4) 2d systems and PEPS

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Some reviews

J. Eisert, Modeling and Simulation 3, 520 (2013), arXiv:1308.3318

N. Schuch, QIP, Lecture Notes of the 44th IFF Spring School 2013, arXiv:1306.5551

R. Orus, arXiv:1306.2164, arXiv:1407.6552

J. I. Cirac, F. Verstraete, J. Phys. A: Math. Theor. 42, 504004 (2009)

F. Verstratete, J. I. Cirac, V. Murg, Adv. Phys. 57,143 (2008)

J. Jordan, PhD thesis, www.romanorus.com/JordanThesis.pdf

G. Evenbly, PhD thesis, arXiv:1109.5424

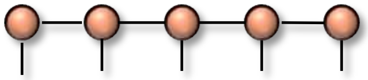
U. Schollwöck, RMP 77, 259 (2005)

U. Schollwöck, Annals of Physics 326, 96 (2011)

Basics and properties

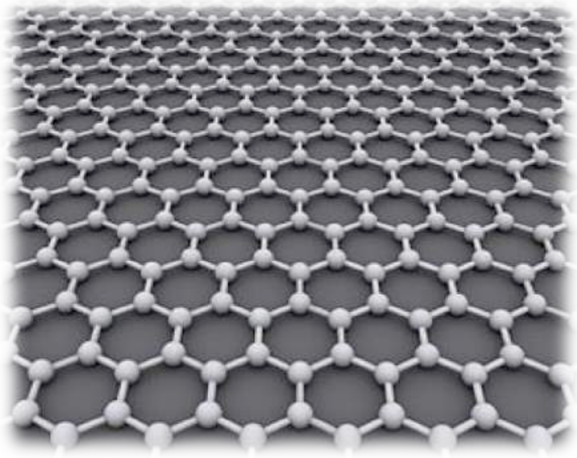
From MPS to PEPS

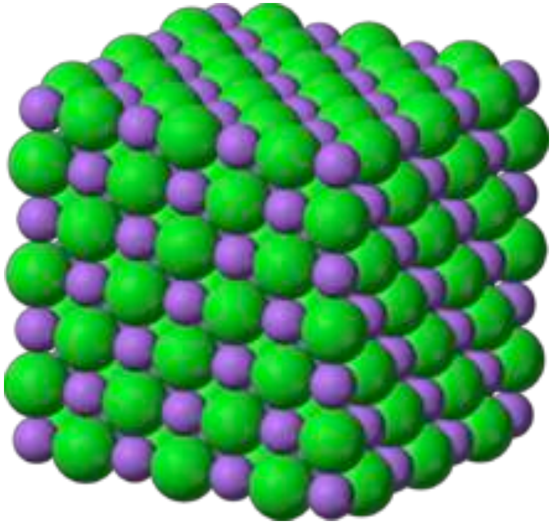
Matrix Product States (MPS)



1d systems

But we want to go beyond 1d systems!!!

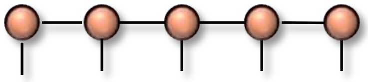




Very painful for DMRG…

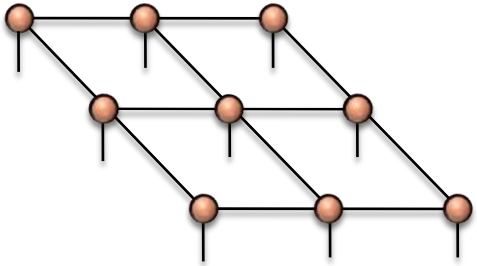
From MPS to PEPS

Matrix Product States (MPS)

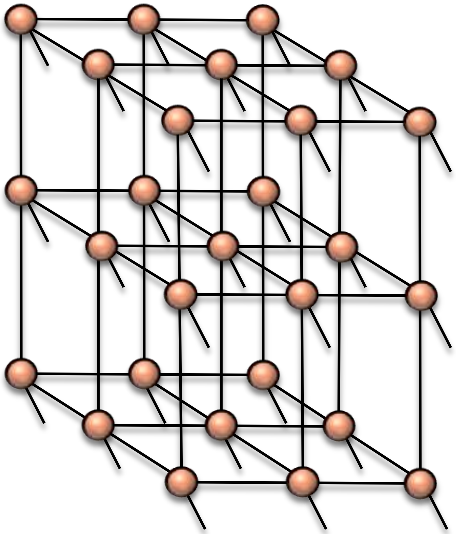


1d systems

Projected Entangled Pair States (PEPS),
Tensor Product States (TPS)



2d systems

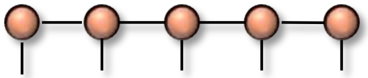


3d systems

and so on…

From MPS to PEPS

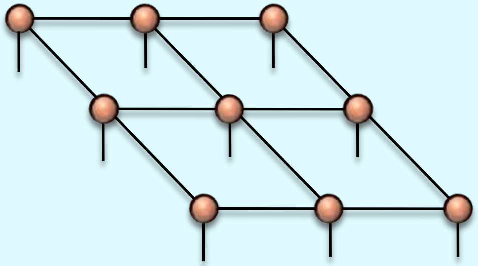
Matrix Product States (MPS)



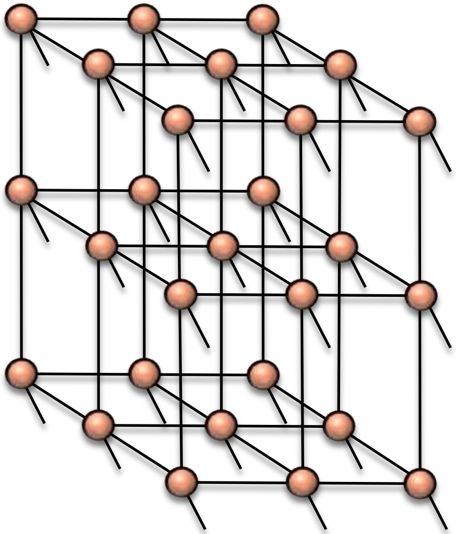
1d systems

Projected Entangled Pair States (PEPS),
Tensor Product States (TPS)

This lecture



2d systems



3d systems

and so on…

PEPS are not your friends... *(M. Lubasch)*

MPS

PEPS are not your friends… *(M. Lubasch)*

MPS

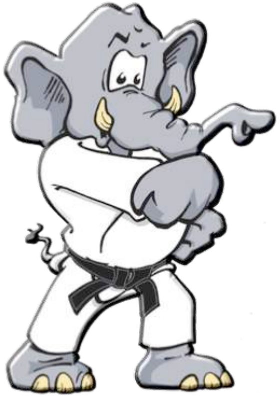
PEPS *(initially)*

PEPS are not your friends... *(M. Lubasch)*

MPS

PEPS *(initially)*

PEPS *(in the end)*



...but, after a lot of gymnastics, they can be your allies!

Two exact examples

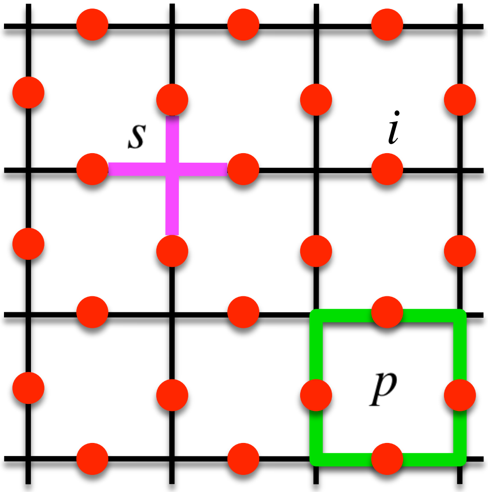
An exact example: Kitaev's Toric Code

Kitaev, 1997

$$H = -J\sum_{s} A_s - J\sum_{p} B_p$$

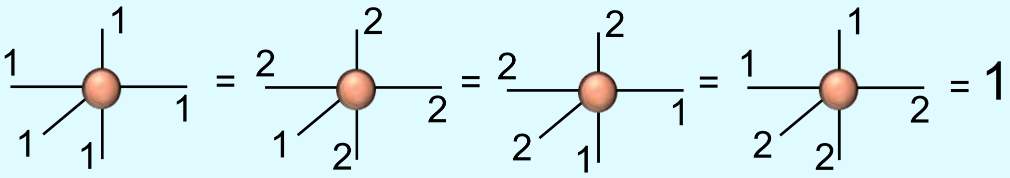
$$A_s = \prod_{i\in s}\sigma_i^x$$ *star operator*

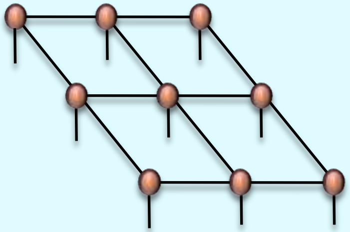
$$B_p = \prod_{i\in p}\sigma_i^z$$ *plaquette operator*



Simplest known model with "topological order"

Ground state (and in fact all eigenstates) are PEPS with D=2

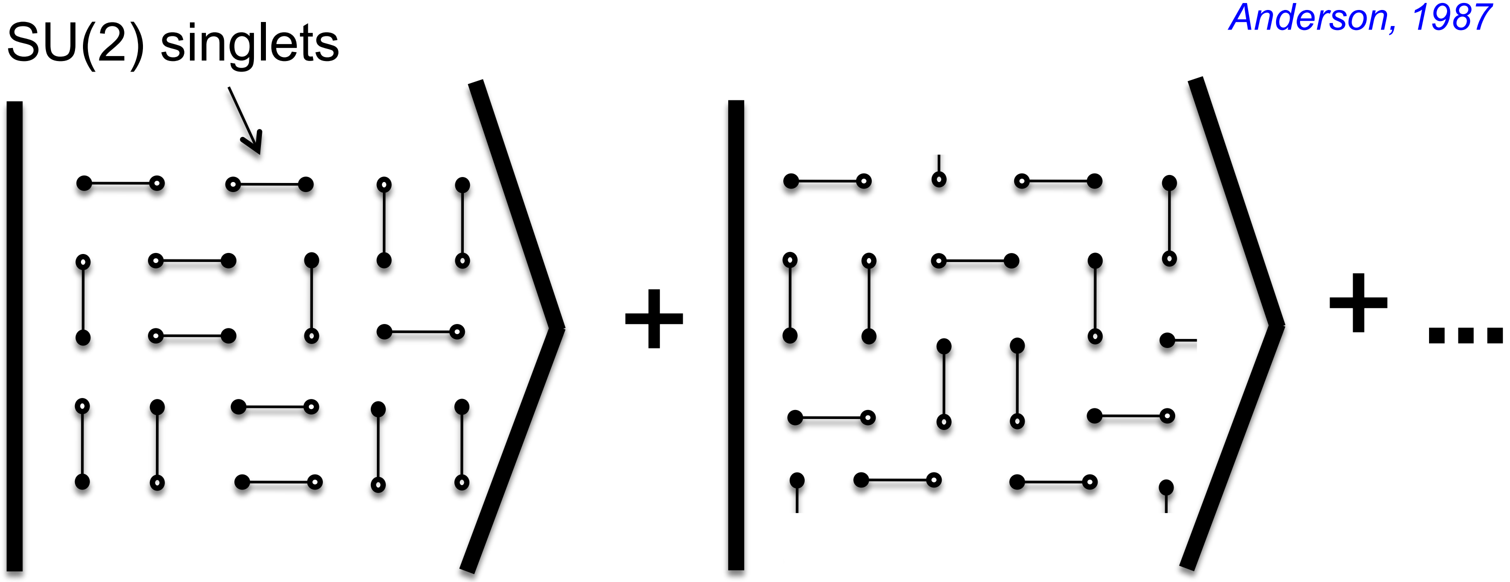




And another tensor rotated 90˚

Resonating Valence Bond State

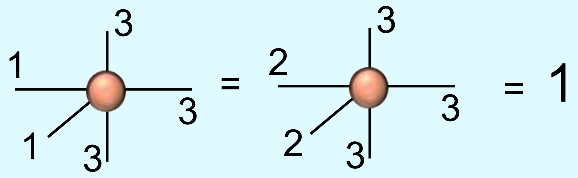
Anderson, 1987



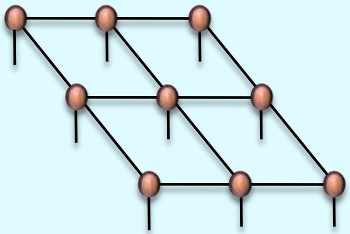
Equal superposition of all possible nearest-neighbor singlet coverings of a lattice (spin liquid)

Proposed to understand high-T_C superconductivity

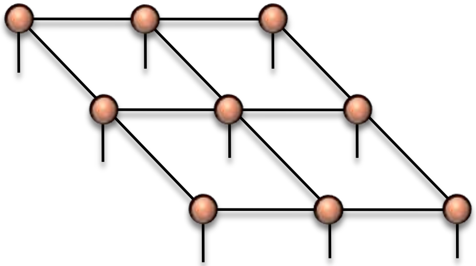
It is a PEPS with D=3



And rotations

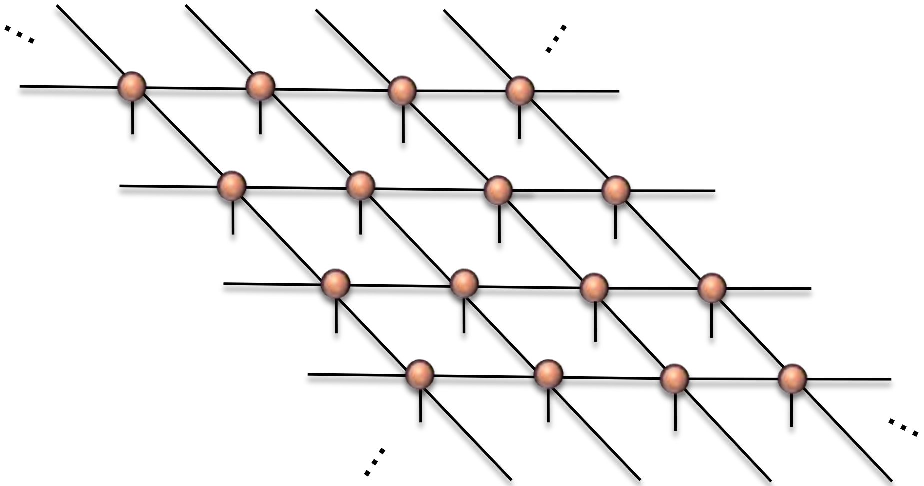


PEPS…



F. Verstraete, I. Cirac, cond-mat/0407066

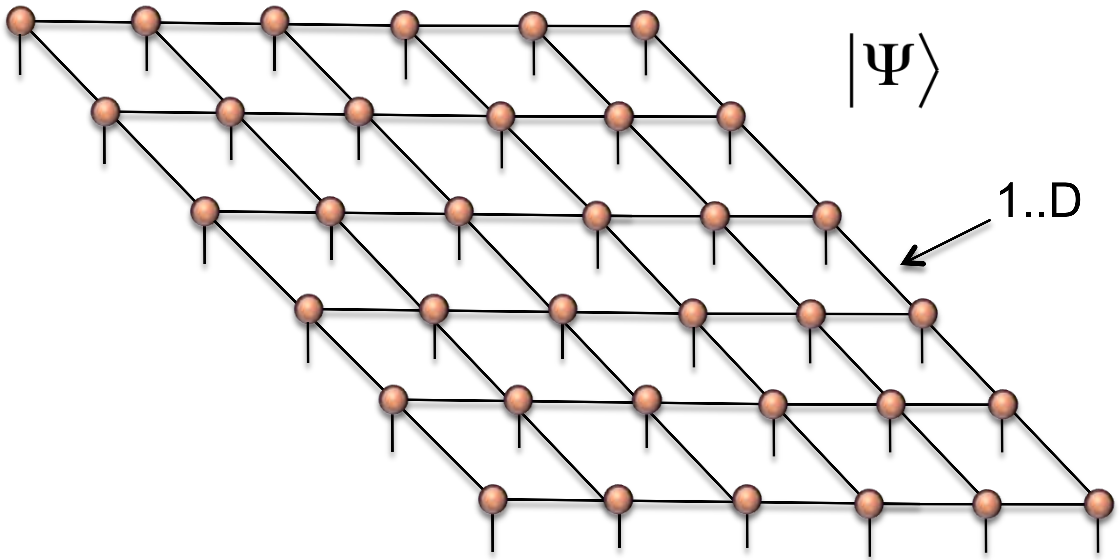
and infinite PEPS (iPEPS)



assuming translation invariance

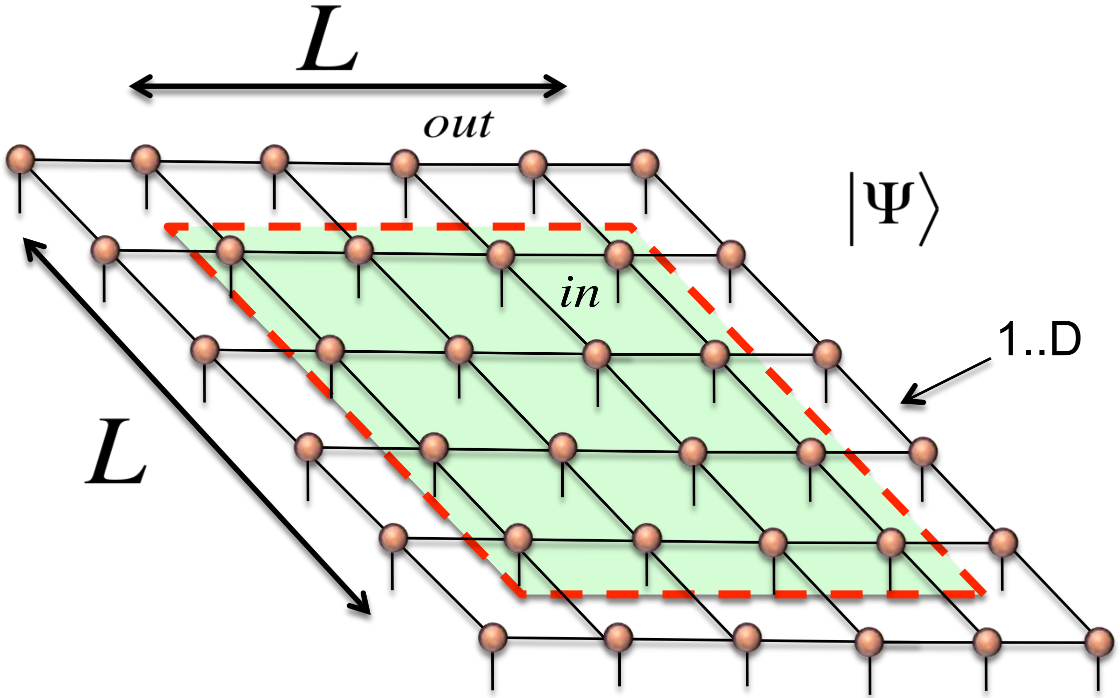
J. Jordan, RO, G. Vidal, F. Verstraete, I. Cirac, Phys. Rev. Lett. 101, 250602 (2008)

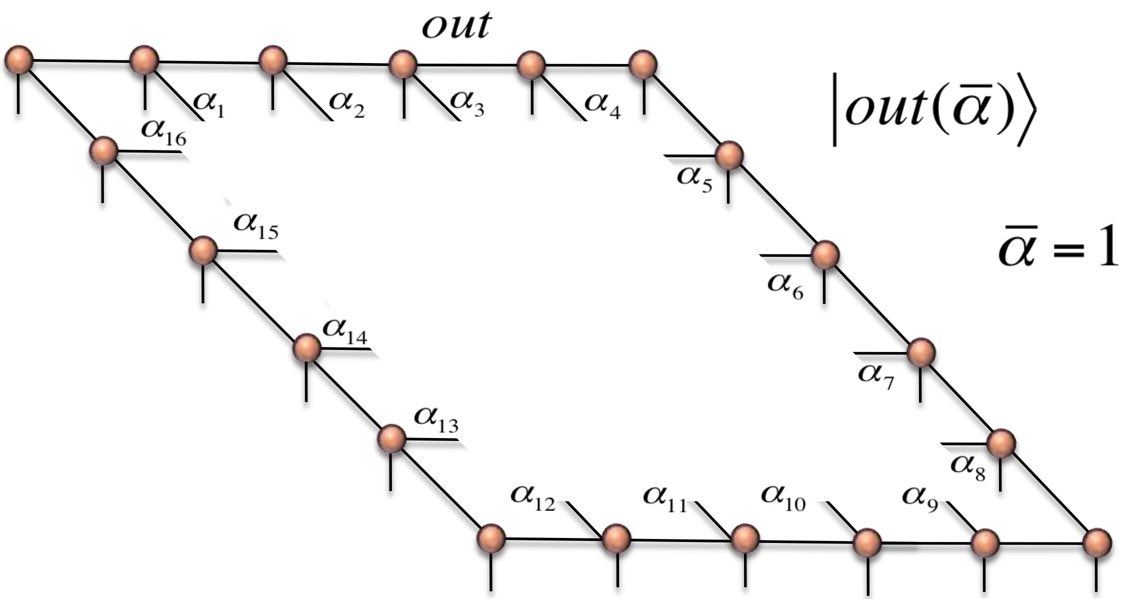
PEPS obey 2d area-law



$|\Psi\rangle$

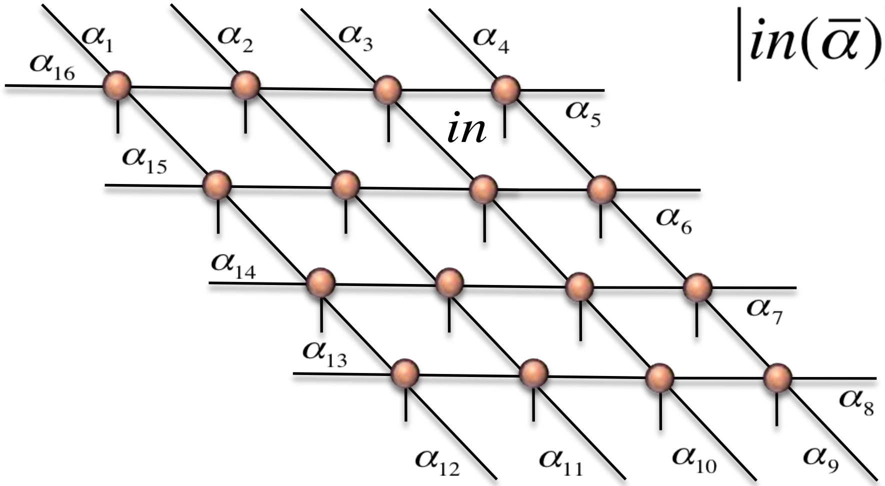
1..D


L
out
$|\Psi\rangle$
in
1..D
L



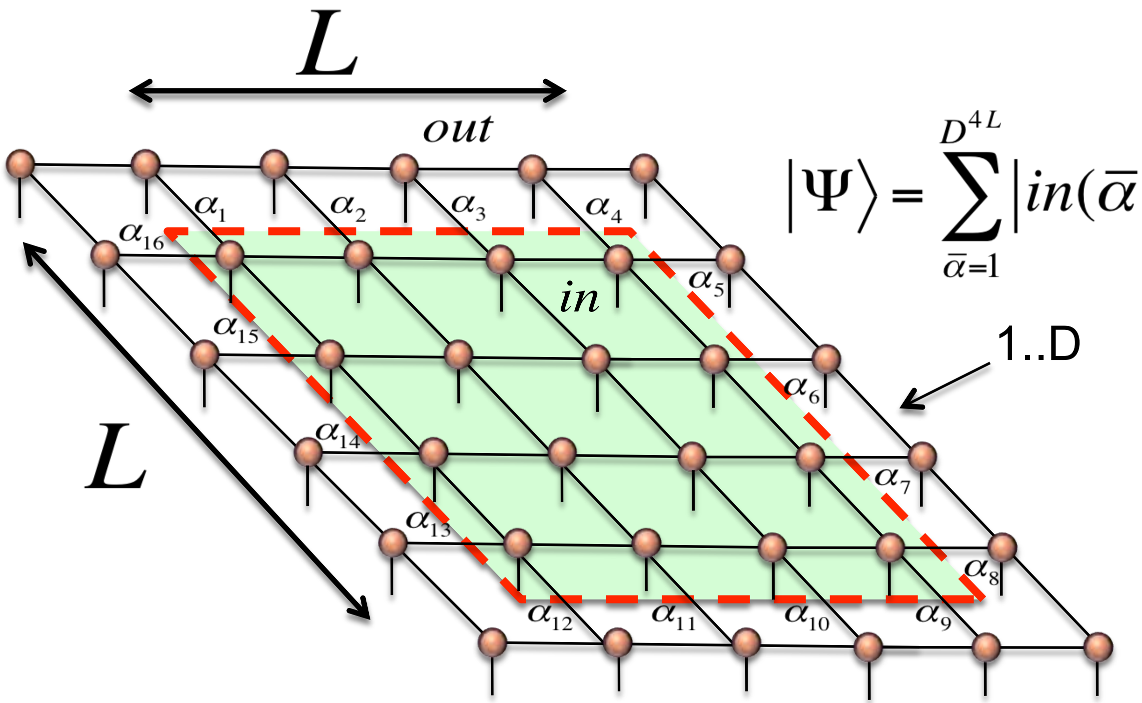
$$\left| out(\bar{\alpha}) \right\rangle$$

$$\bar{\alpha} = 1, 2, \ldots, D^{4L}$$



$$\left|in(\bar{\alpha})\right\rangle$$

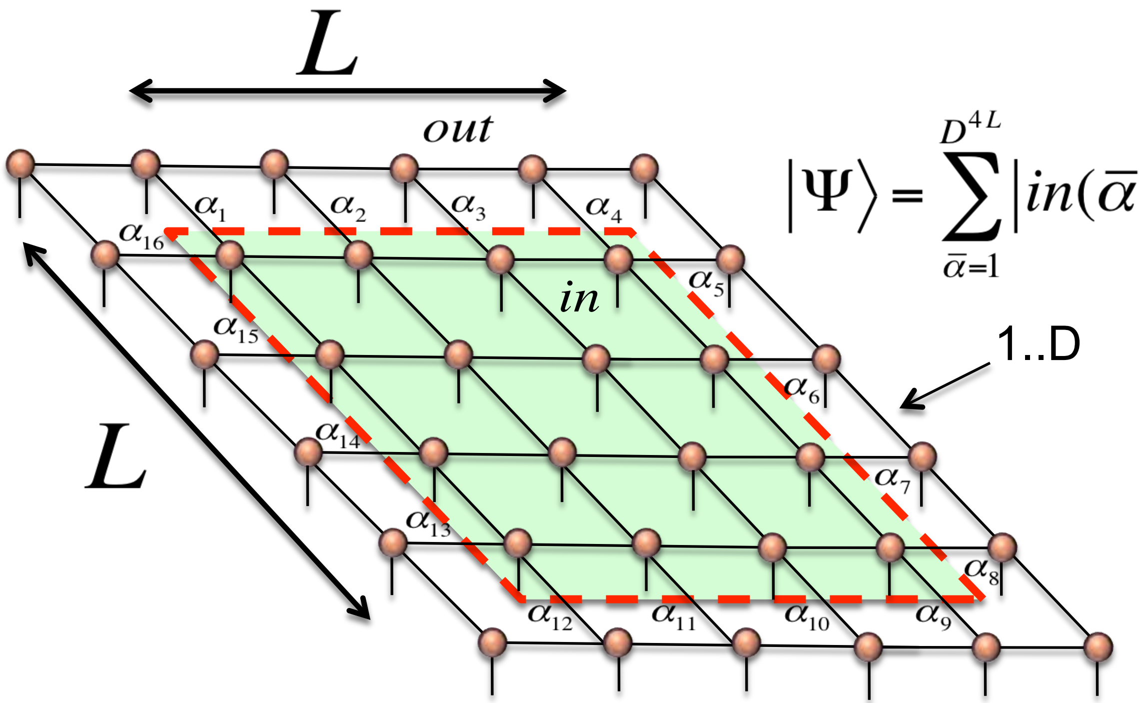
$$\bar{\alpha} = 1, 2, \ldots, D^{4L}$$



$$|\Psi\rangle = \sum_{\bar{\alpha}=1}^{D^{4L}} |in(\bar{\alpha})\rangle |out(\bar{\alpha})\rangle$$

$$\rho_{in} = \mathrm{tr}_{out}\left(|\Psi\rangle\langle\Psi|\right) = \sum_{\bar{\alpha},\bar{\alpha}'} X_{\bar{\alpha},\bar{\alpha}'} |in(\bar{\alpha})\rangle\langle in(\bar{\alpha}')| \qquad X_{\bar{\alpha},\bar{\alpha}'} = \langle out(\bar{\alpha}')|out(\bar{\alpha})\rangle$$

$$rank(\rho_{in}) \leq D^{4L} \qquad S(L) = -\mathrm{tr}\left(\rho_{in}\log\rho_{in}\right) \leq \log(D)4L$$



$$|\Psi\rangle = \sum_{\bar{\alpha}=1}^{D^{4L}} |in(\bar{\alpha})\rangle |out(\bar{\alpha})\rangle$$

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$$X_{\bar{\alpha},\bar{\alpha}'} = \langle out(\bar{\alpha}')|out(\bar{\alpha})\rangle$$

$rank(\rho_{in}) \leq D^{4L}$

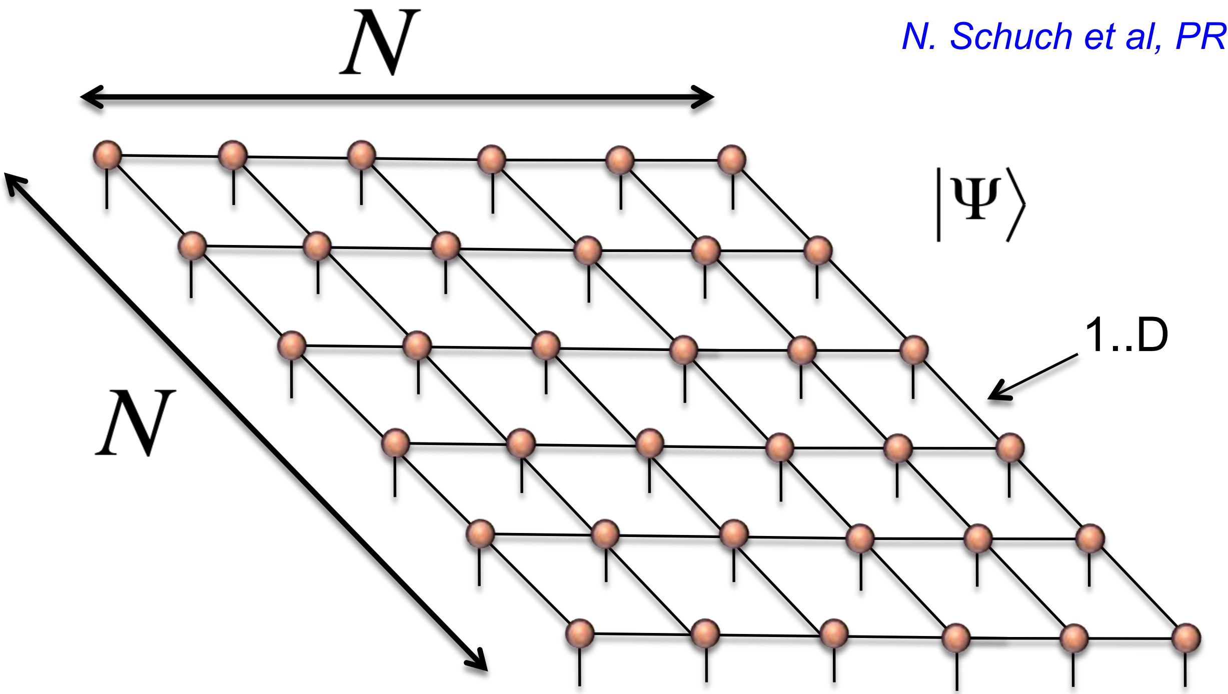
$$S(L) = -\mathrm{tr}\left(\rho_{in} \log \rho_{in}\right) \leq \log(D)\boxed{4L}$$

prefactor

size of the boundary

Exact contraction is inneficient

N. Schuch et al, PRL 98, 140506 (2007)



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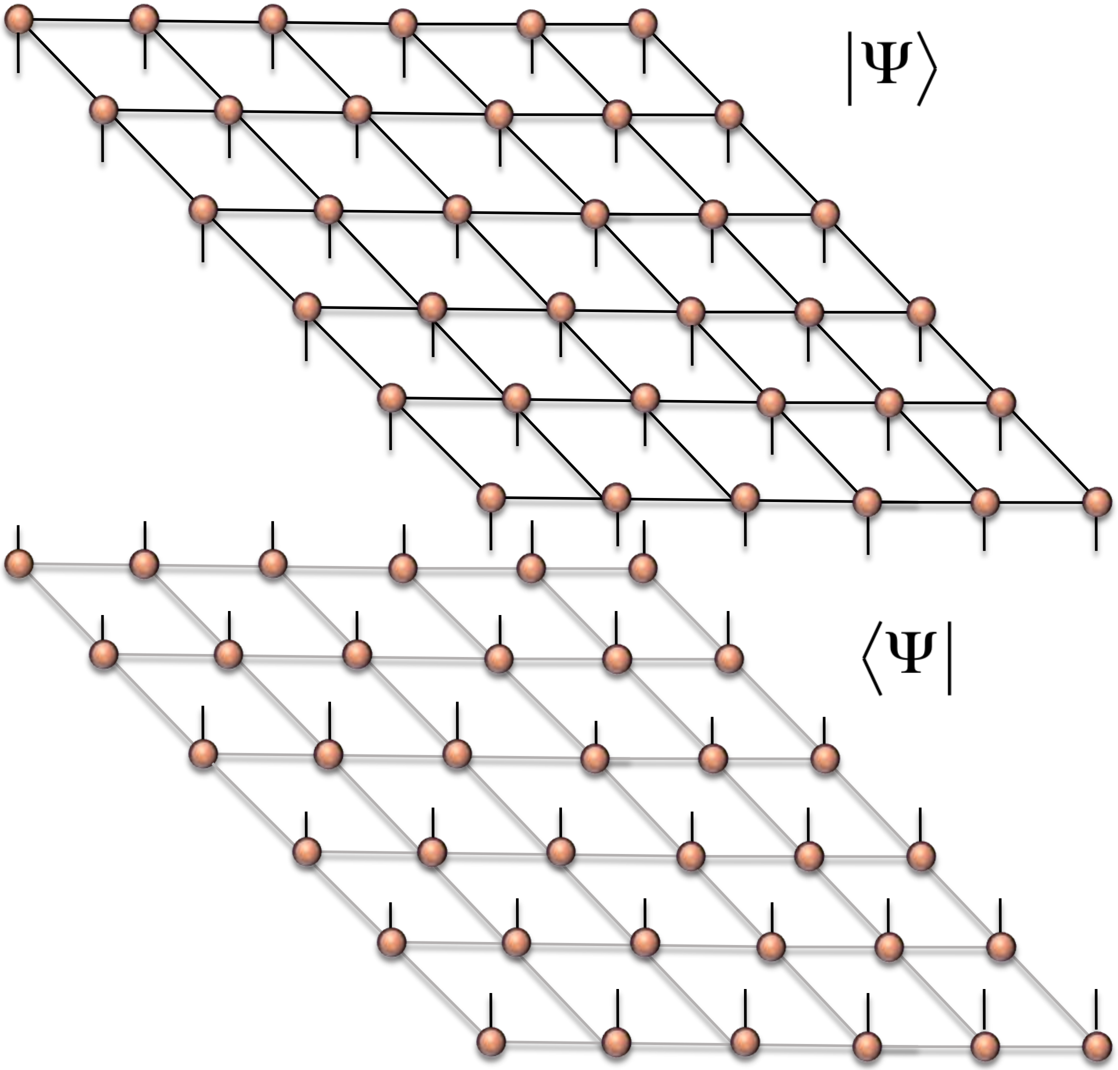

