Anderson localization

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August 2017@ postech

"Geometry and Holography for Quantum Criticality"

outline

- What is Anderson localization and the Anderson transition?
- Analytic approach
  - Kubo formula
  - Weak localization theory (renewed interests due to the discovery of topological materials)
- Analytic approach
  - Renormalization group, scaling theory
  - Nonlinear sigma model
    - 3 and 10 universality classes
    - Higher order expansion, Borel-Pade resummation
  - Harris-Chayes inequality
- Numerical approach
  - Finite size scaling
    - Transfer matrix
      - 2D conformal invariance
    - Geometry dependent critical values
    - Conformal invariance in higher dimensions?
- Colloquium: *Deep learning approach to localization-delocalization transition*

Review articles

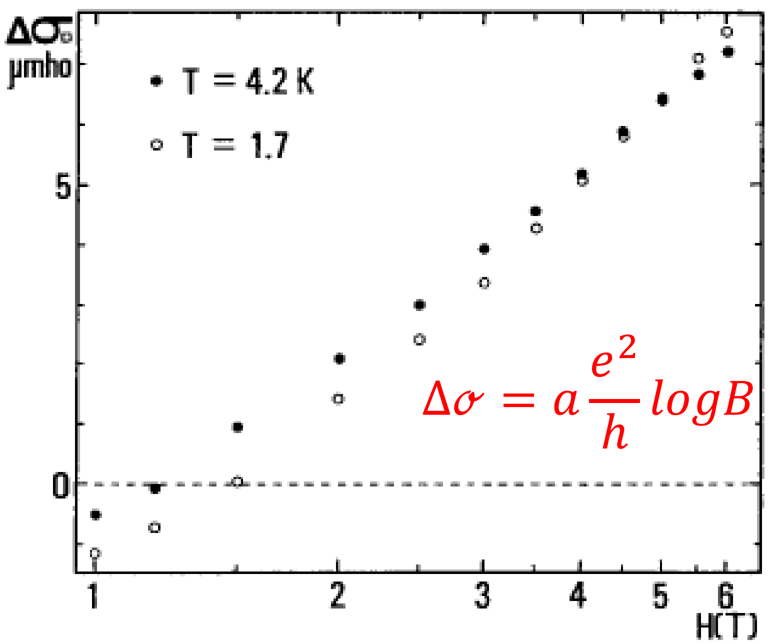
- P.A. Lee, T.V. Ramakrishnan: Rev. Mod. Phys. **57**, 287–337 (1985).
- *Anderson Localization*, Vol. 84 of Prog. Theor. Phys. Suppl.（1985）,open access: https://academic.oup.com/ptps/issue/volume/84
- B. Kramer, A. MacKinnon: Reports on Progress in Physics **56**(12), 1469(1993), numerical approaches.
- F. Evers, A.D. Mirlin, “*Anderson Transitions*”, Rev. Mod. Phys. 80, 1355 (2008), 10 universality classes, multifractality at AT.
- K. Slevin, T. Ohtsuki, “*Critical exponent for the Anderson transition in the three-dimensional orthogonal universality class*”, New J. Phys. 16, 015012(2014), high precision numerical simulations.

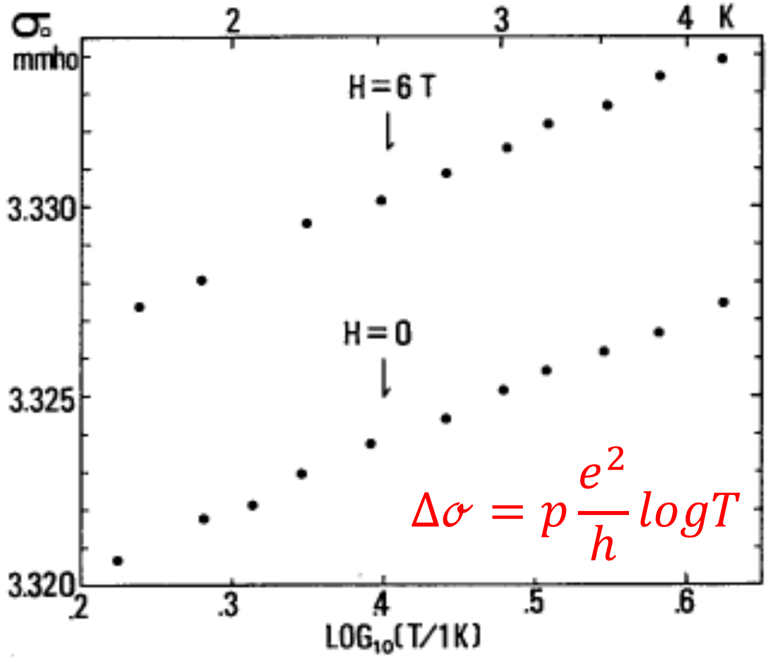
Books

- T. Dittrich et al. : “*Quantum transport and dissipation”*, Wiley-VHC (1998), weak localization theory.
- K. Efetov:“*Supersymmetry in Disorder and Chaos*”, Cambridge Univ. Press (1997), Nonlinear sigma model.
- E. Akkermans and G. Montambaux: “*Mesoscopic physics of electrons and photons*”, Cambridge Univ. Press (2007), weak localization theory.
- “*50 years of Anderson localization*”, ed. Abrahams, World scientific (2010)
- *“Conductor Insulator Quantum Phase Transitions”,* ed. V. Dobrosavljevic, N. Trivedi, J. M. Valles, Jr., Oxford Univ. Press (2012), interaction effect.

Various phenomena

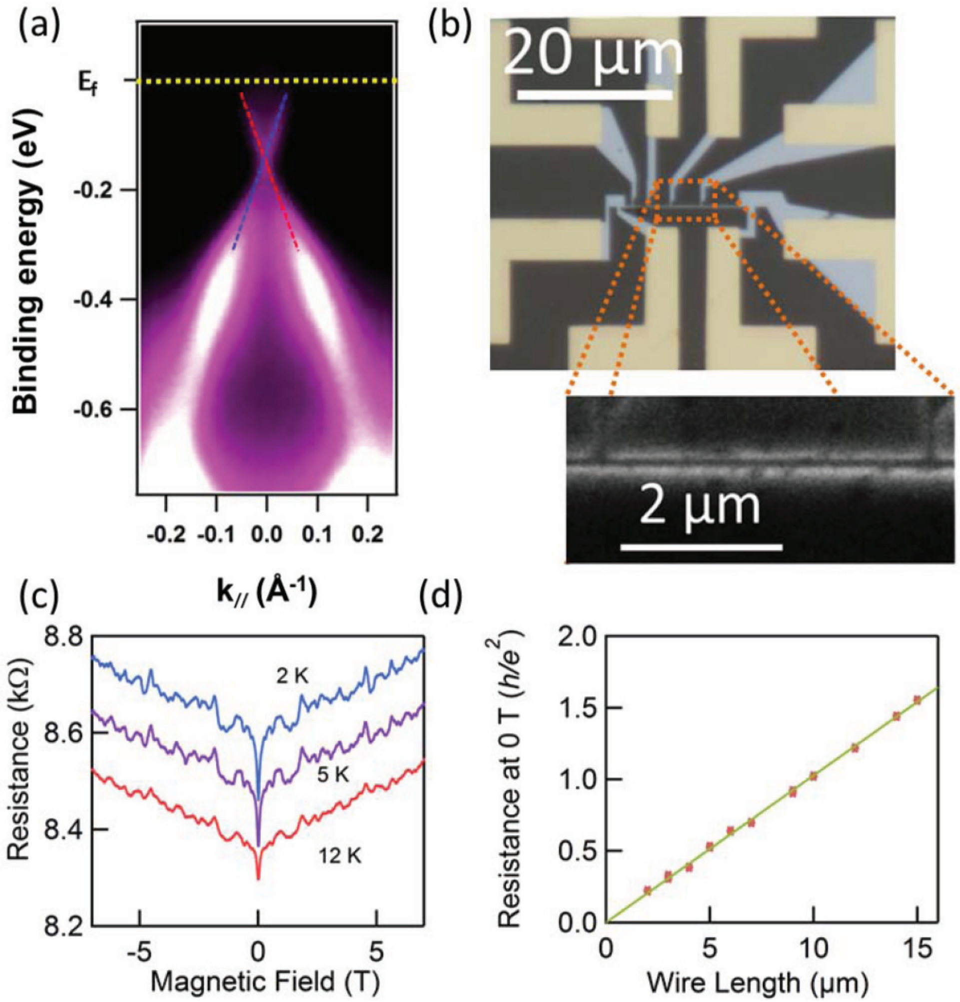
Weak localization: positive magnetoconductance, universal slope (e^2/h)



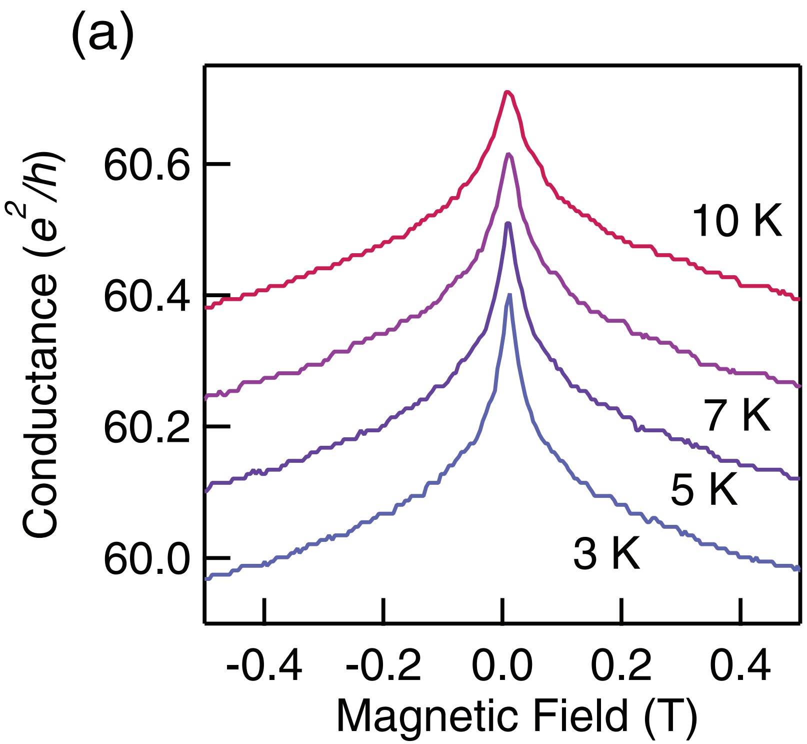


Komori et al, 1981 JPSJ

Renewed interests due to the discovery of topological insulators



BiSe topological insulator
Matsuo et al., PRB ’13



Weyl semimetal

- H-J. Kim, Ki-Seok Kim, et al. PRL (2013)

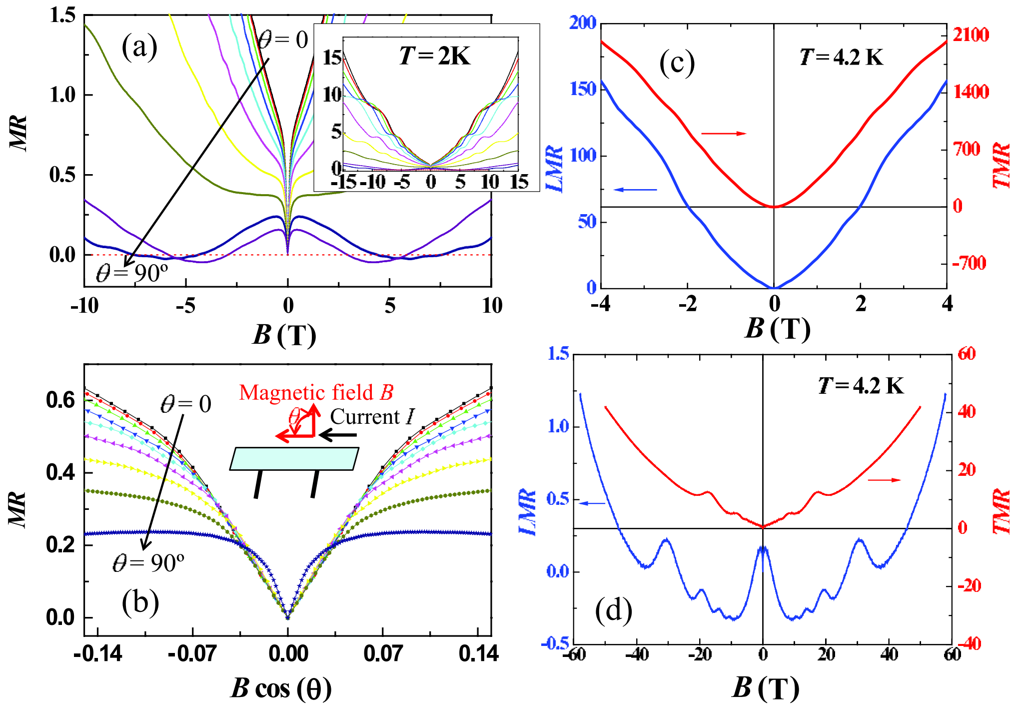
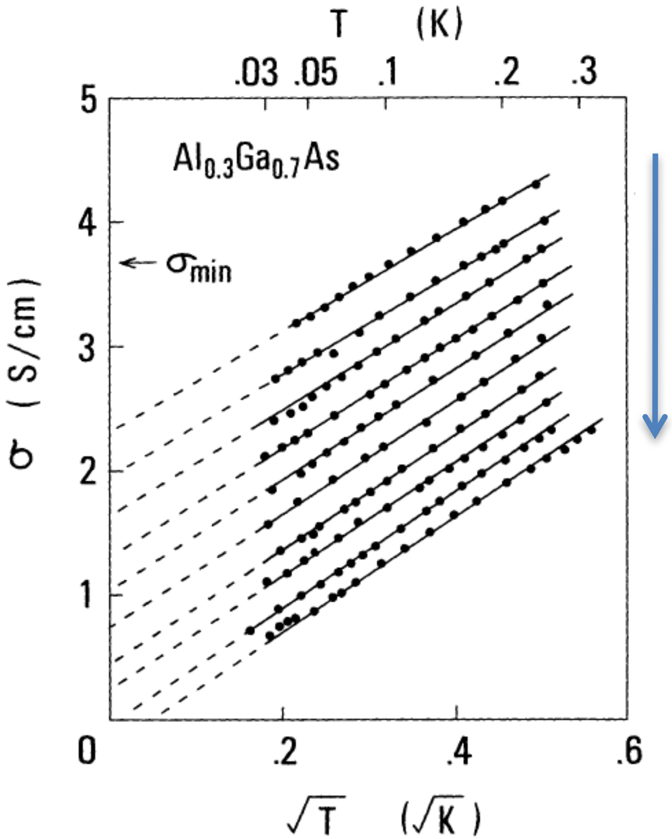


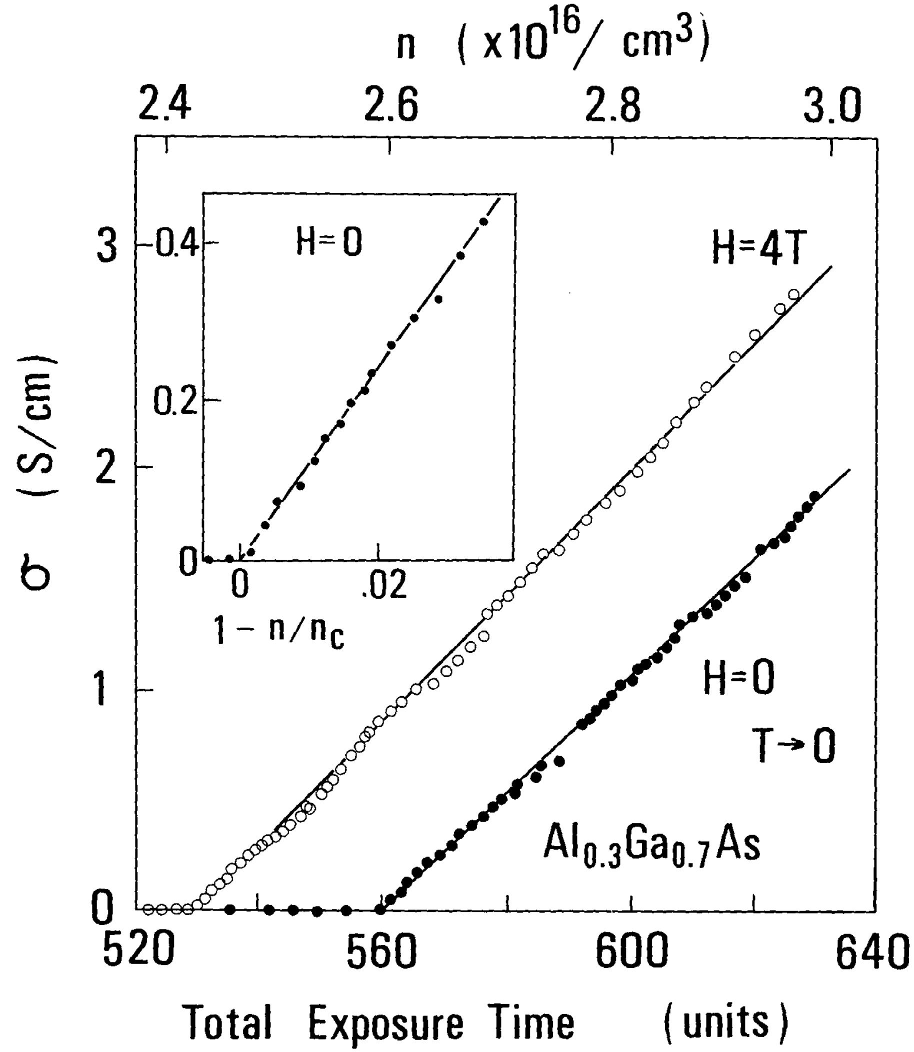
FIG. 2 (color online). (a) Angle-dependent MR. "Negative" MR appears around $\theta = 90°$, which originates from the Adler-Bell-Jackiw anomaly of Weyl fermions. (b) The scaling property of MR in the low-field region. The x axis is the perpendicular field component of $B \cos\theta$. This dip was attributed to the three-dimensional weak antilocalization. (c) Longitudinal and transverse MR for a Bi single crystal. We could not find any negative signals in the longitudinal MR. (d) The transverse and longitudinal MR measured up to 50–60 T in a pulse magnet.

