Theory of metallic transport in strongly coupled matter

2. Memory matrix formalism

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- this lecture: with 'mild' assumptions, we prove that this result is *exact for any QFT*, to leading order in a perturbatively weak amount of disorder.

a (mostly complete) proof: [Hartnoll, Hofman; 1201.3917], but a few subtleties only addressed later...

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- we will show that (here $\hbar = 1$)

$$\frac{1}{\tau} \approx \frac{1}{\mathcal{M}} \lim_{\omega\to 0} \frac{1}{\omega} \mathrm{Im}\left(G^{\mathrm{R}}_{\dot{P}_x\dot{P}_x}(\omega)\right) + \mathrm{O}(\varepsilon^3), \qquad \dot{P}_x = \mathrm{i}[H, P_x].$$

the momentum relaxation time is given by the *spectral weight* of $[H, P_x]$

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- define $(A(t)|B) \equiv \mathcal{C}_{AB}(t)$. basic manipulations give

$$\Theta(t)\partial_t \mathcal{C}_{AB}(t) = -\mathrm{i}T\Theta(t)\langle[A(t), B]\rangle_T = -TG^{\mathrm{R}}_{AB}(t).$$

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- integrate above from $t = 0$ to $t = \infty$:

$$\mathcal{C}_{AB}(t=0) = TG^{\mathrm{R}}_{AB}(\omega = 0) = \underbrace{T\chi_{AB}}_{\text{static susceptibility}}$$

An "Operator Hilbert Space"

- Laplace transform related to conductivity:

$$\mathcal{C}_{AB}(z) = \int_0^\infty \mathrm{d}t \mathrm{e}^{\mathrm{i}zt} \mathcal{C}_{AB}(t) = \frac{T}{\mathrm{i}z} \left(G^{\mathrm{R}}_{AB}(z) - G^{\mathrm{R}}_{AB}(0) \right)$$

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- define the Liouvillian L:

$$\mathrm{i}L|A) = |\dot{A}), \quad \mathrm{e}^{\mathrm{i}Lt}|A) = |A(t))$$

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our goal is to compute $\sigma(\omega) = \frac{1}{T}\mathcal{C}_{J_x J_x}(z = \omega + \mathrm{i}0^+)$

- suppose that $|P)$ is conserved – i.e., $L|P) = 0$. then consider

$$
\begin{aligned}
(A|\mathrm{i}(z-L)^{-1}|B) &\supset \frac{(A|P)(P|\mathrm{i}(z-L)^{-1}|P)(P|B)}{(P|P)^2} \\
&\supset \frac{(A|P)(P|B)}{(P|P)} \times \frac{\mathrm{i}}{z}
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- long lived quantities will lead to nearly singular Green's functions as $\omega \to 0$

assumption: momentum P_x is the only (almost) conserved operator where $(P_x|J_x) \neq 0$.

Outline of the Rest of the Computation

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1. formal re-writing of σ_{AB} (matrix indices *only* include J_x, P_x):

$$\sigma_{AB} = \chi_{AC}(M(\omega) + N - \mathrm{i}\omega\chi)^{-1}_{CD}\chi_{DB}$$

- a component of the memory matrix $M_{PP} \sim \tau^{-1}$
- $N = 0$ until last lecture

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2. show that perturbatively

$$\sigma_{J_x J_x} \approx \frac{\chi^2_{J_x P_x}}{M_{P_x P_x}(\omega = 0) - \mathrm{i}\omega\chi_{P_x P_x}}$$

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3. give more useful expressions for $\chi_{J_xP_x}$, $\chi_{P_xP_x}$, $M_{P_xP_x}$

A Projection Matrix

we wish to separate degrees of freedom into:

slow $(A, B \in \{J_x, P_x\})$

$$\mathfrak{p} = \frac{1}{T} \sum_{AB} |A)\chi^{-1}_{AB}(B|$$

fast (all others)

$$\mathfrak{q} = 1 - \mathfrak{p}$$

e.g. $\mathfrak{p}|J_x) = |J_x)$, and $\mathfrak{q}|J_x) = 0$.