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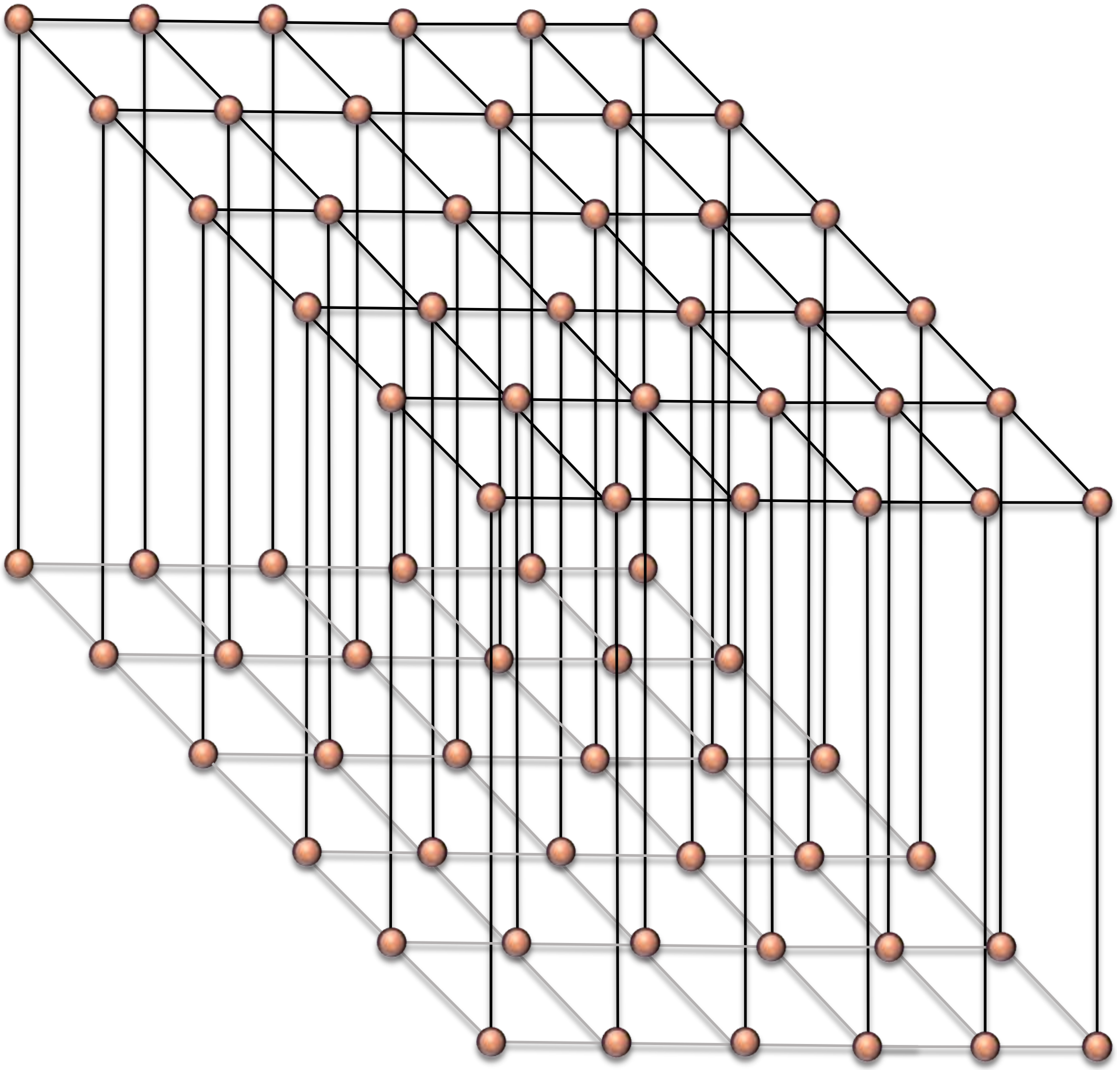
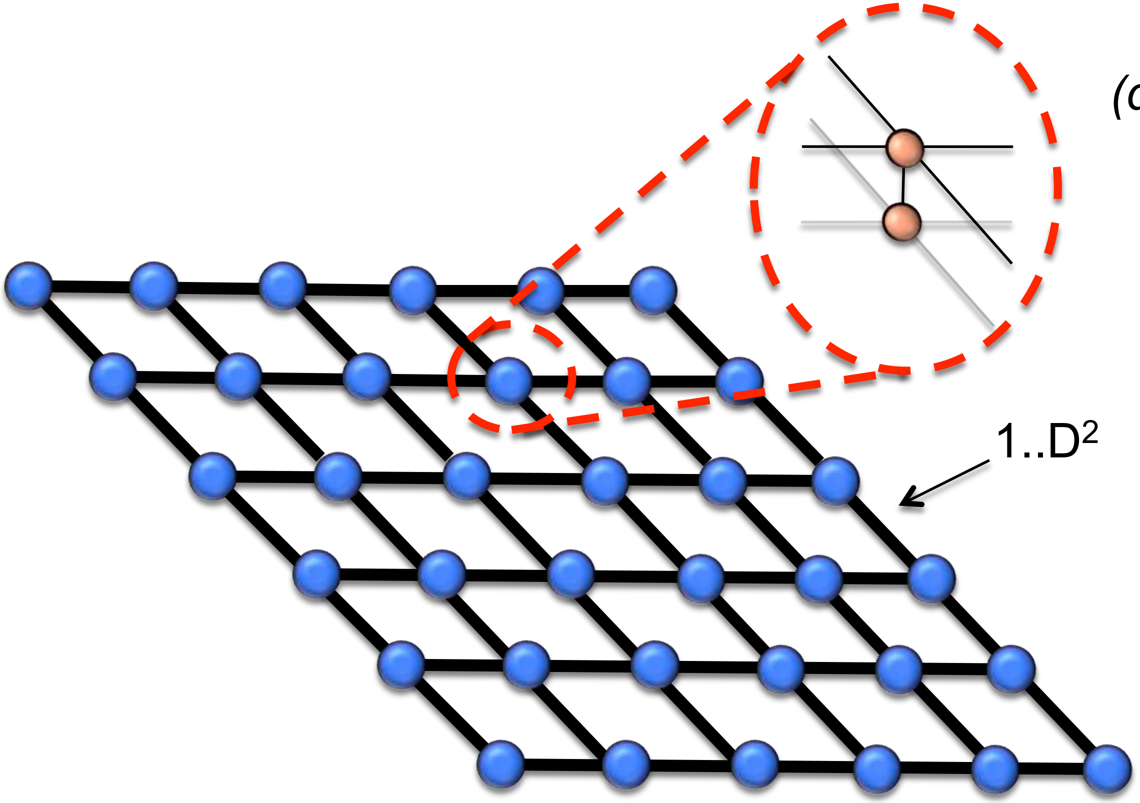


$$\langle\Psi|\Psi\rangle$$

Exact contraction is ineficient

N. Schuch et al, PRL 98, 140506 (2007)

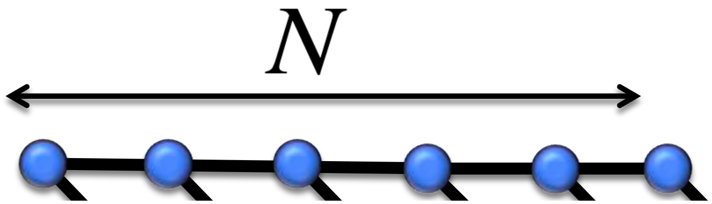


(double indices)

$$\langle\Psi|\Psi\rangle$$

Exact contraction is inneficient

N. Schuch et al, PRL 98, 140506 (2007)



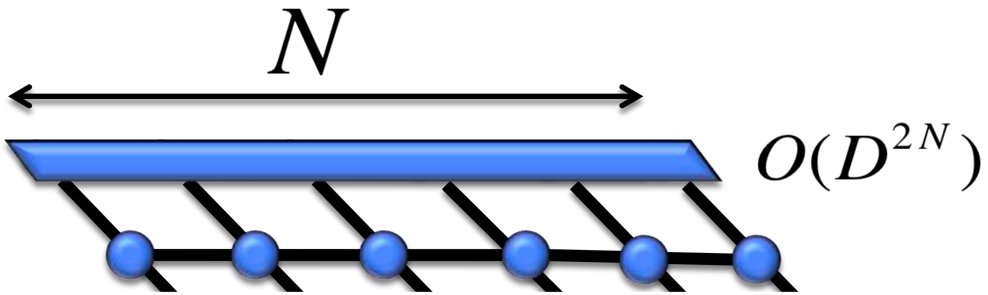
Exact contraction is ineficient

N. Schuch et al, PRL 98, 140506 (2007)

$O(D^{2N})$

Exact contraction is ineficient

N. Schuch et al, PRL 98, 140506 (2007)



Exact contraction is ineficient

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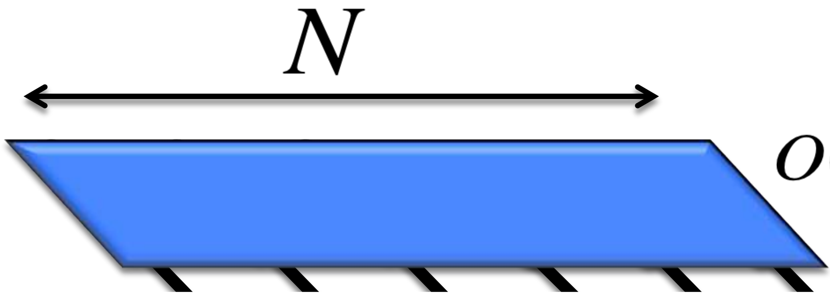
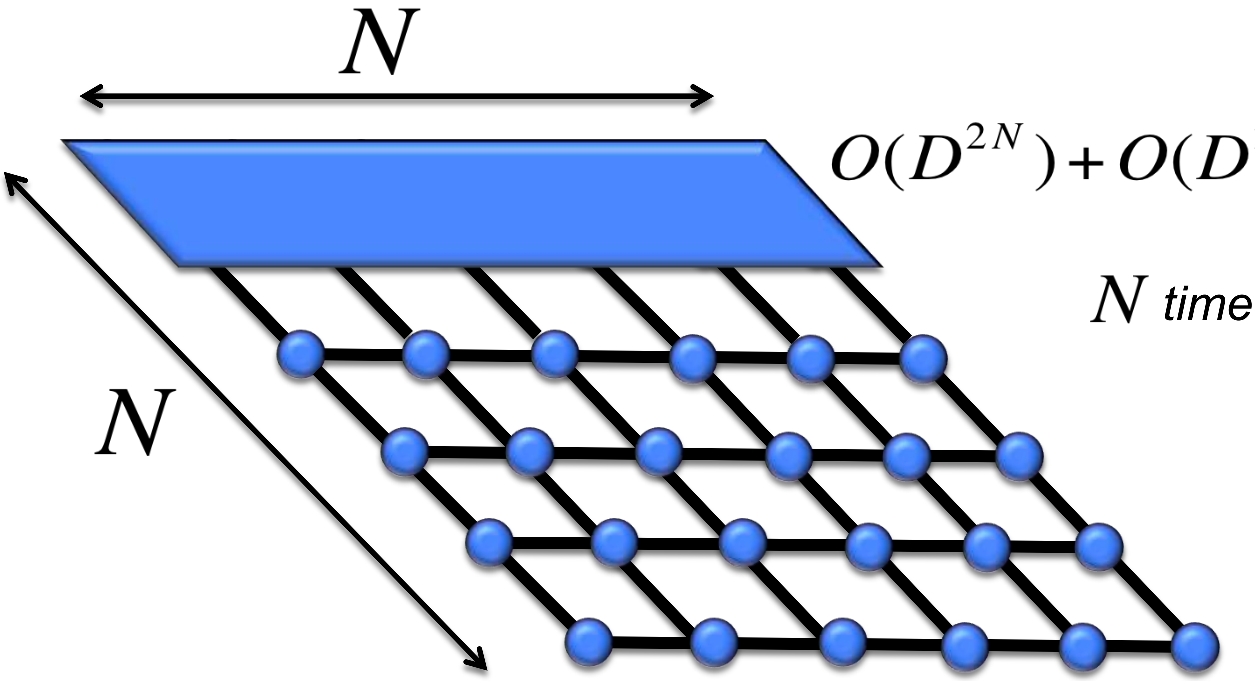


$$O(D^{2N}) + O(D^{2N})$$

Exact contraction is ineficient

N. Schuch et al, PRL 98, 140506 (2007)



$O(D^{2N}) + O(D^{2N}) + \ldots$

N *times*

computing time ~ $O(ND^{2N})$ ***Exponential amount of time!***

Mathematical statement: *exact contraction of a PEPS is a #P-Hard problem (harder than NP-Complete)*

Applies also to expectation values of observables

Critical correlation functions

F. Verstraete et al, PRL 96, 220601 (2006)

$$|\Psi(\beta)\rangle = \frac{1}{\sqrt{Z(\beta)}} \exp\left(\frac{\beta}{2}\sum_{\langle i,j\rangle}\sigma_z^i\sigma_z^j\right)|+,+\ldots+\rangle$$

Expectation values are those of the classical 2d Ising model

$$\langle\sigma_z^r\sigma_z^{r'}\rangle_\beta = \frac{1}{Z(\beta)}\sum_{\{s\}} s^r s^{r'} \exp\left(\beta\sum_{\langle i,j\rangle} s^i s^j\right) \qquad s = \pm 1$$

Critical correlation functions

F. Verstraete et al, PRL 96, 220601 (2006)

$$|\Psi(\beta)\rangle = \frac{1}{\sqrt{Z(\beta)}} \exp\left(\frac{\beta}{2} \sum_{\langle i,j \rangle} \sigma_z^i \sigma_z^j \right) |+,+\ldots+\rangle$$

Expectation values are those of the classical 2d Ising model

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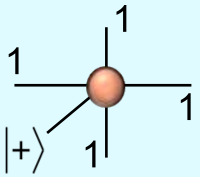
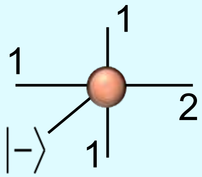
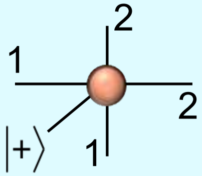
It is a PEPS with D=2 *(left as exercise):*



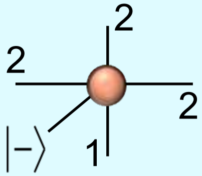
$= \left(\cosh(\beta/2)\right)^4$



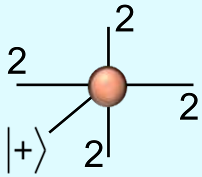
$= \left(\cosh(\beta/2)\right)^3 \left(\sinh(\beta/2)\right)$



$= \left(\cosh(\beta/2)\right)^2 \left(\sinh(\beta/2)\right)^2$



$= \left(\cosh(\beta/2)\right) \left(\sinh(\beta/2)\right)^3$



$= \left(\sinh(\beta/2)\right)^4$

+ permutations

At $\beta_c = \left(\log(1+\sqrt{2})\right)/2$ ***the correlation length is infinite:*** $\langle \sigma_z^r \sigma_z^{r'} \rangle_{\beta_c} \approx \frac{a}{|r-r'|^{1/4}}$

PEPS to/from Hamiltonians

PEPS to/from Hamiltonians

World of Hamiltonians

World of states

PEPS to/from Hamiltonians

World of Hamiltonians

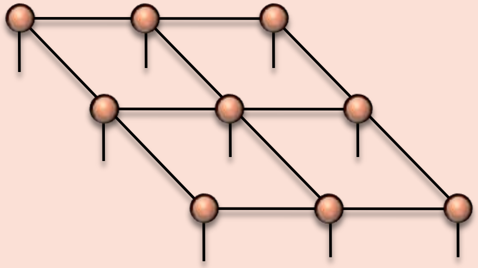
Ground state of a
gapped Parent Hamiltonian
with local interactions

$$H_{parent} = \sum h_{local}$$

e.g. D. Perez-Garcia et al, QIC 8, 0650 (2008)

World of states

Generic PEPS with finite D



PEPS to/from Hamiltonians

World of Hamiltonians

Ground state of a gapped** **Parent Hamiltonian** **with local interactions

$$H_{parent} = \sum h_{local}$$

Hamiltonian with local interactions

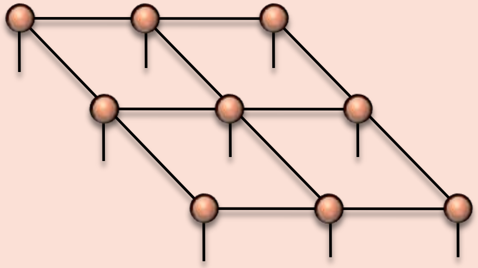
$$H' = \sum h'_{local}$$

e.g. D. Perez-Garcia et al, QIC 8, 0650 (2008)

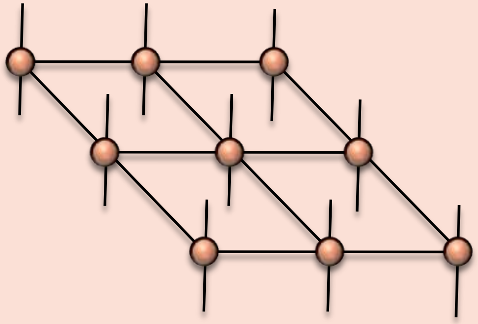
M. Hastings, PRB 73, 085115 (2006)

World of states

Generic PEPS with finite D

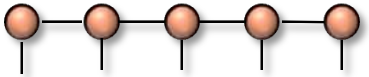
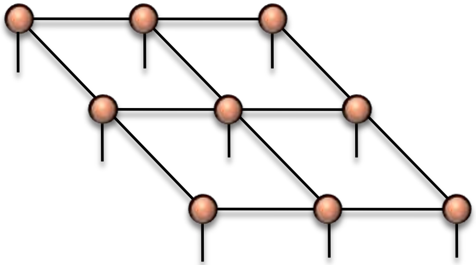
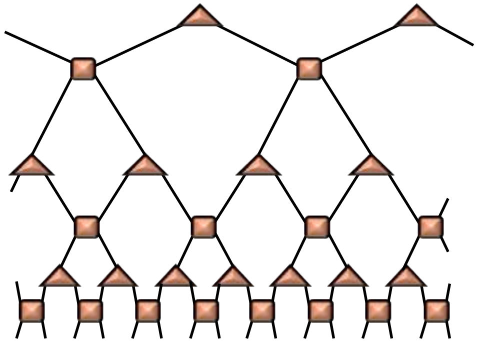


Thermal states can be approx. by a** **PEPS with finite D



PEPS target the relevant corner of the Hilbert space (area-law)

Comparison

	MPS in 1d 	*PEPS in 2d* 	*MERA in 1d* 
Ent. entropy	$S(L)=O(1)$	$S(L)=O(L)$	$S(L)=O(\log L)$
Exact contraction	efficient	inefficient	efficient
Corr. length	finite	finite & infinite	finite & infinite
To/from	1d Ham.	2d Ham.	1d Ham.
Tensors	arbitrary	arbitrary	constrained

PEPS as ansatz:
variational optimization

Variational optimization (e.g. finite PEPS)

e.g. F. Verstraete, I. Cirac, cond-mat/0407066

$$\min\left(\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}\right)$$

Optimize over each tensor individually and sweep over the entire system (as in DMRG)

Variational optimization (e.g. finite PEPS)

e.g. F. Verstraete, I. Cirac, cond-mat/0407066

$$\min\left(\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}\right)$$

Optimize over each tensor individually and sweep over the entire system (as in DMRG)

$|\Psi\rangle$

E

Variational optimization (e.g. finite PEPS)

e.g. F. Verstraete, I. Cirac, cond-mat/0407066

$$\min\left(\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}\right)$$

Optimize over each tensor individually and sweep over the entire system (as in DMRG)

$$|\Psi\rangle \xrightarrow{\min\limits_{A^1}} |\Psi^1\rangle$$

$$E \quad \geq \quad E^1$$

Variational optimization (e.g. finite PEPS)

e.g. F. Verstraete, I. Cirac, cond-mat/0407066

$$\min\left(\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}\right)$$

Optimize over each tensor individually and sweep over the entire system (as in DMRG)

$$|\Psi\rangle \xrightarrow{\min_{A^1}} |\Psi^1\rangle \xrightarrow{\min_{A^2}} \ldots \xrightarrow{\min_{A^N}} |\Psi^N\rangle$$

$$E \geq E^1 \geq \ldots \geq E^N$$

Variational optimization (e.g. finite PEPS)

e.g. F. Verstraete, I. Cirac, cond-mat/0407066

$$\min\left(\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}\right)$$

Optimize over each tensor individually and sweep over the entire system (as in DMRG)

$$\begin{array}{ccccccccc}
|\Psi\rangle & \xrightarrow{\min_{A^1}} & |\Psi^1\rangle & \xrightarrow{\min_{A^2}} & \dots & \xrightarrow{\min_{A^N}} & |\Psi^N\rangle & \xrightarrow{\min_{A^{N-1}}} & |\Psi^{N+1}\rangle \\
E & \geq & E^1 & \geq & \dots & \geq & E^N & \geq & E^{N+1}
\end{array}$$

Variational optimization (e.g. finite PEPS)

e.g. F. Verstraete, I. Cirac, cond-mat/0407066

$$\min\left(\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}\right)$$

Optimize over each tensor individually and sweep over the entire system (as in DMRG)

$$\begin{array}{ccccccccccccc} |\Psi\rangle & \xrightarrow{\min_{A^1}} & |\Psi^1\rangle & \xrightarrow{\min_{A^2}} & \dots & \xrightarrow{\min_{A^N}} & |\Psi^N\rangle & \xrightarrow{\min_{A^{N-1}}} & |\Psi^{N+1}\rangle & \xrightarrow{\min_{A^{N-2}}} & \dots \\ E & \geq & E^1 & \geq & \dots & \geq & E^N & \geq & E^{N+1} & \geq & \dots \end{array}$$

Variational optimization (e.g. finite PEPS)

e.g. F. Verstraete, I. Cirac, cond-mat/0407066

$$\min\left(\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}\right)$$

Optimize over each tensor individually and sweep over the entire system (as in DMRG)

$$|\Psi\rangle \xrightarrow{\min\limits_{A^1}} |\Psi^1\rangle \xrightarrow{\min\limits_{A^2}} \dots \xrightarrow{\min\limits_{A^N}} |\Psi^N\rangle \xrightarrow{\min\limits_{A^{N-1}}} |\Psi^{N+1}\rangle \xrightarrow{\min\limits_{A^{N-2}}} \dots$$

$$E \geq E^1 \geq \dots \geq E^N \geq E^{N+1} \geq \dots$$

$$A^{*i}$$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0$$

$$A^i$$

Minimization of quadratic function

Variational optimization (e.g. finite PEPS)

e.g. F. Verstraete, I. Cirac, cond-mat/0407066

$$\min\left(\frac{\langle\Psi|H|\Psi\rangle}{\langle\Psi|\Psi\rangle}\right)$$

Optimize over each tensor individually and sweep over the entire system (as in DMRG)

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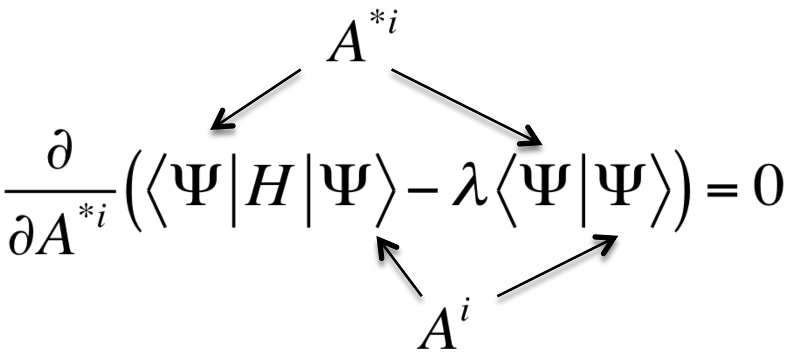
$$E \quad \geq \quad E^1 \quad \geq \quad \ldots \quad \geq \quad E^N \quad \geq \quad E^{N+1} \quad \geq \quad \ldots$$



$$A^{*i}$$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0$$

$$A^i$$

Minimization of quadratic function

$$\mathbf{H}^i_{eff}\vec{A}^i = \lambda\mathbf{N}^i\vec{A}^i$$

Generalized eigenvalue problem

Once $\mathbf{H}^i_{eff}$ and $\mathbf{N}^i$ are known, we can solve this problem efficiently

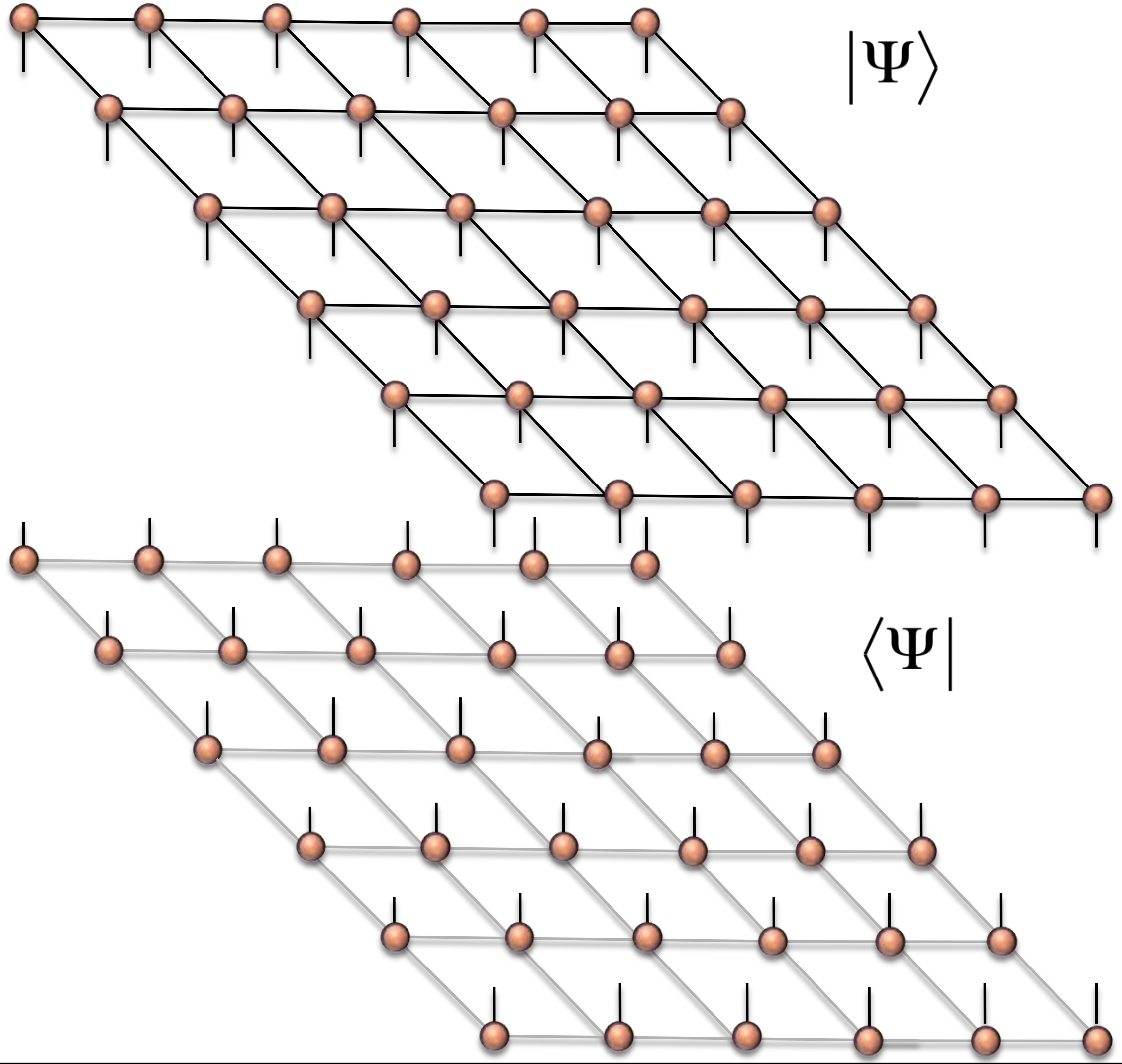
Approximate calculation of $\mathbf{H}^i_{eff}$ and $\mathbf{N}^i$

e.g. calculation of $\mathbf{N}^i\vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle-\lambda\langle\Psi|\Psi\rangle\right)=0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i=\lambda\mathbf{N}^i\vec{A}^i$$