Anderson Transition



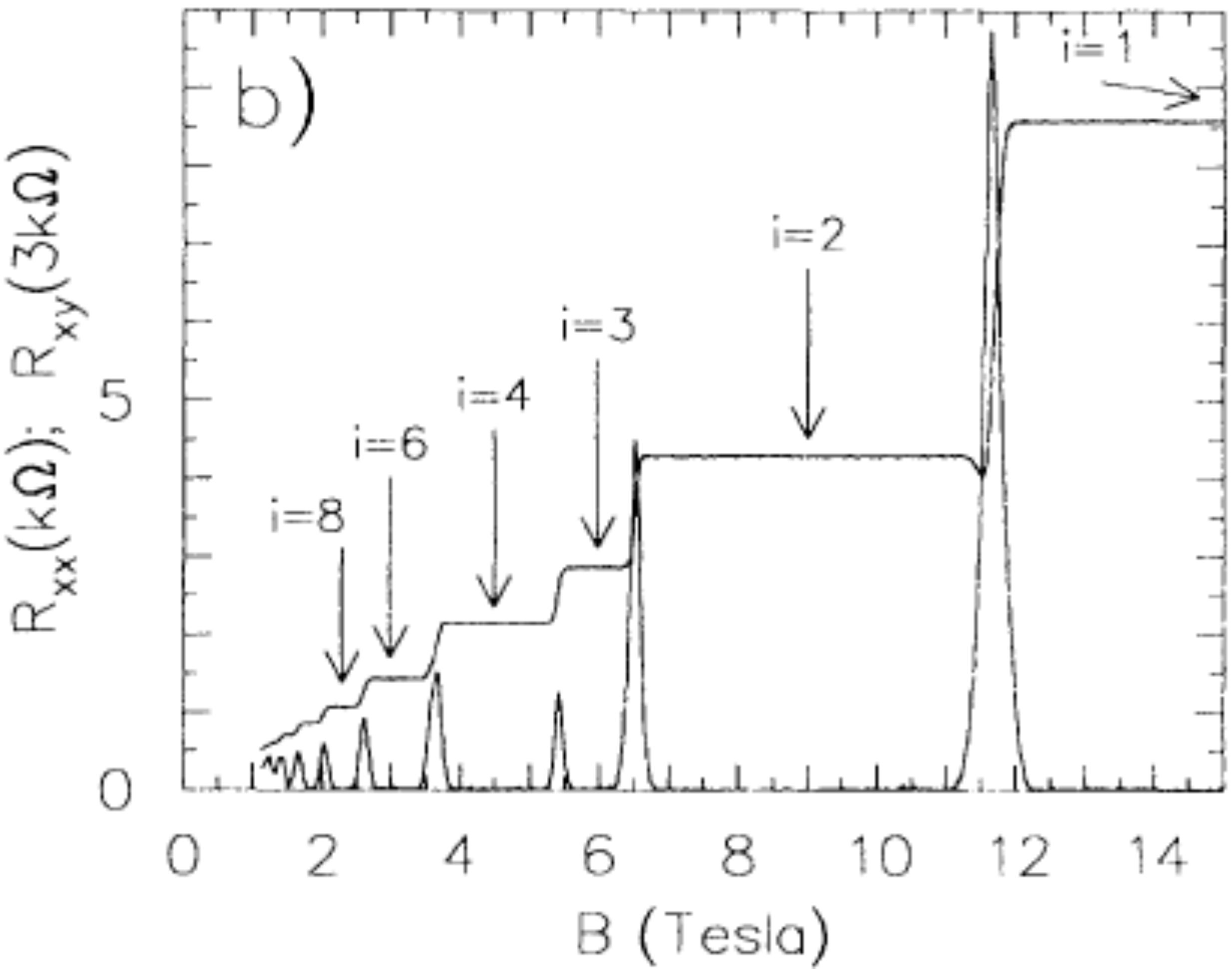
σ(*T*=0)

smaller



Katsumoto et al,
1987 JPSJ

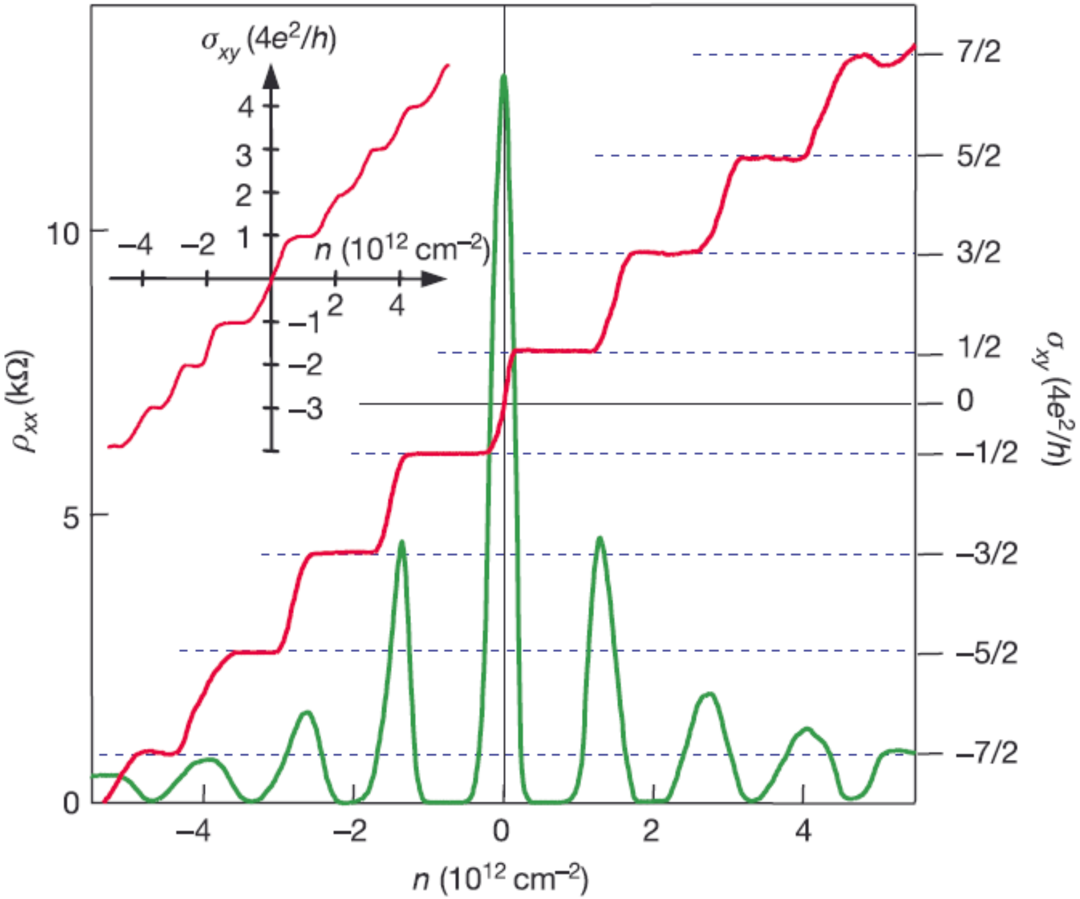
Quantum Hall Effect

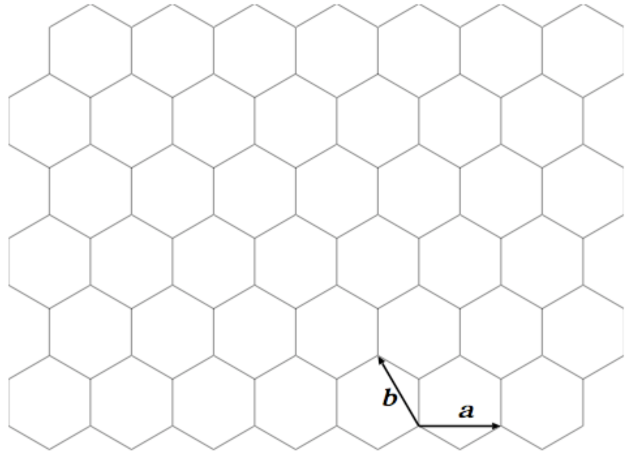


Engel et al, 1993 PRL

Quantization of Hall conductivity is due to the topological origin (‘16 Nobel prize).
Plateau is due to localization.

Quantum Hall effect in graphene:
evidence of single layer graphene



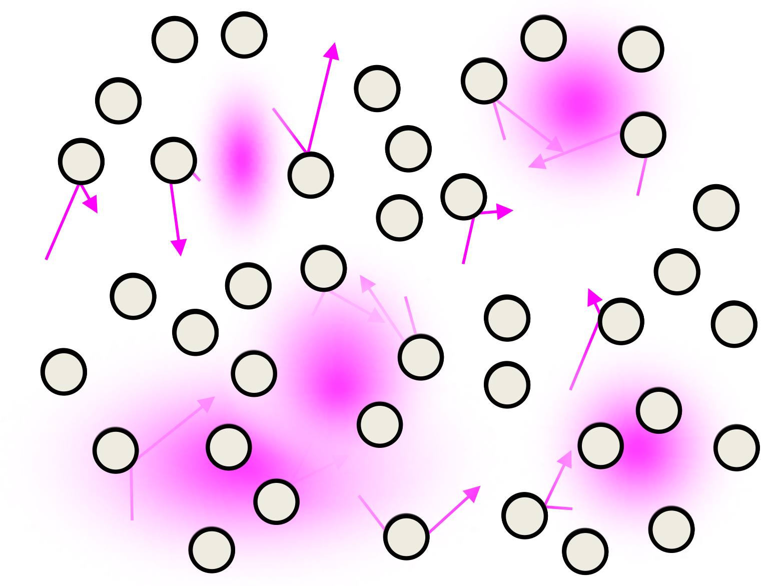


Novoselov et al, Nature 2005

Anderson localization

- Wave propagation in random medium → constructive and destructive interferences (wave can be electron, e-m, sound, matter waves)

→localization of wave (P.W. Anderson, '58)



Ubiquitous phenomena in many fields of physics

Anderson transition in QCD?

PHYSICAL REVIEW LETTERS

Universality and the QCD Anderson Transition

Matteo Giordano, Tamás G. Kovács, and Ferenc Pittler
Phys. Rev. Lett. **112**, 102002 – Published 12 March 2014

ABSTRACT

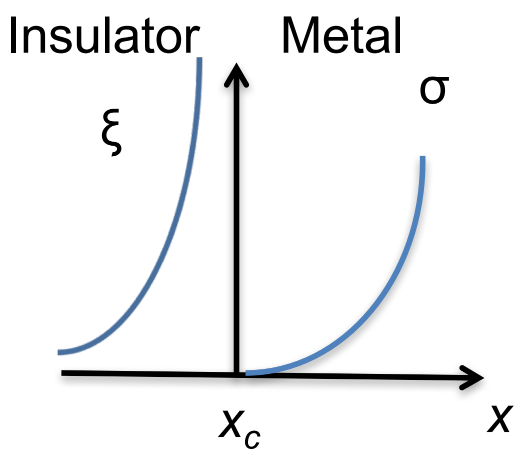
We study the Anderson-type transition previously found in the spectrum of the QCD quark Dirac operator in the high-temperature, quark-gluon plasma phase. Using finite size scaling for the unfolded level spacing distribution, we show that in the thermodynamic limit there is a genuine mobility edge, where the spectral statistics changes from Poisson to Wigner-Dyson statistics in a nonanalytic way. We determine the correlation length critical exponent ν and find that it is compatible with that of the unitary Anderson model.

Localization-delocalization transition (Anderson transition)

- With the change of parameter *x* (such as Fermi energy, electron density, randomness, magnetic field, pressure,…….)
  - In insulating phase,

