathrm{ee}} \sim \frac{\mu}{T^2}, \qquad \eta \sim \epsilon\tau_{\mathrm{ee}} \sim \frac{\mu^{d+2}}{T^2}$$

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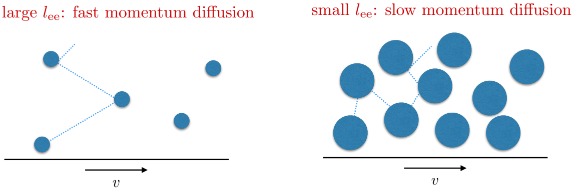

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$$R \sim \eta \sim \frac{1}{T^2}$$

Experimental Evidence for Viscous Flow

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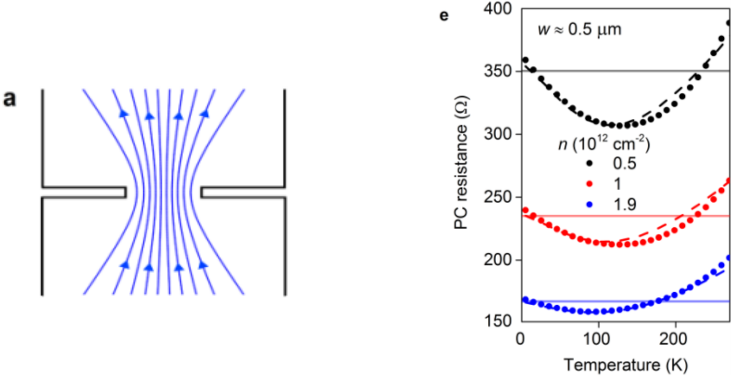
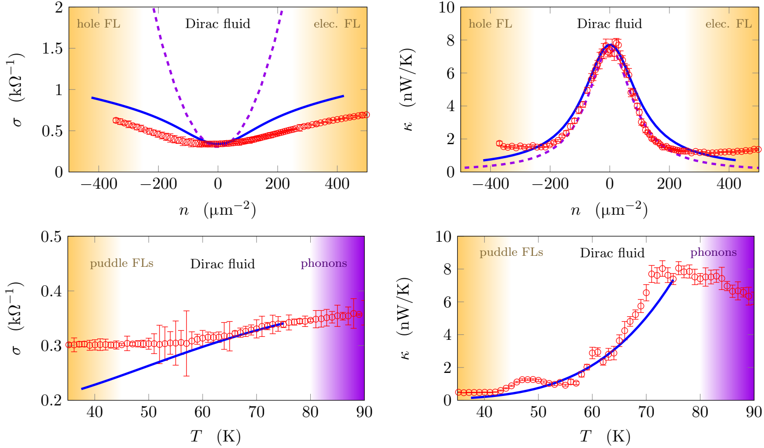
- non-monotonic $\rho(T)$ observed in graphene Fermi liquid

[Kumar *et al*; 1703.06672]



Thermal Transport at Charge Neutrality



[Lucas, Crossno, Fong, Kim, Sachdev; 1510.01738]

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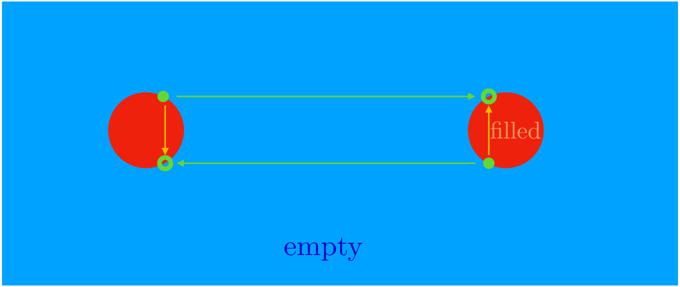

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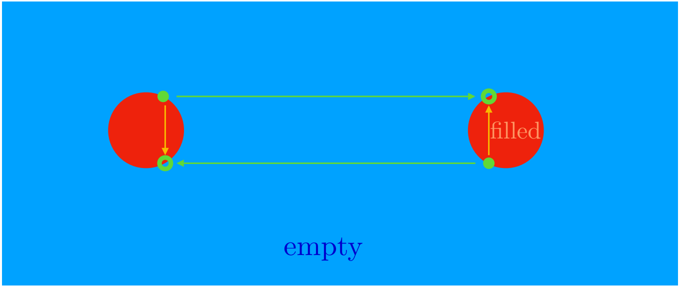