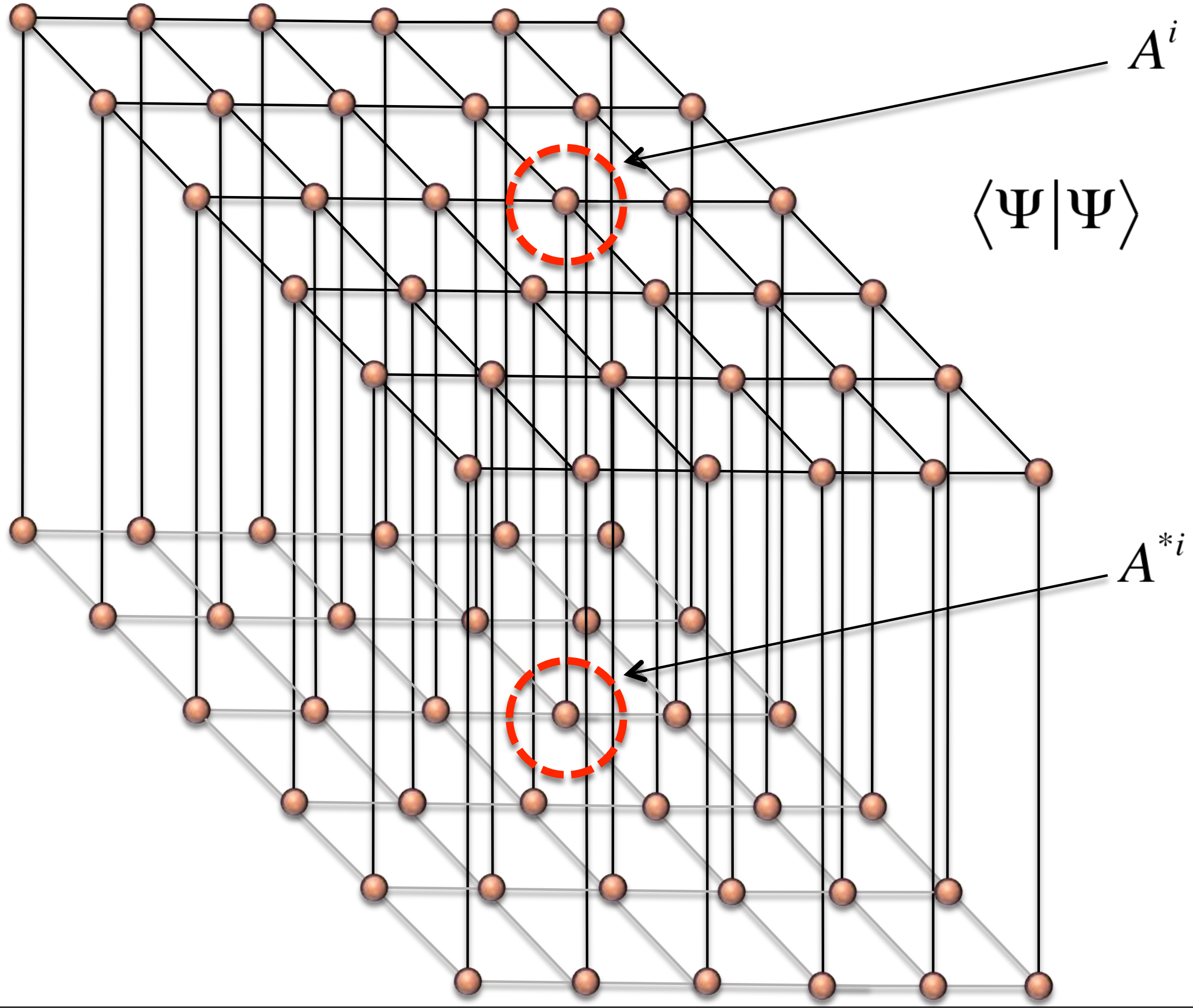
e.g. calculation of $\mathbf{N}^i\vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle-\lambda\langle\Psi|\Psi\rangle\right)=0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i=\lambda\mathbf{N}^i\vec{A}^i$$



e.g. calculation of $\mathbf{N}^i \vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i = \lambda \mathbf{N}^i \vec{A}^i$$



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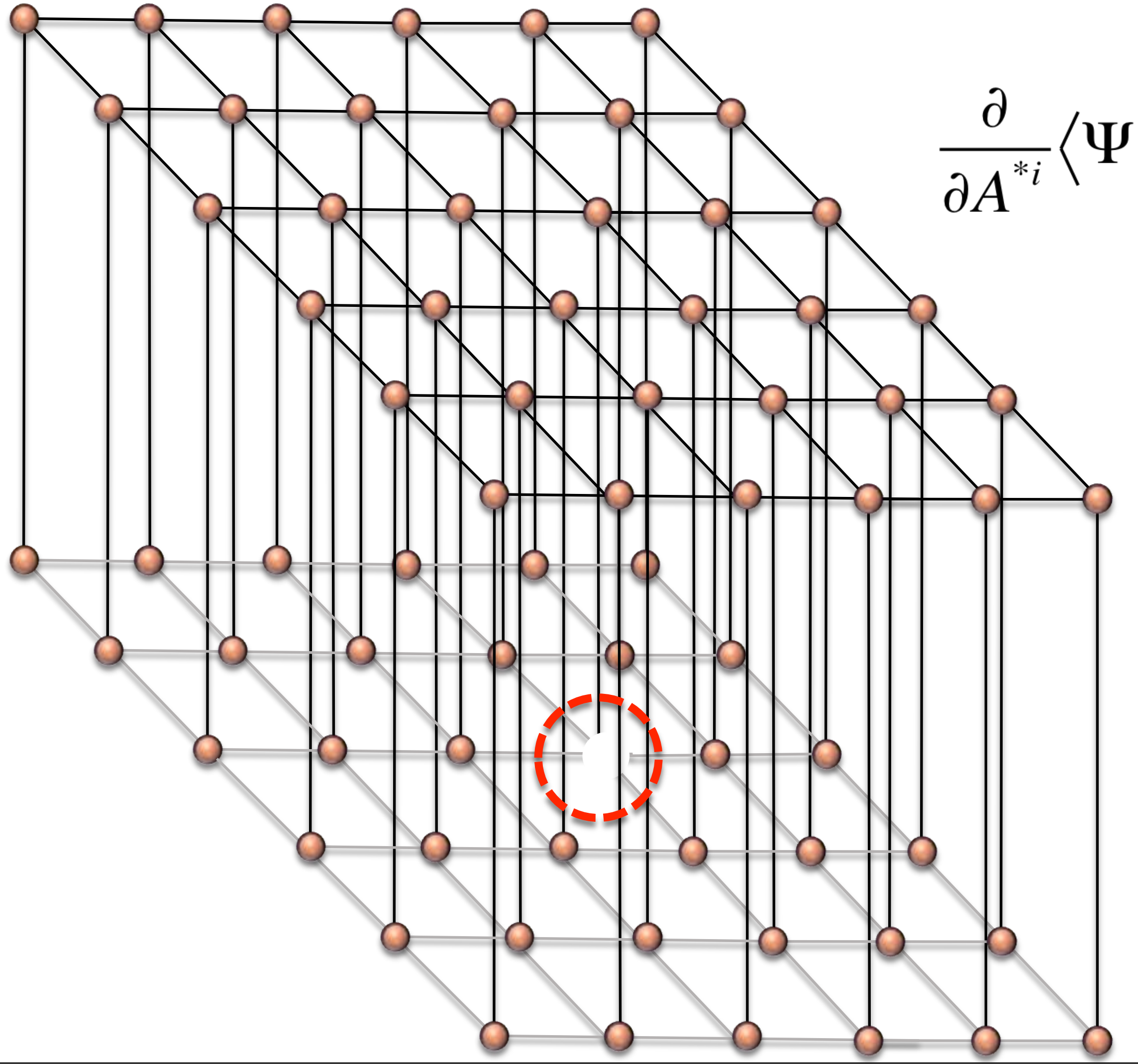

$$\frac{\partial}{\partial A^{*i}}\langle\Psi|\Psi\rangle=\mathbf{N}^i\vec{A}^i$$

e.g. calculation of $\mathbf{N}^i \vec{A}^i$

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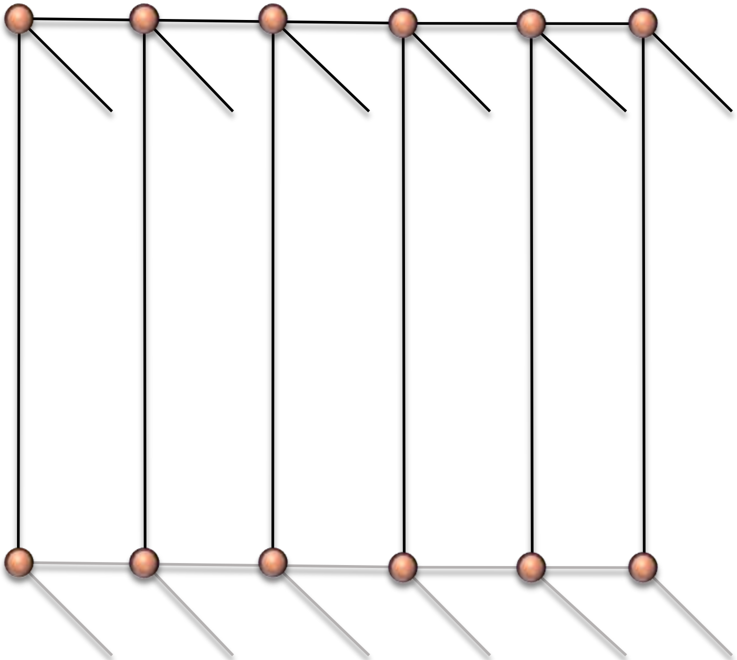

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e.g. calculation of $\mathbf{N}^i \vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \Rightarrow \mathbf{H}^i_{eff}\vec{A}^i = \lambda\mathbf{N}^i\vec{A}^i$$

MPS

$$\frac{\partial}{\partial A^{*i}}\langle\Psi|\Psi\rangle = \mathbf{N}^i\vec{A}^i$$

e.g. calculation of $\mathbf{N}^i \vec{A}^i$

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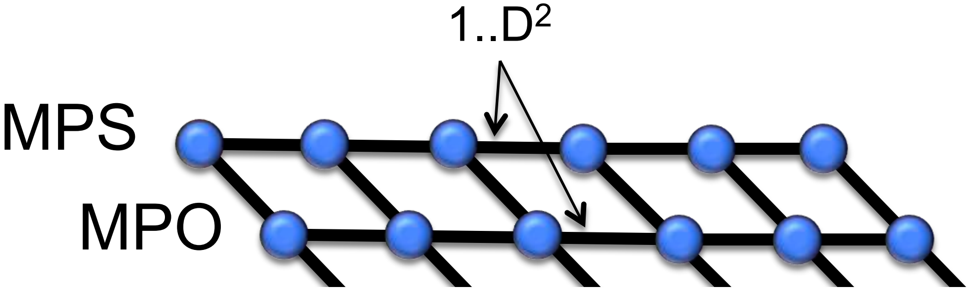

$$\frac{\partial}{\partial A^{*i}}\langle\Psi|\Psi\rangle=\mathbf{N}^i\vec{A}^i$$

1d problem: use a 1d method for MPS (e.g., DMRG or TEBD)

e.g. calculation of $\mathbf{N}^i \vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle-\lambda\langle\Psi|\Psi\rangle\right)=0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i=\lambda\mathbf{N}^i\vec{A}^i$$

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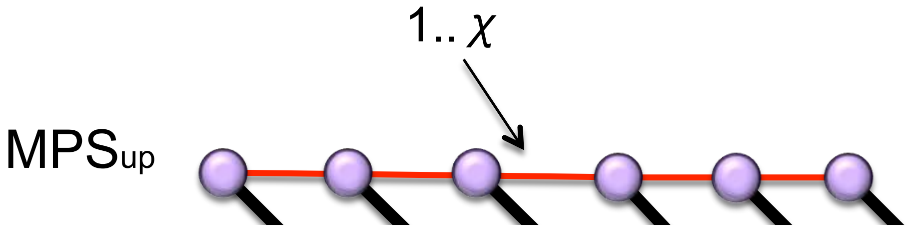

e.g. calculation of $\mathbf{N}^i \vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i = \lambda \mathbf{N}^i \vec{A}^i$$

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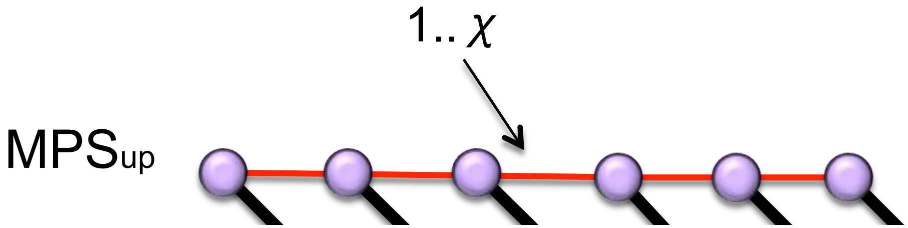

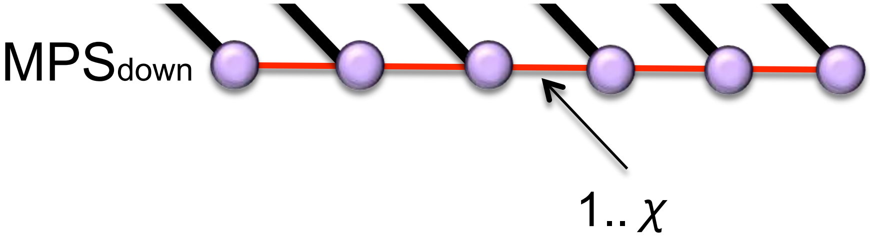

e.g. calculation of $\mathbf{N}^i \vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i = \lambda\mathbf{N}^i\vec{A}^i$$

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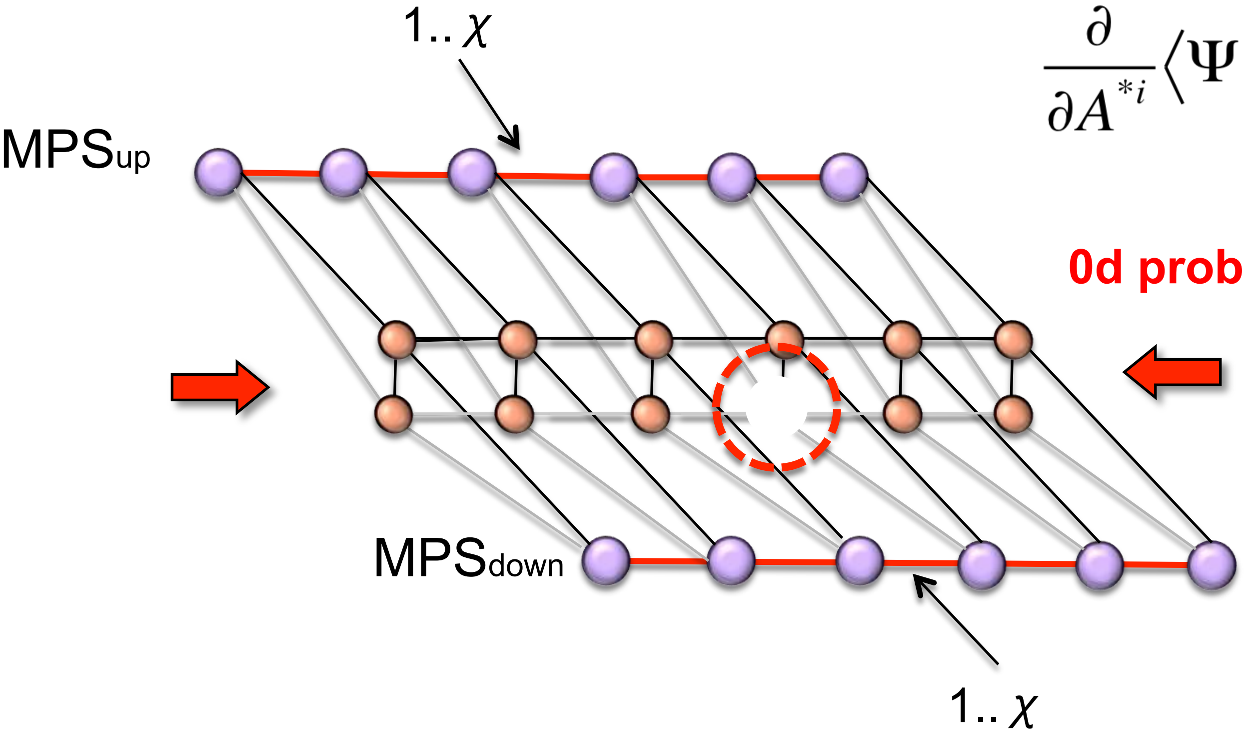


0d problem: exact!

e.g. calculation of $\mathbf{N}^i \vec{A}^i$

$$\frac{\partial}{\partial A^{*i}}\left(\langle\Psi|H|\Psi\rangle - \lambda\langle\Psi|\Psi\rangle\right) = 0 \quad \Rightarrow \quad \mathbf{H}^i_{eff}\vec{A}^i = \lambda \mathbf{N}^i \vec{A}^i$$

$$\frac{\partial}{\partial A^{*i}}\langle\Psi|\Psi\rangle = \mathbf{N}^i \vec{A}^i$$

Dimensional reduction

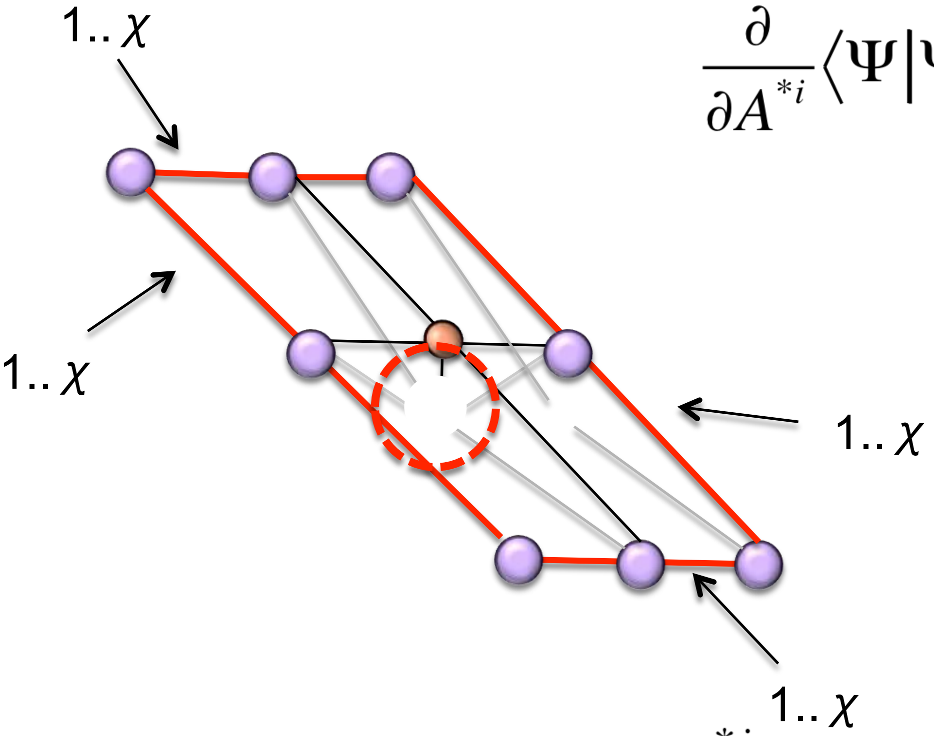
2d problem

⬇

1d problem: use DMRG!

⬇

0d problem: exact!



$\mathbf{N}^i \vec{A}^i$ is the ***environment*** of tensor A^{*i}

$\mathbf{H}^i_{eff}\vec{A}^i$ is computed similarly, but sandwitching with the Hamiltonian

Valid also for any expectation value

Time evolution

(real, imaginary)

Time evolution (e.g. imaginary)

e.g. J. Jordan et al, PRL 101, 250602 (2008)

$$|\Psi_0\rangle = \lim_{\tau\to\infty} \frac{e^{-\tau H}|\Psi\rangle}{\left\| e^{-\tau H}|\Psi\rangle \right\|}$$

Divide into small time-steps $\delta\tau << 1$

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Divide into small time-steps $\delta\tau << 1$

$$\begin{array}{ccccccccccccc} \left|\Psi\right\rangle & \xrightarrow{e^{-\delta\tau H}} & \left|\Psi^1\right\rangle & \xrightarrow{e^{-\delta\tau H}} & \dots & \xrightarrow{e^{-\delta\tau H}} & \left|\Psi^m\right\rangle & \xrightarrow{e^{-\delta\tau H}} & \left|\Psi^{m+1}\right\rangle & \xrightarrow{e^{-\delta\tau H}} & \dots \\ E & \geq & E^1 & \geq & \dots & \geq & E^m & \geq & E^{m+1} & \geq & \dots \end{array}$$

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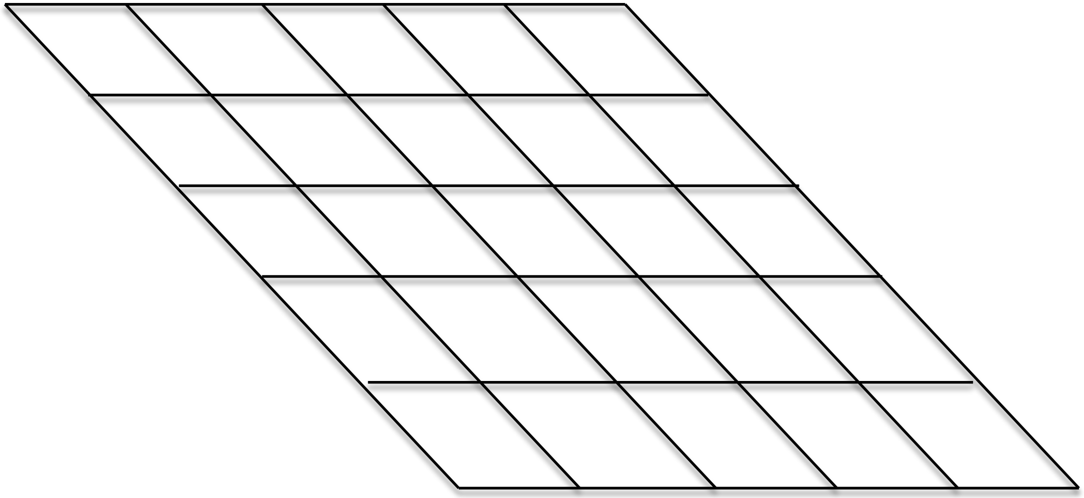
Split the Hamiltonian
(e.g. 2-body n.n.)

$$H = H_{hor}^{even} + H_{hor}^{odd} + H_{ver}^{even} + H_{ver}^{odd}$$



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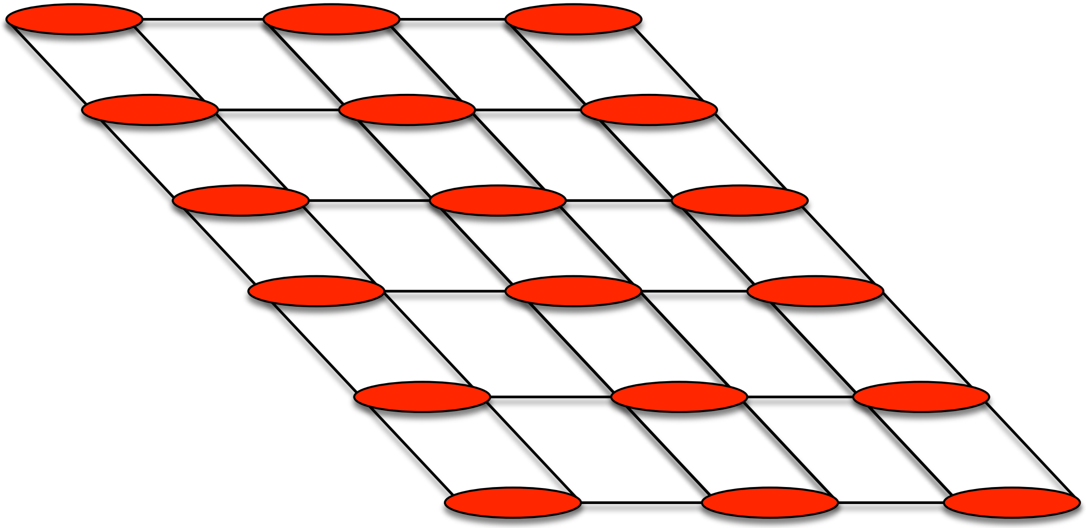

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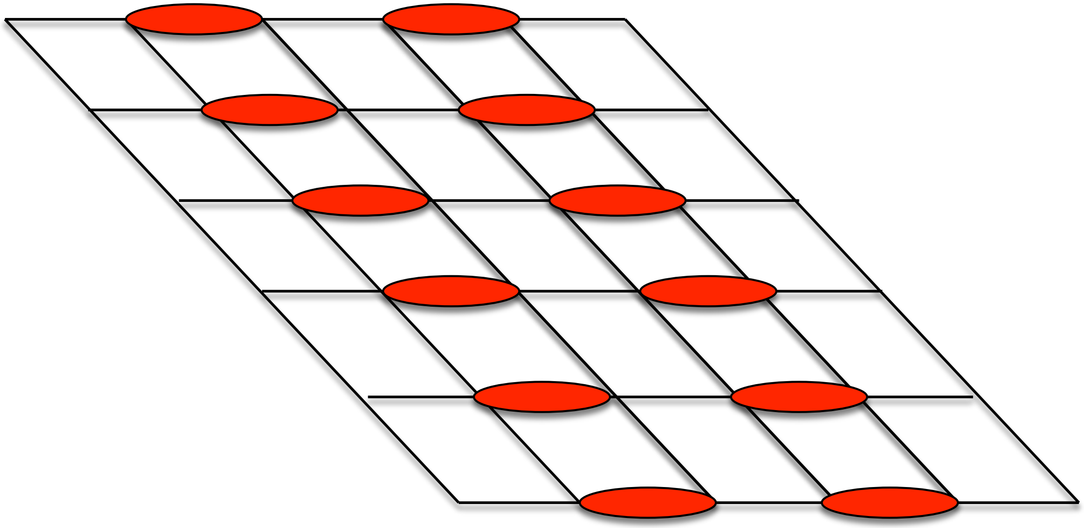

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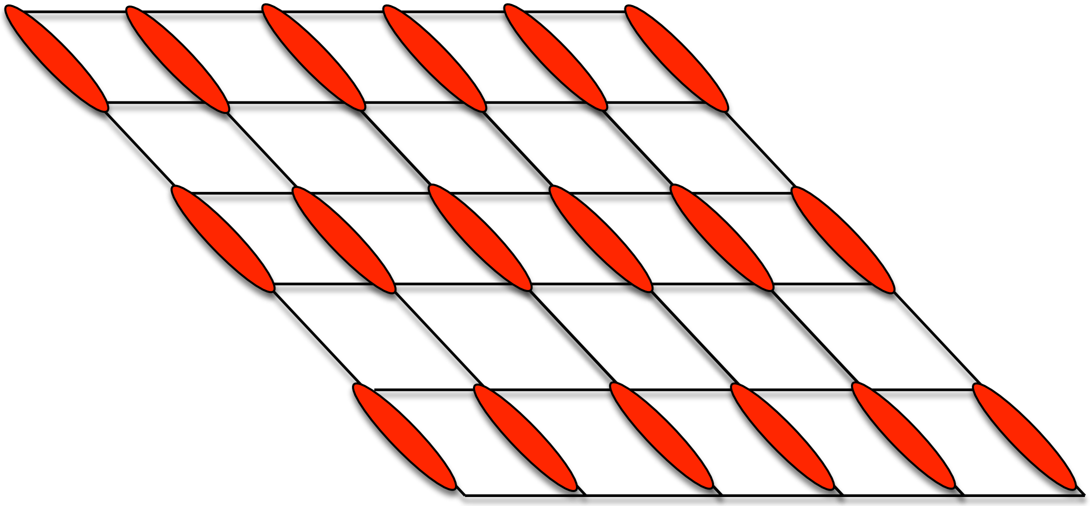

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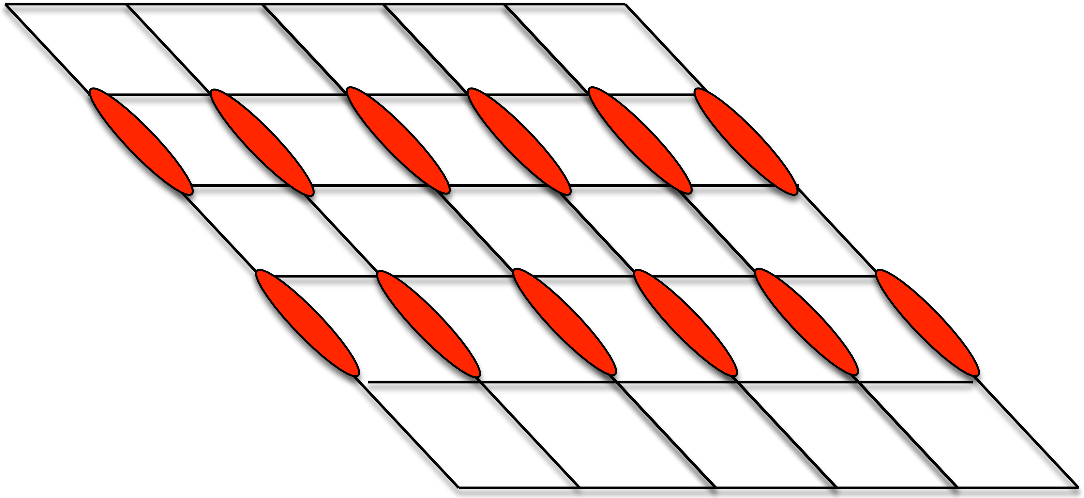
$$E \geq E^1 \geq \dots \geq E^m \geq E^{m+1} \geq \dots$$

Split the Hamiltonian
(e.g. 2-body n.n.)

$$H = H^{even}_{hor} + H^{odd}_{hor} + H^{even}_{ver} + \boxed{H^{odd}_{ver}}$$

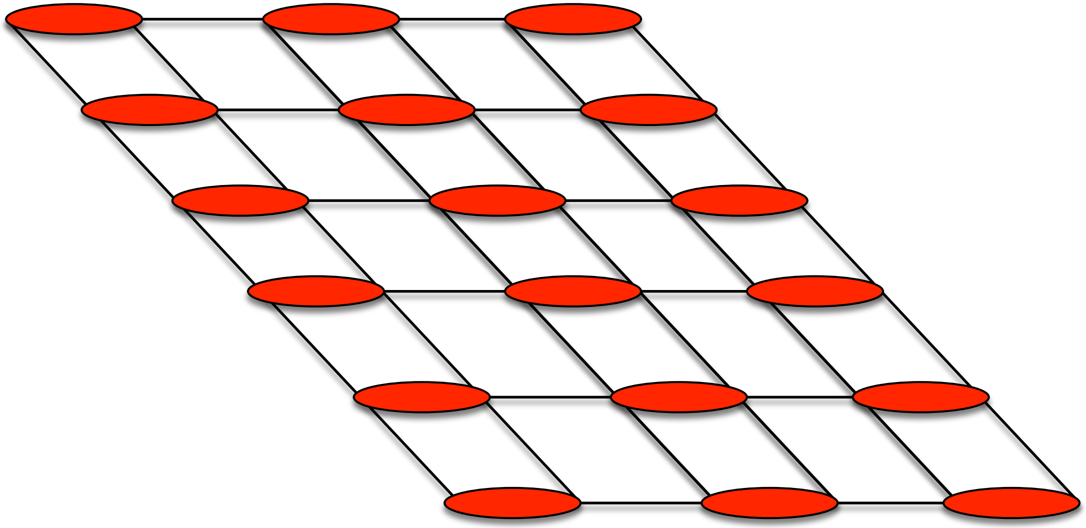


Time evolution (e.g. imaginary)

e.g. J. Jordan et al, PRL 101, 250602 (2008)

$$e^{-\delta\tau H} \approx e^{-\delta\tau H_{hor}^{even}} e^{-\delta\tau H_{hor}^{odd}} e^{-\delta\tau H_{ver}^{even}} e^{-\delta\tau H_{ver}^{odd}} + O(\delta\tau^2)$$

$$e^{-\delta\tau H_{hor}^{even}}$$



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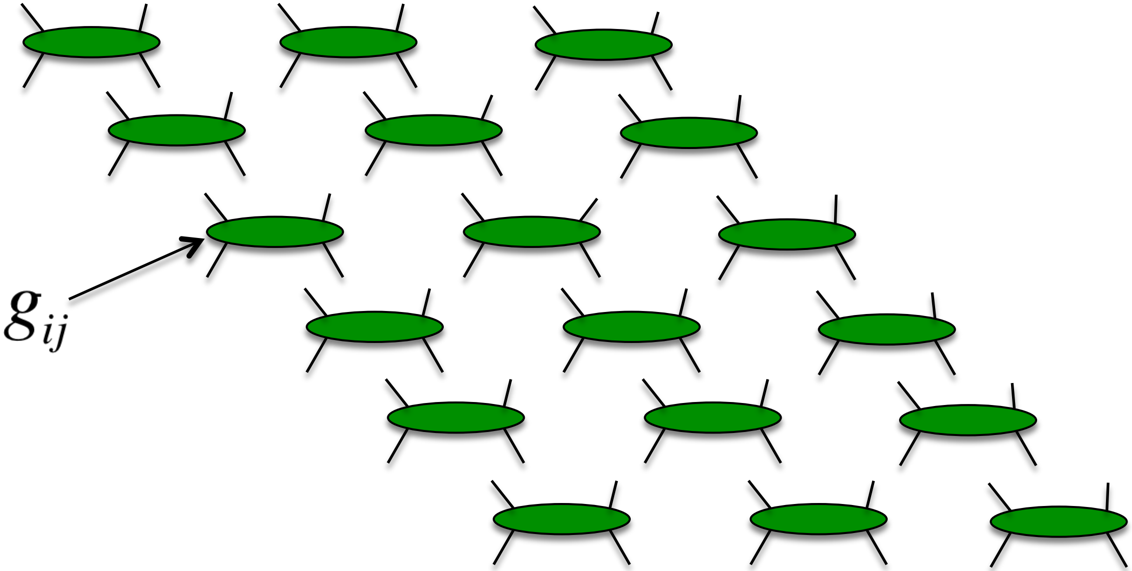
$$e^{-\delta\tau H_{hor}^{even}} = \bigotimes_{<i,j>} e^{-\delta\tau h_{ij}} = \bigotimes_{<i,j>} g_{ij}$$