$$\sigma \approx \exp(-aL/\xi) \ , \ \xi \propto \frac{1}{(x_c - x)^{\nu}}$$

  - In metallic phase, $\sigma \propto (x - x_c)^s$



- Wegner's relation $s = (d-2)\nu$

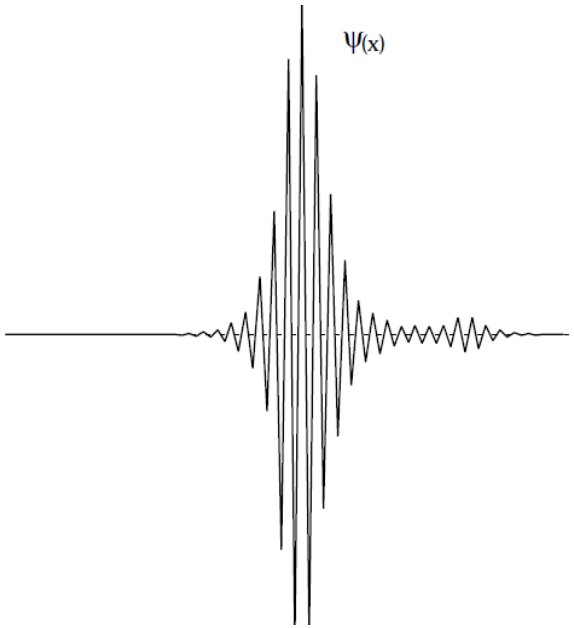
d: dimension of the system

- s: experimentally determined, ν: numerically

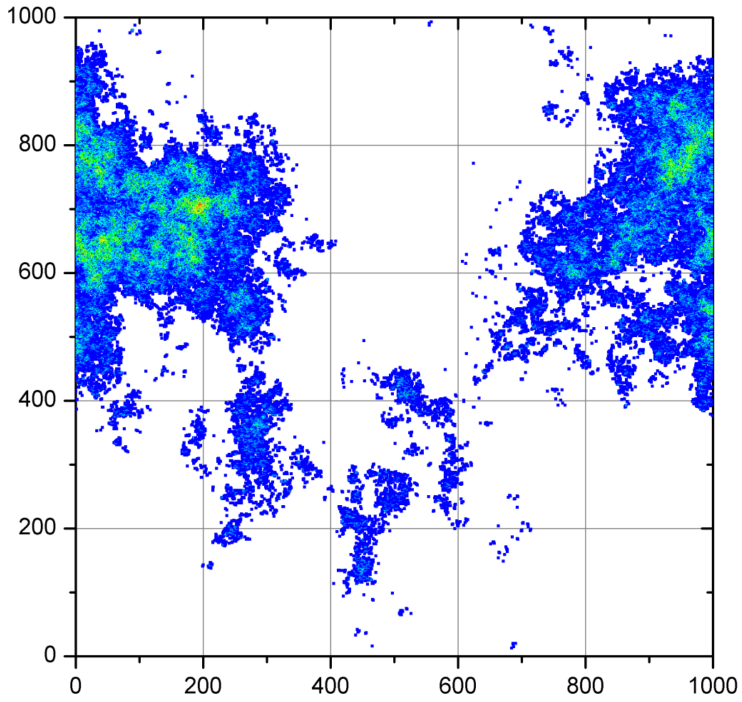
Anderson localization 2

- Eigenfunctions are exponentially localized

$$\Psi(x) = a(x)\exp\left(-\frac{|x - x_0|}{\xi}\right)$$



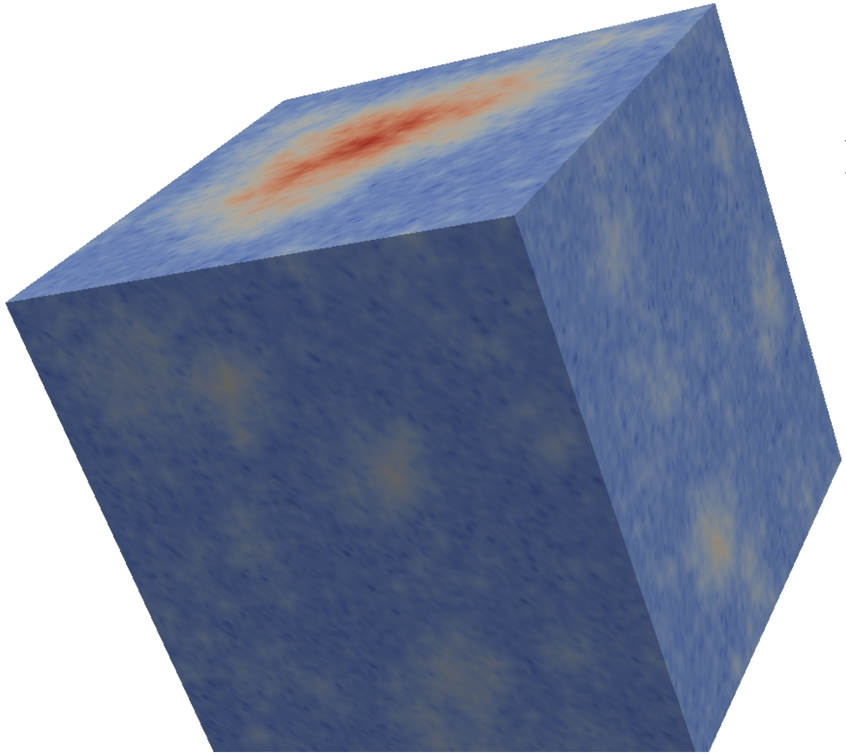
$|\psi(x,y)|^2$



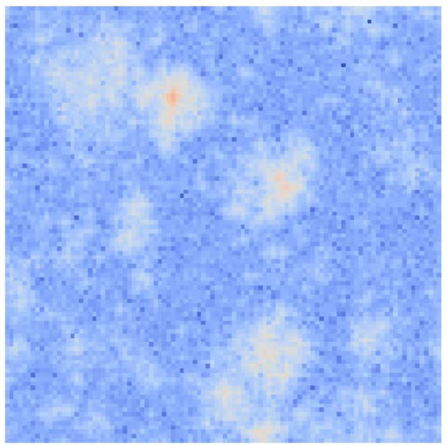
Red-green, 80%. Red-green-blue, 99%

Slevin –Ohtsuki '12

Strongly localized: W=35, 100x100x100 cubic lattice



$$\log|\psi(x,y,z)|^2$$

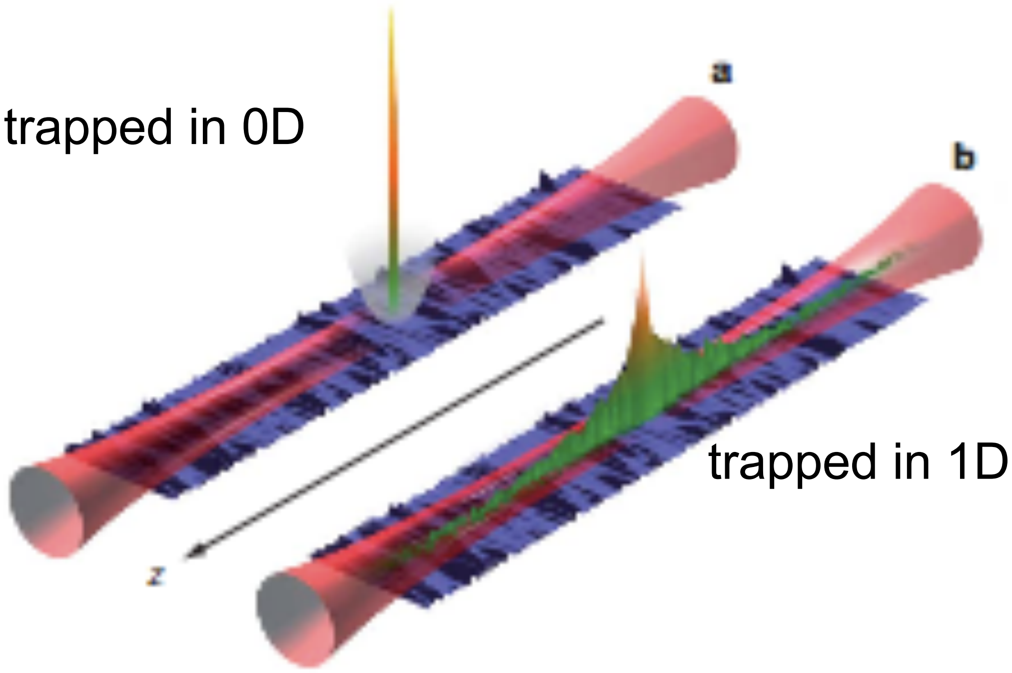


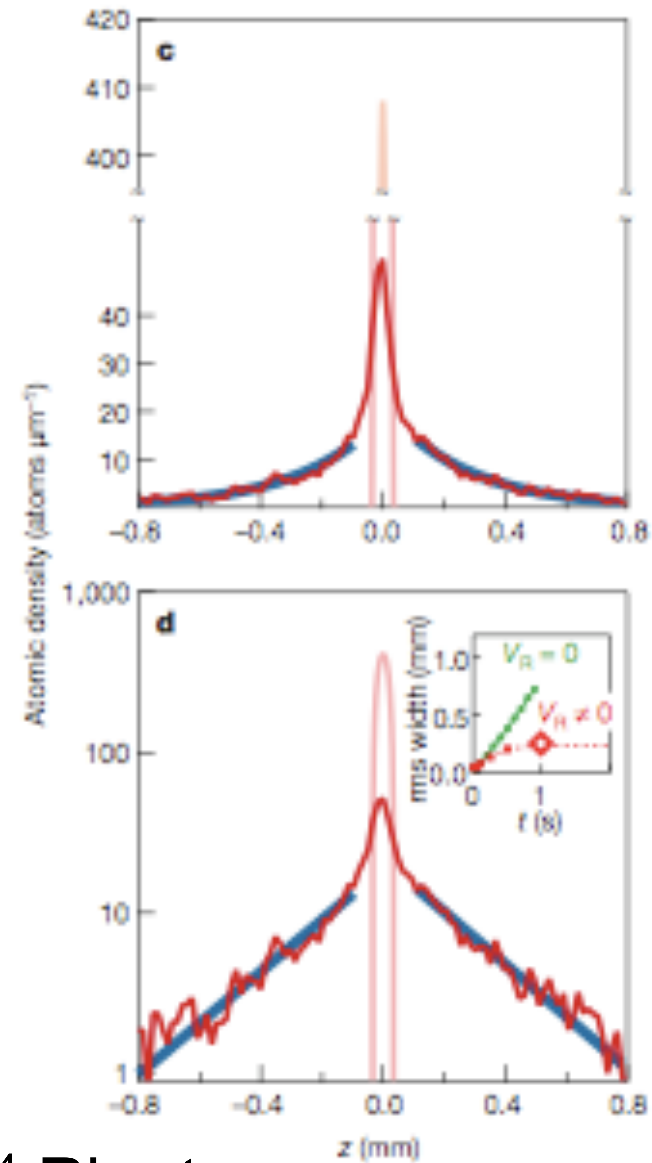
How do we observe the AT experimentally?

- Metal insulator transition in doped semi-conductor → electron-electron interaction is inevitable.
- Sound wave, electromagnetic wave → absorption is inevitable. Even if the transmission of wave decays exponentially with size L, $T(L) \sim \exp(-L/\lambda)$, this may be due to absorption.

New technique using cold atom systems

Matter wave (real space)





Billy et al., Nature Vol 453 (2008) 891

Direct observation of 1d localization, 10^4 Rb atoms

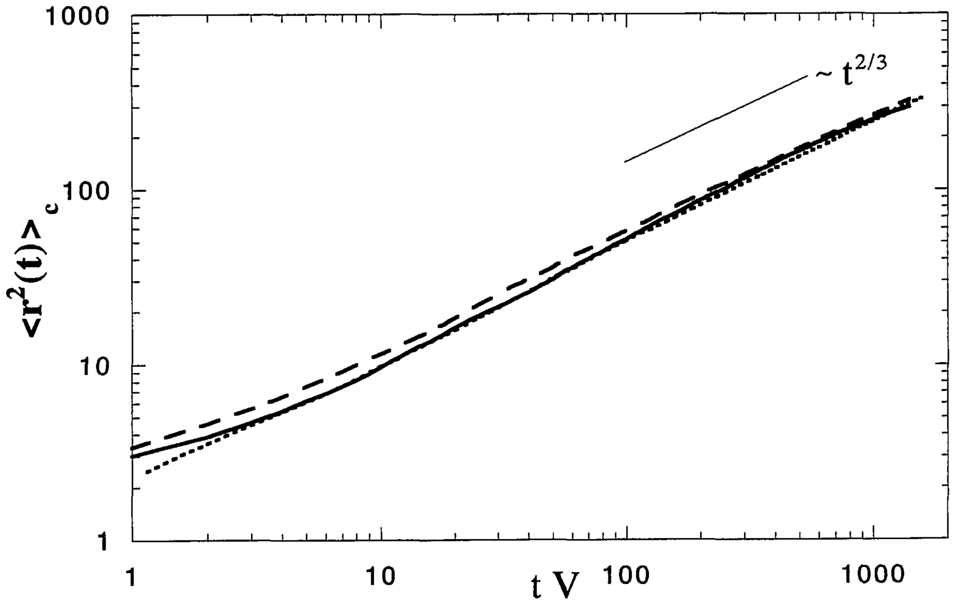
Wave packet dynamics

$$\int dr|\psi|^2 r^2 = \langle r^2 \rangle = \begin{cases} 2dDt \propto (x-x_c)^s t & \text{(metal)} \\ \xi^2 \propto (x_c - x)^{-2\nu} & \text{(insulator)} \end{cases} = t^{k_1} f((x-x_c)t^{k_2}) = t^{k_1} F\left(\frac{t}{\xi^z}\right)$$

$$\Rightarrow k_1 + sk_2 = 1 \;,\; k_1 - 2\nu k_2 = 0 \Rightarrow k_1 = \frac{2\nu}{s+2\nu} \;,\; k_2 = \frac{1}{s+2\nu}$$

$$\Rightarrow k_1 = \frac{2}{d},\; k_2 = \frac{1}{d\nu},\; \because s = (d-2)\nu$$

Note: dynamical exponent $z=d$



TO, Kawarabayashi '97

Matter wave (momentum space)

$$H = \frac{p^2}{2} + K\cos x[1 \quad]\sum_{n=0}^{N-1}\delta(t-n),$$

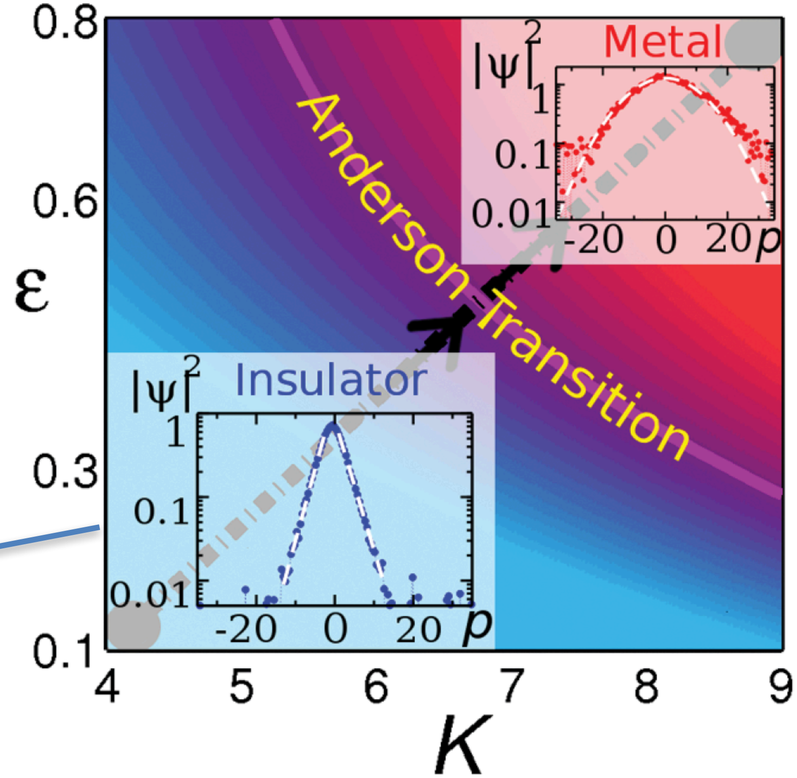
$$H = \sum_j \varepsilon_j |j\rangle\langle j| + \sum_{j',j} V_{j',j} |j'\rangle\langle j|$$

(Anderson model in momentum space)

Mapping kicked rotor to 3D Anderson model in momentum space

10^7 Cs atoms, 3.2 microkelvin

$$\exp(-|p|/p_{loc})$$

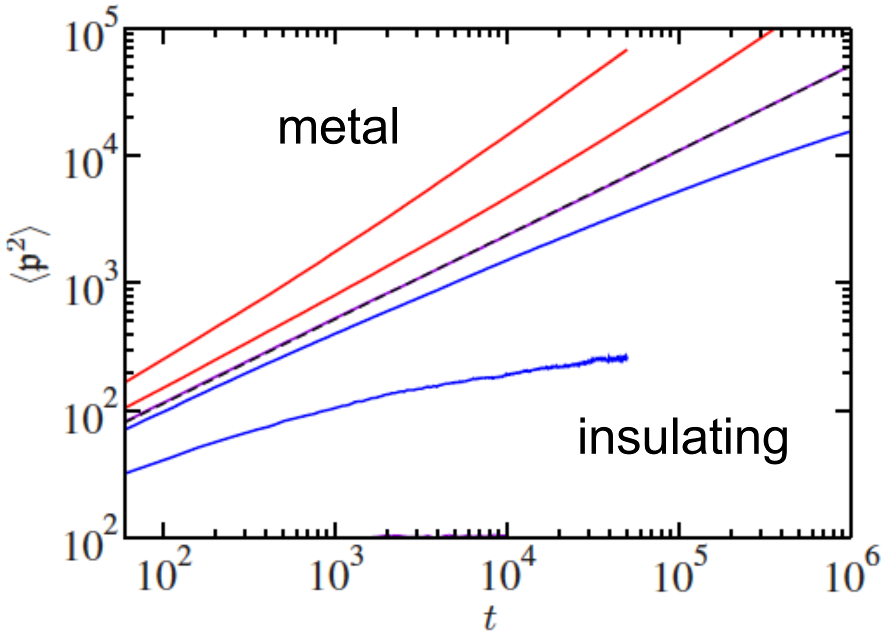


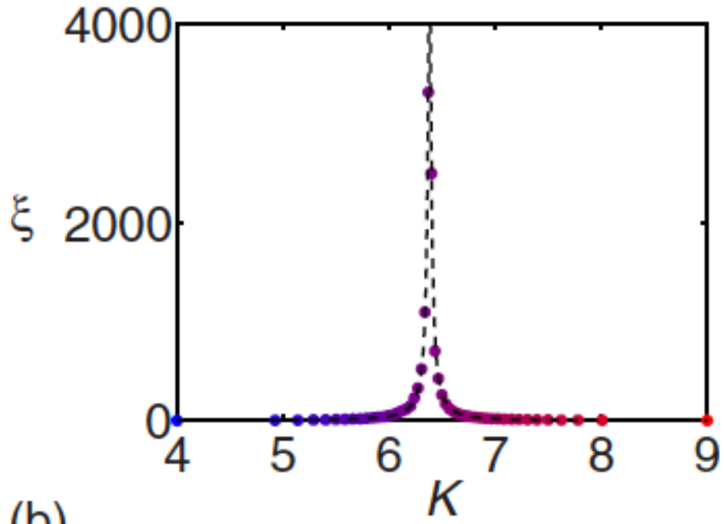
Chabe et al., PRL '08

Critical exponent from cold atom systems

- Chabe et al. PRL 2008, Lemarie et al., PRA 2009, , Tian, PRL ‘11

$$\langle p^2 \rangle = t^{k_1} F[(K - K_c) t^{k_2}], \quad k_1 = 2/3; \quad k_2 = 1/3\nu.$$





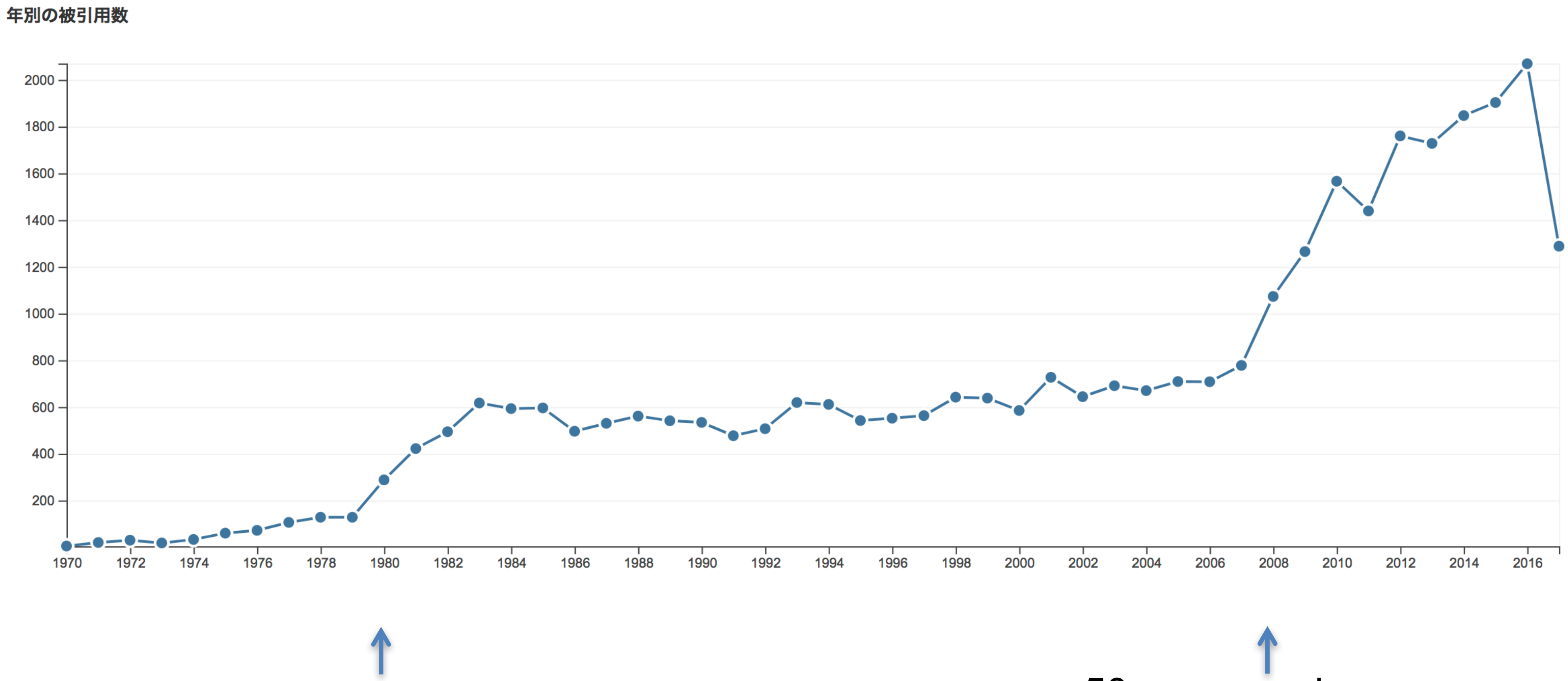
(b)

$\nu = 1.59 \pm 0.01$

Experimental estimate 1.4 ± 0.3

Still actively studied field

Number of citations of papers that include “scaling theory of localiz(s)ation”, “Anderson localiz(s)ation” or “Anderson transition” in titles. (Web of Science, August ‘17)



Perturbation theory
$H = H_0 + V$, V: random potential

Impurity potential $$V(\boldsymbol{r}) = \sum_i^{N_{\mathrm{imp}}} v(\boldsymbol{r} - \boldsymbol{r}_i)$$