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- if we choose $|J_x)$ *to be slow*:

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- *schematically*: perform block matrix inversion and "integrating out" fast degrees of freedom

Integrating Out the Fast Modes

- since our basis $|J_x)$ is not orthogonal, it is easier to proceed differently. note the identity

  $$(z-L)^{-1} = (z - L\mathfrak{p} - L\mathfrak{q})^{-1} = (z - L\mathfrak{q})^{-1}(1 + L\mathfrak{p}(z-L)^{-1})$$

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$$\underbrace{\sigma_{AB}}_{\text{slow only}} - \frac{\mathrm{i}\chi_{AB}}{z} = \frac{\mathrm{i}}{T}(A|(z - L\mathfrak{q})^{-1}L\mathfrak{p}(z-L)^{-1}|B)$$

$$= \frac{\mathrm{i}}{T}\sum_{CD}(A|(z - L\mathfrak{q})^{-1}L|C)\chi^{-1}_{CD}\mathcal{C}_{DB}$$

$$= \frac{\mathrm{i}}{Tz}\sum_{CD}(A|L + L\mathfrak{q}(z - L\mathfrak{q})^{-1}L|C)\chi^{-1}_{CD}\mathcal{C}_{DB}$$

- the antisymmetric matrix

$$N_{AB} = \frac{\mathrm{i}}{T}(A|L|B) = \frac{1}{T}(A|\dot{B}) = -\frac{1}{T}(\dot{A}|B)$$

describes how slow operators mix among themselves

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- $N_{AB} = 0$ for us: J_x, P_x both time reversal odd

Simplifications

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- what we will show: if $\dot{P}_x \sim \epsilon$, for small ϵ:

$$\begin{pmatrix} M_{J_x J_x} & M_{J_x P_x} \\ M_{P_x J_x} & M_{P_x P_x} \end{pmatrix} \sim \begin{pmatrix} \epsilon^0 & \epsilon^2 \\ \epsilon^2 & \epsilon^2 \end{pmatrix}$$

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- taking $\omega \sim \epsilon^2$ small:

$$\sigma_{J_x J_x} = \frac{\chi^2_{J_x P_x}}{M_{P_x P_x} - \mathrm{i}\omega \chi_{P_x P_x}} \sim \frac{1}{\epsilon^2}$$

$M_{P_x P_x}$

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- the general form of H_{imp}:

$$H_{\mathrm{imp}} = \sum_\alpha \int \mathrm{d}^d\mathbf{x}\; h_\alpha(\mathbf{x})\mathcal{O}_\alpha(\mathbf{x})$$

  and as P_x generates translations:

$$\dot{P}_x = \mathrm{i}[\varepsilon H_{\mathrm{imp}}, P_x] = -\mathrm{i}\sum_\alpha \int \mathrm{d}^d\mathbf{x}\; h_\alpha(\mathbf{x})\partial_x\mathcal{O}_\alpha(\mathbf{x})$$

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- thus we write

$$|\dot{P}_x) = \epsilon \sum_\alpha \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} h_\alpha(-\mathbf{k}) k_x |\alpha(\mathbf{k}))$$

- translation invariance implies

$$(A(\mathbf{k})|B(\mathbf{q})) \propto \delta(\mathbf{k}+\mathbf{q})(A(\mathbf{k})|B(\mathbf{q}))$$

  and from above, $|\dot{P}_x)$ consists of $\mathbf{k} \neq \mathbf{0}$ operators

- thus we find:

$$\begin{aligned} M_{PP} &= \epsilon^2 \frac{\mathrm{i}}{T}(\dot{P}_x|(\omega - \mathfrak{q}L\mathfrak{q})^{-1}|\dot{P}_x) \\ &\approx \epsilon^2 \frac{\mathrm{i}}{T} \sum_{\alpha\beta} \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} k_x^2 h_\alpha(\mathbf{k})(\alpha(-\mathbf{k})|(\omega - L)^{-1}|\beta(\mathbf{k}))h_\beta(-\mathbf{k}) \\ &= \epsilon^2 \sum_{\alpha\beta} \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} k_x^2 h_\alpha(\mathbf{k}) h_\beta(-\mathbf{k}) \lim_{\omega\to 0} \frac{1}{\omega} \mathrm{Im}\left(G^{\mathrm{R}}_{\alpha\beta}(\mathbf{k},\omega)\right). \end{aligned}$$

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- similarly, using translation invariance we find:

$$M_{P_x J_x} \sim \epsilon \sum_\alpha \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d}(\alpha(\mathbf{k})|(\omega - \mathfrak{q}L\mathfrak{q})^{-1}|\dot{J}_x) = \epsilon\times 0 + \mathrm{O}(\epsilon^2)$$