2-body gates



Time evolution (e.g. imaginary)

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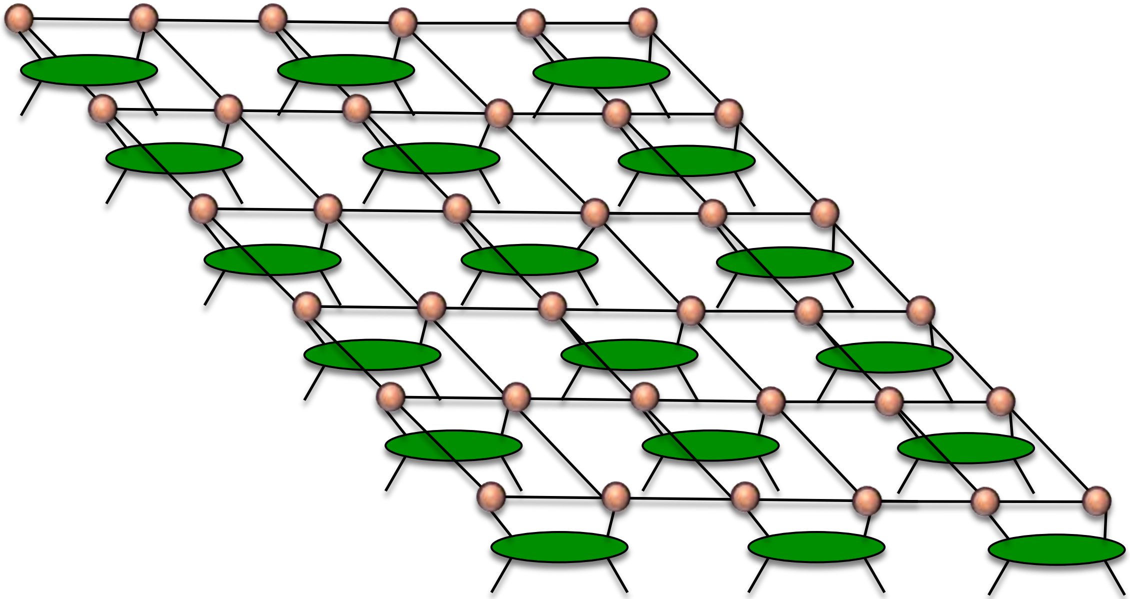
$$e^{-\delta\tau H} \approx e^{-\delta\tau H_{hor}^{even}} e^{-\delta\tau H_{hor}^{odd}} e^{-\delta\tau H_{ver}^{even}} e^{-\delta\tau H_{ver}^{odd}} + O(\delta\tau^2)$$

$$e^{-\delta\tau H_{hor}^{even}} \left|\Psi\right\rangle = \left|\tilde{\Psi}\right\rangle$$

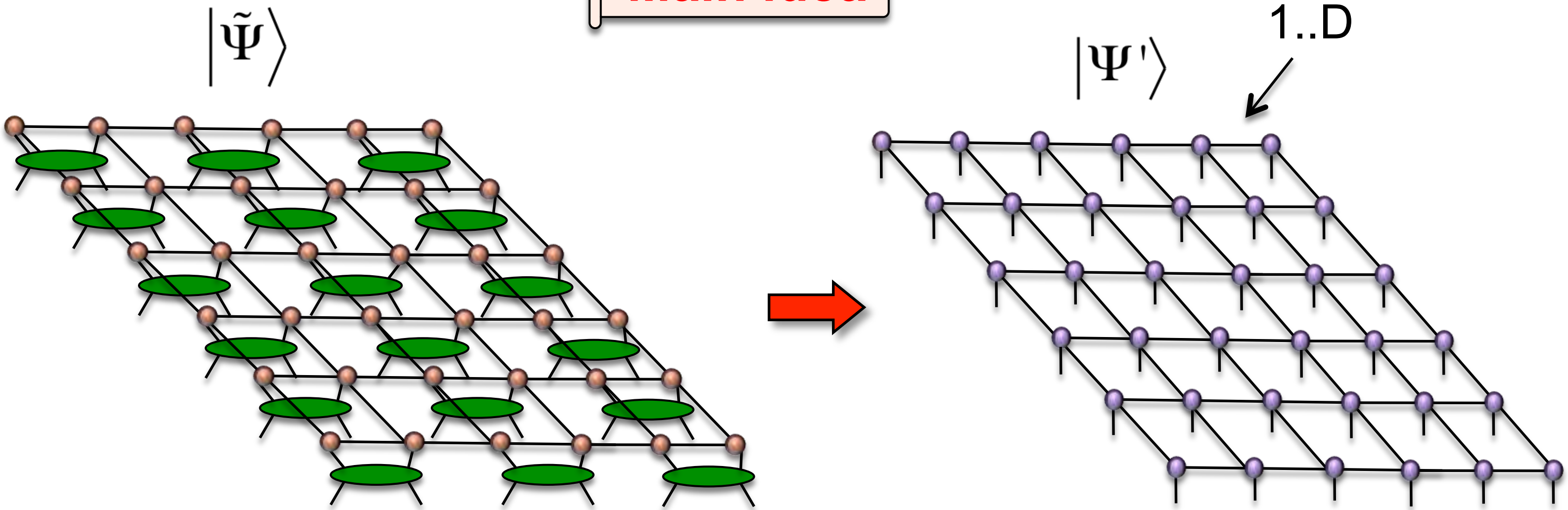
evolved state



Time evolution (e.g. imaginary)

e.g. J. Jordan et al, PRL 101, 250602 (2008)

Main idea



Different approaches to this problem: (fast) full update, simplified update, TPVA...

Full update: $\min \left\| |\tilde{\Psi}\rangle - |\Psi'\rangle \right\|^2$

Finite systems: optimize over all tensors in the PEPS (as before)

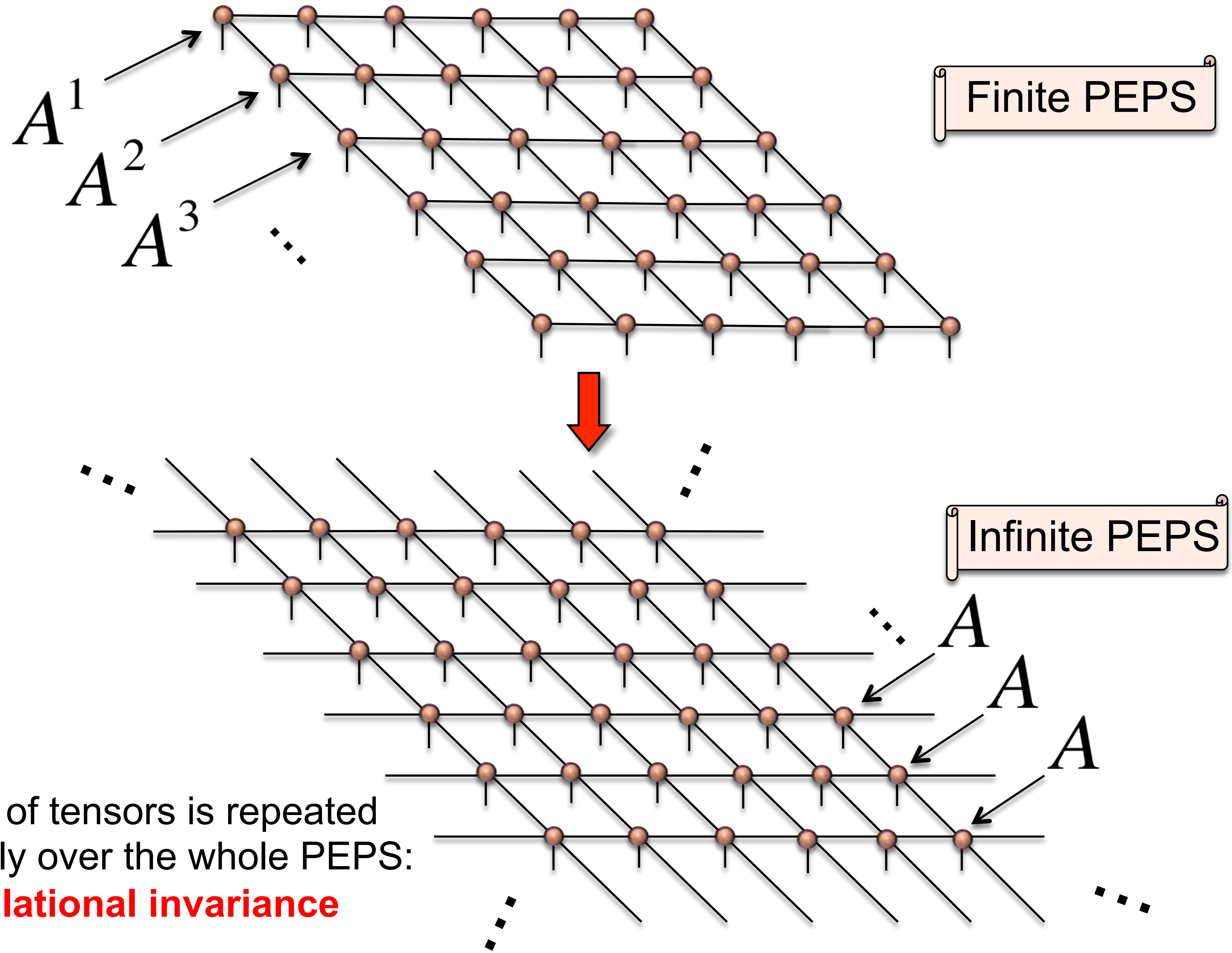
Infinite systems: optimize over tensors in the PEPS unit cell (iPEPS)

Require ***calculations of environments***, like the one shown before.

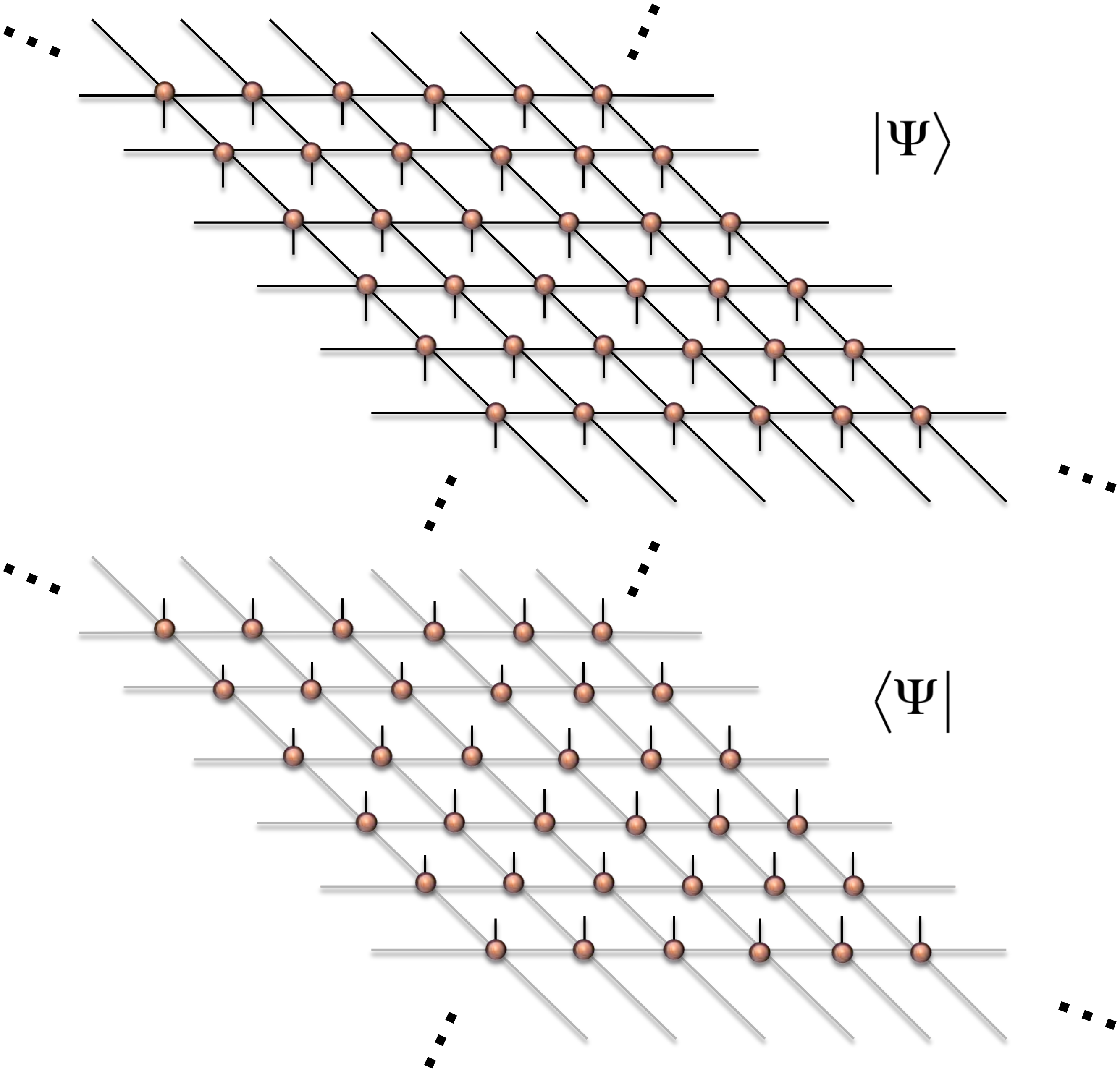
Environments with infinite PEPS

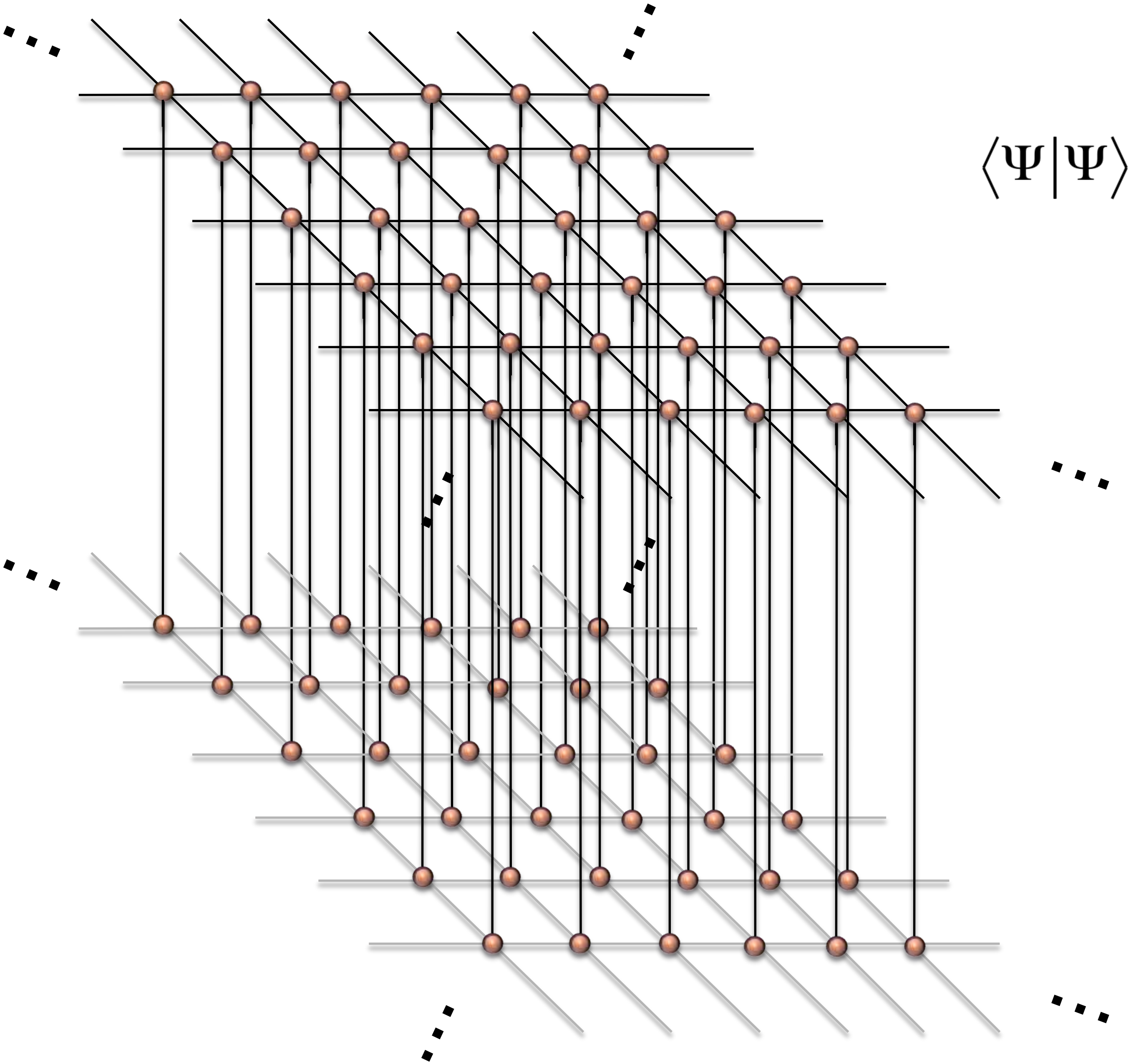
Infinite systems

e.g. J. Jordan et al, PRL 101, 250602 (2008)
R. Orús, G. Vidal, PRB 80 094403 (2009)



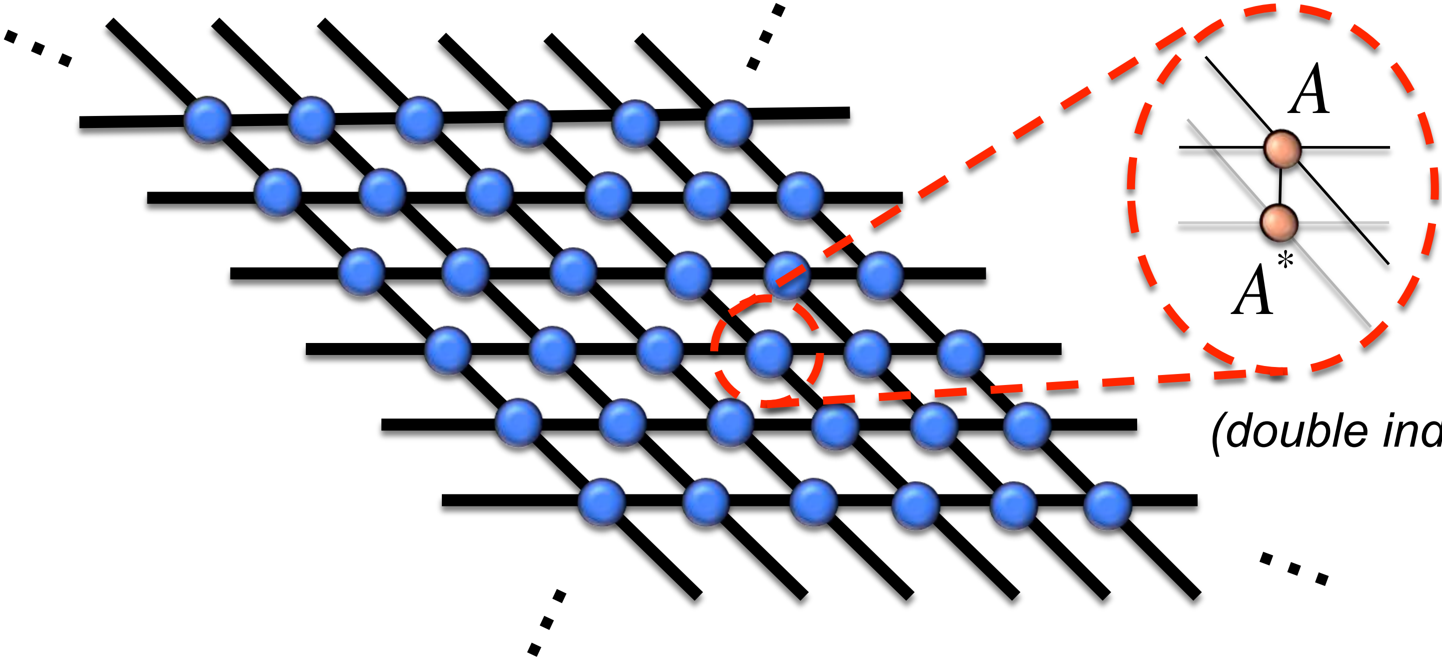
Unit cell of tensors is repeated periodically over the whole PEPS: **translational invariance**


$|\Psi\rangle$
$\langle\Psi|$



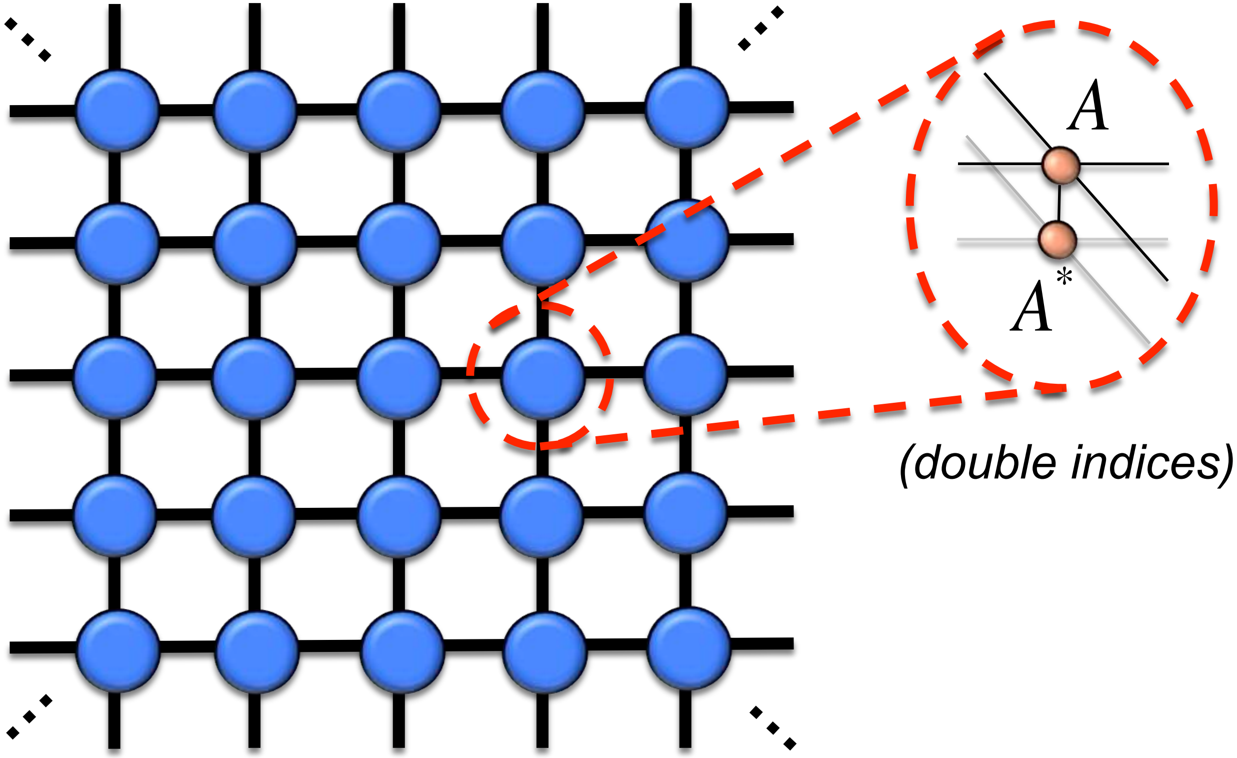
$$\langle \Psi | \Psi \rangle$$

$\langle\Psi|\Psi\rangle$



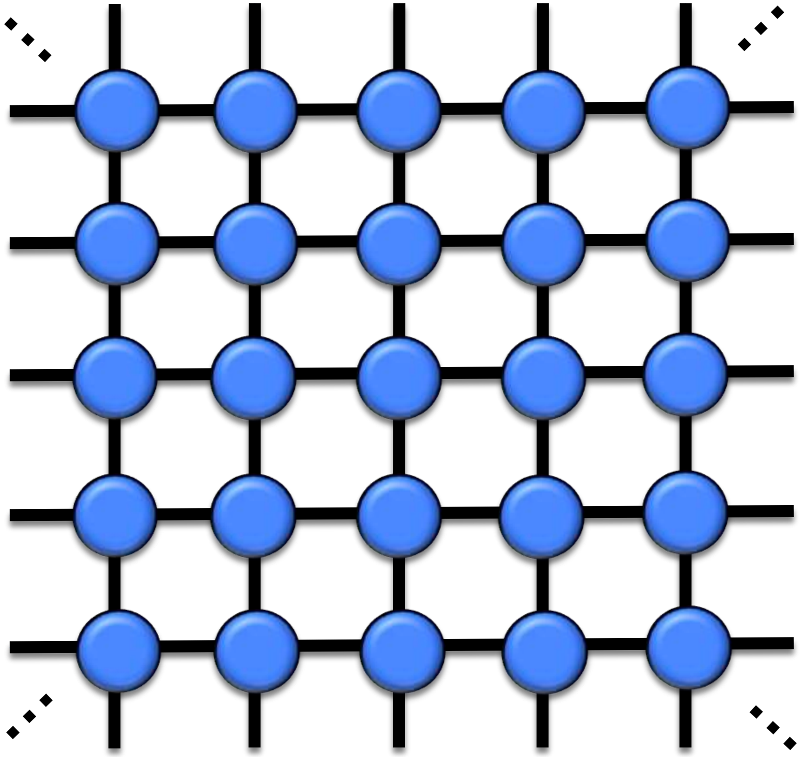
(double indices)

Let's put it on the plane of the screen!

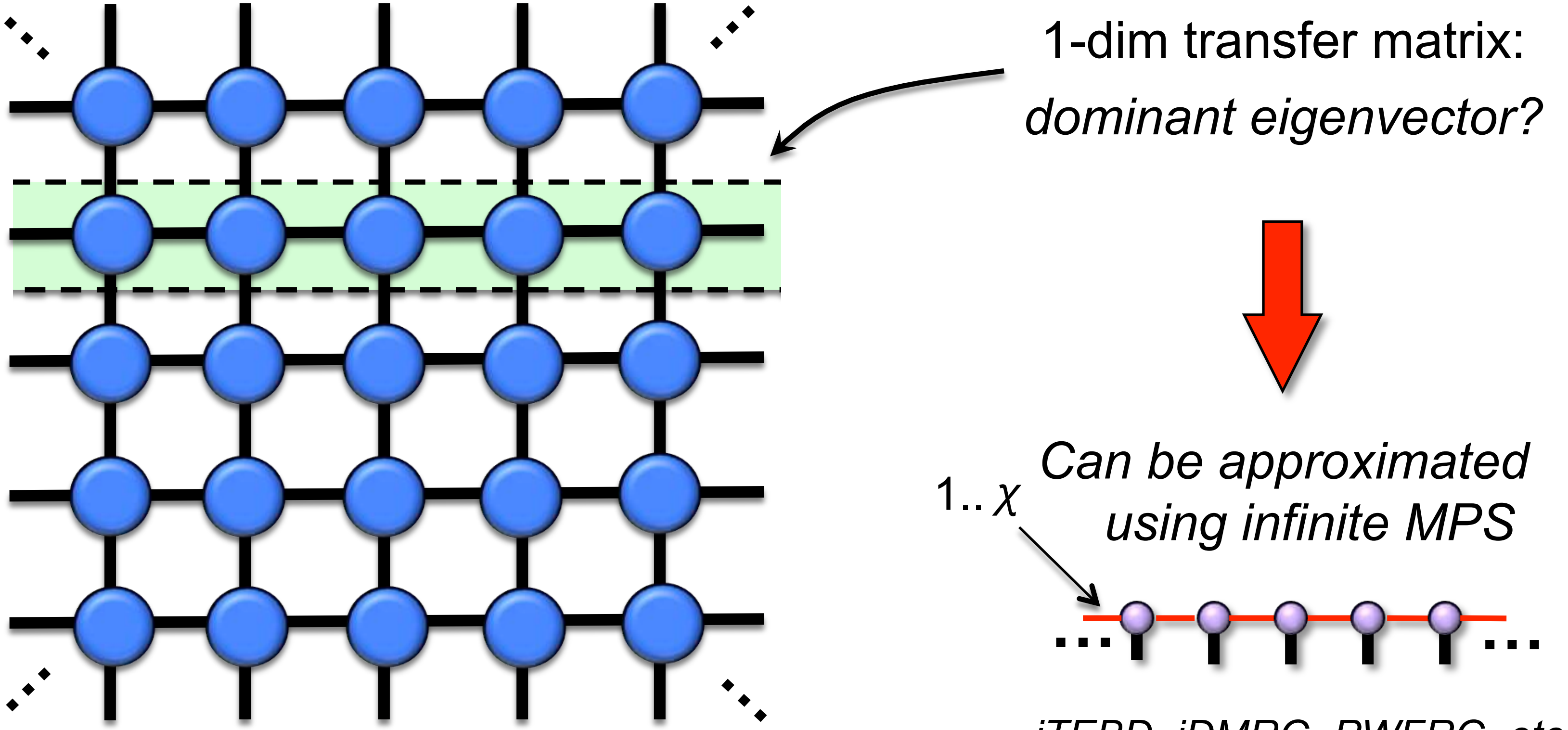


Environment calculations ⟷ ***Contraction of this infinite lattice***

Contracting the infinite 2d lattice



Contracting the infinite 2d lattice



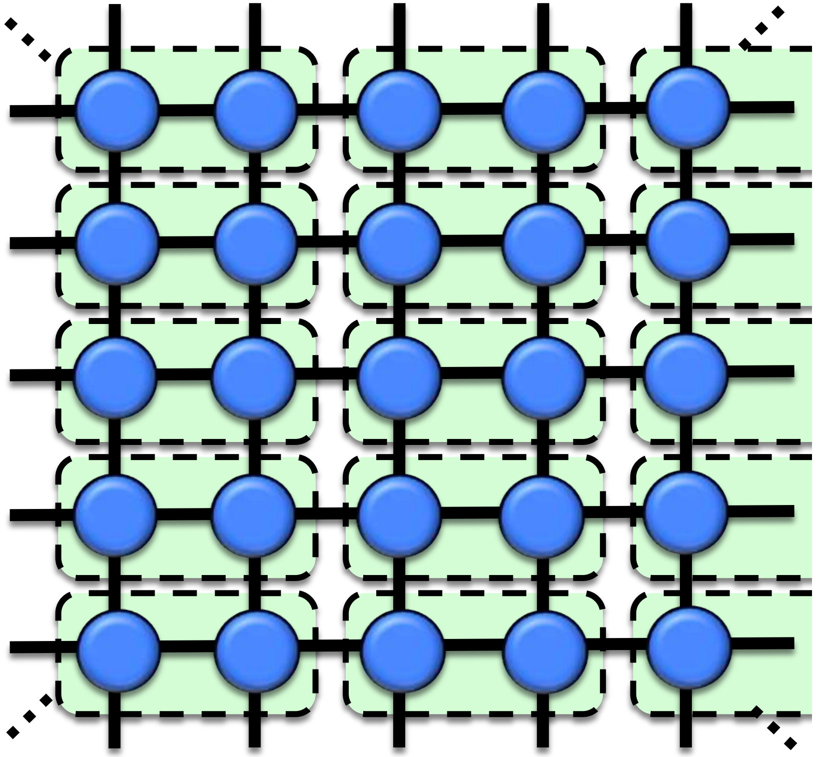
1-dim transfer matrix:

dominant eigenvector?