Fermi's Golden rule

$$\longrightarrow \quad W(\boldsymbol{k}, \boldsymbol{k}') = \frac{2\pi}{\hbar} \left\langle \left| \langle \boldsymbol{k} | V | \boldsymbol{k}' \rangle \right|^2 \right\rangle \times \delta\left(E_{\boldsymbol{k}} - E_{\boldsymbol{k}'}\right)$$

$$\langle \boldsymbol{k} | V | \boldsymbol{k}' \rangle = \frac{1}{\Omega} \int \mathrm{d}^d \boldsymbol{r} V(\boldsymbol{r}) \mathrm{e}^{\mathrm{i}(\boldsymbol{k}' - \boldsymbol{k}) \cdot \boldsymbol{r}}$$

$$\left\langle \left| \langle \boldsymbol{k} | V | \boldsymbol{k}' \rangle \right|^2 \right\rangle = \frac{1}{\Omega^2} \int \mathrm{d}^d \boldsymbol{r} \mathrm{d}^d \boldsymbol{r}' \underbrace{\langle V(\boldsymbol{r}) V(\boldsymbol{r}') \rangle}_{C(\boldsymbol{r}-\boldsymbol{r}')} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}' - \boldsymbol{k}) \cdot (\boldsymbol{r} - \boldsymbol{r}')},$$

which contains the correlation function

$$C(\boldsymbol{r}, \boldsymbol{r}') = \langle V(\boldsymbol{r}) V(\boldsymbol{r}') \rangle_{\mathrm{imp}} = C(\boldsymbol{r}-\boldsymbol{r}') = C_0 \delta(\boldsymbol{r}-\boldsymbol{r}')$$

Kubo formula

Calculates conductivity (in the presense of electric field) using states in equilibrium (without electric field).

$$\mathcal{H}_{\rm total} = \mathcal{H} + \mathcal{H}_{\rm ext}\,, \qquad \hat{\rho}_{\rm eq} = \frac{\mathrm{e}^{-\beta\mathcal{H}}}{\mathrm{Tr}\,\mathrm{e}^{-\beta\mathcal{H}}}\,.$$

Equation of motion for density matrix $\dfrac{\partial\hat{\rho}}{\partial t} = \dfrac{1}{\mathrm{i}\hbar}[\mathcal{H}_{\rm total}, \hat{\rho}],$

$$\longrightarrow \quad \hat{\rho}(t) = \hat{\rho}_{\rm eq} + \int_{-\infty}^{t} \mathrm{d}t'\,\mathrm{e}^{-\mathrm{i}(t-t')\mathcal{H}/\hbar}\frac{[\mathcal{H}_{\rm ext}(t), \hat{\rho}_{\rm eq}]}{\mathrm{i}\hbar}\mathrm{e}^{\mathrm{i}(t-t')\mathcal{H}/\hbar} + O(\mathcal{H}_{\rm ext}^2)$$

$$\begin{aligned}\delta B(t) &= \mathrm{Tr}\,\hat{\rho}(t)B - \mathrm{Tr}\,\hat{\rho}_{\rm eq}B \\ &= \int_{-\infty}^{t} \mathrm{d}t'\,\mathrm{Tr}\,\mathrm{e}^{-\mathrm{i}(t-t')\mathcal{H}/\hbar}\frac{[\mathcal{H}_{\rm ext}(t'), \hat{\rho}_{\rm eq}]}{\mathrm{i}\hbar}\mathrm{e}^{\mathrm{i}(t-t')\mathcal{H}/\hbar}B\,.\end{aligned}$$

$$B(t) \equiv \mathrm{e}^{\mathrm{i}t\mathcal{H}/\hbar}B\mathrm{e}^{-\mathrm{i}t\mathcal{H}/\hbar}\,,$$

$$\delta B(t) = \int_{-\infty}^{t} \mathrm{d}t'\,\mathrm{Tr}\,\frac{[\mathcal{H}_{\rm ext}(t'), \hat{\rho}_{\rm eq}]}{\mathrm{i}\hbar}B(t-t')$$

For conductivity,

$$\mathcal{H}_{\rm ext} = -\boldsymbol{J}_0 \cdot \boldsymbol{A}(t)\,, \quad \boldsymbol{E}(t) = -\frac{\partial \boldsymbol{A}}{\partial t}\,.$$

$$\boldsymbol{J}_0 = -\frac{e}{m}\sum_i \boldsymbol{p}_i\,,$$

$$J_\mu = \frac{-1}{\mathrm{i}\hbar}\int_{-\infty}^{t} \mathrm{d}t' \left\langle [J_\mu(t-t'), J_\nu] \right\rangle_{\rm eq} A_\nu(t') - \frac{N_{\rm e}e^2}{m} A_\nu(t'),$$

$$A_\nu(t) = A_{\omega,\nu}\mathrm{e}^{-\mathrm{i}\omega t + \varepsilon t} \qquad E_{\omega,\nu} = \mathrm{i}\omega A_{\omega,\nu}.$$

$$\begin{aligned} j_\mu &= \left(\frac{-1}{\Omega \mathrm{i}\hbar}\int_{-\infty}^{t} \mathrm{d}t' \left\langle [J_\mu(t-t'), J_\nu] \right\rangle_{\rm eq} \mathrm{e}^{-\mathrm{i}\omega t'} - \frac{ne^2}{m}\mathrm{e}^{-\mathrm{i}\omega t}\delta_{\mu,\nu}\right) A_{\omega,\nu} \\ &= \left(\frac{-1}{\Omega \mathrm{i}\hbar}\int_{-\infty}^{t} \mathrm{d}t' \left\langle [J_\mu(t-t'), J_\nu] \right\rangle_{\rm eq} \mathrm{e}^{\mathrm{i}\omega (t-t')} - \frac{ne^2}{m}\delta_{\mu,\nu}\right) A_{\omega,\nu}\mathrm{e}^{-\mathrm{i}\omega t} \\ &= (K_{\mu,\nu}(\omega) - K_{\mu,\nu}(0))\frac{E_{\omega,\nu}}{\mathrm{i}\omega}\mathrm{e}^{-\mathrm{i}\omega t}, \end{aligned}$$

$$\boldsymbol{j}(t) = \boldsymbol{j}_\omega \mathrm{e}^{-\mathrm{i}\omega t + \varepsilon t}$$

$$(\boldsymbol{j}_\omega)_\mu = \sigma_{\mu,\nu}(\omega) E_{\omega,\nu}\,,$$

$$K_{\mu,\nu} = \frac{-1}{\Omega \mathrm{i}\hbar} \int_0^\infty \mathrm{d}t' \, \langle [J_\mu(t), J_\nu] \rangle_{\mathrm{eq}} \, \mathrm{e}^{\mathrm{i}\omega t - \varepsilon t},$$

$$\sigma_{\mu,\nu} = \frac{K_{\mu,\nu}(\omega) - K_{\mu,\nu}(0)}{\mathrm{i}\omega}$$

$$\langle [J_\mu(t), J_\nu] \rangle_{\mathrm{eq}} = 2 \times \frac{e^2}{m^2} \sum_{\alpha,\beta} f(E_\alpha) \langle \alpha | p_x | \beta \rangle \langle \beta | p_x | \alpha \rangle \times (\mathrm{e}^{\mathrm{i}(E_\alpha - E_\beta)t/\hbar} - \mathrm{e}^{\mathrm{i}(E_\beta - E_\alpha)t/\hbar}),$$

$$K(\omega) = -\frac{2}{\Omega} \frac{e^2}{m^2} \sum_{\alpha,\beta} \frac{f(E_\alpha) - f(E_\beta)}{E_\alpha - E_\beta + \hbar\omega + \mathrm{i}\hbar\varepsilon} \langle \alpha | p_x | \beta \rangle \langle \beta | p_x | \alpha \rangle \; .$$

$$K(\omega) \approx K(0) + \frac{2\pi \mathrm{i}}{\Omega} \frac{e^2}{m^2} \sum_{\alpha,\beta} (f(E_\alpha) - f(E_\beta)) \langle \alpha | p_x | \beta \rangle \langle \beta | p_x | \alpha \rangle \times \delta(E_\alpha - E_\beta + \hbar\omega),$$

Example

PRL **98,** 076602 (2007) PHYSICAL REVIEW LETTERS week ending 16 FEBRUARY 2007

Quantum Transport of Massless Dirac Fermions

Kentaro Nomura and A. H. MacDonald

MDF model finite-size Kubo formula.—Our numerical results, obtained by evaluating the finite-size Kubo formula

$$\sigma = -\frac{i\hbar e^2}{L^2}\sum_{n,n'}\frac{f(E_n) - f(E_{n'})}{E_n - E_{n'}}\,\frac{\langle n|v_x|n'\rangle\langle n'|v_x|n\rangle}{E_n - E_{n'} + i\eta}, \quad (2)$$

PHYSICAL REVIEW B **72**, 165316 (2005)

Dependence of the intrinsic spin-Hall effect on spin-orbit interaction character

K. Nomura,[1] Jairo Sinova,[2] N. A. Sinitsyn,[1] and A. H. MacDonald[1]

$$\sigma_{xy}^{z} = -\frac{i\hbar}{L^2}\sum_{\alpha,\alpha'}\frac{f(E_\alpha)-f(E_{\alpha'})}{E_\alpha - E_{\alpha'}}\frac{\langle\alpha|j_x^z|\alpha'\rangle\langle\alpha'|j_y|\alpha\rangle}{E_\alpha - E_{\alpha'} + i\eta},$$

VOLUME 49, NUMBER 6 PHYSICAL REVIEW LETTERS

Quantized Hall Conductance in a Two-Dimensional Periodic Potential

D. J. Thouless, M. Kohmoto,[(a)] M. P. Nightingale, and M. den Nijs
Department of Physics, University of Washington, Seattle, Washington 98195
(Received 30 April 1982)

$$\sigma_H = \frac{ie^2}{A_0\hbar}\sum_{\epsilon_\alpha < E_F}\sum_{\epsilon_\beta > E_F}\frac{(\partial\hat{H}/\partial k_1)_{\alpha\beta}(\partial\hat{H}/\partial k_2)_{\beta\alpha} - (\partial\hat{H}/\partial k_2)_{\alpha\beta}(\partial\hat{H}/\partial k_1)_{\beta\alpha}}{(\epsilon_\alpha - \epsilon_\beta)^2},$$

Green function expression for conductivity

$$\sigma(0) = \frac{2\pi}{\Omega}\frac{e^2}{m^2}\sum_{\alpha,\beta}\frac{f(E_\alpha)-f(E_\beta)}{\omega}\langle\alpha|p_x|\beta\rangle\langle\beta|p_x|\alpha\rangle \times\delta(E_\alpha - E_\beta + \hbar\omega)$$

Static limit → $$= \frac{2\pi\hbar}{\Omega}\frac{e^2}{m^2}\int_0^\infty \mathrm{d}E\left(-\frac{\partial f(E)}{\partial E}\right)\sum_{\alpha,\beta}\langle\alpha|p_x|\beta\rangle\langle\beta|p_x|\alpha\rangle \times\delta(E-E_\alpha)\delta(E-E_\beta).$$

$$\sigma(0) = \frac{2\pi\hbar}{\Omega}\frac{e^2}{m^2}\int_0^\infty \mathrm{d}E\left(-\frac{\partial f(E)}{\partial E}\right)\langle \mathrm{Tr}\,\delta(E-H_0)p_x\delta(E-H_0)p_x\rangle_{\mathrm{imp}}$$

$$\delta(x) = \frac{1}{2\pi i} \lim_{\delta \to +0} \left(\frac{1}{x - \mathrm{i}\delta} - \frac{1}{x + \mathrm{i}\delta} \right)$$

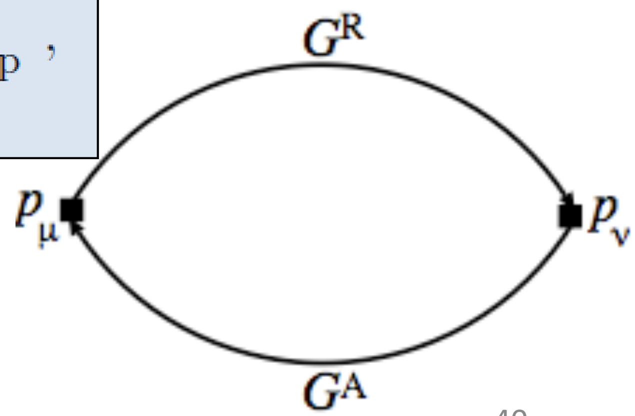
$$\sigma(0) = \frac{\hbar^3 e^2}{2\pi m^2 \Omega} \int_0^\infty \mathrm{d}E \left(-\frac{\partial f(E)}{\partial E} \right) \times \sum_{\boldsymbol{k},\boldsymbol{k}'} k_x k'_x$$
$$\times \left\langle 2G^{\mathrm{R}}(\boldsymbol{k},\boldsymbol{k}')G^{\mathrm{A}}(\boldsymbol{k}',\boldsymbol{k}) - G^{\mathrm{R}}(\boldsymbol{k},\boldsymbol{k}')G^{\mathrm{R}}(\boldsymbol{k}',\boldsymbol{k}) - G^{\mathrm{A}}(\boldsymbol{k},\boldsymbol{k}')G^{\mathrm{A}}(\boldsymbol{k}',\boldsymbol{k}) \right\rangle_{\mathrm{imp}}$$

$|\langle G^R G^R \rangle|, |\langle G^A G^A \rangle| \ll | \langle G^R G^A \rangle |$

$$\longrightarrow \sigma(0) = \frac{\hbar^3 e^2}{\pi m^2 \Omega} \int \mathrm{d}E \left(-\frac{\partial f(E)}{\partial E} \right) \times \sum_{\boldsymbol{k},\boldsymbol{k}'} k_x k'_x \times \left\langle G^{\mathrm{R}}(\boldsymbol{k},\boldsymbol{k}')G^{\mathrm{A}}(\boldsymbol{k}',\boldsymbol{k}) \right\rangle_{\mathrm{imp}} .$$

Low temperature limit:

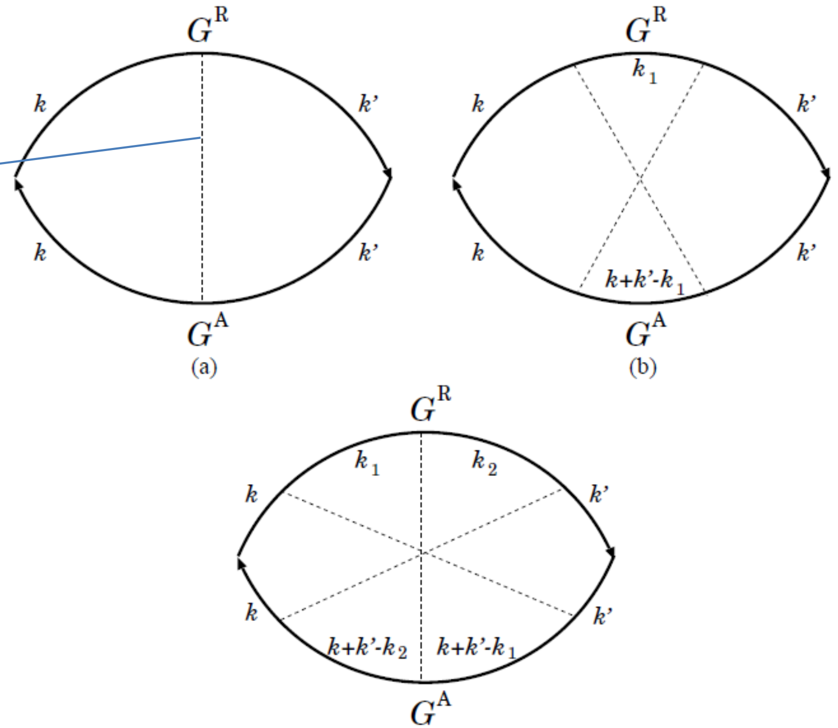
$$\sigma(0) = \frac{\hbar^3 e^2}{\pi m^2 \Omega} \sum_{\boldsymbol{k},\boldsymbol{k}'} k_x k'_x \times \left\langle G^{\mathrm{R}}(\boldsymbol{k},\boldsymbol{k}')G^{\mathrm{A}}(\boldsymbol{k}',\boldsymbol{k}) \right\rangle_{\mathrm{imp}} ,$$



Weak localization theory:

summation of maximally crossed diagrams

$\langle V(r)\, V(r')\rangle_{imp} = C(\boldsymbol{r}-\boldsymbol{r}')$



$$\Pi(\boldsymbol{q}) \equiv \sum_{\boldsymbol{k}} G^{\mathrm{R}}(\boldsymbol{k}) G^{\mathrm{A}}(\boldsymbol{k}-\boldsymbol{q})$$

$$\sum_{\boldsymbol{k}} G^{\mathrm{R}}(\boldsymbol{k})^m G^{\mathrm{A}}(\boldsymbol{k})^n = \Omega \int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} G^{\mathrm{R}}(\boldsymbol{k})^m G^{\mathrm{A}}(\boldsymbol{k})^n$$

Ω: volume

$$= \Omega \int_{-\infty}^{\infty} \mathrm{d}\epsilon \rho(\epsilon) \frac{1}{(E_{\mathrm{F}} - \epsilon + \mathrm{i}\hbar/2\tau)^m} \frac{1}{(E_{\mathrm{F}} - \epsilon - \mathrm{i}\hbar/2\tau)^n}$$

$$\approx 2\pi\Omega\rho(E_{\mathrm{F}}) \left(\frac{\tau}{\hbar}\right)^{m+n-1} \mathrm{i}^{n-m} \frac{(m+n-2)!}{(m-1)!(n-1)!} ,$$

$$\Pi(\boldsymbol{q}) = \Omega\rho(E_{\mathrm{F}}) \int_{-\infty}^{\infty} \mathrm{d}\epsilon$$

$$\left\langle \frac{1}{(E_{\mathrm{F}} - \epsilon + \mathrm{i}\hbar/2\tau)(E_{\mathrm{F}} - \epsilon + \hbar q v_{\mathrm{F}} \cos\theta - \mathrm{i}\hbar/2\tau)} \right\rangle_\theta$$

$$= \frac{2\pi\rho\Omega\tau}{\hbar} \left\langle \frac{1}{1 + \mathrm{i} q v_{\mathrm{F}} \tau \cos\theta} \right\rangle_\theta$$

$$= \frac{2\pi\rho\Omega\tau}{\hbar} (1 - Dq^2\tau) ,$$

Assumption

$$q v_{\mathrm{F}} \tau = ql \ll 1$$

$$D = \frac{v_{\mathrm{F}}^2 \tau}{d} .$$ Diffusion constant

$$\Gamma(q) = \frac{C}{\Omega} + \frac{C}{\Omega}\left(\frac{\Pi(q)C}{\Omega}\right) + \frac{C}{\Omega}\left(\frac{\Pi(q)C}{\Omega}\right)^2 + \cdots = \frac{C}{1 - \Pi(q)C/\Omega}$$

$$\xrightarrow{2\pi C\rho\tau/\hbar = 1} \quad \Gamma(q) = \frac{C}{\Omega}\frac{1}{Dq^2\tau}$$

$$\Delta\sigma(0) = \frac{e^2\hbar}{\pi\Omega}\left(\frac{\hbar}{m}\right)^2 \sum_{\boldsymbol{k},\boldsymbol{k}'} k_x k'_x G^{\mathrm{R}}(\boldsymbol{k})G^{\mathrm{A}}(\boldsymbol{k})G^{\mathrm{R}}(\boldsymbol{k}')G^{\mathrm{A}}(\boldsymbol{k}')\Gamma(\boldsymbol{k}+\boldsymbol{k}') .$$

$$\Delta\sigma(0) = -\frac{e^2\hbar}{\pi\Omega}\left(\frac{\hbar}{m}\right)^2 \frac{k_{\mathrm{F}}^2}{d} \sum_{\boldsymbol{k}} (G^{\mathrm{R}}(\boldsymbol{k})G^{\mathrm{A}}(\boldsymbol{k}))^2 \sum_{\boldsymbol{q}} \Gamma(\boldsymbol{q})$$

$$= -\frac{2e^2}{\pi\hbar\Omega}\sum_{\boldsymbol{q}}\frac{1}{q^2}$$

$$= -\frac{2e^2}{\pi\hbar}\int \frac{\mathrm{d}^d\boldsymbol{q}}{(2\pi)^d}\frac{1}{q^2} .$$

In 2D

$$\Delta\sigma(0) = -\frac{e^2}{\pi^2\hbar}\log\left(\frac{L}{l}\right) = -\frac{2e^2}{h}\frac{1}{\pi}\log\left(\frac{L}{l}\right) .$$

magnitude of WL correction

Drude-Boltzmann value (classical) $\sigma_0 = \frac{2E_{\mathrm{F}}\rho(E_{\mathrm{F}}e^2\tau)}{m} = \frac{2e^2}{h}\frac{E_{\mathrm{F}}\tau}{\hbar}$

weak localization correction (quantum) $-\frac{2e^2}{h}\frac{1}{\pi}\log\left(\frac{L}{l}\right)$

$\hbar/E_{\mathrm{F}}\tau \ll 1$ → The latter is small but singular.