- quasiparticle 2-body collisions conserve momentum $\Longrightarrow$ number of quasiparticles on each Fermi surface (approximately) conserved
  - (weak?) higher order effects spoil this new conservation law...

hydrodynamics of conserved 'scalar' quantities n^A (charge, heat, "imbalance", etc.), and momentum:

- charges:

$$0 = \partial_i \left(n^A v_i + \Sigma^{AB} \left(E_i - \partial_i \mu^B \right) \right)$$

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(n^A shows up in both places – second law)
- recall that $\eta, \Sigma \sim \tau_{\mathrm{ee}}$; we are interested in theories with multiple charges with
$$\lim_{T\to 0} n^A > 0$$
(this leads to ≥ 1 diffusive modes with $D(T\to 0)\to\infty$)

Entropy Production

- Joule heating:

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- theorem: [Lucas; 1506.02662], [Lucas, Hartnoll; 1704.07384]

$$\mathcal{R}[J_i, Q_i] \equiv \frac{1}{V_d}\int \mathrm{d}^d\mathbf{x}\; T\dot{s}[J_i^A, v_i]$$

obeys

$$\frac{\mathcal{R}[J_i^A, v_i]}{(V_d^{-1}\int J_x^{\text{charge}})^2} \geq \frac{1}{\sigma}, \quad \text{for } \textit{arbitrary } J_i^A \text{ if } \partial_i J_i^A = 0.$$

- first step: show that

$$\min \mathcal{R}[J_i^A, v_i] = J_i^A \left(\sigma^{-1}\right)_{ij}^{AB} J_j^B, \quad \text{if} \quad \int \mathrm{d}^d\mathbf{x} J_i^A = \text{fixed}.$$

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- consider $J_i^A = \overline{J}_i^A + \hat{J}_i^A$, $v_i = \overline{v}_i + \hat{v}_i$ ($\overline{J}_i^A$, $\overline{v}_i$ solve EOM):

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- similarly, observe that

$$\mathcal{R}[\overline{J}_i^A, \overline{v}_i] = \frac{1}{V_d}\int \mathrm{d}^d\mathbf{x}\, E_i^A \overline{J}_i^A.$$

- $\mathcal{R}$ is positive definite $\implies \mathcal{R}[J_i^A, v_i] \geq \mathcal{R}[\overline{J}_i^A, \overline{v}_i]$

- remains to minimize over the $\int J_i^A$, but this follows from basic linear algebra:

$$\frac{\partial}{\partial x_k}\frac{M_{ij}x_ix_j}{(x_la_l)^2} = \frac{2}{(x_la_l)^2}\left[M_{ki}x_i - \frac{x_iM_{ij}x_j}{x_la_l}a_k\right]$$

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- simple derivation of perturbative result:
  - plug in x-independent J_i^A, v_i, and minimize

- consider the case with one conserved current:

$$T\dot{s} = \int \frac{\mathrm{d}^d\mathbf{x}}{V_d} \left[\frac{(J_i - nv_i)^2}{\Sigma} + \eta_{ijkl}\partial_i v_j \partial_k v_l \right]$$

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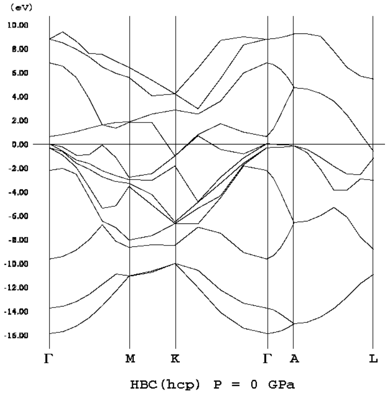

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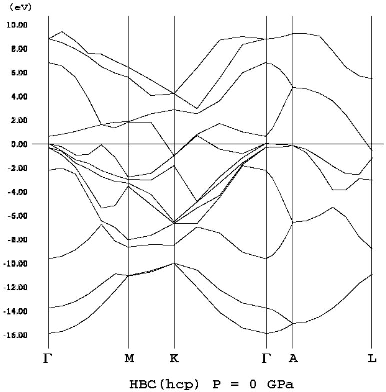