Can be approximated using infinite MPS

iTEBD, iDMRG, PWFRG, etc

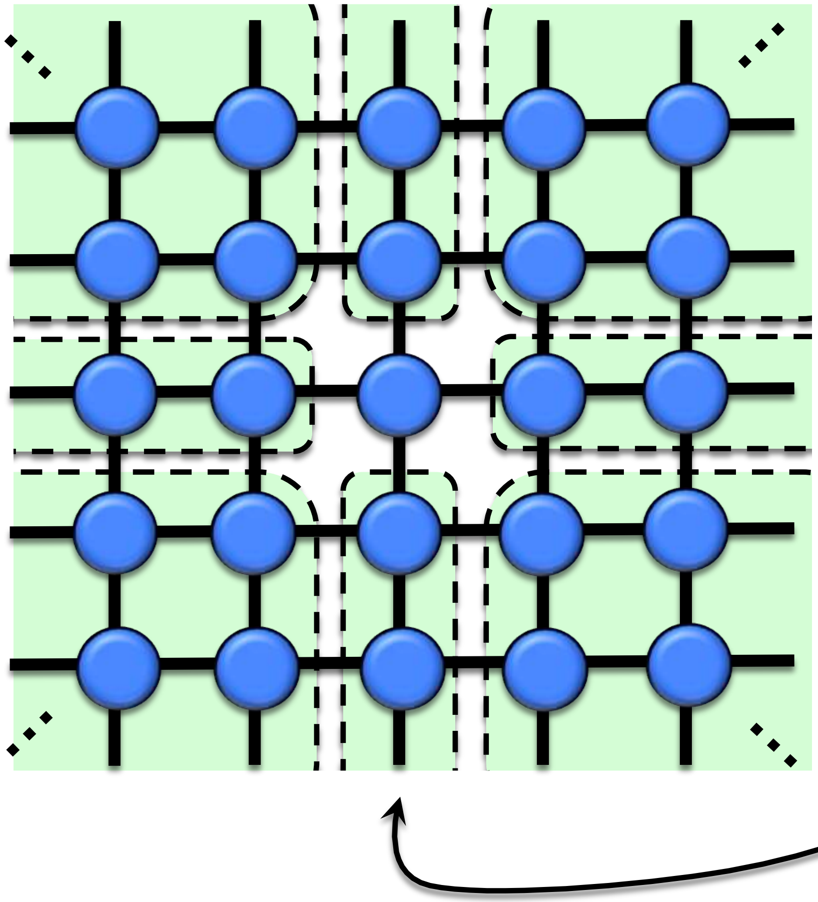
Contracting the infinite 2d lattice



Coarse-graining approaches:

TRG/SRG, HOSVD, TNR

Contracting the infinite 2d lattice



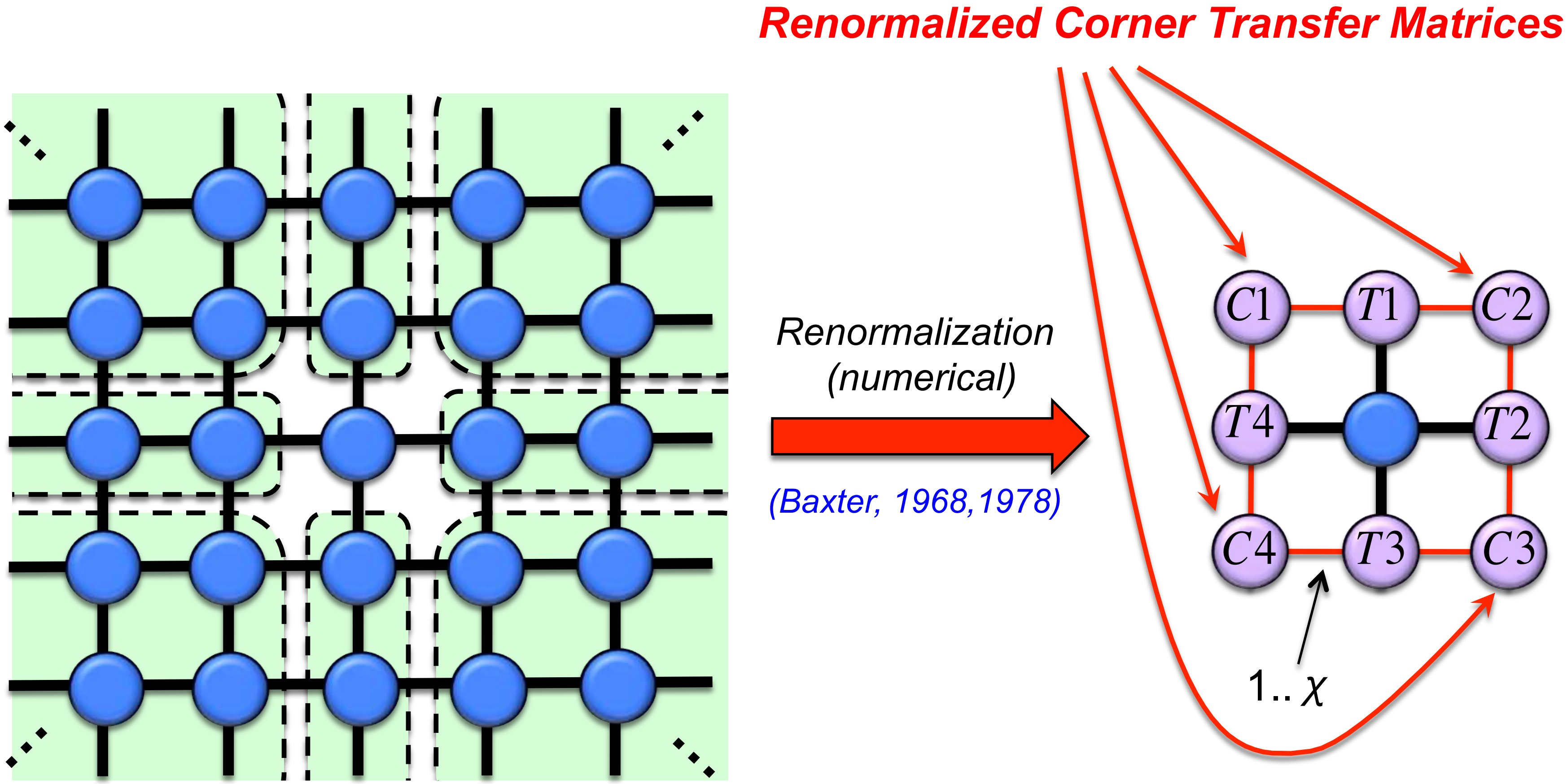
corner transfer matrix *(Baxter, 1968)*

(nice spectral properties)

half-row transfer matrix

half-column transfer matrix

Contracting the infinite 2d lattice



Renormalized Corner Transfer Matrices

Renormalization (numerical)

(Baxter, 1968,1978)

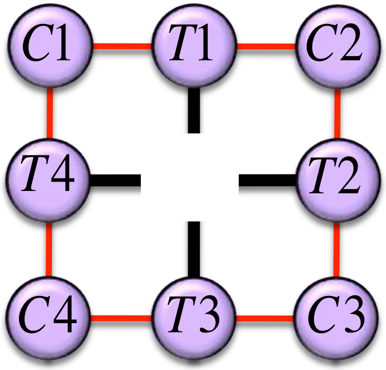
Directional version *of the* ***corner transfer matrix renormalization group***

(faster than 1d transfer matrix methods) *T. Nishino, K. Okunishi, JPS Jpn. 65, 891 (1996)*

Corner transfer matrix and iPEPS

R. Orús, G. Vidal, PRB 80 094403 (2009), R. Orús, PRB 85, 205117 (2012)

Example: left move



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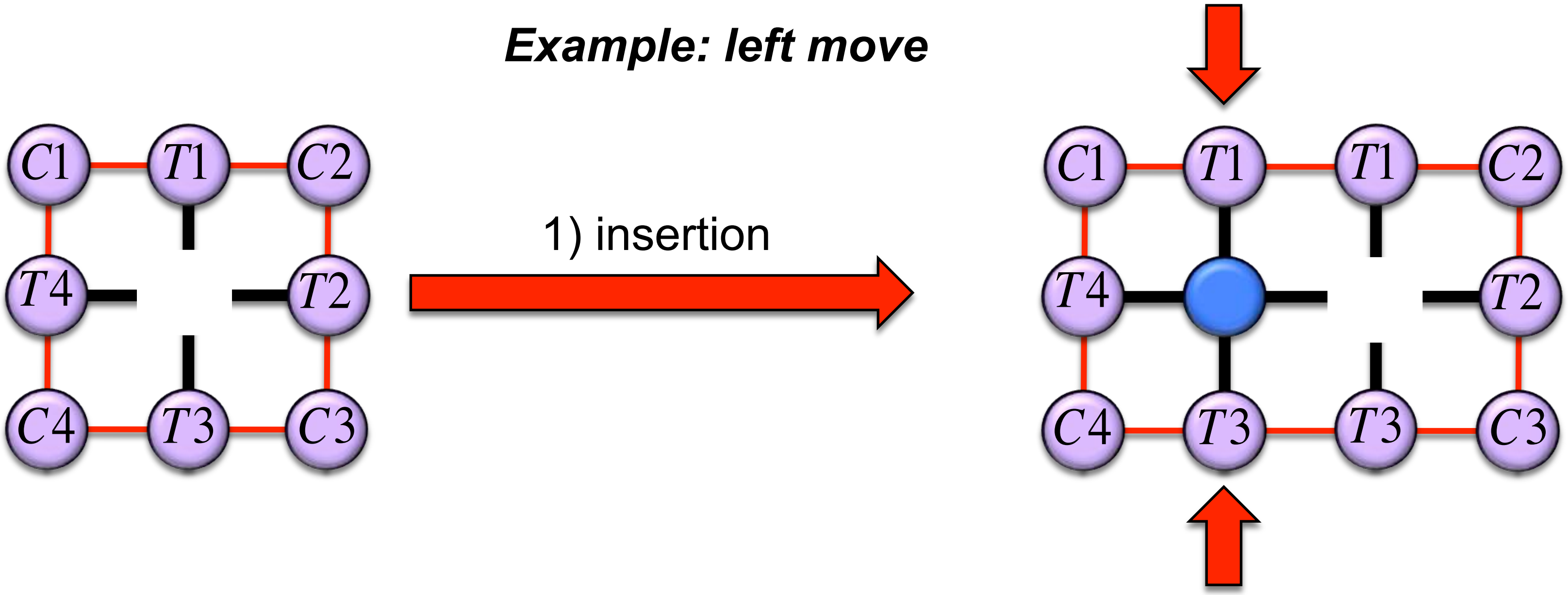

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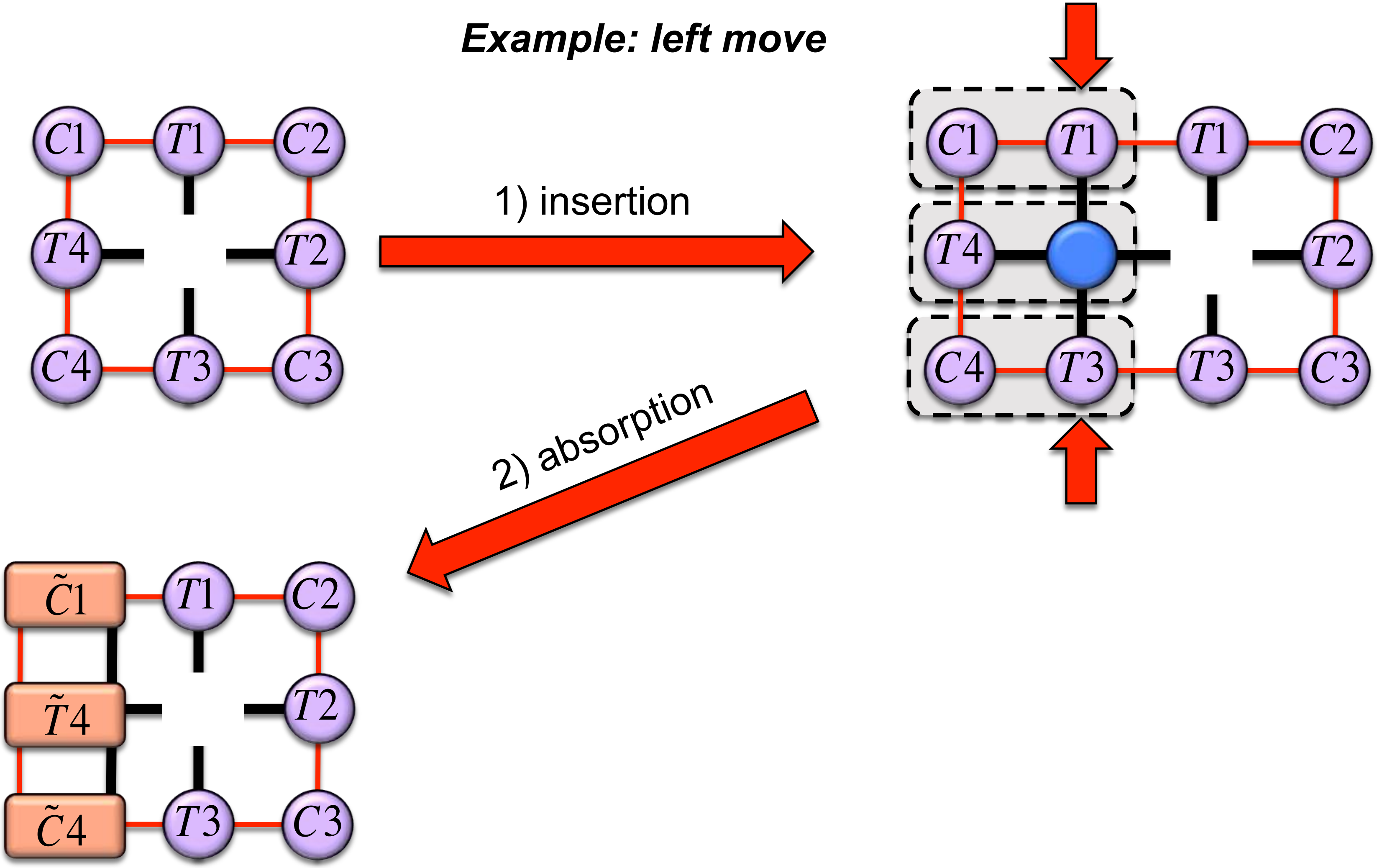

Corner transfer matrix and iPEPS

R. Orús, G. Vidal, PRB 80 094403 (2009), R. Orús, PRB 85, 205117 (2012)

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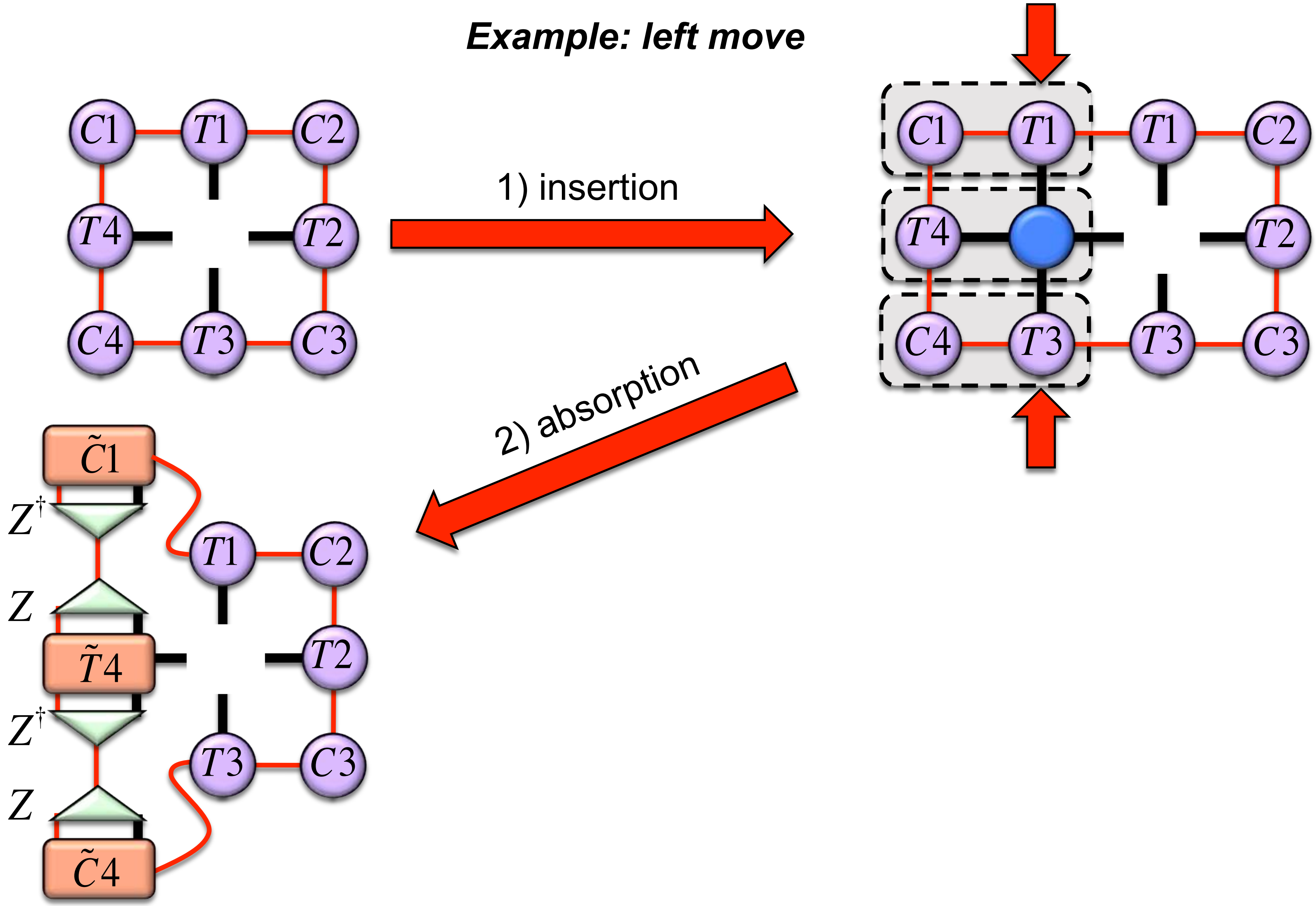

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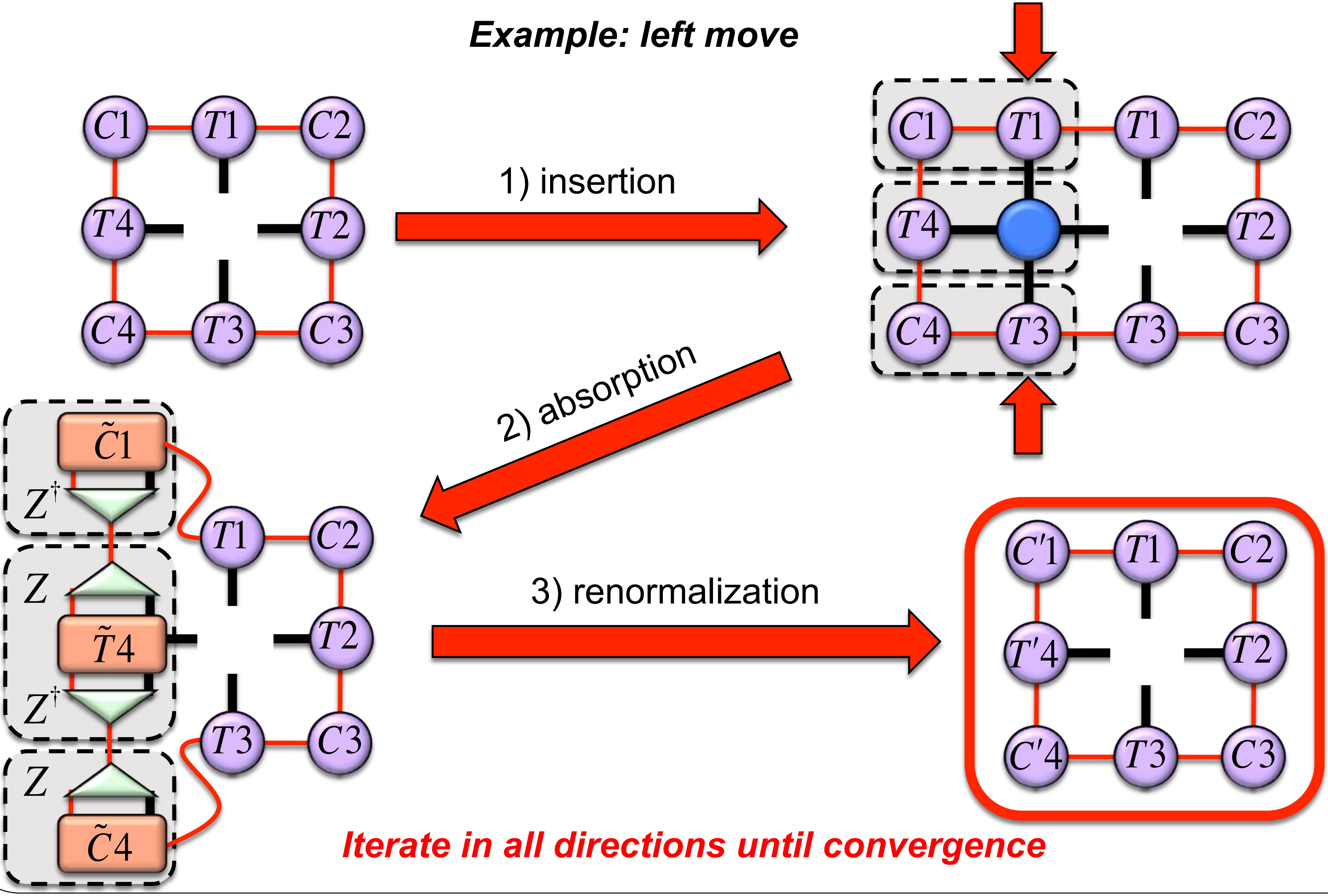


Example: left move

1) insertion

2) absorption

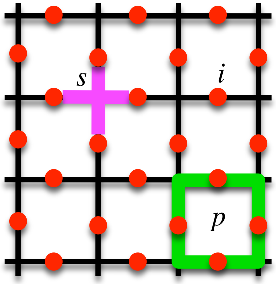
3) renormalization

Iterate in all directions until convergence

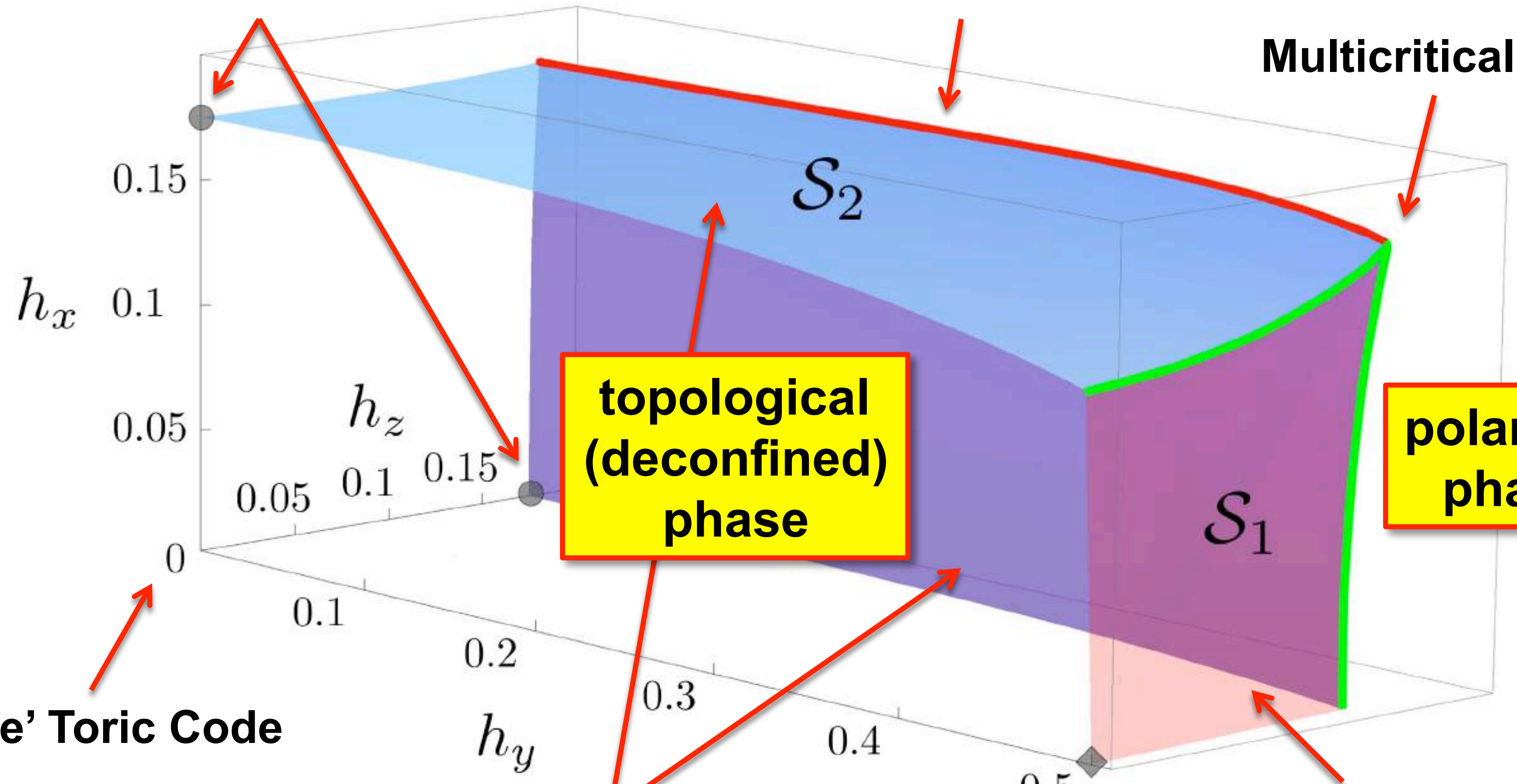
A typical example:
Toric Code in arbitrary field

S. Dusuel et al., PRL 106, 107203 (2011)

$$H = -J\sum_s A_s - J\sum_p B_p - h_x \sum_i \sigma_i^x - h_y \sum_i \sigma_i^y - h_z \sum_i \sigma_i^z$$



Phase diagram



Critical quantum Ising points

Multicritical line

Multicritical point

topological (deconfined) phase

polarized phase

'bare' Toric Code

2nd order sheets, quantum Ising universality class

Self-dual point

1st order sheet

5) Fermionic PEPS, and the MERA

Román Orús

University of Mainz

November 2nd 2017

Fermions with 2d PEPS

e.g., P. Corboz, R. Orús, B. Bauer, G. Vidal, PRB 81, 165104 (2010)

Fermionic 2d systems

Fermionic systems are extremely interesting physical systems, e.g. the **2d fermionic Hubbard model** may be the key to understand the emergence of **high-T_c superconductivity**

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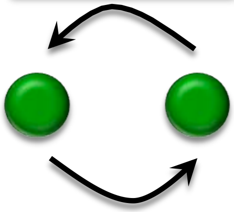
Unfortunately, fermionic systems are also **amongst the most difficult** to simulate, because of the **sign problem in Quantum MonteCarlo (sampling of negative probabilities)**

Fermions are a NUMERICAL MONSTER

for Quantum MonteCarlo because of the sign problem!

but there is hope...

Bosons



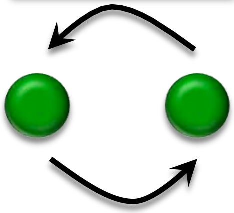
$$\Psi_B(x_1,x_2) = \Psi_B(x_2,x_1)$$

Symmetric wavefunction

$$b_i b_j = b_j b_i$$

Operators commute

Bosons



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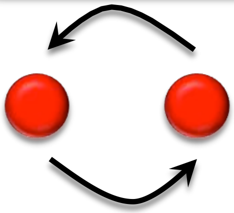


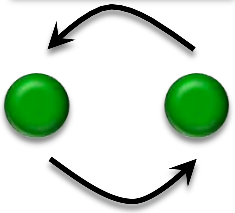
$$\Psi_F(x_1,x_2) = -\Psi_F(x_2,x_1)$$

Antisymmetric wavefunction

$$c_i c_j = -c_j c_i$$

Operators anticommute

Bosons



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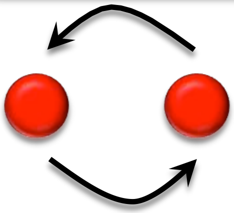

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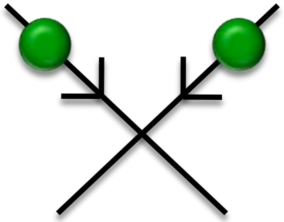
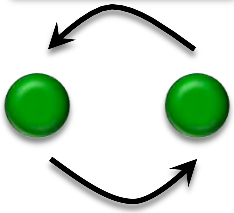
Crossings in a TN



+

Ignore crossings

Bosons



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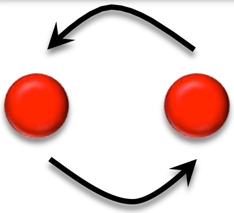

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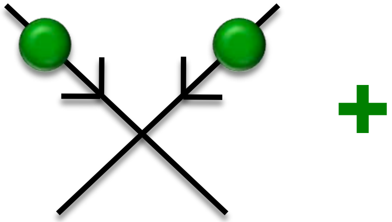
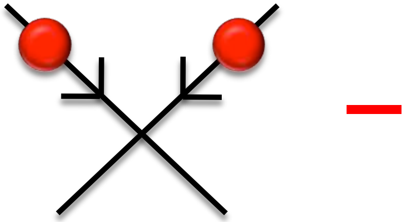


Ignore crossings



Careful!!!!