↑

equivalent to the fact that the mean free path >> Fermi wave length

Spin-orbit interaction (c.f. topological insulators)

For moving electron $\boldsymbol{\mathcal{B}}' = -\gamma \frac{\boldsymbol{v} \times \boldsymbol{\mathcal{E}}}{c^2}$. $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$,

$$H_{\rm so} = -\boldsymbol{\mu} \cdot \boldsymbol{\mathcal{B}}' .$$

$$\boldsymbol{\mu} = -g_e \frac{e}{2m_e} \boldsymbol{s} .$$

$$\boldsymbol{s} = \frac{\hbar}{2} \boldsymbol{\sigma} ,$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix}$$

$$\boldsymbol{\mu} = -\frac{e\hbar}{2m_e} \boldsymbol{\sigma} = -\mu_B \boldsymbol{\sigma} \longrightarrow \mathcal{H}_{\rm so} = -\frac{\mu_{\rm B}}{c^2} \boldsymbol{\sigma} \cdot \boldsymbol{v} \times \boldsymbol{\mathcal{E}} .$$

or

$$\mathcal{H}_{\rm so} = -\frac{\mu_{\rm B}}{c^2} \boldsymbol{\mathcal{E}} \cdot \boldsymbol{\sigma} \times \boldsymbol{v} .$$

Rashba term

$$V(z) = +e\mathcal{E}_z z \ .$$

$$\mathcal{H}_{\rm so} = -\frac{\mu_{\rm B}\mathcal{E}_z}{c^2}\left(\sigma_x v_y - \sigma_y v_x\right)$$

$$\longrightarrow \quad \mathcal{H}_{\rm so} = -v_{\rm so}\left(\sigma_x p_y - \sigma_y p_x\right) \ ,$$

$$v_{\rm so} = \frac{\mu_{\rm B}\mathcal{E}_z}{mc^2} \ .$$

$$\boldsymbol{v} = \frac{[\boldsymbol{x}, H]}{\mathrm{i}\hbar} \ . \quad \longrightarrow \quad \begin{aligned} v_x &= p_x/m + v_{\rm so}\sigma_y \\ v_y &= p_y/m - v_{\rm so}\sigma_x \ . \end{aligned}$$

$$\mathcal{H}_{\rm so} = -v_{\rm so}\left(\sigma_x p_y - \sigma_y p_x\right) + m v_{\rm so}^2\left(\sigma_x^2 + \sigma_y^2\right) \ .$$

Unitary transformation $e^{\mathrm{i}\pi\sigma_x/2} \longrightarrow \mathcal{H}_{\rm so} = -v_{\rm so}\left(\sigma_x p_y + \sigma_y p_x\right)$

Weak localization in the presence of Rashaba term

$$H_{\rm SO} = \hbar\boldsymbol{\sigma}\cdot\boldsymbol{\omega}_{\boldsymbol{k}}\,,\qquad \boldsymbol{\omega}_{\boldsymbol{k}} = \alpha(k_y, -k_x, 0)$$

$$G^{\rm R(A)}(E_{\rm F}, \boldsymbol{k}, \boldsymbol{\omega}_{\boldsymbol{k}}) = \frac{1}{E_{\rm F} - E_{\boldsymbol{k}} \pm \frac{{\rm i}\hbar}{2\tau} - \hbar\boldsymbol{\sigma}\cdot\boldsymbol{\omega}_{\boldsymbol{k}}}\,,$$

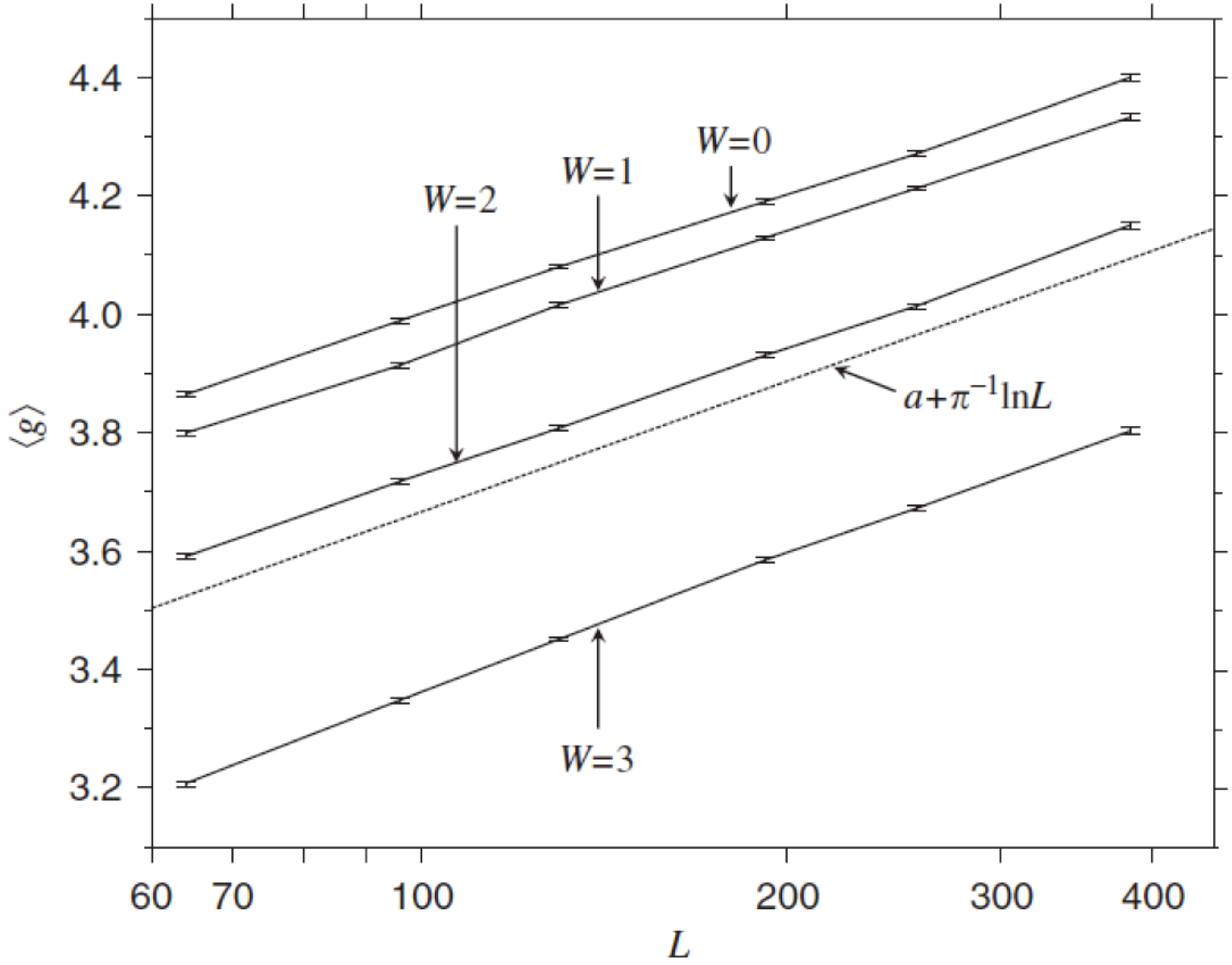
$$\Pi(\boldsymbol{q}) = \sum_{\boldsymbol{k}} \frac{1}{E_{\rm F} - E_{\boldsymbol{k}} + \frac{{\rm i}\hbar}{2\tau} - \hbar\boldsymbol{\sigma}\cdot\boldsymbol{\omega}_{\boldsymbol{k}}} \cdot \frac{1}{E_{\rm F} - E_{\boldsymbol{q}-\boldsymbol{k}} - \frac{{\rm i}\hbar}{2\tau} + \hbar\boldsymbol{\sigma}'\cdot\boldsymbol{\omega}_{\boldsymbol{k}}}\,.$$

Skipping the detail

$$\Delta\sigma(0) = -\frac{e^2}{2\pi^2\hbar}\left(-\log\left(\frac{L}{l}\right) + 3\log\left(\frac{L_{\rm SO}}{l}\right)\right)\,,$$

Increases with *L* infinitely!

Numerical simulation



Asada ’06 Physica E

In the presence of magnetic field

$$\Delta\sigma(B) = -\frac{2e^2}{\pi\hbar\Omega}\sum_{\boldsymbol{q}}\frac{1}{(\boldsymbol{q}+2e\boldsymbol{A}/\hbar)^2}$$

$$E_n = \left(n+\frac{1}{2}\right)\hbar\omega_{\mathrm{c}} \ , \ n = 0, 1, 2. \cdots$$

$$\Delta\sigma(B) = -\frac{e^2}{2\pi^2\hbar}\sum_{n=0}^{n_{\mathrm{c}}}\frac{1}{n+1/2} \, .$$

upper cutoff:

$$q^2 < l^{-2} \rightarrow \frac{2m}{\hbar^2}(n+1/2)\hbar\frac{2eB}{m} < l^{-2} \rightarrow n_{\mathrm{c}} = \frac{\hbar}{4eBl^2} = \frac{\ell_B^2}{4l^2} \, .$$

$$\Delta\sigma(B) = -\frac{e^2}{2\pi^2\hbar}\left(\psi\left(\frac{\ell_B^2}{4l^2}+\frac{1}{2}\right)-\psi\left(\frac{1}{2}\right)\right)$$

$$\boxed{\Delta\sigma(B) = \frac{e^2}{2\pi^2\hbar}\log B \, .}$$

Summary of WL corrections

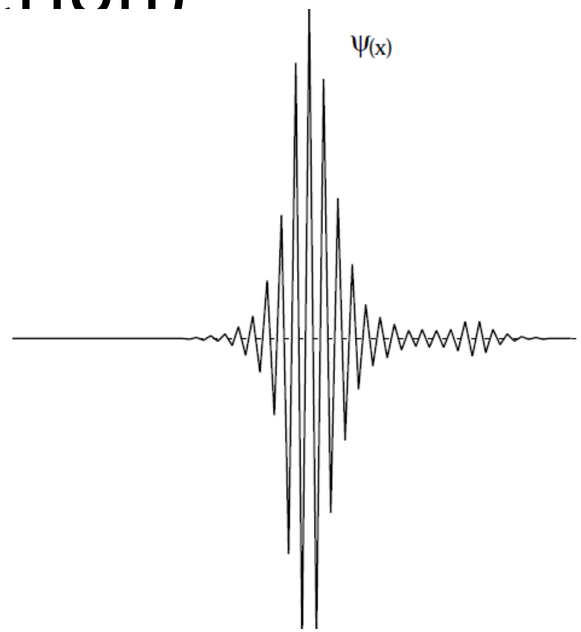
Note: For finite temperature T, L is replaced with T^{-p}

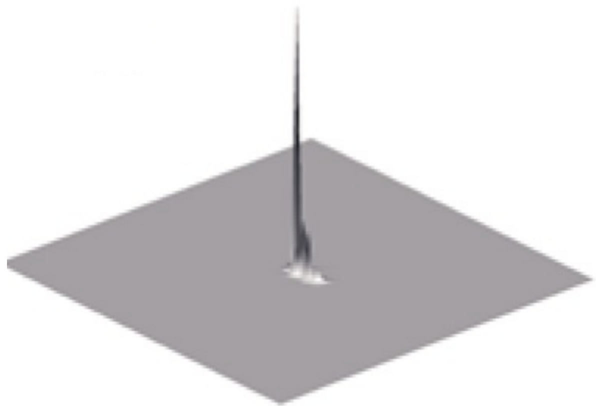
$\Delta\sigma/(e^2/2\pi^2\hbar)$	Regime
$-2\log(L/l)$	$\ell_B, L_{SO} \gg L$
$\log(L/l) - 3\log(L_{SO}/l)$	$\ell_B \gg L \gg L_{SO}$
$-2\log(\ell_B/l)$	$L_{SO} \gg L \gg \ell_B$
$\log(\ell_B/l) - 3\log(L_{SO}/l)$	$L \gg \ell_B \gg L_{SO}$
$-2\log(\ell_B/l)$	$L \gg L_{SO} \gg \ell_B$

Toward strong localization

- Weak localization: mean free path is much longer than the Fermi wave length.
- When they become comparable, the Anderson metal-insulator transition occurs. (Ioffe-Regel criterion)

$$\psi(\boldsymbol{r}) \sim \exp\left(-\frac{|\boldsymbol{r}-\boldsymbol{r}_0|}{\xi}\right)$$





Scaling theory of localization

$$\psi_{\mathrm{rel}} = \frac{\sigma_{\mathrm{D}} - \sigma_c}{\sigma_c}$$

$$\psi'_{\mathrm{rel}} = b^{1/\nu} \psi_{\mathrm{rel}}$$

Physical quantity $$Q_{\mathrm{P}} = f\left(\psi_{\mathrm{rel}}, L\right)$$

$\xrightarrow{\text{RG}}$ $$Q_{\mathrm{P}} = f\left(b^{1/\nu}\psi_{\mathrm{rel}}, \frac{L}{b}\right)$$

$\xrightarrow{\text{RG}}$ $$Q_{\mathrm{P}} = f\left(\left(\frac{L}{L_0}\right)^{1/\nu}\psi_{\mathrm{rel}}, L_0\right)$$

$$\boxed{Q_{\mathrm{P}} = F\left(L^{1/\nu}\psi_{\mathrm{rel}}\right)}$$

$$\frac{dQ_p}{dlogL} = f\left(L^{\frac{1}{\nu}}\psi_{rel}\right) = h(Q_p)$$

Anderson transition

- Scaling theory of localisation
  - E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, PRL **42**, 673 (1979).
- β-function

$$\beta(g) = \frac{\mathrm{d}\ln g}{\mathrm{d}\ln L} = \begin{cases} d-2=\epsilon, g \gg 1 \\ \log g, g \ll 1 \end{cases}$$

- Critical point

$$\beta(g_c) = 0$$

- Critical exponent

$$\left.\frac{\mathrm{d}\beta(g)}{\mathrm{d}\ln g}\right|_{g_c} = \frac{1}{\nu} \qquad \xi \sim |x - x_c|^{-\nu}$$

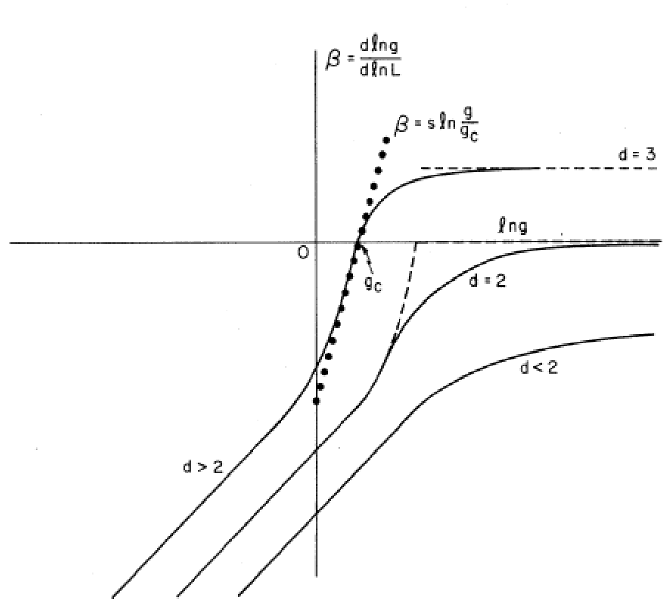


FIG. 1. Plot of $\beta(g)$ vs $\ln g$ for $d>2$, $d=2$, $d<2$. $g(L)$ is the normalized "local conductance." The approximation $\beta = s\ln(g/g_c)$ is shown for $g>2$ as the solid-circled line; this unphysical behavior necessary for a conductance jump in $d=2$ is shown dashed.