- from a deformed thermal density matrix

$$\rho_{\mathbf{v}} = \exp[-\beta(H_0 - \mu Q - \mathbf{v} \cdot \mathbf{P})]$$

we may define susceptibilities via linear response:

$$\mathrm{tr}[\rho_{\mathbf{v}} J_x(\mathbf{x})] = \chi_{J_x P_x} v_x + \cdots \,, \qquad \mathrm{tr}[\rho_{\mathbf{v}} P_x(\mathbf{x})] = \chi_{P_x P_x} v_x + \cdots \,.$$

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- from a hydrodynamic limit, we identify

$$\chi_{J_x P_x} = \underbrace{\rho}_{\text{charge density}}, \qquad \chi_{P_x P_x} = \underbrace{\mathcal{M}}_{\text{generalized mass density}}$$

- thus we have derived

$$\sigma(\omega) = \frac{\rho^2}{\mathcal{M}} \times \frac{1}{\frac{1}{\tau} - \mathrm{i}\omega}.$$

for *any QFT* where only almost conserved operator that overlaps with J_x is P_x

- replace charge current J_x with heat current Q_x:

$$\chi_{Q_x P_x} = \underbrace{Ts}_{\sim \text{entropy density}}$$

$$\begin{pmatrix} \sigma & T\alpha \\ T\alpha & T\bar{\kappa} \end{pmatrix} = \begin{pmatrix} \rho^2 & Ts\rho \\ Ts\rho & (Ts)^2 \end{pmatrix} \times \frac{1}{\dfrac{\mathcal{M}}{\tau} - \mathrm{i}\omega\mathcal{M}}$$

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- broken time-reversal symmetry (last lecture)

- conformal field theory at finite T, deformed by scalar operator $\mathcal{O}$ of dimension $\mathit{\Delta}$ coupled to "random field":

$$H = H_{\mathrm{CFT}} - \int \mathrm{d}^d\mathbf{x}\, h(\mathbf{x})\mathcal{O}(\mathbf{x}).$$

$$\langle h(\mathbf{x})\rangle_{\mathrm{dis}} = 0, \qquad \langle h(\mathbf{x})h(\mathbf{y})\rangle_{\mathrm{dis}} = \varepsilon^2\delta(\mathbf{x}-\mathbf{y}).$$

QFT Deformed by One Operator

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$$\langle h(\mathbf{x})\rangle_{\mathrm{dis}} = 0, \qquad \langle h(\mathbf{x})h(\mathbf{y})\rangle_{\mathrm{dis}} = \varepsilon^2\delta(\mathbf{x}-\mathbf{y}).$$

- momentum relaxation: [Lucas, Sachdev, Schalm; 1401.7993]

$$\frac{\mathcal{M}}{\tau} = \frac{\epsilon + P}{\tau} \sim \varepsilon^2 \int \mathrm{d}^d\mathbf{k}\; k_x^2 \lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\left(G^{\mathrm{R}}_{\mathcal{O}\mathcal{O}}(\mathbf{k},\omega)\right) \sim \varepsilon^2 T^{2\Delta}$$

$$\sim T^{d+2}$$

$$\sim \frac{1}{T} \times T^{2\Delta-d-1}$$

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- **Harris criterion**: disorder is relevant if

$$\Delta < \frac{d}{2} + 1$$

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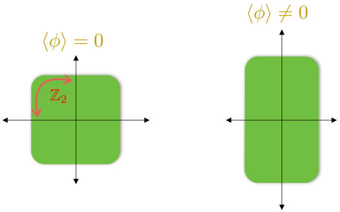

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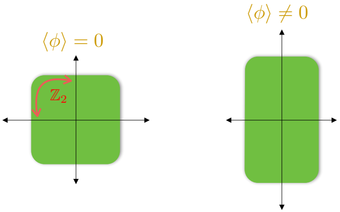

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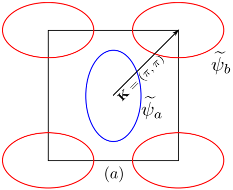
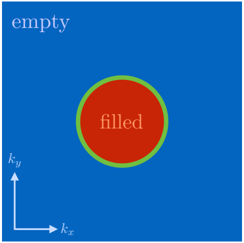
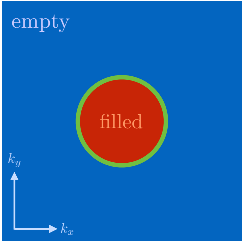
- spin density wave:
  [Patel, Sachdev; 1408.6549]



$$\rho \sim \frac{1}{\tau} \sim V_{\text{imp}}^2 + m_{\text{imp}}^2 T$$

Kinetic Theory: A Single Fermi Surface





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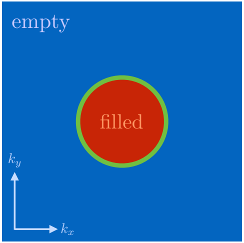