Tensor Network “fermionization” rules

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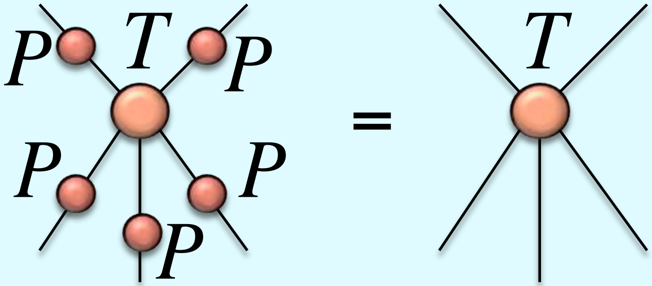
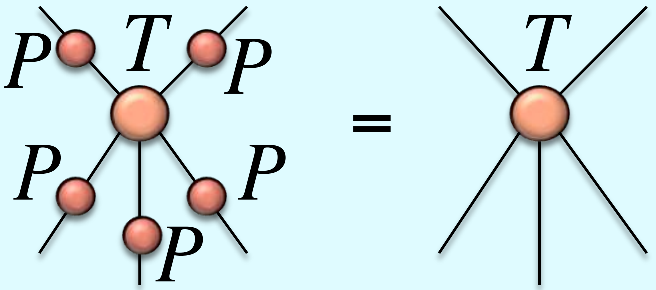


Use **parity-preserving** tensors

$$T_{i_1 i_2 \ldots i_M} = 0 \ \text{ if } \ P(i_1)P(i_2)\ldots P(i_M) \neq 1$$

Symmetry of the Hamiltonian

Tensor Network "fermionization" rules



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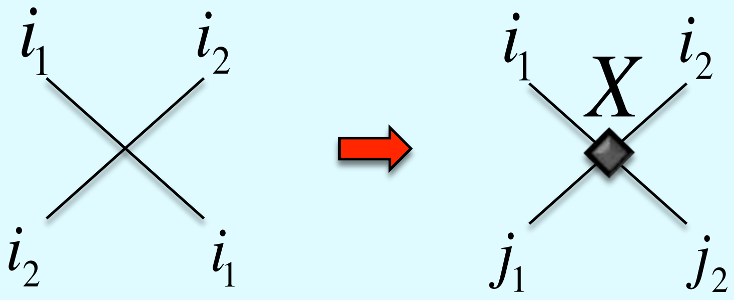

Replace crossings by

fermionic swap gates

$$X_{i_2 i_1 j_1 j_2} = \delta_{i_1 j_1}\delta_{i_2 j_2} S(P(i_1),P(i_2))$$

$$S(P(i_1),P(i_2)) = \begin{cases} -1 & \text{if } \; P(i_1) = P(i_2) = -1 \\ +1 & \text{otherwise} \end{cases}$$

*Fermionic operators **anti**commute*

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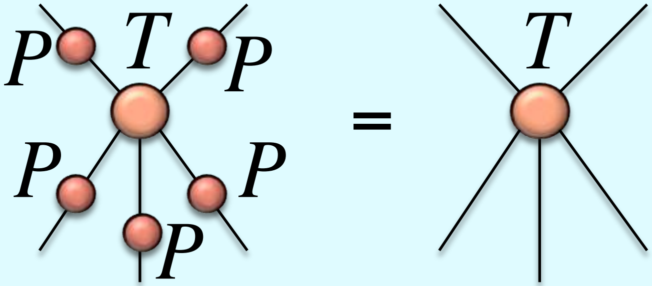

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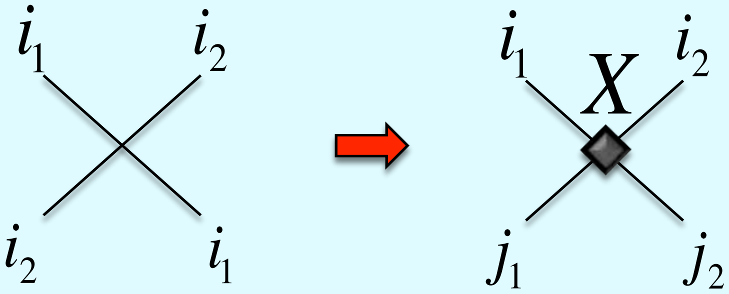

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The leading order of the computational cost is the same as in the bosonic case

fermionic order ~ graphical projection of a PEPS

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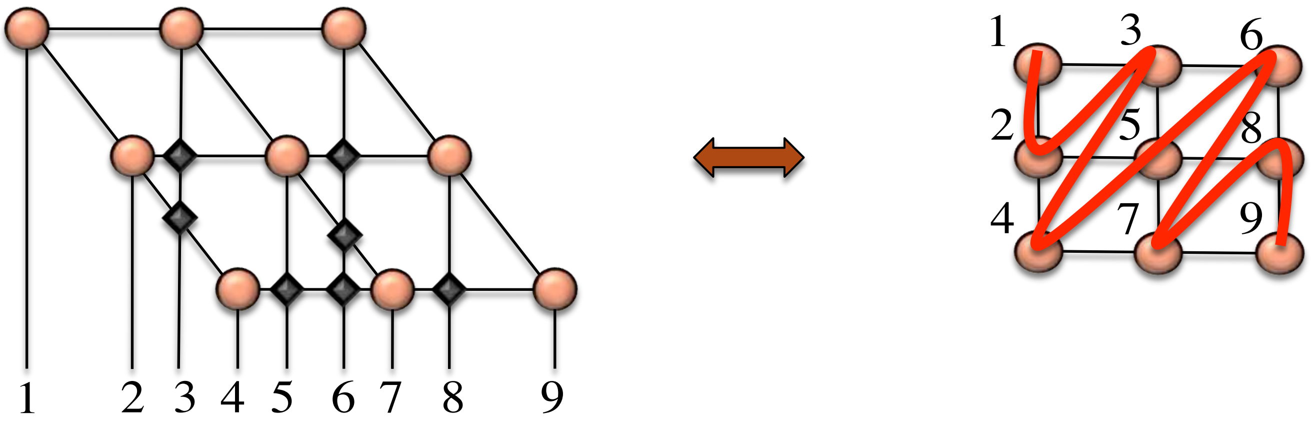
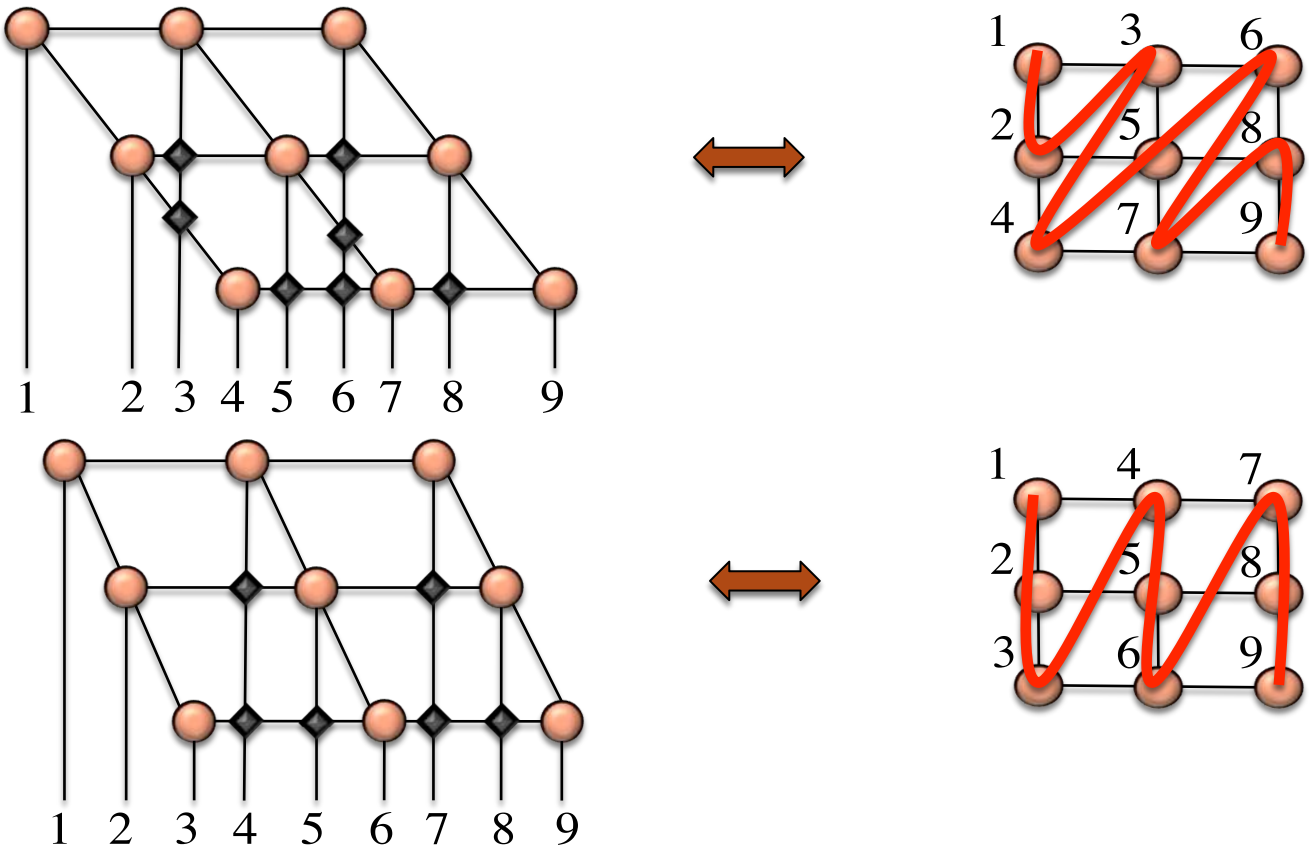


fermionic order ~ graphical projection of a PEPS



physics is independent of the order

⇕

physics is independent of graphical projection

(different choices of Jordan-Wigner transformation, if mapping to a spin system)

Example: *scalar product of 3x3 PEPS*

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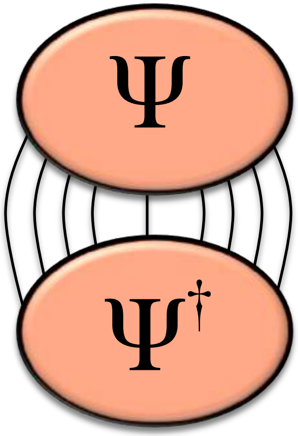
$$\langle \Psi | \Psi \rangle$$

Example: *scalar product of 3x3 PEPS*

$$\langle \Psi | \Psi \rangle \Rightarrow$$



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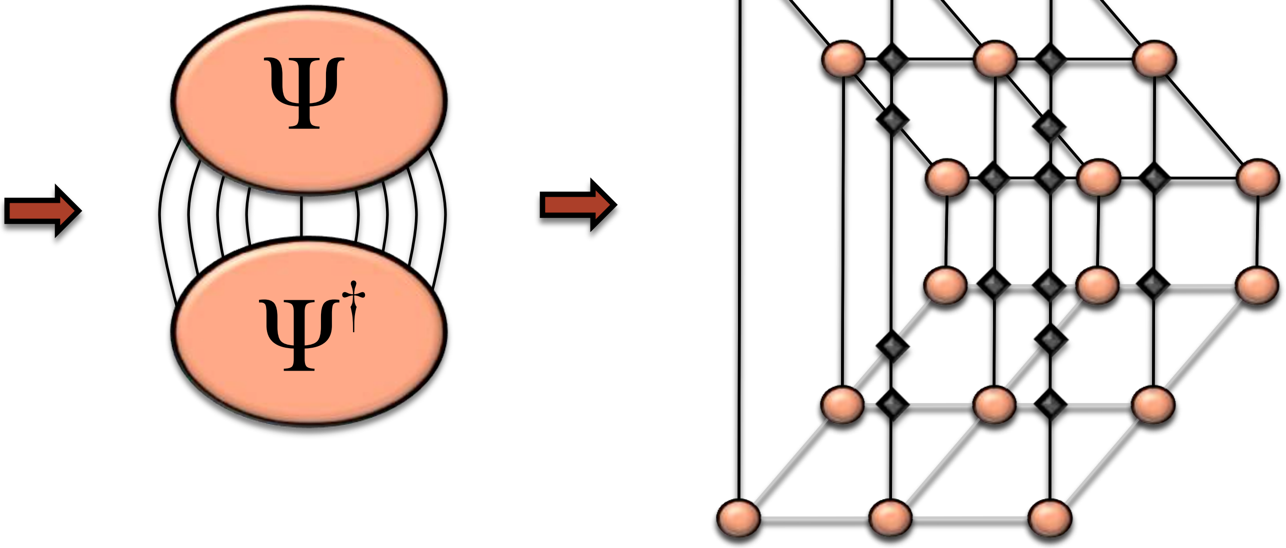
$$\langle\Psi|\Psi\rangle$$



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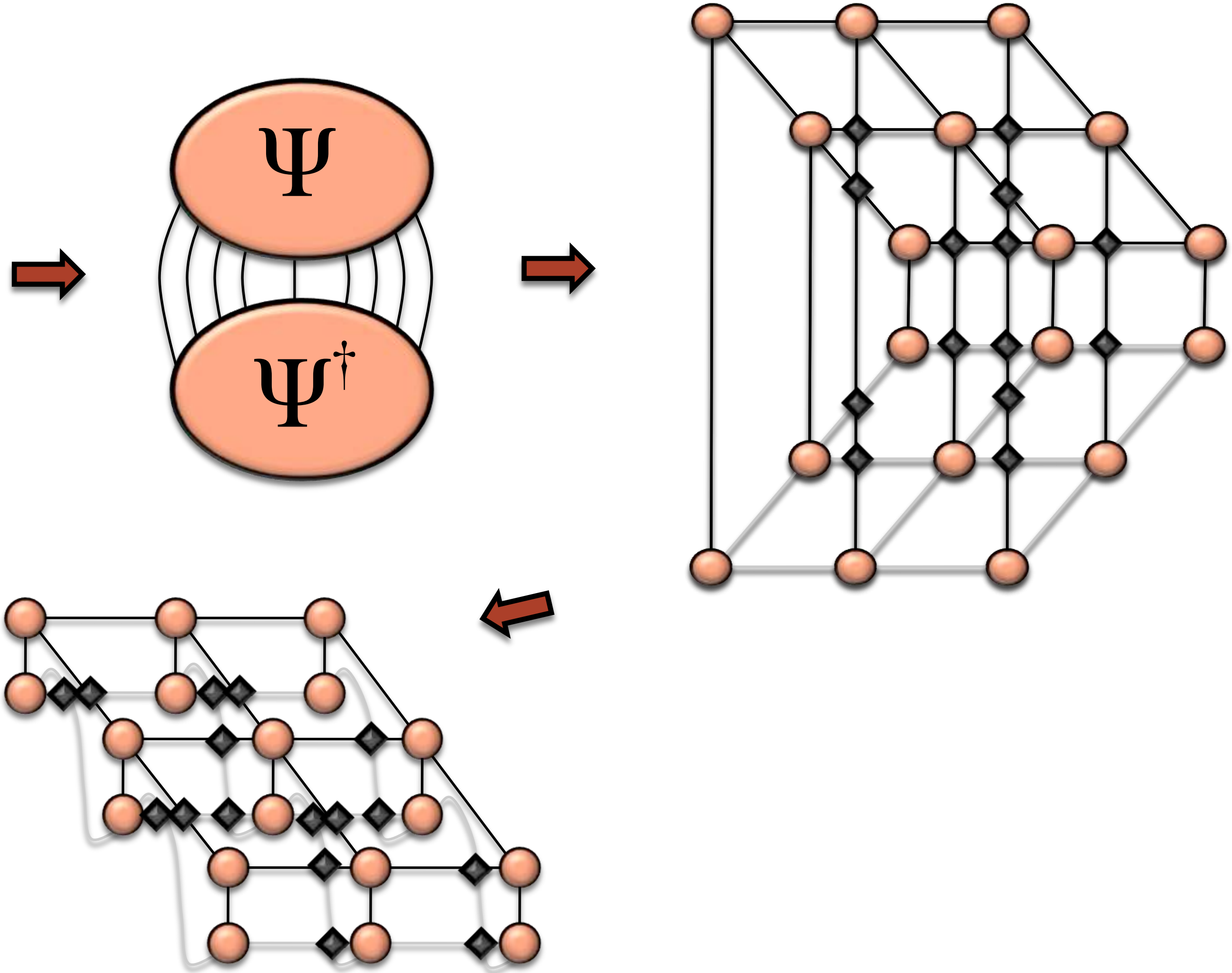

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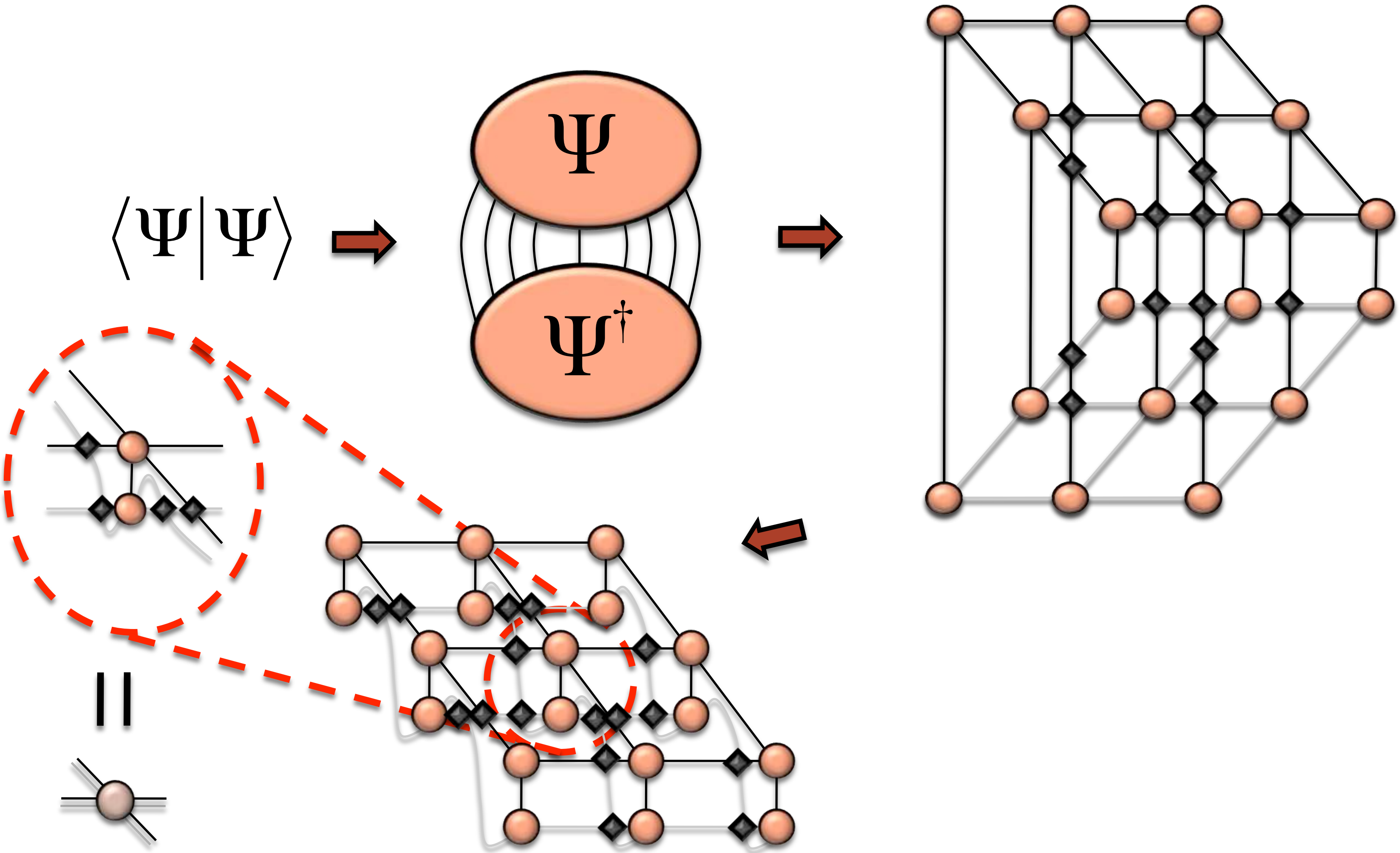

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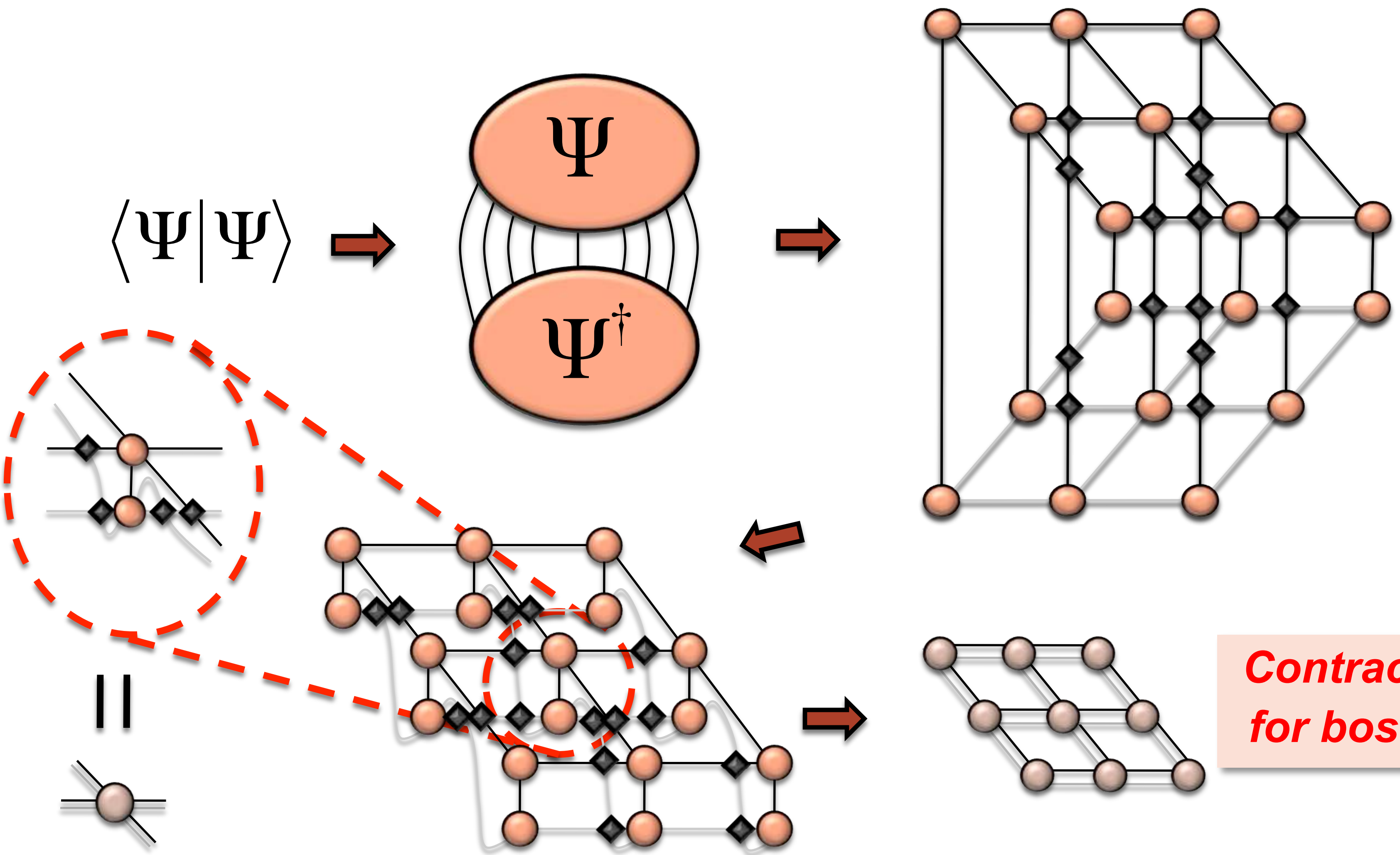


Contract as for bosons!

But... does it work?

But... does it work?

„Tensor networks provide today the best variational energies for the Hubbard model in the strong coupling limit. iPEPS has really made it“.

Matthias Troyer (at the Korrelationstage 2015)

J. Jordan, RO, G. Vidal, F. Verstraete, I. Cirac, PRL ***101*** *250602 (2008)*
P. Corboz, RO, B. Bauer, G. Vidal, PRB ***81*** *165104 (2010)*

YES, it does

P. Corboz, PRB ***93*** *045116 (2016)*

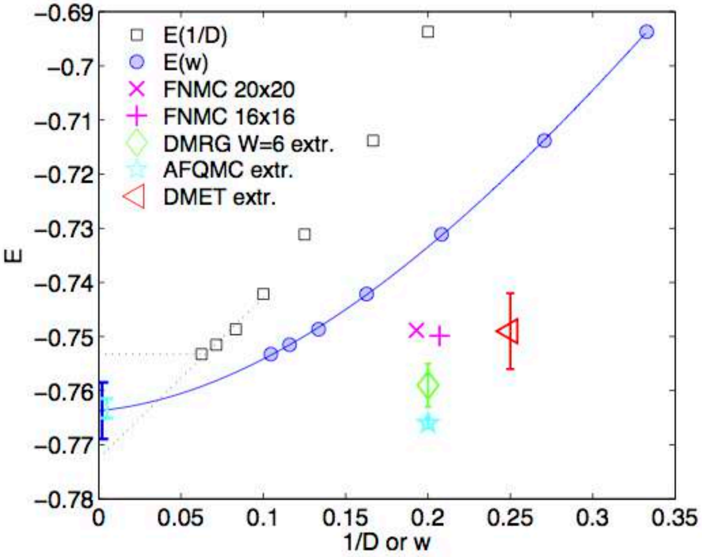
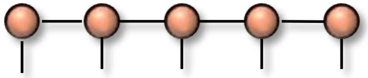


FIG. 4. (Color online) iPEPS energy of a period-5 stripe in the doped case in the strongly correlated regime ($U/t = 8$, $n = 0.875$) in comparison with other methods.

Multiscale Entanglement Renormalization Ansatz (MERA)

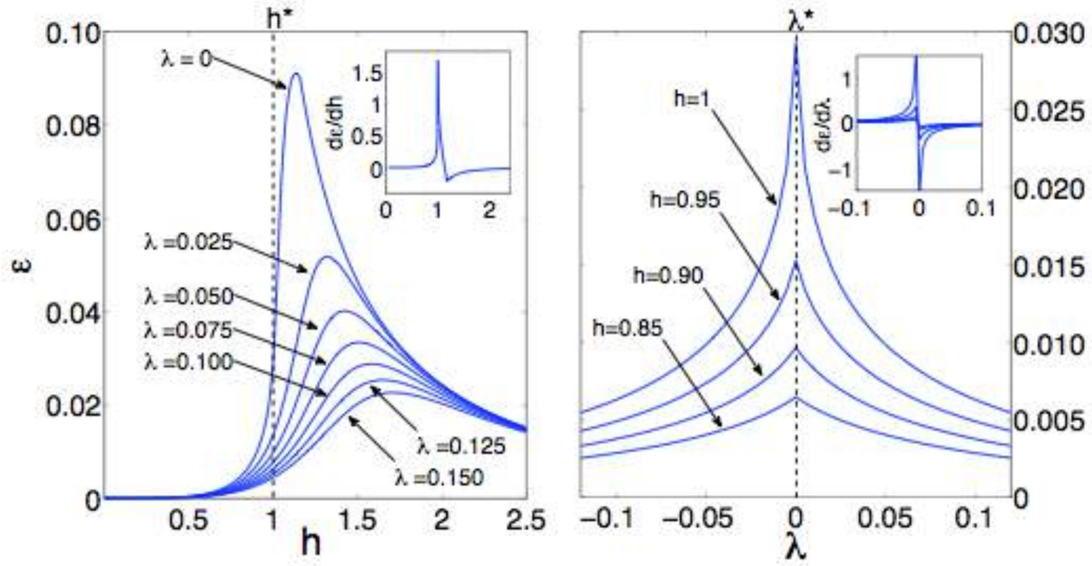
From MPS to MERA

Matrix Product States (MPS)



1d systems

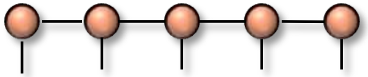
But we want to do critical systems!!!



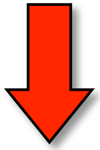
Also very painful for DMRG...

From MPS to MERA

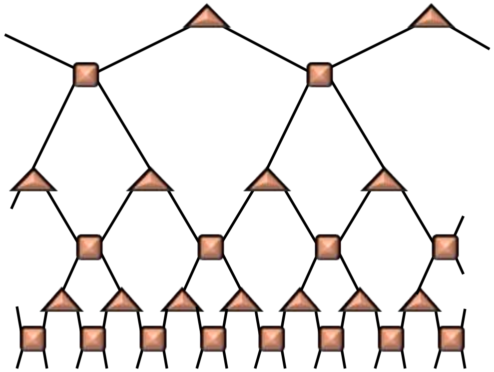
Matrix Product States (MPS)



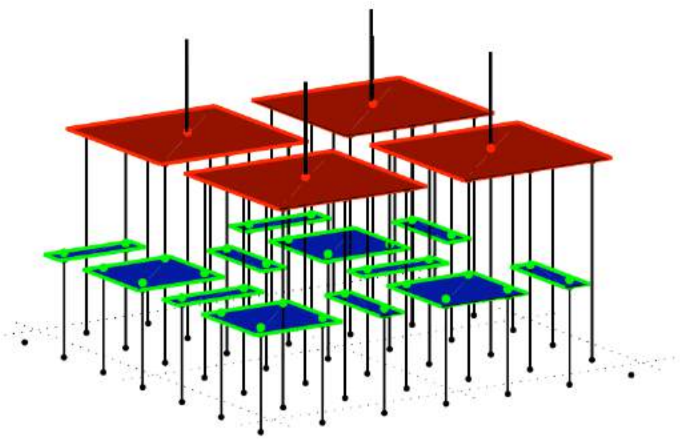
1d systems



Multiscale Entanglement Renormalization Ansatz (MERA)



1d systems

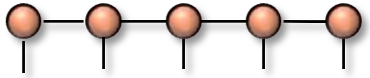


2d systems

and so on…

From MPS to MERA

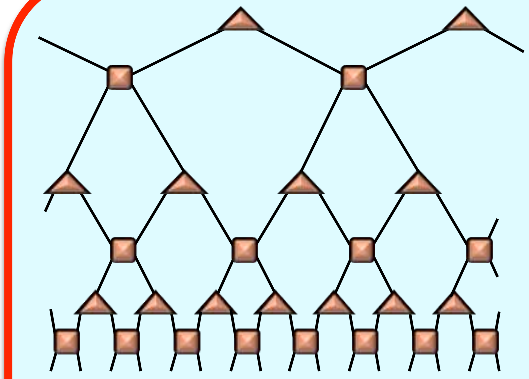
Matrix Product States (MPS)



1d systems

Multiscale Entanglement Renormalization Ansatz (MERA)

This lecture



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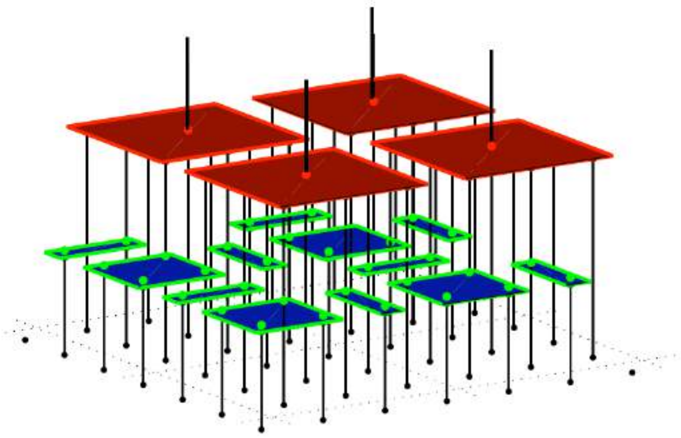
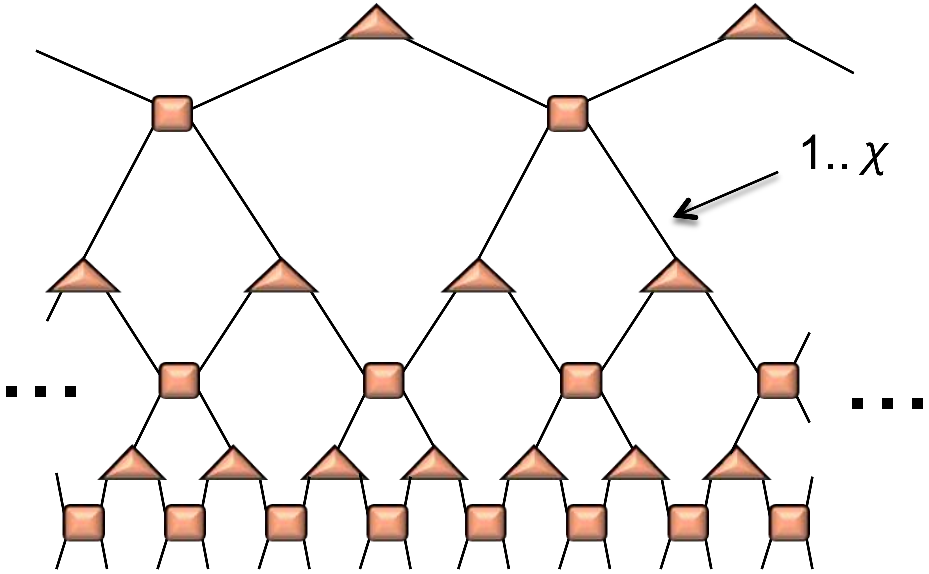