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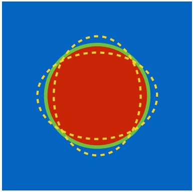

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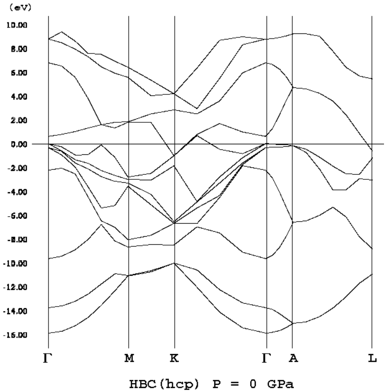

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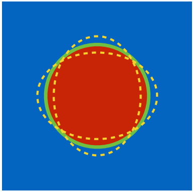

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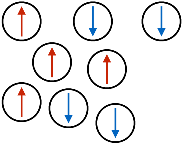
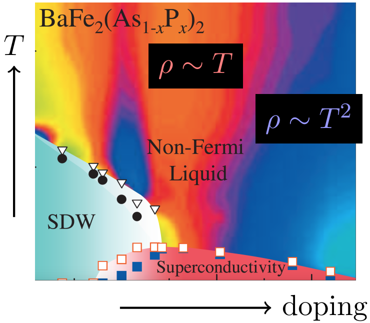


Phenomenology: Enhanced Resistivity near Criticality

- sample phase diagram:



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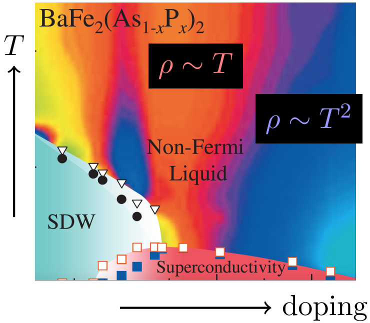

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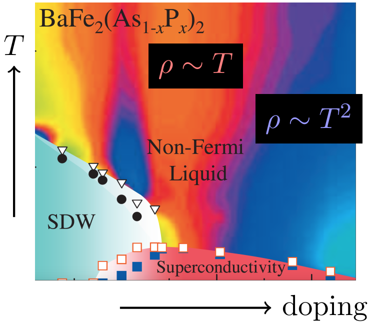

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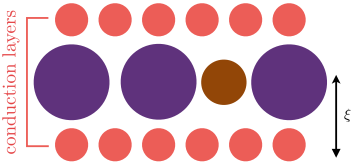
- ρ *increases* near quantum critical points?

- out of plane impurities for quasi-2d metal?



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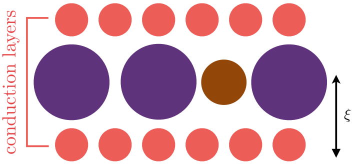

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- failure of Mattheisen's rule at weak disorder?
  - observed in some heavy fermions? [Kadowaki, Woods (1986)]

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- $\sigma_{\mathrm{dc}} > 0$ at $k = \infty$ ('infinite' disorder)

- using "Drude" momentum relaxation time approximation:

$$0 = \rho E_i - \frac{\epsilon + P}{\tau} v_i,$$

but using that

$$J_i = \sigma_{\mathrm{Q}} E_i + \rho v_i,$$

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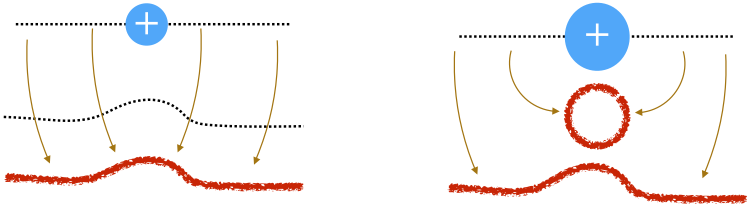
- this is *not* really describing holographic result...

$$\sigma(\omega) = \frac{1}{1 - \mathrm{i}\omega\tau} \left(C_1 - \sigma_{\mathrm{Q}} + \frac{C_2 \rho^2}{k^2} \right) + \sigma_{\mathrm{Q}} + \mathrm{O}(\omega, \tau^{-1})$$

with $\tau^{-1} \sim a_2 k^2 + a_4 k^4 + \cdots$

[Davison, Goutéraux; 1505.05092], [Blake; 1505.06992]

a more sophisticated holographic approach:



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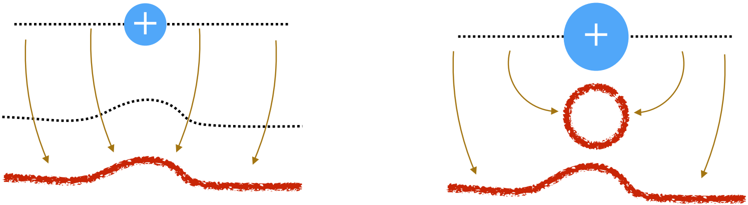

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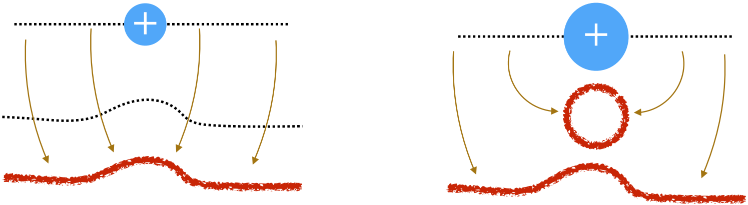

- we consider holographic black holes with inhomogeneous ‘hair’, but connected black hole horizon at finite temperature T
- but the possibility of “floating” black holes is also interesting...and may describe non-metallic physics
  [Horowitz, Iqbal, Santos, Way; 1412.1830]

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$$\partial_M \left(\sqrt{-g} Z F^{MN}\right) = 0.$$

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$$\sqrt{-g} Z F^{ir} \to \mathcal{I}^i \equiv \gamma^{ij} Z (E_j - \partial_j \tilde{\mu}), \quad \nabla_i \mathcal{I}^i = 0.$$

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Membrane Paradigm with Inhomogeneity

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- diffusion in emergent horizon fluid governs transport!

[Donos, Gauntlett; 1506.01360]

Emergent Horizon Fluid

- even more generally, using near horizon solution

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}r^2 - (2\pi Tr)^2\mathrm{d}t^2 + \gamma_{ij}\mathrm{d}x^i\mathrm{d}x^j + \cdots , \\ A &= \pi T r^2 \mathcal{Q}(\mathbf{x})\mathrm{d}t + \cdots , \end{aligned}$$

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- Thomson's principle for horizon fluid:
[Grozdanov, Lucas, Sachdev, Schalm; 1507.00003]

$$\mathcal{P}[\mathcal{I},\mathcal{J}] = \mathbb{E}\left[\frac{\sqrt{\gamma}}{Z}\left(\mathcal{I} - \frac{\mathcal{Q}\mathcal{J}}{4\pi T}\right)^2 + \frac{2\sqrt{\gamma}}{(4\pi T)^2}\nabla^{(i}\mathcal{J}^{j)}\nabla_{(i}\mathcal{J}_{j)} + \cdots\right]$$

$$\underbrace{\nabla_i \mathcal{I}^i = 0}_{\text{charge conservation}} \qquad \underbrace{\nabla_i \mathcal{J}^i = 0}_{\text{heat conservation}}$$

consider the Einstein-Maxwell-AdS_4 theory (M2 branes). if $\sigma_{ij} = \sigma\delta_{ij}$, [Grozdanov, Lucas, Sachdev, Schalm; 1507.00003]

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- mean field approaches:
  [Blake, Tong; 1308.4970]

$$\sigma = 1 + \frac{4\pi\mathcal{Q}^2}{\mathcal{S}m^2} \geq 1.$$

Thermal Conductivity Bounds

in M2 brane theory, if $\bar{\kappa}_{ij} = \bar{\kappa}\delta_{ij}$,

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$$\bar{\kappa} = \frac{4\pi \mathcal{S} T}{m^2}, \quad \mathcal{S} \geq \frac{\pi m^2}{3}.$$

- EMD theory: [Grozdanov, Lucas, Schalm; 1511.05970]

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- these bounds are not ‘universal’ to all QFT as they depend on V and Z, but can be universal for certain QFTs (changing V, Z changes dual QFT)