Abrahams et al., PRL '79

Comments on the scaling theory

- if we assume the monotonic behavior of beta-function, g is renormalized to 0 in 2d, hence all the states are localized.
  – monotonicity is not always true: QHE, spin-orbit, …
- Perturbative expansion:

$$g = g_0 - a \log L \; - b \; \frac{\log L}{g^3}$$

$$\beta(g) = -a - \frac{b}{g^3} = f(g)$$

Inequality of the exponent from single parameter scaling: B. Kramer PRB '93

$$Q_{\rm p} = \int {\rm d}W_1 \cdots {\rm d}W_N Q_{\rm p}(L, \{W_i\}) P_W(W_1) \cdots P_W(W_N)\,.$$

$$\left.\frac{{\rm d}f(\psi_{\rm rel}L^{1/\nu})}{{\rm d}W}\right|_{W_{\rm c}} = A_1 L^{1/\nu}$$
$$= \int {\rm d}W_1 \cdots {\rm d}W_N Q_{\rm p}(L, \{W_i\}) P_{\rm total} \left.\frac{{\rm d}\ln P_{\rm total}}{{\rm d}W}\right|_{W_{\rm c}}\,,$$

with $P_{\rm total} = \prod_i P_W(W_i)$. Here $A_1 = \psi'_{\rm rel}(W_{\rm c}) f'(0)$.

Using $$\left(\int {\rm d}x\, g(x) h(x)\right)^2 \leq \left(\int {\rm d}x\, g(x)^2\right)\left(\int {\rm d}x\, h(x)^2\right)$$

$$A_1^2 L^{2/\nu} \leq \int \mathrm{d}W_1 \cdots \mathrm{d}W_N P_{\mathrm{total}} \left(Q_{\mathrm{p}}(L, \{W_i\})\right)^2$$

$$\times \int \mathrm{d}W_1 \cdots \mathrm{d}W_N P_{\mathrm{total}} \left(\frac{\mathrm{d} \ln P_{\mathrm{total}}}{\mathrm{d}W} \right)^2 \Bigg|_{W_{\mathrm{c}}}$$

The first term is $\langle Q_{\mathrm{p}}(L, \{W_i\})^2 \rangle_{W_{\mathrm{c}}}$, which gives a factor independent of L. The second factor is expanded as

$$\int \mathrm{d}W_1 \cdots \mathrm{d}W_N P_{\mathrm{total}} \left(\frac{\mathrm{d} \ln P_{\mathrm{total}}}{\mathrm{d}W} \right)^2$$

$$= \int \mathrm{d}W_1 \cdots \mathrm{d}W_N P_{\mathrm{total}} \left(\sum_i \frac{\mathrm{d}P_W(W_i)/\mathrm{d}W}{P_W(W_i)} \right)^2$$

$$= \int \mathrm{d}W_1 \cdots \mathrm{d}W_N P_{\mathrm{total}} \sum_i \left(\frac{\mathrm{d}P_W(W_i)/\mathrm{d}W}{P_W(W_i)} \right)^2$$

$$+ \int \mathrm{d}W_1 \cdots \mathrm{d}W_N P_{\mathrm{total}} \sum_{i \neq j} \frac{\mathrm{d}P_W(W_i)/\mathrm{d}W}{P_W(W_i)} \frac{\mathrm{d}P_W(W_j)/\mathrm{d}W}{P_W(W_j)}$$

$$= N \int \mathrm{d}W_1 P_W(W_1) \left(\frac{\mathrm{d}P_W(W_1)/\mathrm{d}W}{P_W(W_1)} \right)^2 + N(N-1) \left(\int \mathrm{d}W_1 \frac{\mathrm{d}P_W(W_1)}{\mathrm{d}W} \right)^2 .$$

Due to normaliozation $\int \mathrm{d}W_1 P_W(W_1) = 1$ the 2nd term vanishes.

$$\longrightarrow \quad A_1^2 L^{2/\nu} \le B\langle Q_\mathrm{p}(L,\{W_i\})^2\rangle_{W_\mathrm{c}} N \propto L^d$$

with $$B = \int \mathrm{d}W_1 P_W(W_1)\left(\frac{\mathrm{d}P_W(W_1)/\mathrm{d}W}{P_W(W_1)}\right)^2\Bigg|_{W_\mathrm{c}} = \left\langle\left(\frac{\mathrm{d}P_W(W_1)/\mathrm{d}W}{P_W(W_1)}\right)^2\right\rangle_{W_\mathrm{c}}$$

$$\boxed{\nu \ge \frac{2}{d}}$$

Chayes, Chayes, Fisher, Spencer: PRL 1986, cf. Harris criterion
B. Kramer PRB '93

Very useful when checking the numerical calculations.

Non-linear sigma model for spin system

$$\mathcal{H} = \frac{J_1 a^2}{2} \int \frac{\mathrm{d}^d \boldsymbol{r}}{a^d} \sum_{k=1}^{N} (\nabla \sigma^k(\boldsymbol{r}))^2 = J \int \mathrm{d}^d \boldsymbol{r} (\nabla \boldsymbol{\sigma}(\boldsymbol{r}))^2$$

$$\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \cdots, \sigma_n) = (\sigma_1, \sigma_2, \cdots, \pm\sqrt{1 - \sigma_1^2 - \sigma_2^2 - \cdots - \sigma_{n-1}^2})$$

Non-linear sigma model for disordered electron system:

Efetov: "*Supersymmetry in Disorder and Chaos*", Cambridge Univ. Press

$$F[Q] = \frac{\pi \rho(E_\mathrm{F})}{8} \mathrm{Str} \int \mathrm{d}\boldsymbol{r} \left[D_0 (\nabla Q)^2 + 2\mathrm{i}(\omega + \mathrm{i}\delta) \Lambda Q \right]$$

$$Q = W + \Lambda (1 - W^2)^{1/2}, W = \begin{bmatrix} 0 & Q_{12} \\ Q_{21} & 0 \end{bmatrix}$$

Q: 16 x 16 supermatrix or N x N c-number

From NLsM,

1) universality classes

2) higher perturbative expansion

Classification of the Anderson localization

(Hikami ’80, Efetov et al. ‘80)

- Classified to 3 universality classes according to whether the time reversal symmetry (TRS) and spin rotation symmetry (SRS) are preserved or not.（Wigner-Dyson classes: c.f. random matrix theory）
  - Broken TRS due to magnetic field or magnetic impurities→Unitary class
  - TRS but no SRS due to spin-orbit interaction → Symplectic class
  - TRS and SRS→Orthogonal class

10 universality classes

(Altland-Zirnbauer, PRB '97)

- Wigner-Dyson classes (3 classes)
  - Unitary (no TRS, SRS irrelevant, QHE)
  - Orthogonal (TRS+SRS)
  - Symplectic (TRS but no SRS, QSH, 3D TI)
- chiral class (3 classes)
  - Wigner-Dyson classes + chiral symmetry
- BdG class (4 classes)
  - TRS, SRS

Borel-Pade approach

t=1/π*g* (resistance), $\beta(t) = -\frac{\mathrm{d}t}{\mathrm{d}\ln L}$. $\beta(t_c) = 0$, $\left.\frac{\mathrm{d}\beta}{\mathrm{d}t}\right|_{t_c} = -\frac{1}{\nu}$

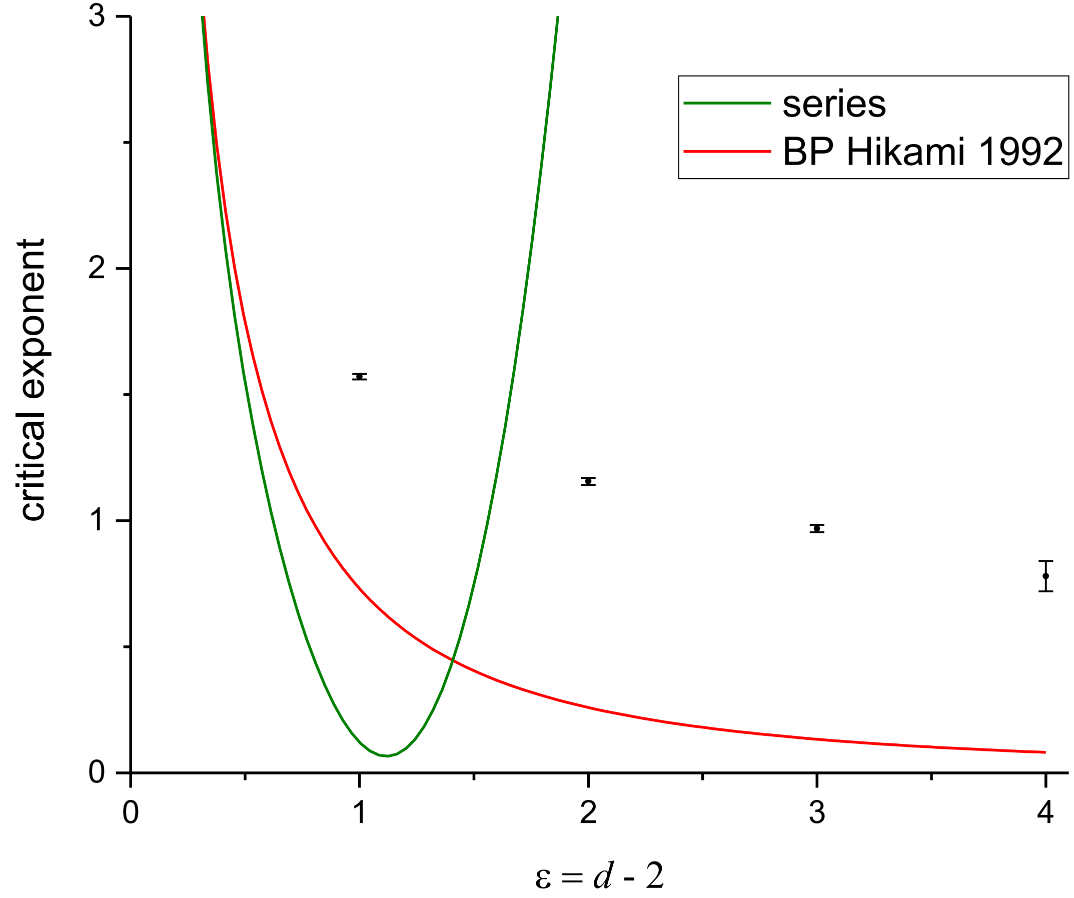
$$\beta(t) = \epsilon t - 2t^2 - 12\zeta(3)t^5 + \frac{27}{2}\zeta(4)t^6 + \mathcal{O}(t^7) \,,$$ Orthogonal class

$$\beta(t) = \epsilon t + t^2 - \frac{3}{4}\zeta(3)t^5 - \frac{27}{64}\zeta(4)t^6 + \mathcal{O}(t^7) \,,$$ Symplectic class

$$\beta(t) = \epsilon t - 2t^3 - 6t^5 + \mathcal{O}(t^7) \,.$$ Unitary class

Comparison with numerical estimates

- simple series expansion, Borel-Pade approx.→not so good



Limiting behaviour at upper critical dimension

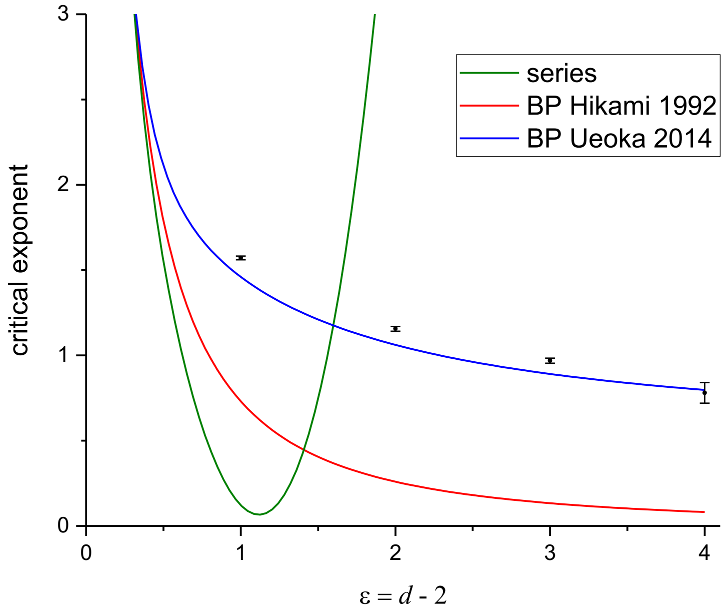
Upper critical dimension and limiting value of exponent

$$\nu(\epsilon) \to \frac{1}{2} \quad (\epsilon \to \infty) \ .$$

- K. B. Efetov, Physica A 167, 119 (1990).
- A. D. Mirlin and Y. V. Fyodorov, PRL 72, 526 (1994).
- M. Schreiber and H. Grussbach, PRL 76, 1687 (1996).
- Mard, Hoyos, Miranda, Dobrosavljevic, arXiv:1412.3793v1

Resum the series for ν taking into account the limiting behavior ν→1/2

- Much better



Ueoka, Y. and K. Slevin (2014). Journal of the Physical Society of Japan

Re-summation of β-functions

- ϵ-expansion β-functions obey

$$\beta(g,\epsilon) = \epsilon + \beta(g, \epsilon = 0) \ ,$$

- Then since

$$g_c \to 0 \ \ (\epsilon \to \infty) \ ,$$

- Re-summing so that

$$\beta(g) \sim A \ln g \quad (g \to 0)$$

- Will give $\left.\dfrac{\mathrm{d}\beta(g)}{\mathrm{d}\ln g}\right|_{g_c} \to A \quad (\ \to \infty)$

- ϵ-expansion β-functions have form

$$\beta(t) = \epsilon t - t f(t) \ .$$

- Define new function $h(t)$

$$t\frac{\mathrm{d}f(t)}{\mathrm{d}t} = A + h(t)$$

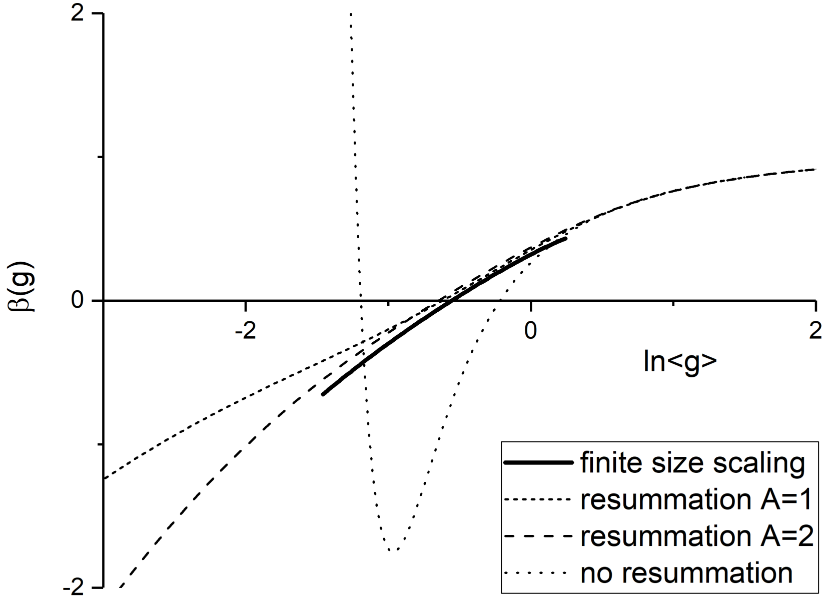
- Apply Borel-Pade to $h(t)$

$$f(t) \approx \sum_{j=1}^{n} \frac{a_j}{\lambda_j} B\left(\lambda_j / t\right)$$

Ueoka, Y. and K. Slevin (2017). To appear in Journal of the Physical Society of Japan. arxiv:1708.00152

Orthogonal d=3

- Comparison with finite size scaling of simulation data



Ueoka, Y. and K. Slevin (2017). To appear in Journal of the Physical Society of Japan. arxiv:1708.00152

Comparison with numerics

d	g_c	ν	Ref. 11 ν	numerical estimate ν
3	5.23×10^{-1}	1.64	1.46	$1.571 \pm .004$ Ref. 17
4	1.37×10^{-1}	1.06	1.06	$1.156 \pm .014$ Ref. 11
5	5.60×10^{-2}	0.775	0.891	$0.969 \pm .015$ Ref. 11
6	2.76×10^{-2}	0.655	0.798	$0.78 \pm .06$ Ref. 18

d	g_c	ν	Ref.[22,23)]
2	1.2×10^{-1}	0.874	$2.73 \pm .02$
3	6.13×10^{-2}	0.565	$1.375 \pm .008$
4	3.64×10^{-2}	0.492	
5	2.26×10^{-2}	0.466	
6	1.42×10^{-2}	0.459	

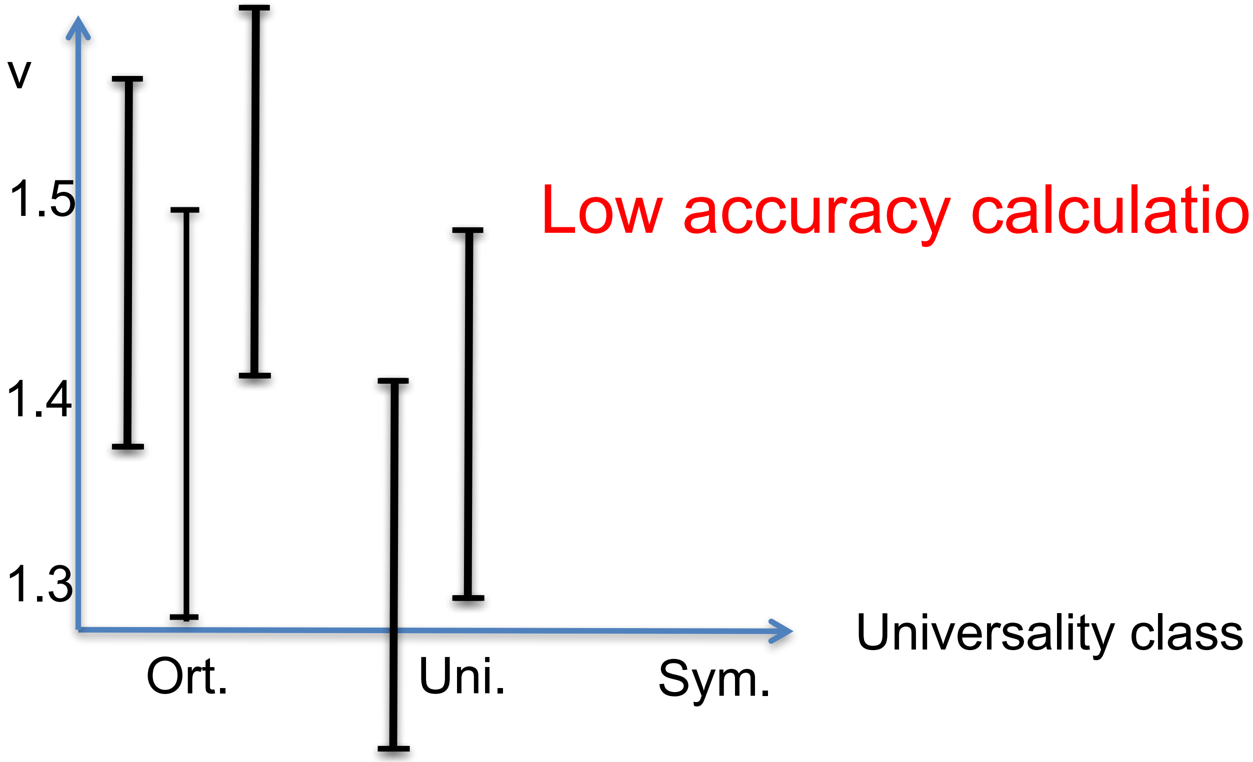
d	Eq.(C·2)	Eq.(C·3)	g_c	ν	Ref.[26–28)]
3	0.26	0.71	2.82×10^{-1}	0.969	1.437[.426, .448]
4	0.1	0.58	1.27×10^{-1}	0.687	1.1[.09, .12]
5	0.05	0.54	6.76×10^{-2}	0.595	
6	0.03	0.53	3.82×10^{-2}	0.552	