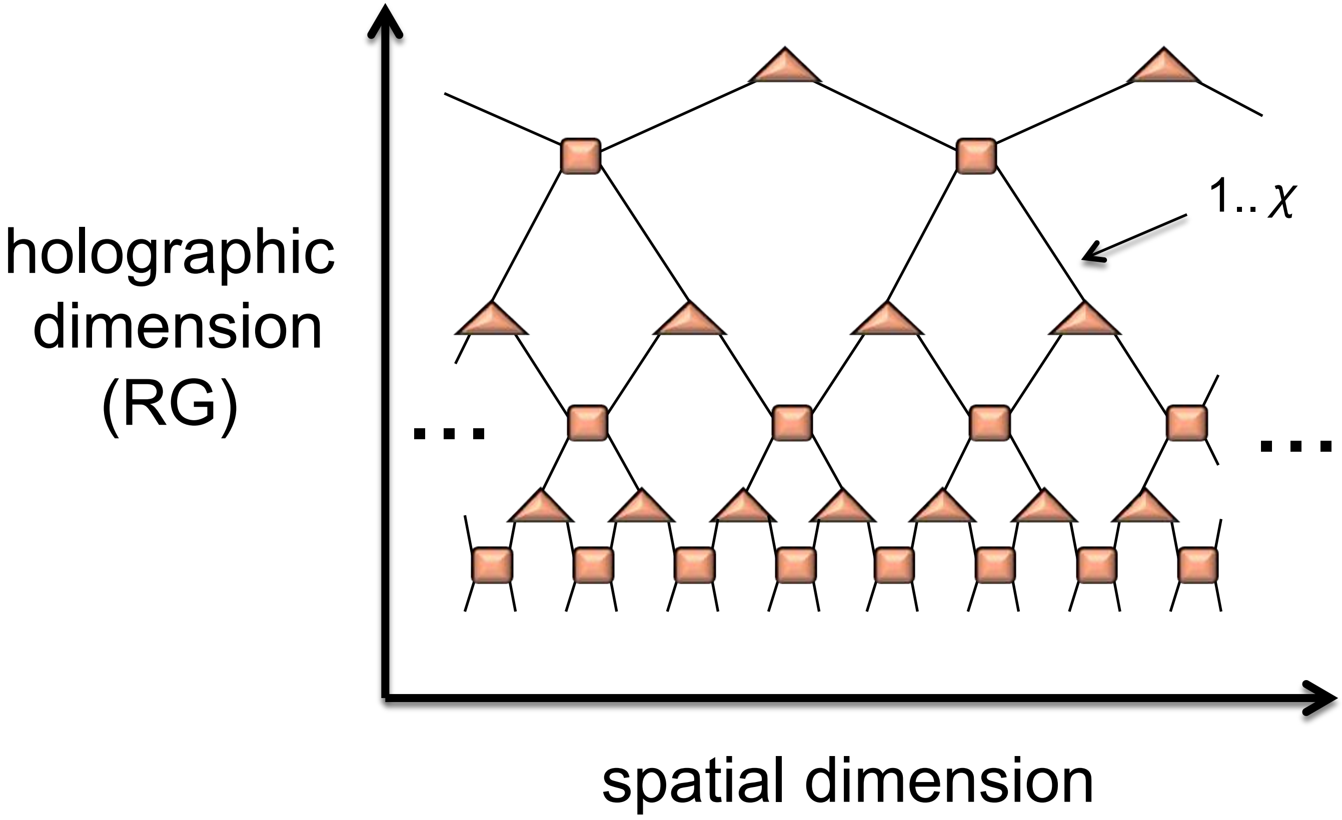
2d systems

and so on…

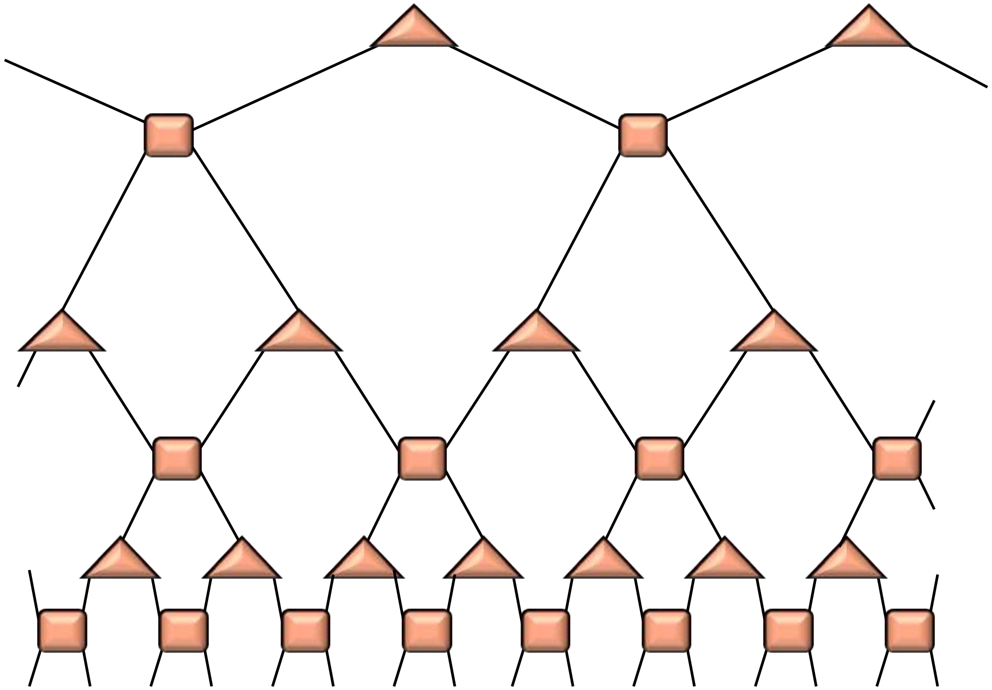
1d MERA

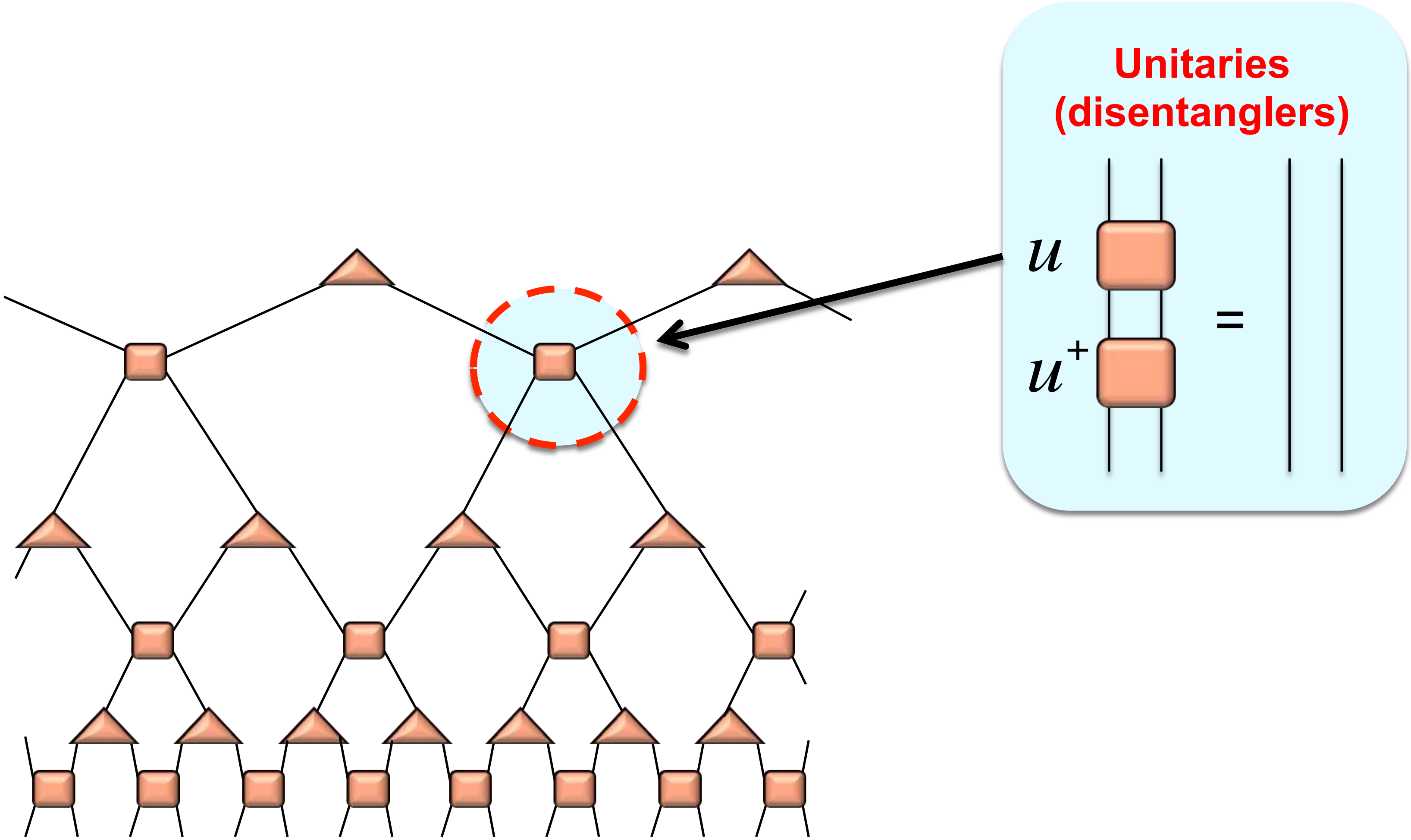


1d MERA

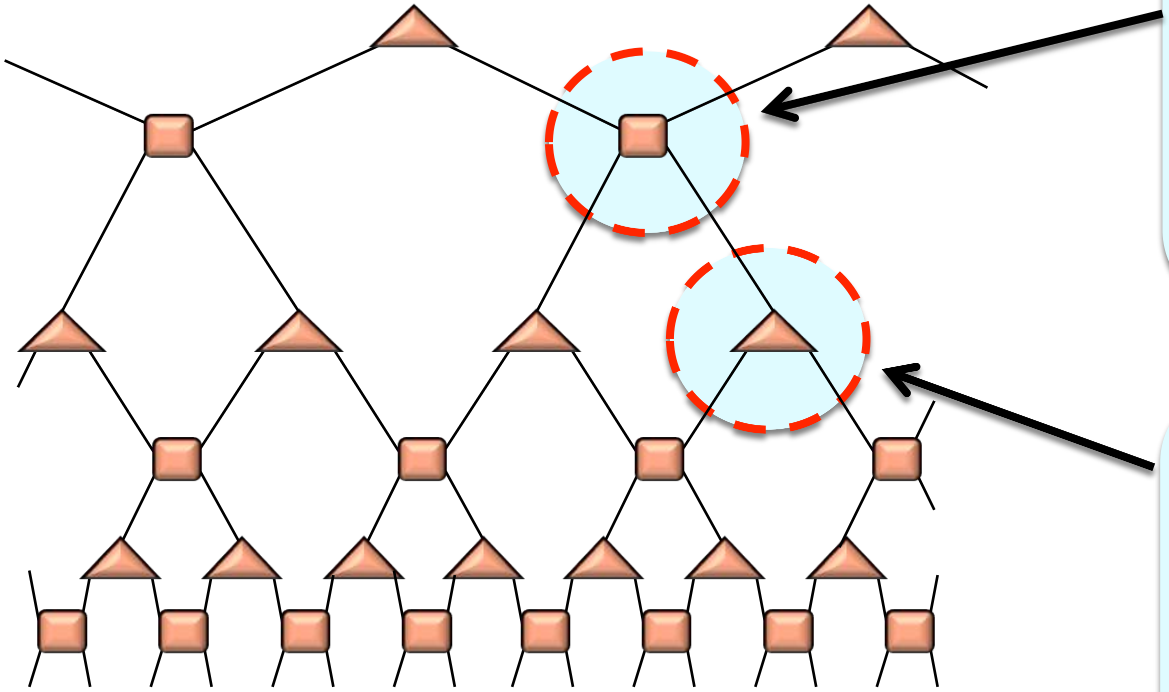


Tensors obey constraints





Unitaries (disentanglers)



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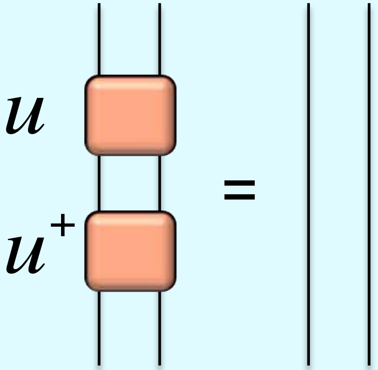
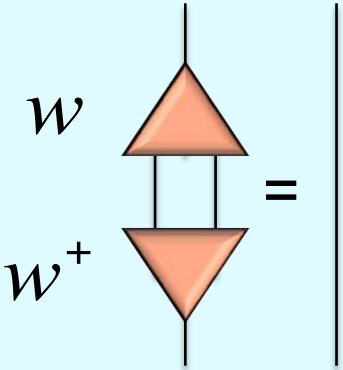
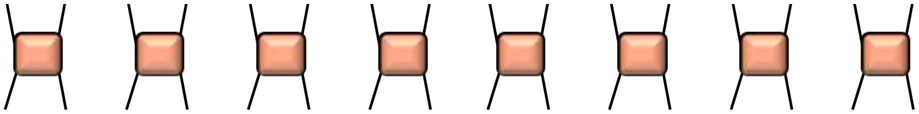


Isommetries (coarse-grainings)



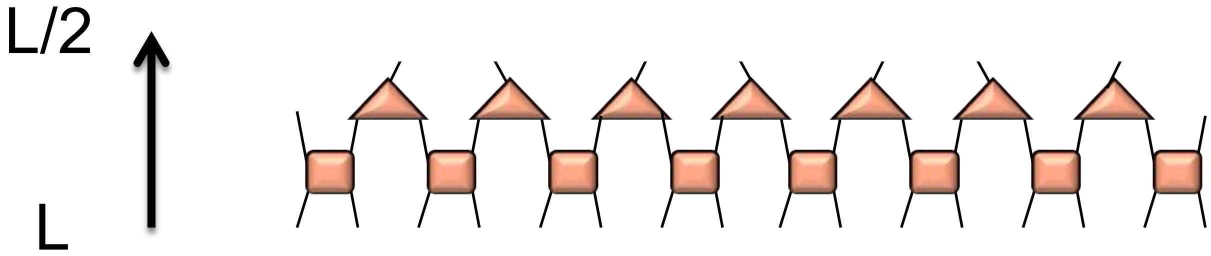
Reason:

entanglement is built locally
at all length scales



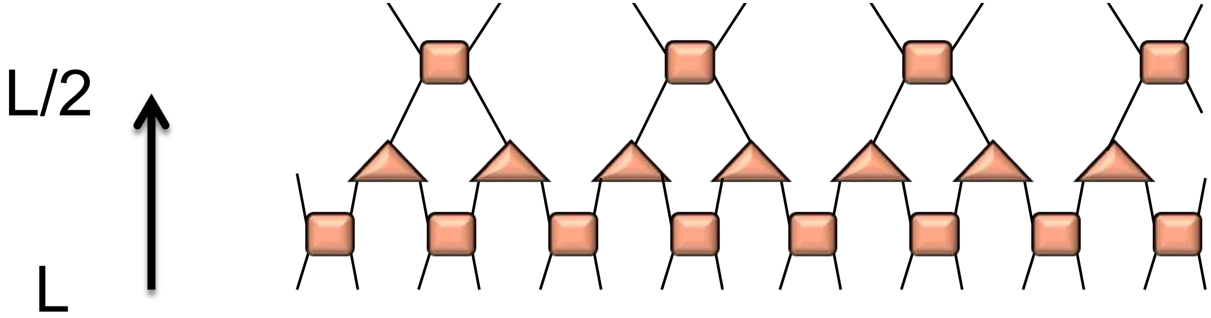
L

entangle locally



coarse-grain

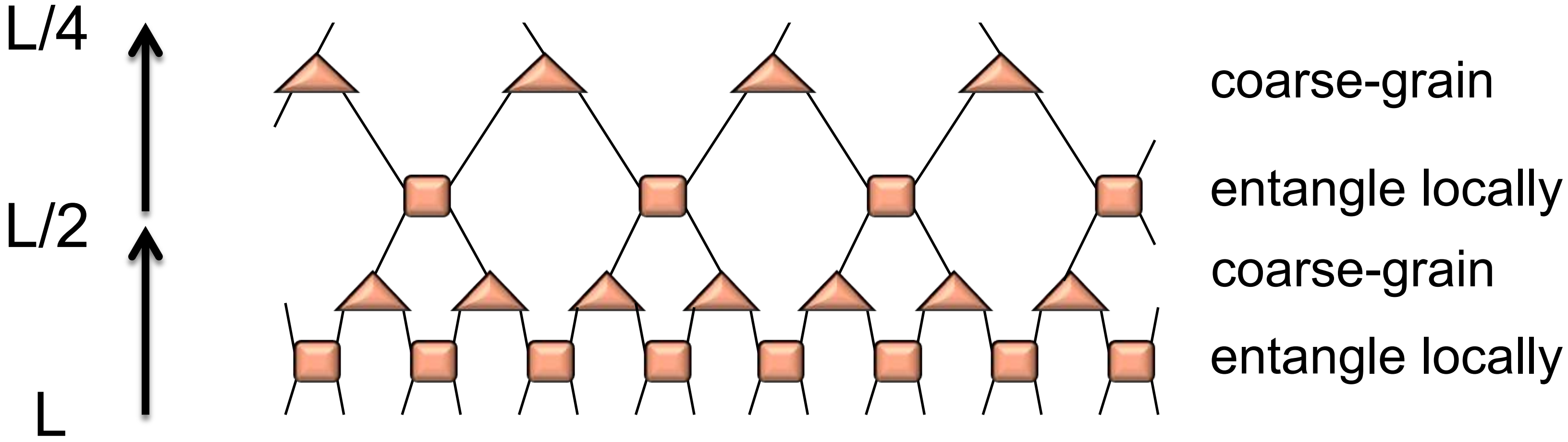
entangle locally



entangle locally

coarse-grain

entangle locally



L/4

L/2

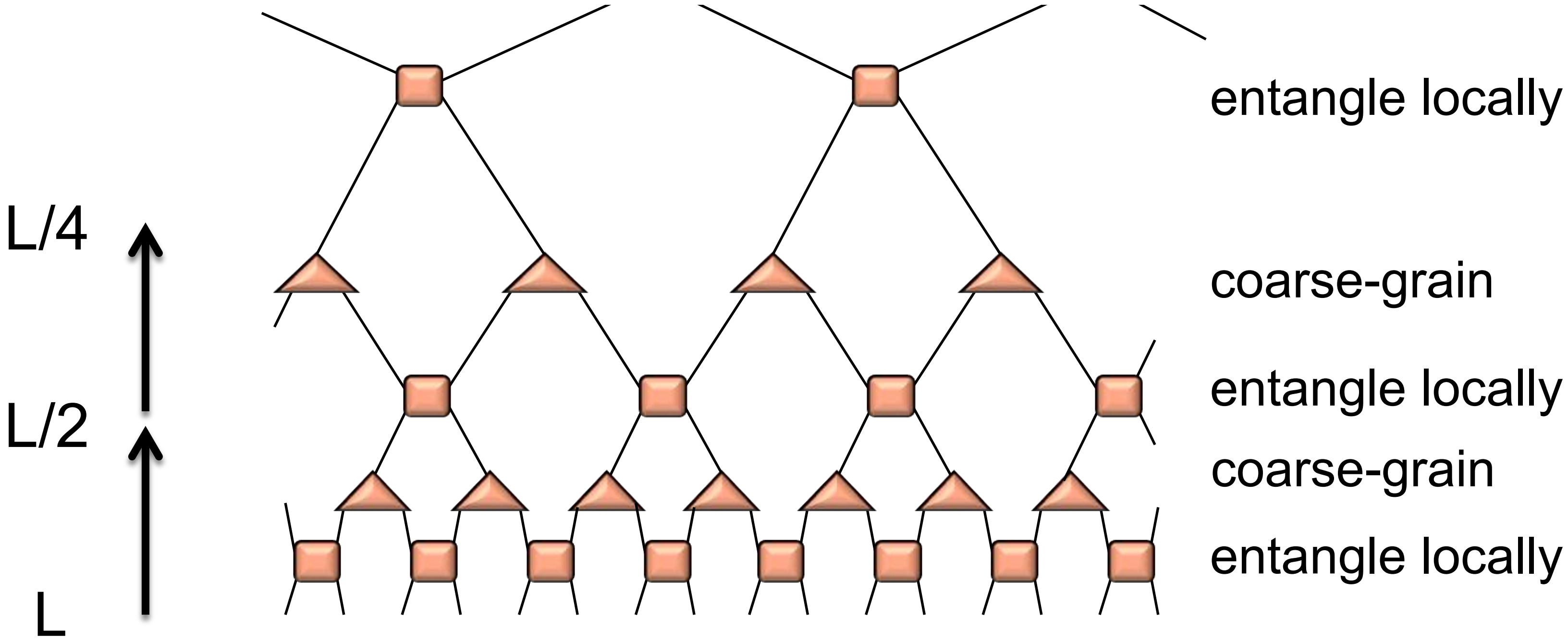
L

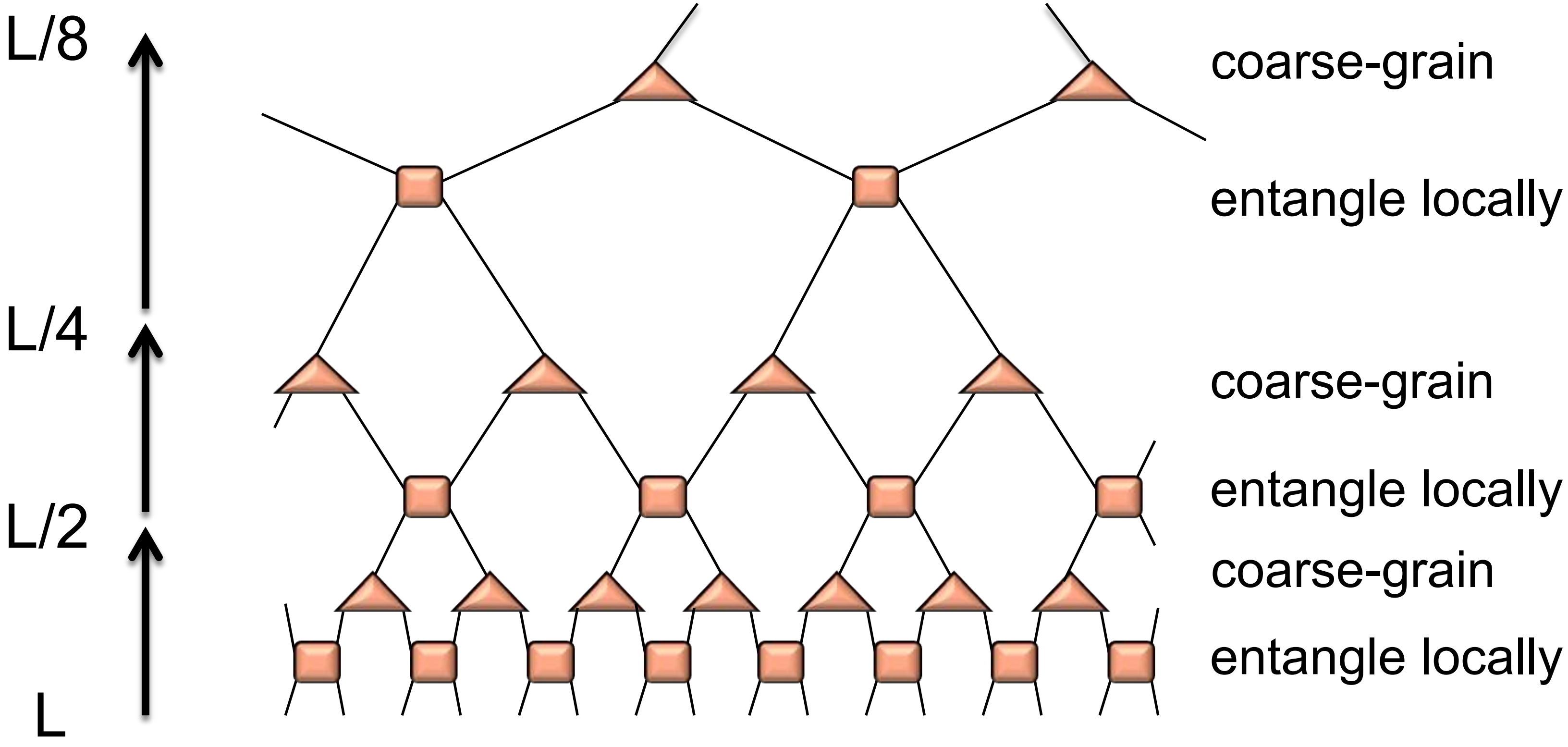
coarse-grain

entangle locally

coarse-grain

entangle locally


L/4
L/2
L
entangle locally
coarse-grain
entangle locally
coarse-grain
entangle locally



L/8

L/4

L/2

L

coarse-grain

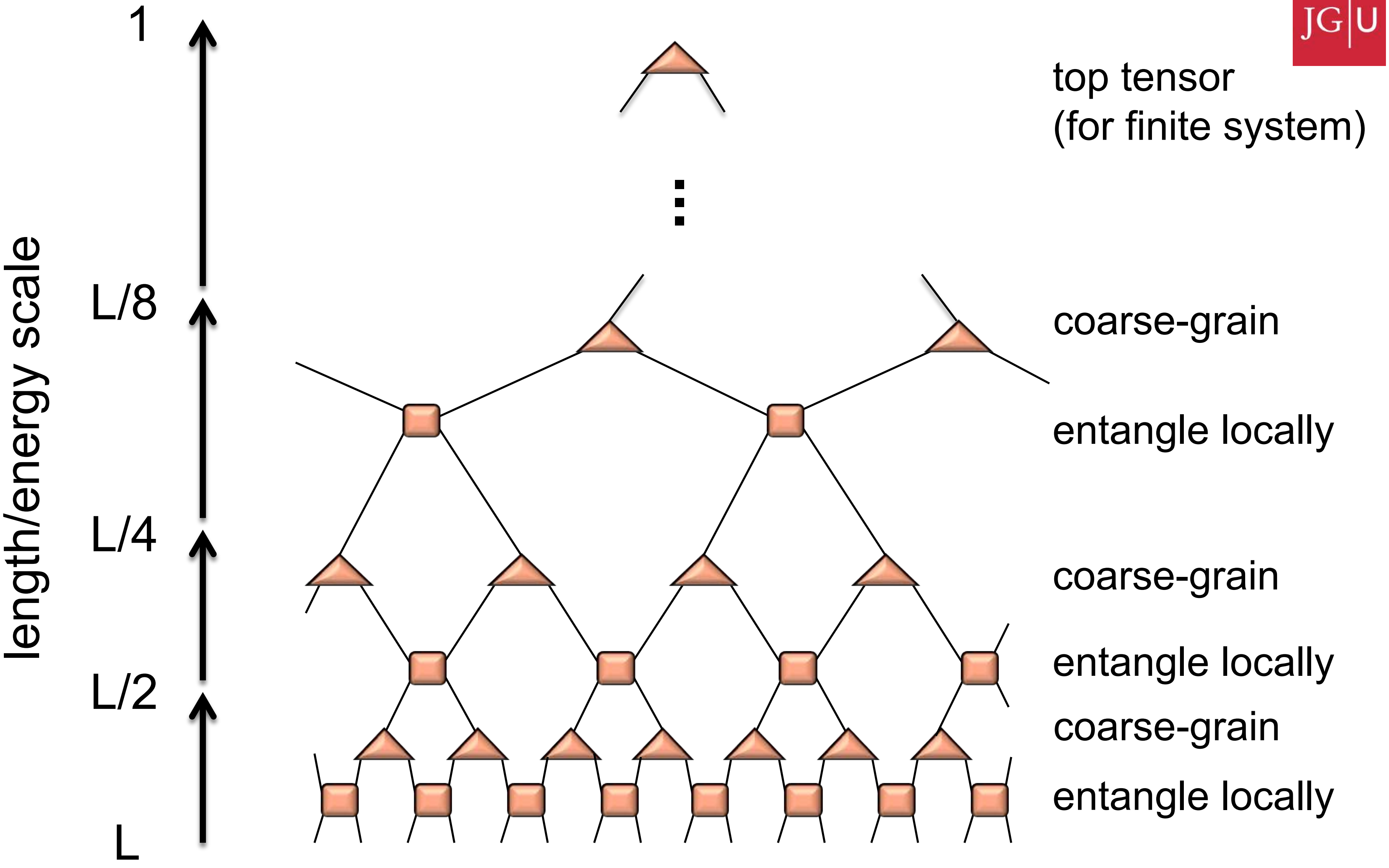
entangle locally

coarse-grain

entangle locally

coarse-grain

entangle locally



length/energy scale

1

L/8

L/4

L/2

L

top tensor
(for finite system)

coarse-grain

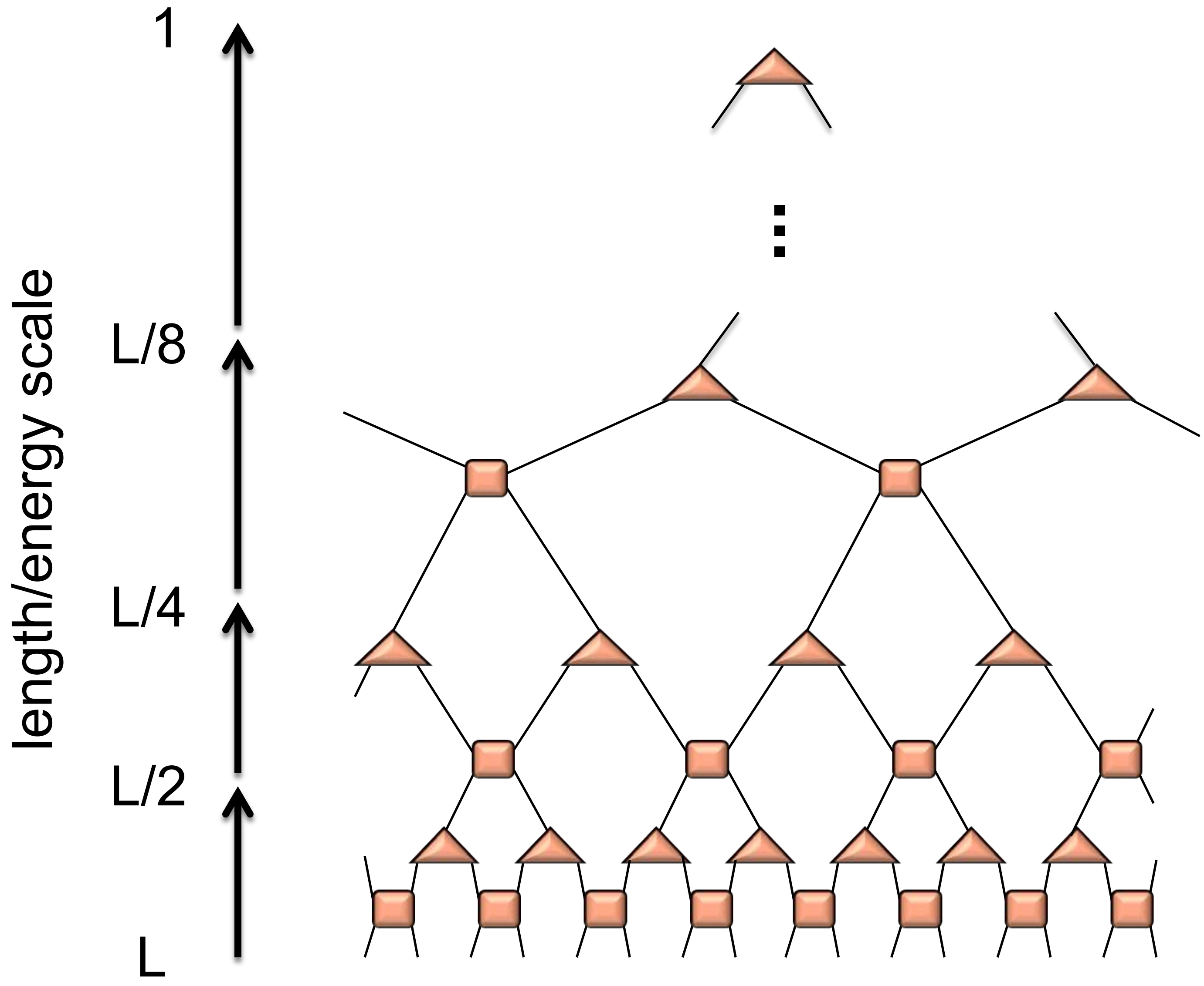
entangle locally

coarse-grain

entangle locally

coarse-grain

entangle locally



top tensor
(for finite system)

coarse-grain

entangle locally

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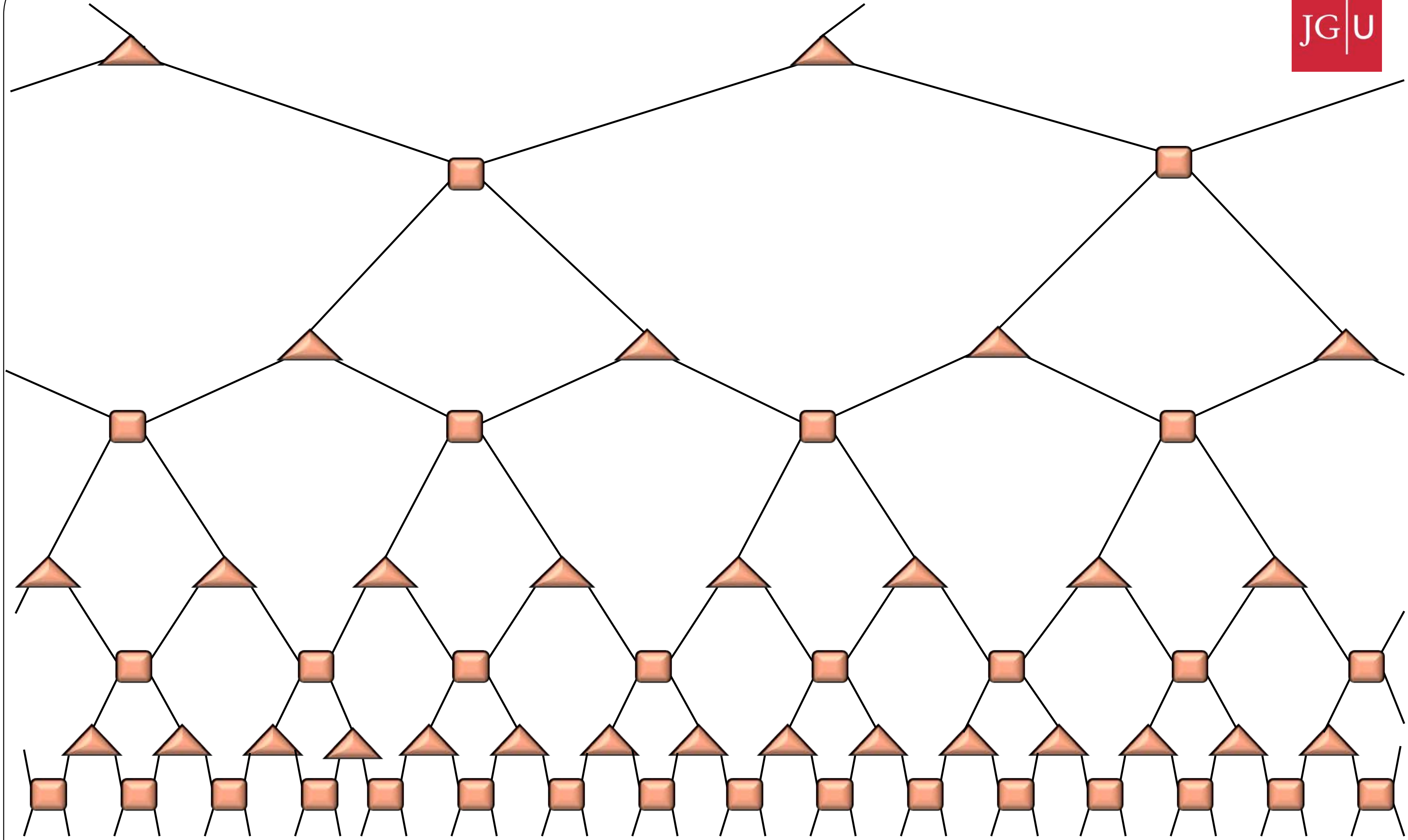
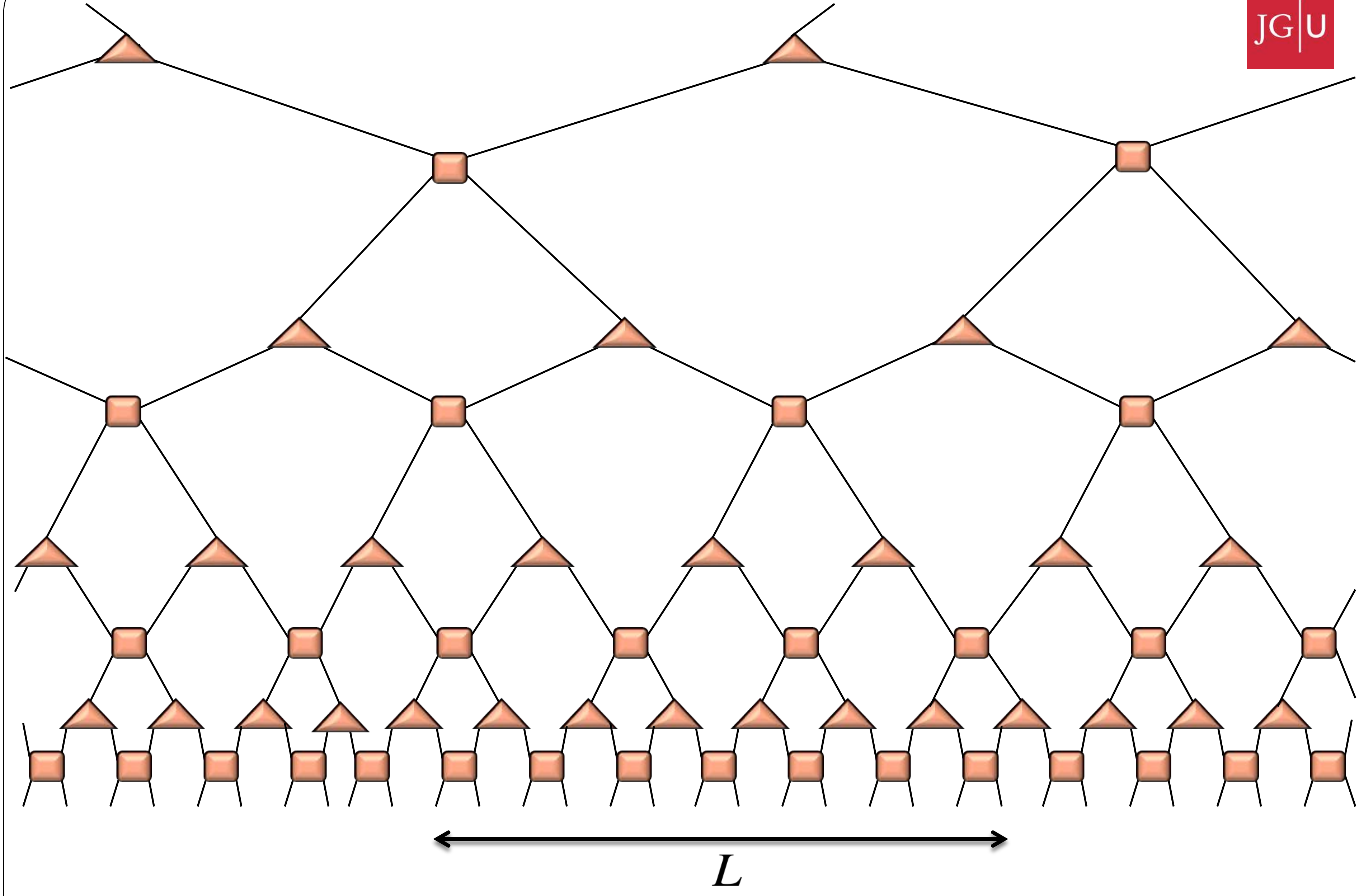
coarse-grain