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[Lucas, Hartnoll; 1706.04621]

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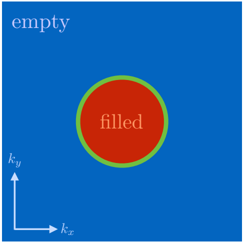

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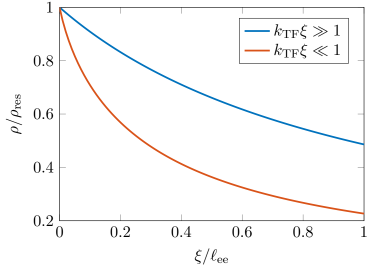
- interactions *always decrease* ρ:

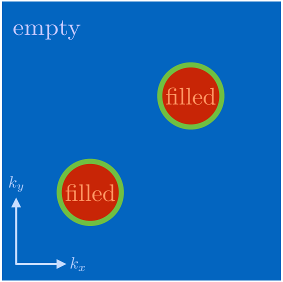


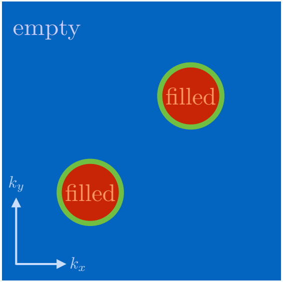
why? (next lecture)

toy model of single FS kinetics from:

[Guo *et al*; *PNAS*, 1607.07269]

Kinetic Theory: Two Fermi Surfaces





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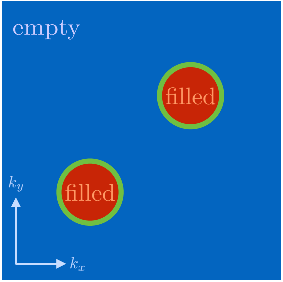

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  - charge in pocket 1
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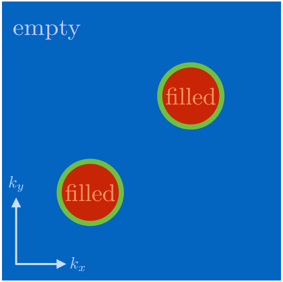

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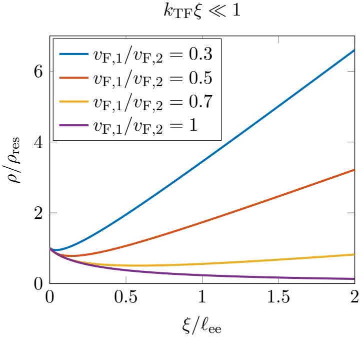
- numerical computation gives:



- why does this happen? (next lecture)

a brief holographic aside: [Lucas; 1501.05656]

- consider
  - scalar operator $\mathcal{O}$ dual to a field Φ in the bulk of AdS
  - a planar black hole in the bulk, with horizon $r = r_+$
  - the regular solution $\Phi(\mathbf{k}, r)$ to linearized bulk equations of motion

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$$\lim_{\omega \to 0} \frac{1}{\omega} \mathrm{Im}\left(G^{\mathrm{R}}_{\mathcal{O}\mathcal{O}}(\mathbf{k}, \omega)\right) \sim \frac{s}{4\pi} \Phi(\mathbf{k}, r_+)^2$$

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- early holographic derivations of conductivity: [Blake, Tong, Vegh; 1310.3832]

$$\sigma(\omega = 0) \sim \Phi(\mathbf{k}, r_+)^{-2}$$

  and are equivalent to more general formalism

- we perturbatively derived universal Drude peak:

$$\sigma(\omega) = \frac{\rho^2}{\mathcal{M}} \times \frac{1}{\frac{1}{\tau} - \mathrm{i}\omega}.$$

and showed τ is the momentum relaxation time:

$$\frac{\mathcal{M}}{\tau} = \epsilon^2 \sum_{\alpha\beta} \int \frac{\mathrm{d}^d\mathbf{k}}{(2\pi)^d} h_\alpha(\mathbf{k}) h_\beta(-\mathbf{k}) \lim_{\omega\to 0} \frac{1}{\omega} \mathrm{Im}\left(G^{\mathrm{R}}_{\alpha\beta}(\mathbf{k},\omega)\right).$$

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- τ^{-1} *sensitive to microscopic details*: non-universal T-scaling, possibly non-monotonic
- controlled (and useful!) but ultimately must go beyond