What is analytically known

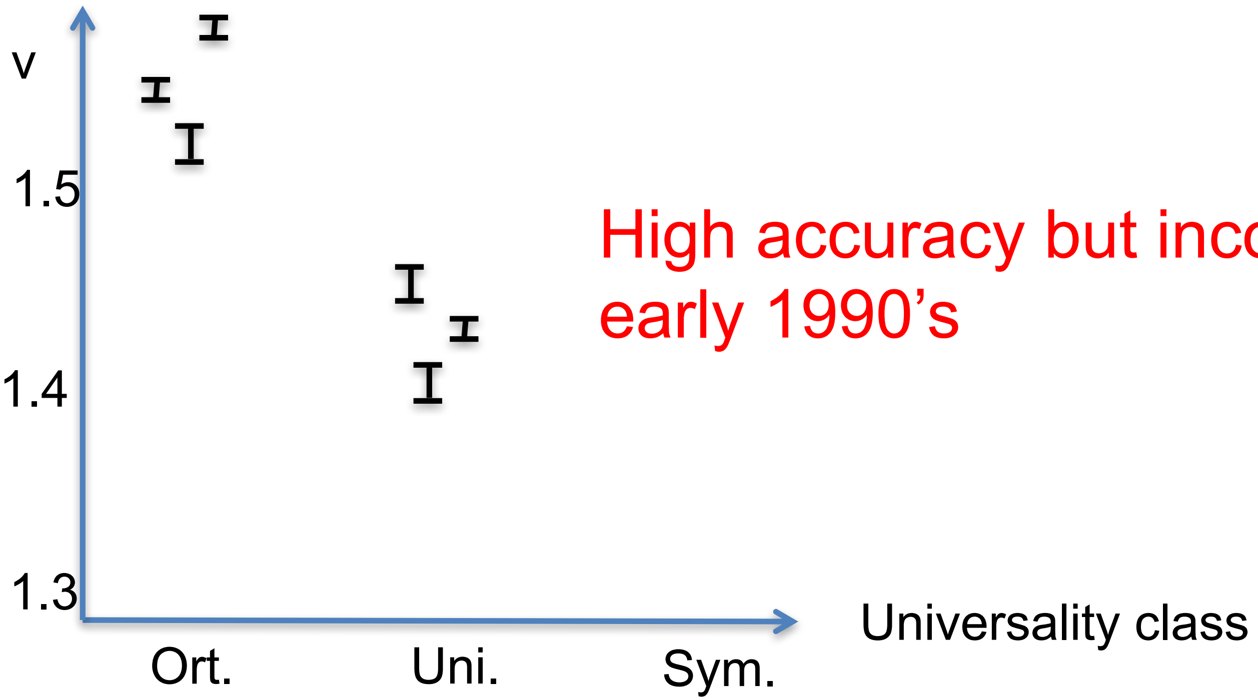
- Weak localization corrections: experimentally verified
- Existence of the Anderson transition
- inequalities and limiting behavior for the exponent, $\nu \geqq 2/d$, $\nu(d=\infty)=1/2$
- NLσM
  - higher order perturbation+Borel-Pade
  - universality classes
- Conformal invariance → next lecture

Simulating Anderson transition

- Analytic approach (ε(=*d*-2)-expansion) to estimate the critical exponent fails.
  - Without proper BP even violates Chayes et al.'s inequality
- To confirm the classification to universality classes, the critical exponent should be
  - the same for the same universality classes (within error bars)
  - different for different universality classes
- In fact, the critical exponents are close to each other even for different universality classes.
  - High precision raw data sets are needed.
  - Corrections to single parameter scaling should be taken into account.

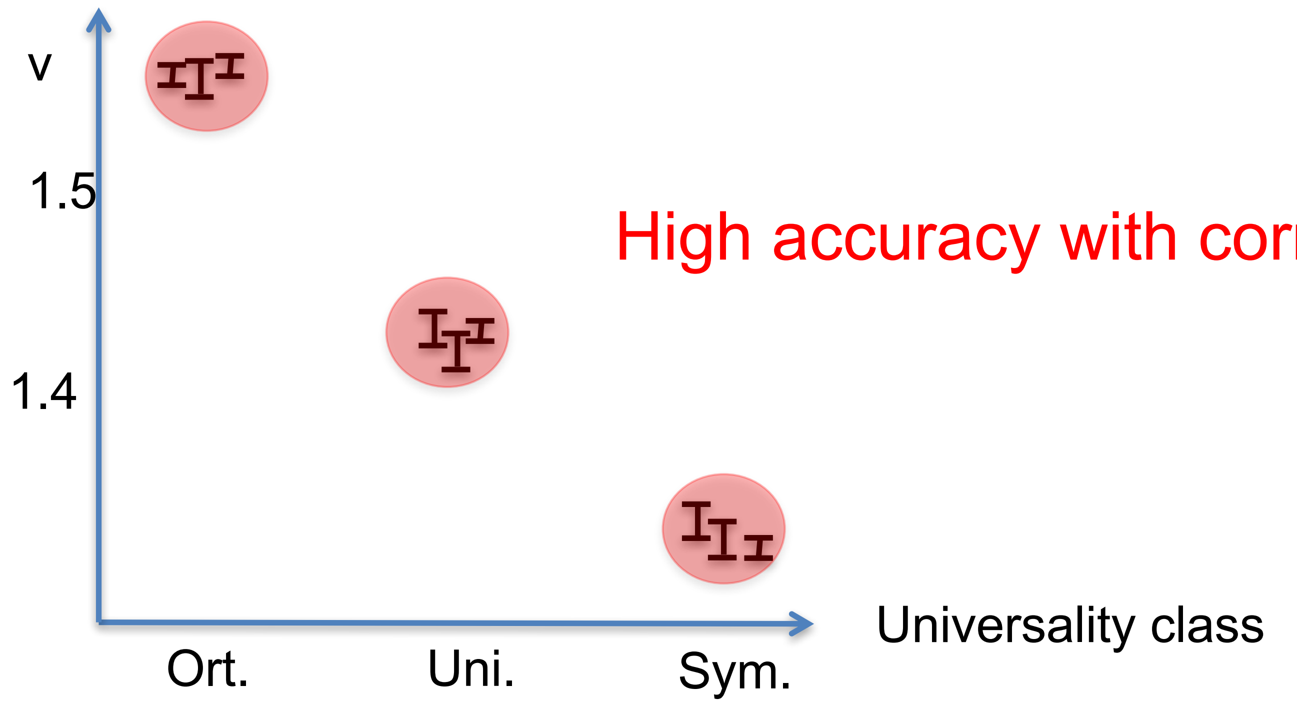


Low accuracy calculation, 1980's



High accuracy but incorrect analyses, early 1990's

To confirm the universality of AT and the concept of universality class



High accuracy with correct analyses

Anderson Model

Non-interacting, random Hamiltonian

$$H = \sum_j \varepsilon_j |j\rangle\langle j| + \sum_{j',j} V_{j',j} |j'\rangle\langle j| \quad , \qquad -\frac{W}{2} < \varepsilon_j < \frac{W}{2}$$

In magnetic fields $V_{jj'} = V \exp\left[\mathrm{i}\frac{e}{\hbar}\int_j^{j'} \mathbf{A}\cdot \mathrm{dx}\right]$

In the presence of spin-orbit interaction

$$H = \sum_{j,\sigma} \varepsilon_j |j,\sigma\rangle\langle j,\sigma| + \sum_{j',j,\sigma,\sigma'} V_{j',\sigma',j,\sigma} |j',\sigma'\rangle\langle j,\sigma|$$

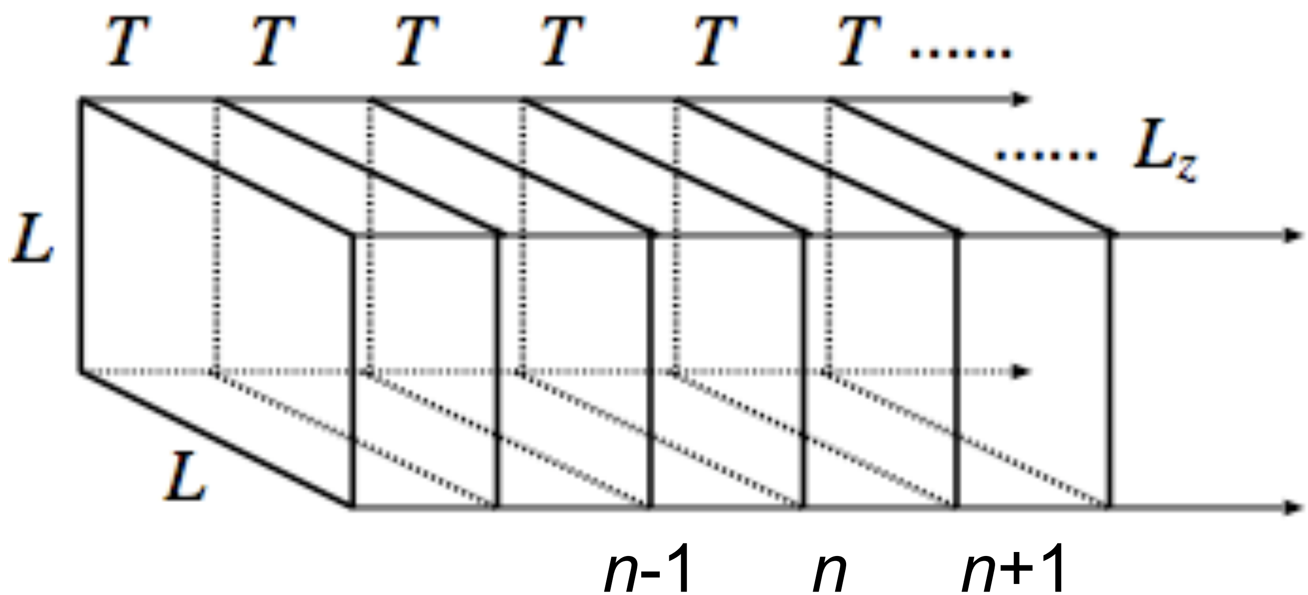
SU(2) matrix

Lyapunov exponent

Transfer matrix

$$E\psi_n = H_n\psi_n + V_{n,n+1}\psi_{n+1} + V_{n,n-1}\psi_{n-1}$$

$$\begin{pmatrix} \psi_{n+1} \\ V_{n+1,n}\psi_n \end{pmatrix} = T_n \begin{pmatrix} \psi_n \\ V_{n,n-1}\psi_{n-1} \end{pmatrix} \qquad \boldsymbol{T}_n = \begin{pmatrix} \boldsymbol{V}_{n,n+1}{}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{V}^{\dagger}_{n,n+1} \end{pmatrix} \begin{pmatrix} E\boldsymbol{I} - \boldsymbol{H}_n & -\boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{0} \end{pmatrix}$$



$$\psi_n \approx \exp\left(-\frac{n}{\xi_{q1D}(L)}\right)$$

Scaling analysis of Lyapunov exponents

- Properties of *T*: psudounitarity $T^{+}\Sigma_2 T=\Sigma_2 \quad , \quad \Sigma_2=\begin{bmatrix} 0 & -I_N \\ I_N & 0 \end{bmatrix}$
  - Reciprocal pairs of eigenvalues for $T^{+}T$, +- pairs of Lyapunov exponents

$$\Omega=\ln T^{+}T \Rightarrow \{\nu_1,\nu_2,\cdots,\nu_N,-\nu_N,\cdots,-\nu_1\} \Rightarrow \lambda_i=\frac{\nu_i}{2L_z}$$

- MacKinnon-Kramer (MK) parameter (MacKinnon-Kramer, '81 PRL, '83 Z. Phys., Pichard-Sarma, '81 J. Phys. C)
  - Quasi-1D localization length=1/smallest LE, normalized by the dimension *L*

$$\Lambda=\frac{\xi_{q1D}}{L}=\frac{1}{\lambda_N L} \quad , \quad \Gamma=\lambda_N L$$

Actual calculation of Lyapunov exponent

- Product of the transfer matrices causes unstable numerical estimate.
- We need to make QR decomposition every *l* (typically 10) times.

$$
\begin{aligned}
M_l \cdots M_1 Q_0 &= Q_1 R_1 \\
M_{2\,l} \cdots M_{l+1} Q_1 &= Q_2 R_2 \\
&\vdots \\
M_L \cdots M_{(p-1)\,l+1} Q_{p-1} &= Q_p R_p \ .
\end{aligned}
$$

The matrix MQ_0 can then be written as

$$MQ_0 = Q_p R_p \cdots R_1 \ .$$

The QR factorization of MQ_0 is thus given by

$$Q = Q_p \ ,$$

and

$$R = R_p \cdots R_1 \ .$$

Properties of MK parameter

$$\Gamma(L) = F(L/\xi) = f(L^{1/\nu}(W - W_c)) = \begin{cases} L^{2-d} \text{ (delocalized, metal)} \\ \Gamma_c \text{ (critical)} \\ L/\xi \text{ (localized, insulator)} \end{cases}$$

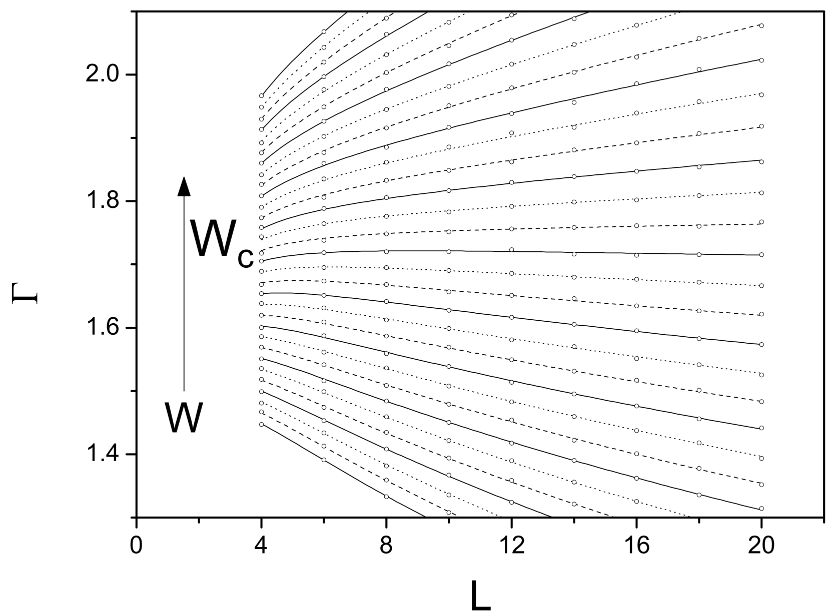
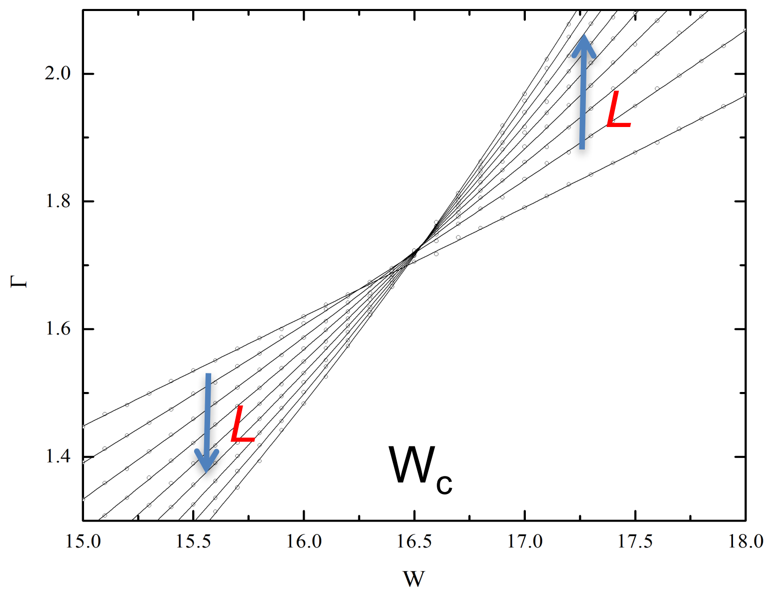


Fig. 1.1. Numerical data for the three dimensional Anderson model with box distributed random potential, width $W = 15 - 18$ in steps of 0.1. The precision of the data is 0.1%. The lines are a finite size scaling fit that includes corrections to scaling.

Kramer et al. 2010



Slevin-Ohtsuki 2012

Comments on the simulation

- To achieve a few percent accuracy of critical exponents, typically 0.2 to 0.05 % accuracy is required for raw data.
- With such high accuracy, deviation from single parameter scaling is observed.
- For 3d, the computation time is proportional to L^7, while for 2d, it is L^4.
- Accuracy of data should be of the same order.
  - If the accuracy of raw data for small system is high, while it is low for large system, the weight of small system data becomes much bigger in the chi square analysis.
- Nonlinear fitting is involved. The confidence intervals of critical exponents are estimated by
  - Bootstrap method
  - Psuedo-data sets analysis

Analysis of MK parameter

- Single parameter scaling

$$\Lambda = f(L^{1/\nu} u_{\mathrm{rel}}) = F(L/\xi) \;\; , \;\; \xi \propto \frac{1}{u_{\mathrm{rel}}^{\nu}} \;\; , u_{\mathrm{rel}} = u_{\mathrm{rel}}(E,W)$$

– plus irrelevant variables $\Lambda = f(L^{1/\nu} u_{\mathrm{rel}}, L^{y} u_{\mathrm{irr}}) \;\; , y < 0$

– We don't know u_{rel}, u_{irr}, f

$$u_{\mathrm{rel}}(E,W) = u_1 \left(\frac{W - W_c}{W_c} \right) + u_2 \left(\frac{W - W_c}{W_c} \right)^2 + \cdots$$

$$u_{\mathrm{irr}}(E,W) = u'_0 + u'_1 \left(\frac{W - W_c}{W_c} \right) + u'_2 \left(\frac{W - W_c}{W_c} \right)^2 + \cdots$$

$$f(L^{1/\nu} u_{\mathrm{rel}}, L^{y} u_{\mathrm{irr}}) = f_0(L^{1/\nu} u_{\mathrm{rel}}) + L^{y} u_{\mathrm{irr}} f_1(L^{1/\nu} u_{\mathrm{rel}}) + (L^{y} u_{\mathrm{irr}})^2 f_2(L^{1/\nu} u_{\mathrm{rel}}) + \cdots$$

$$f_0(L^{1/\nu} u_{\mathrm{rel}}) = f_{00} + f_{01} L^{1/\nu} u_{\mathrm{rel}} + f_{02} (L^{1/\nu} u_{\mathrm{rel}})^2 + \cdots$$

single parameter behavior of Γ=1/Λ

$$\Gamma_{\text{corrected}}(W, L) = \Gamma(W, L) - \Delta(W, L).$$

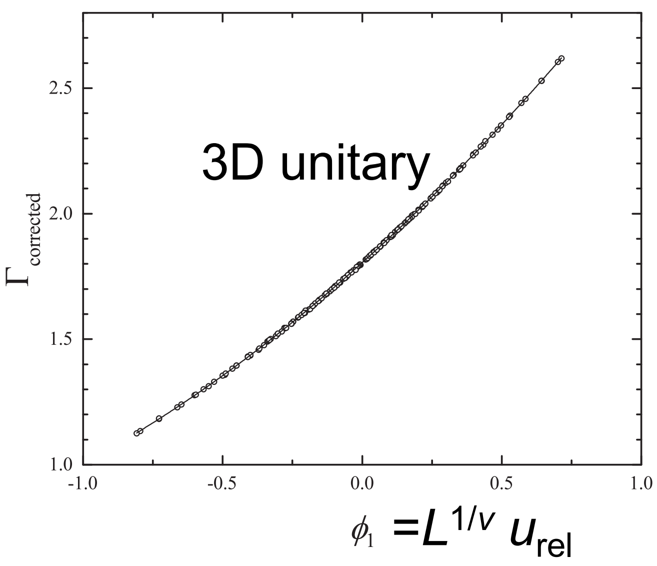


Fig. 3. The numerical data with the corrections to scaling subtracted (circles) are plotted versus the relevant scaling variable. The scaling function F_1 (sold line) is also shown. The plot demonstrates the collapse of all the data onto a single curve that is required by the single parameter scaling hypothesis.

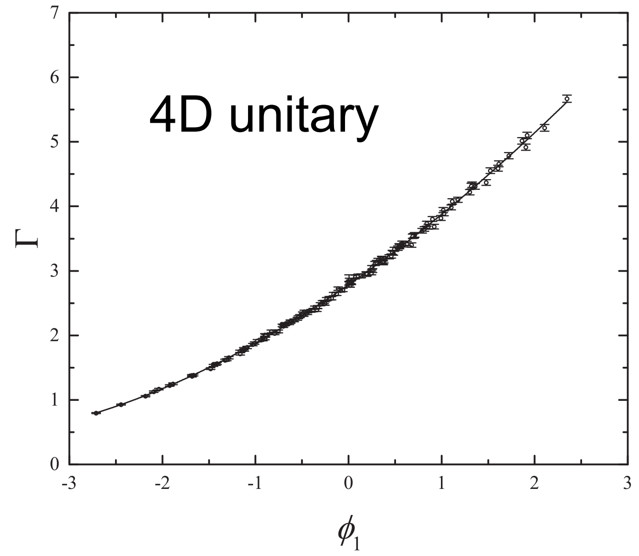


Fig. 4. The numerical data (circles) are plotted versus the relevant scaling variable. The scaling function F (sold line) is also shown.

Slevin-Ohtsuki, '16 JPSJ

Corrections to scaling, smaller for higher dimensions like 4D.

Critical exponents for the Anderson transitions (note: $\nu \geq \frac{2}{d}$)

2D

Universality class	ν
Unitary (QHE)	2.59±0.01
symplectic	2.75±0.01

3D

Universality class	ν
orthogonal	1.57±0.01
unitary	1.44±0.01
symplectic	1.37±0.01

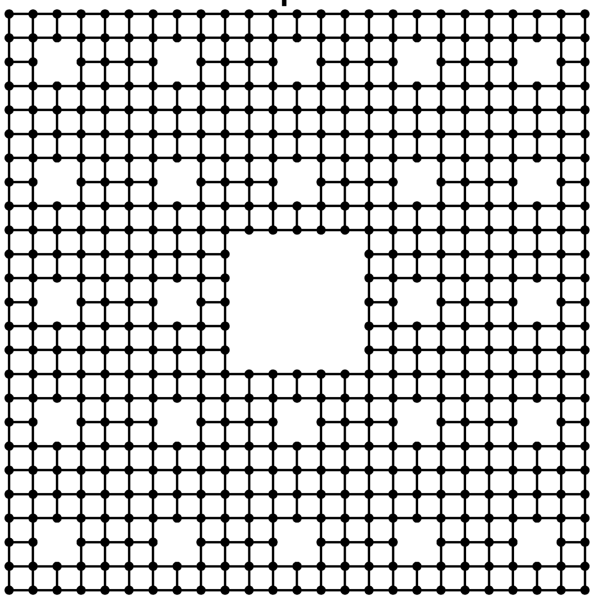
4D

Universality class	ν
orthogonal	1.156±0.014
unitary	1.12±0.05
symplectic	???

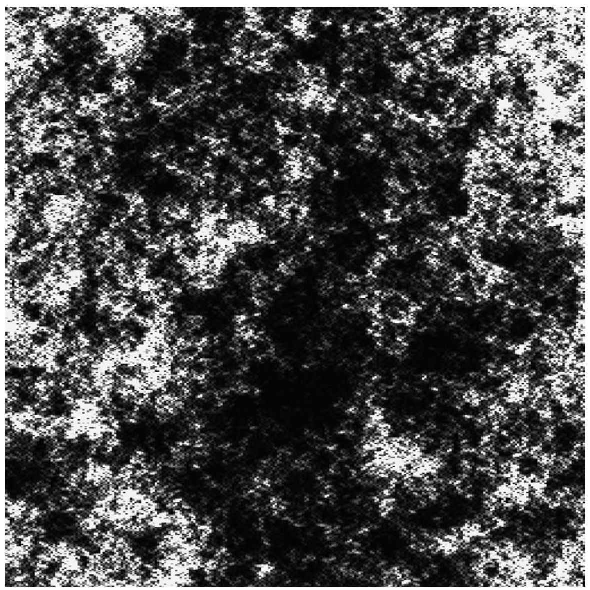
Wave function multifractality

- At plateau-plateau transition (Landau band center), localization length diverges → scale invariance and self-similarity.
- Due to random nature of systems, self-similarity is observed as multifractality.

Simple fractal



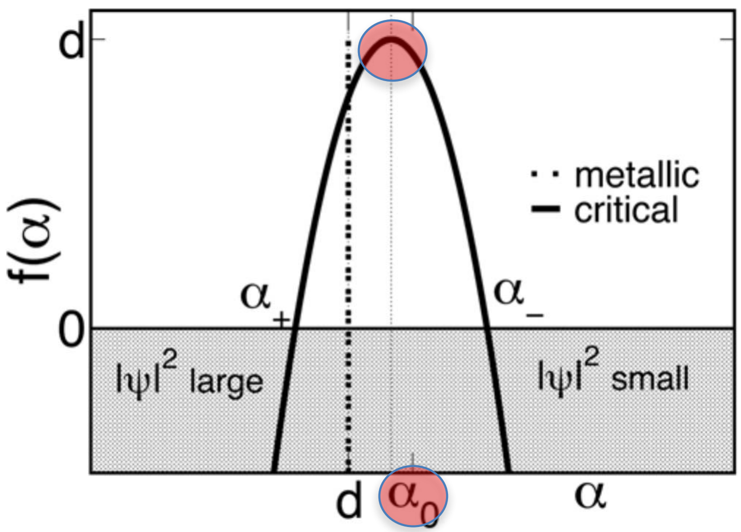
multifractal

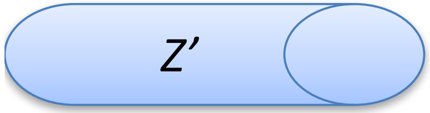


Multifractal analysis

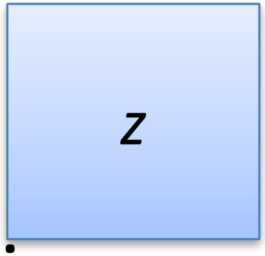
Define a box of l^d, $\lambda \equiv \frac{l}{L}$. $\mu_k \equiv \sum_{j \in \text{box } k} |\psi_j|^2$, $\alpha \equiv \frac{\ln \mu}{\ln \lambda}$, (random variables)

The number of boxes *N* corresponding to (random) variable α $N(\alpha) \sim \lambda^{-f(\alpha)}$,





Conformal invariance?



- LE (Γ_c) in quasi 1D is related to 2D correlation function if conformal invariance exists.
- 2D correlation reflects multifractality of w.f.
- The proposal by Janssen ('98)

$$\Gamma_c = \lim_{N\to\infty} F(N^{\alpha} u_{\mathrm{rel}}, N^{y} u_{\mathrm{irr}})\Big|_{x=0} = \pi(\alpha_0 - 2)$$

- In reality, estimate of *y*, varies from -0.6 to -0.2, so it is very difficult to estimate Γ_c

PHYSICAL REVIEW B **78**, 115301 (2008)

Boundary criticality at the Anderson transition between a metal and a quantum spin Hall insulator in two dimensions

Hideaki Obuse,[1,*] Akira Furusaki,[1] Shinsei Ryu,[2] and Christopher Mudry[3]

$$\Lambda^{(\mathrm{PBC})}(X_s) = 1.81 \pm 0.01\,, \qquad \alpha_0^{(2)} = 2.174 \pm 0.001\,,$$

$1/\Lambda = 0.552$ $\qquad$ $\pi(\alpha_0 - 2) = 0.547$

PRL **101,** 116802 (2008) PHYSICAL REVIEW LETTERS week ending 12 SEPTEMBER 2008

Boundary Multifractality at the Integer Quantum Hall Plateau Transition: Implications for the Critical Theory

H. Obuse,[1,*] A. R. Subramaniam,[2] A. Furusaki,[1] I. A. Gruzberg,[2] and A. W. W. Ludwig[3]

Estimates of $1/\Lambda$ are at variance due to large corrections to scaling.

$$\alpha_0^b - 2 = 0.2617 \pm 0.0006$$

$\pi(\alpha_0 - 2) = 0.822$ $\qquad$ $1/\Lambda = ???$

Critical value g_c, Γ_c, etc. depends on the geometry (Slevin, TO, Kawarabayashi)

VOLUME 84, NUMBER 17 PHYSICAL REVIEW LETTERS 24 APRIL 2000

Topology Dependent Quantities at the Anderson Transition

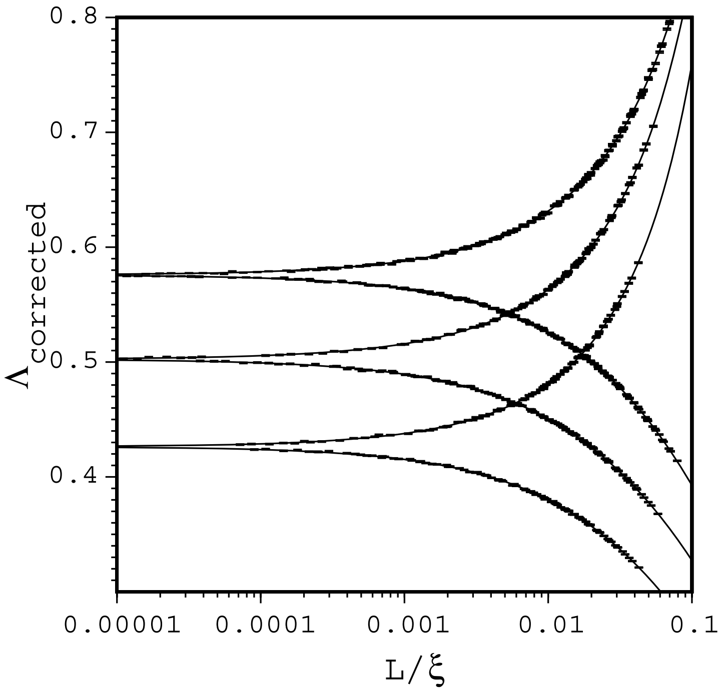


FIG. 3. The scaling functions for different boundary conditions.

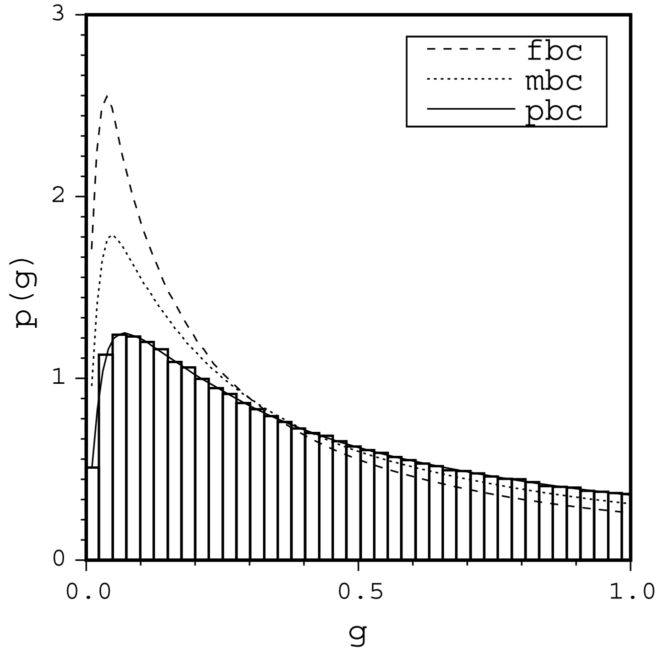


FIG. 4. The dependence of the critical conductance distribution on the choice of boundary conditions. Here the Fermi energy $E_F = 0$, the system size $L = 10$, and the disorder $W = 16.54$ independent of the choice of boundary conditions.

Limit of transfer matrix

- If the matrix elements between sites on n-th layer and n+1th layer, $V_{n,n+1} = \langle n|H|n+1\rangle$, is not invertible, we can not use transfer matrix.
- This happens in the case of bcc and fcc lattices, quantum percolation and of localization on fractal lattice.

Other numerical technique

- Level statistics and Multifractal scaling
  - Sparse matrix diagonalization: Lanczos, …,
  - JADAMILU (**JA**cobi-**DA**vidson method with **M**ultilevel **ILU** preconditioning), developed by Belgian group, is suited for 2D and 3D Anderson models
- Equation of motion method
  - Tchebyshev polynomial, more efficient
  - Kernel Polynomial method
  - Trotter-Suzuki formula, less efficient, but can be used for time dependent Hamiltonian
- Mapping the Anderson model to kicked rotor
  - Successful for 3d orthogonal, and QHE
- Machine learning method

Remaining problems

- quantum Hall experiment
- analytic expression for critical behaviors? ν, multifractality, Λ_c , etc.
- conformal invariance for the quantum Hall effect?
- conformal invariance for higher dimensions: what should be checked?

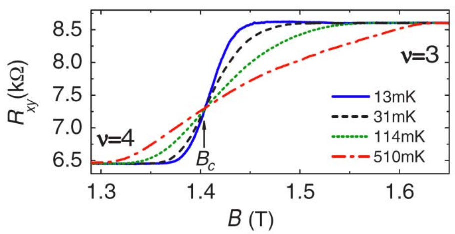


FIG. 1 (color online). Hall resistance around the 4-3 transition at different temperatures. A critical field of $B_c = 1.4$ T is observed.

$$\frac{1}{z\nu} = 0.42 \pm 0.01$$

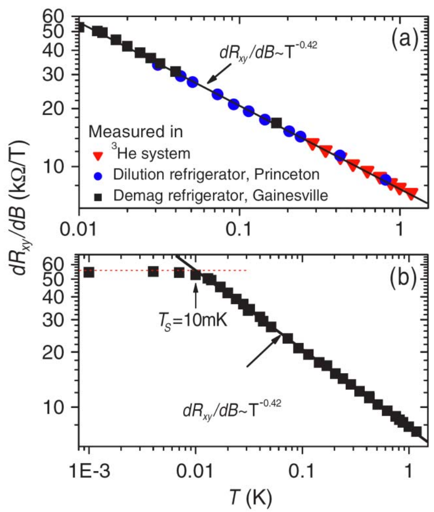


FIG. 2 (color online). (a) Perfect temperature scaling $(dR_{xy}/dB)|_{B_c} \propto T^{-0.42}$ of the 4-3 transition over two decades of temperature between 1.2 K and 12 mK. (b) Saturation of $(dR_{xy}/dB)|_{B_c}$ at low temperatures. The saturation temperature $T_s = 10$ mK is obtained from the cross point between extrapolations of the higher temperature data (black line) and the lower temperature saturated data (horizontal dotted line).

Li, W., C. L. Vicente, et al. (2009). Physical Review Letters **102(21): 216801-4.**