entangle locally

Extra dimension defines an RG flow: Entanglement Renormalization

Entropy of 1d MERA

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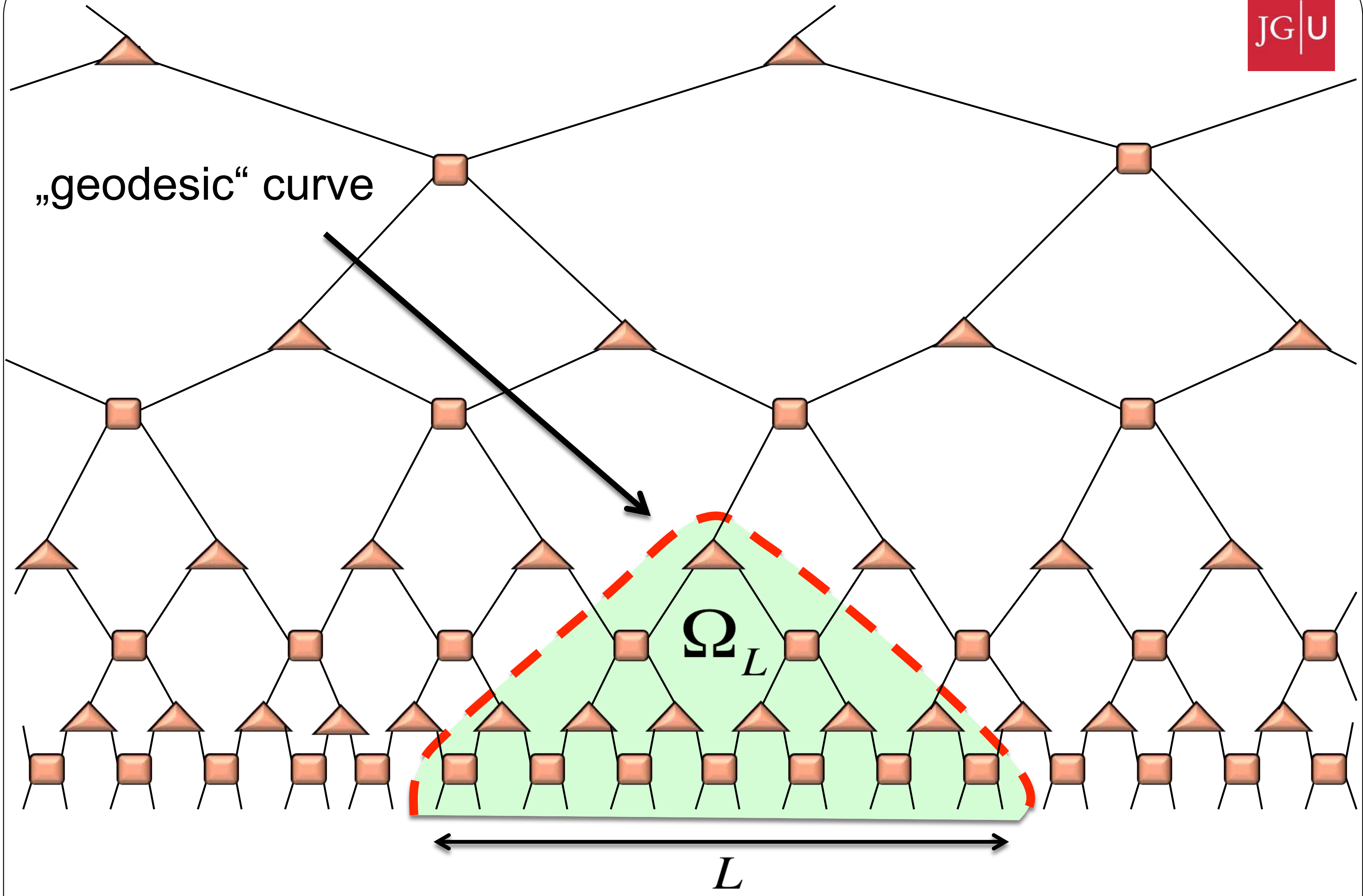
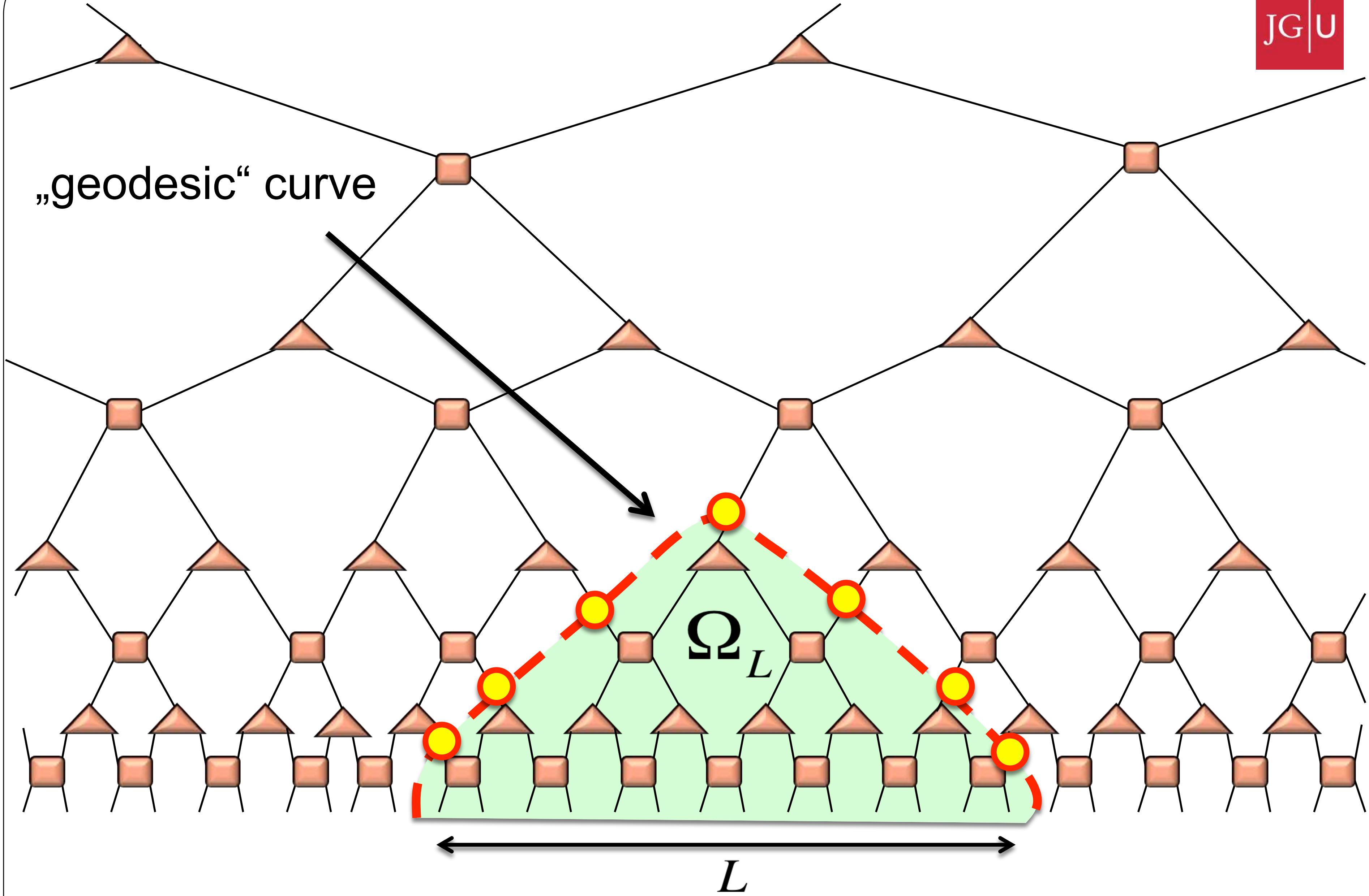
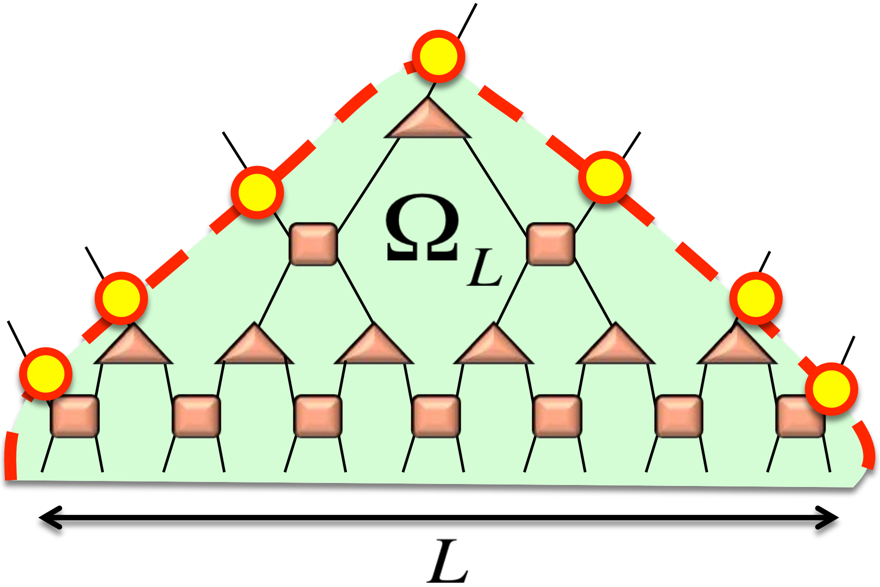


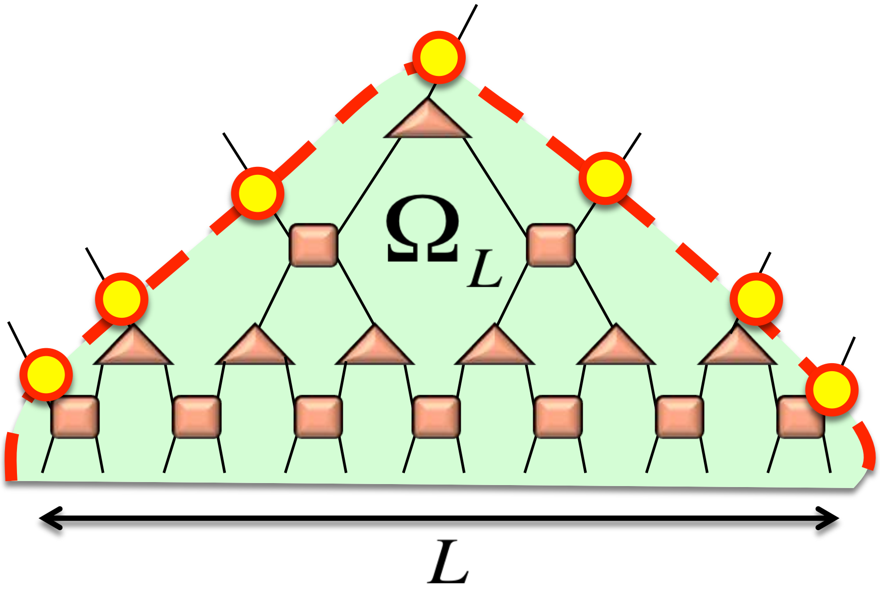
„geodesic“ curve



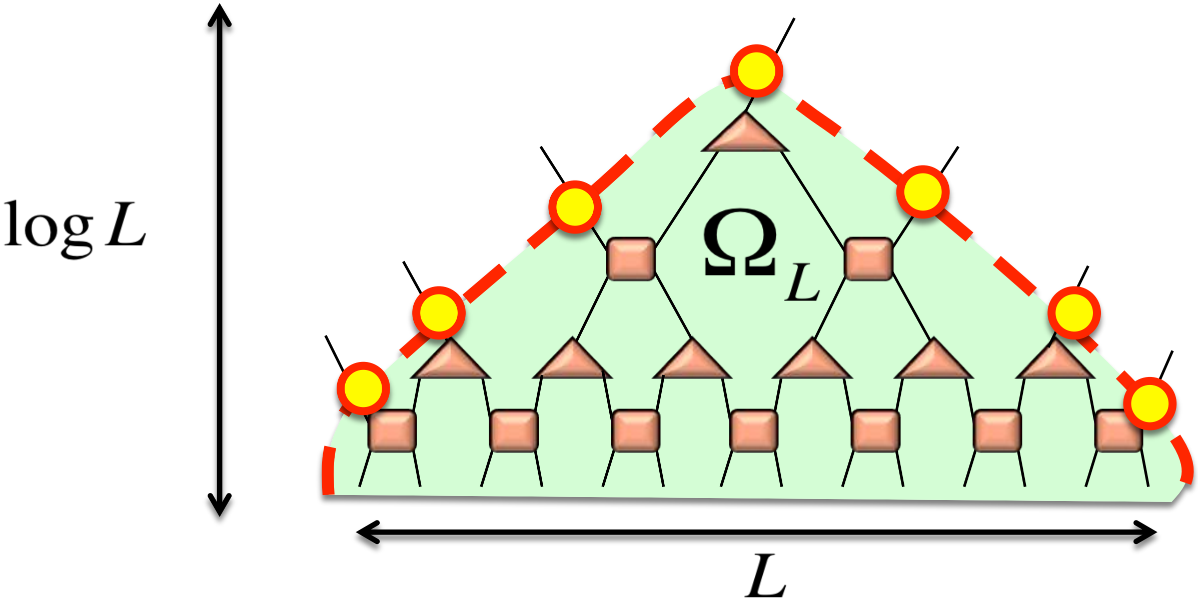
Entanglement as boundary in holographic geometry: $S(L) \leq \log(\chi)\left|\partial\Omega_L\right|$



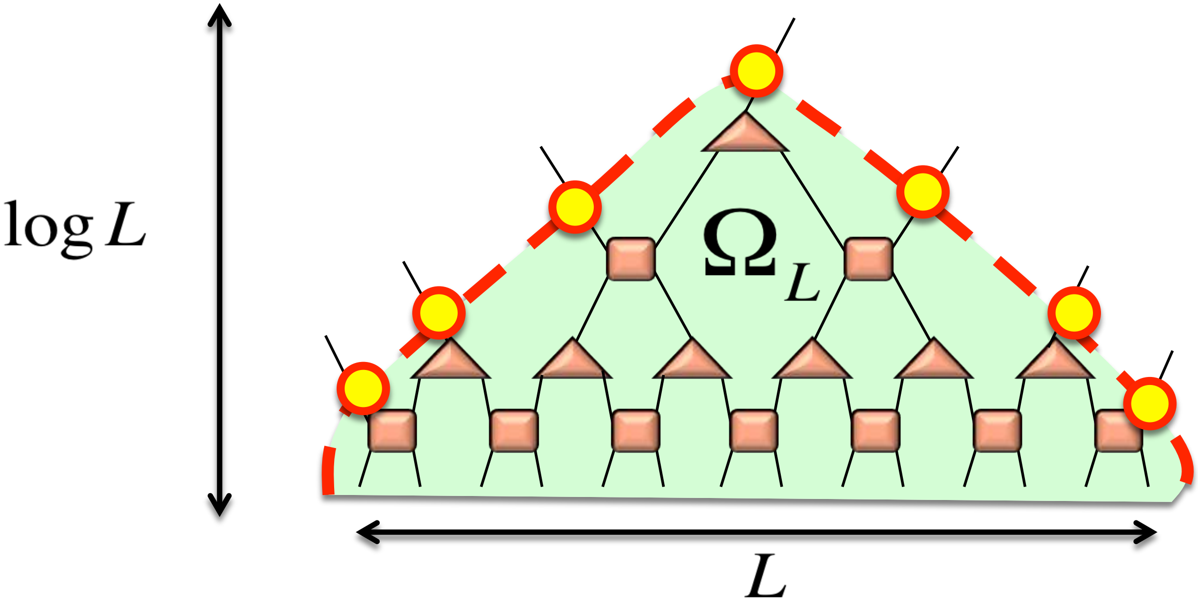
Entanglement as boundary in holographic geometry: $S(L) \leq \log(\chi)\left|\partial\Omega_L\right|$



Entanglement as boundary in holographic geometry: $S(L) \leq \log(\chi)\left|\partial\Omega_L\right|$



Entanglement as boundary in holographic geometry: $S(L) \leq \log(\chi)\left|\partial\Omega_L\right|$



Entanglement as boundary in holographic geometry: $S(L) \leq \log(\chi)\left|\partial\Omega_L\right|$

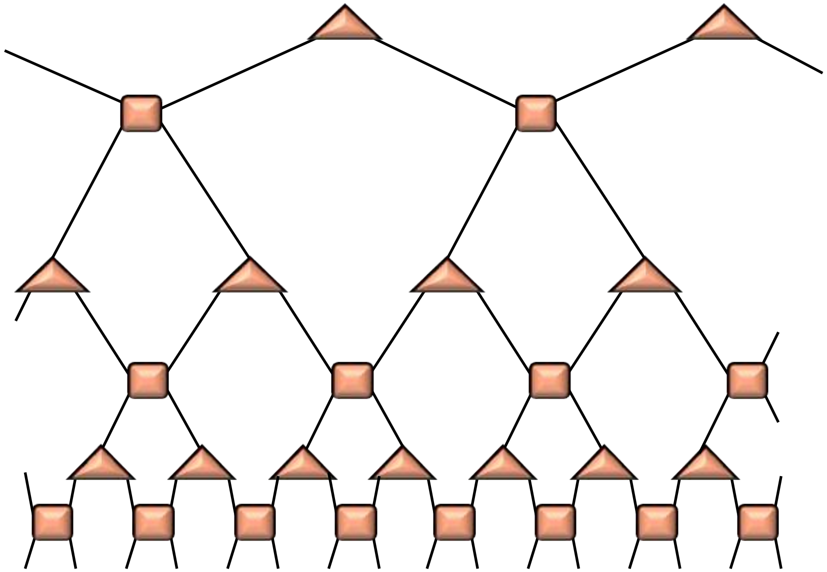
Constant contribution at every layer

1d MERA can produce logarithmic violations to the area-law: $S(L) \approx \log L$

(like 1d critical systems!)

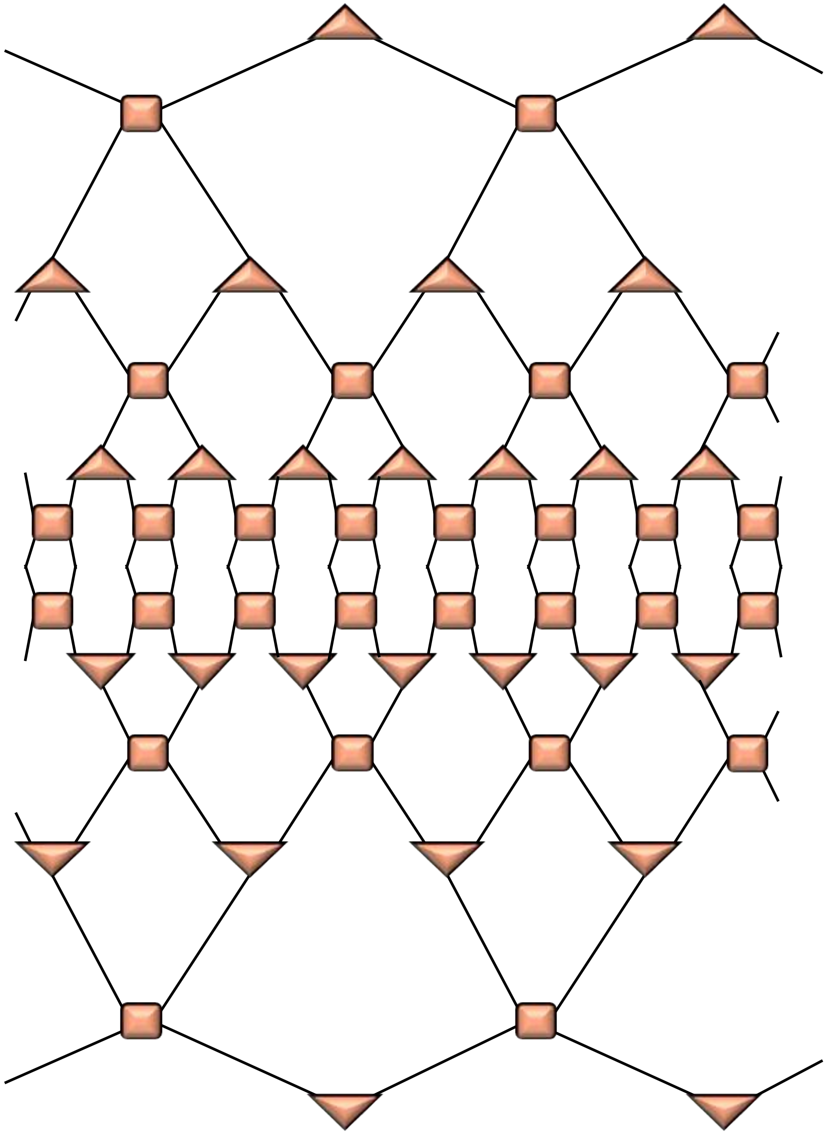
Norm of MERA

Norm of MERA



$|\Psi\rangle$

Norm of MERA



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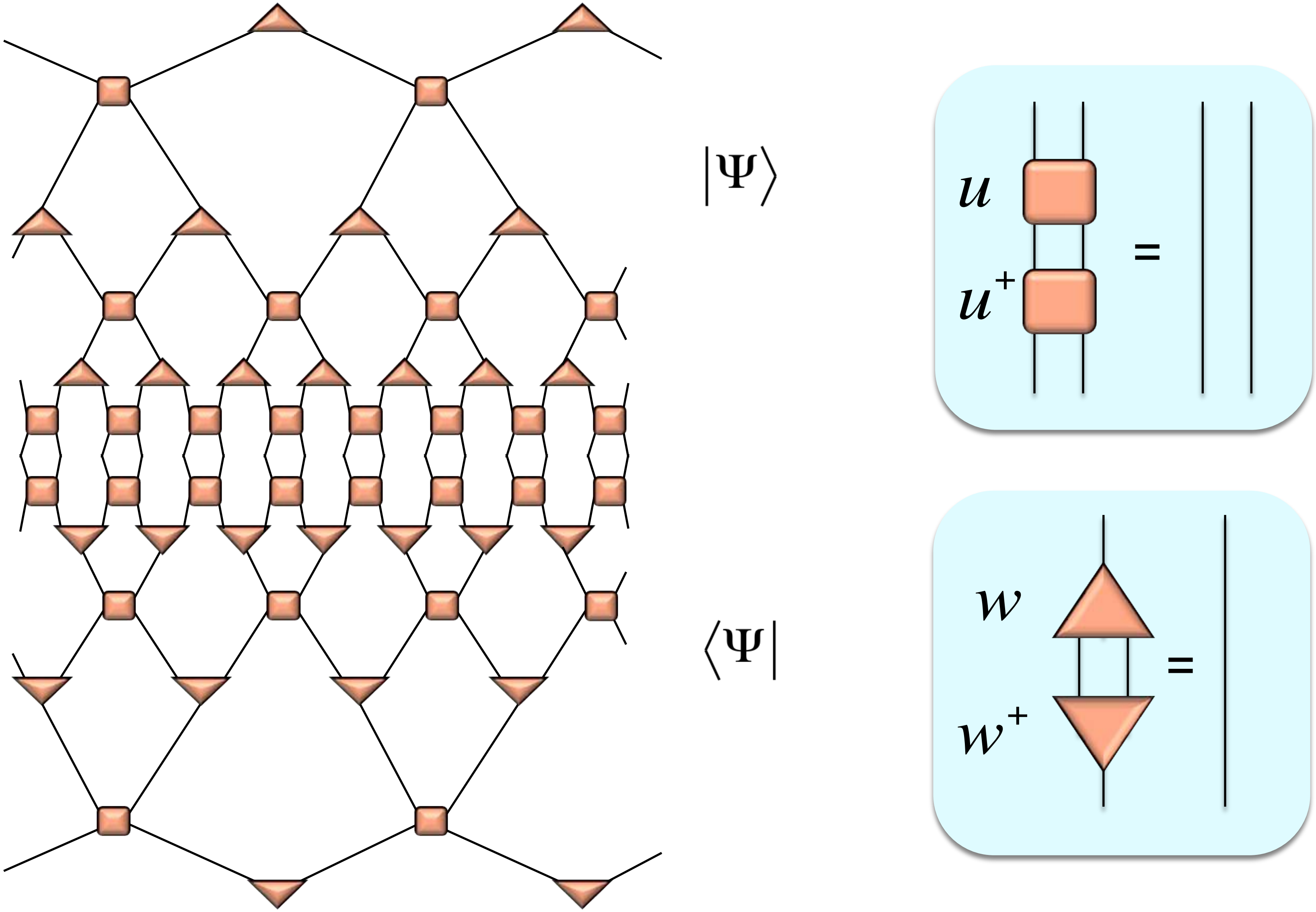
Norm of MERA



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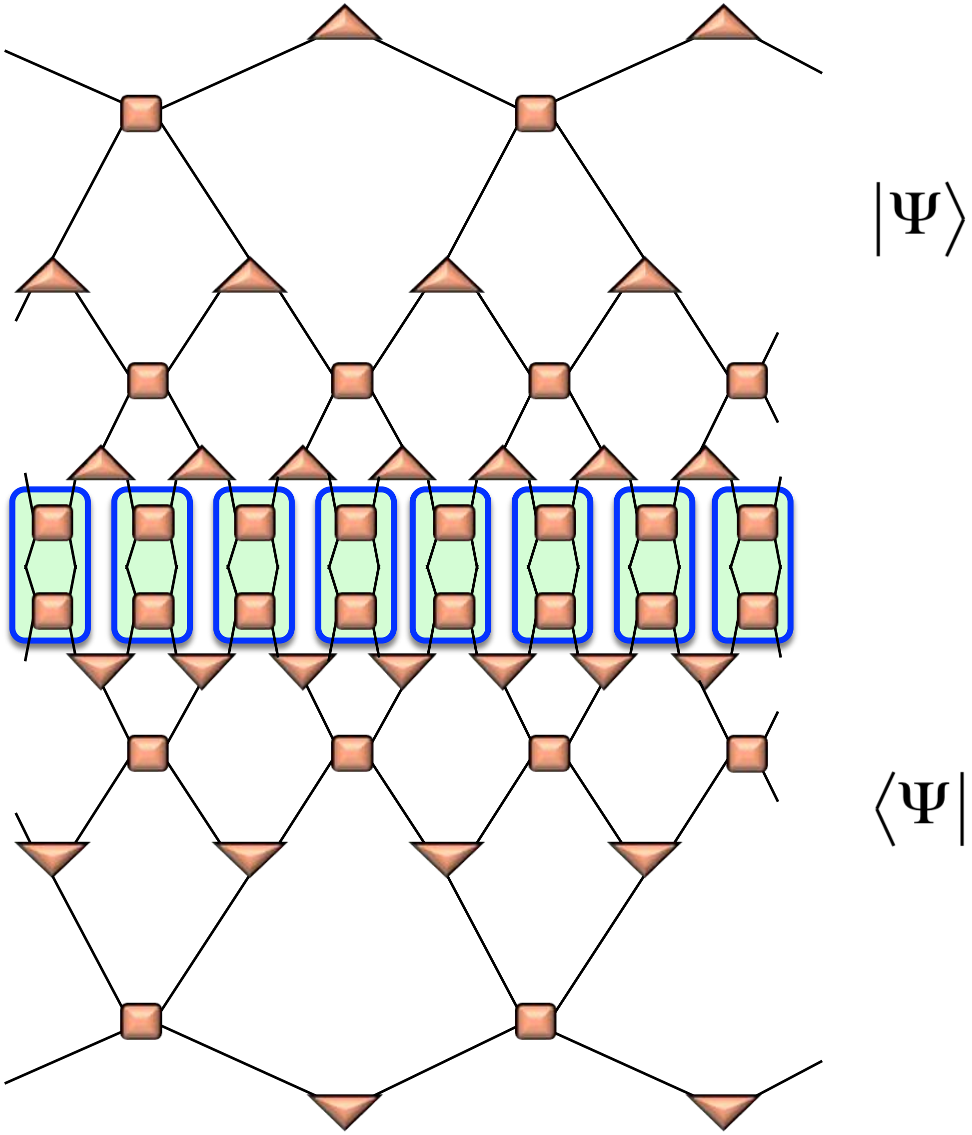

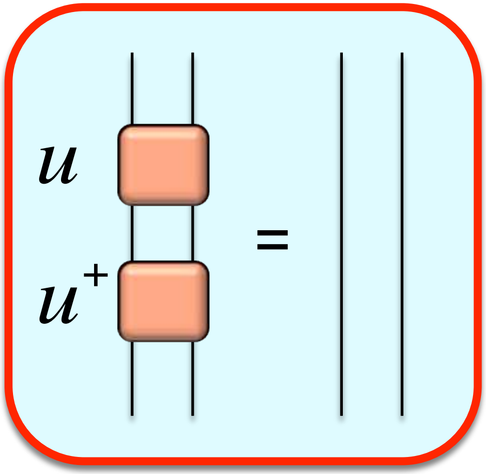

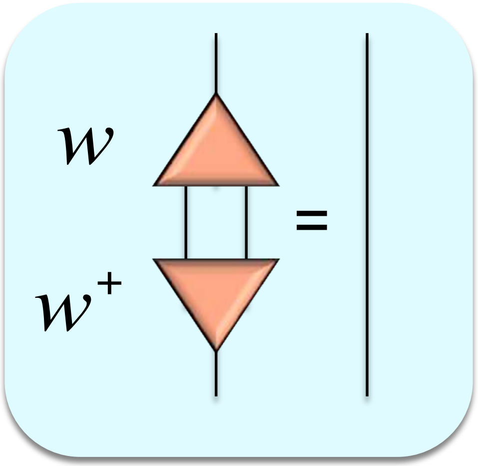

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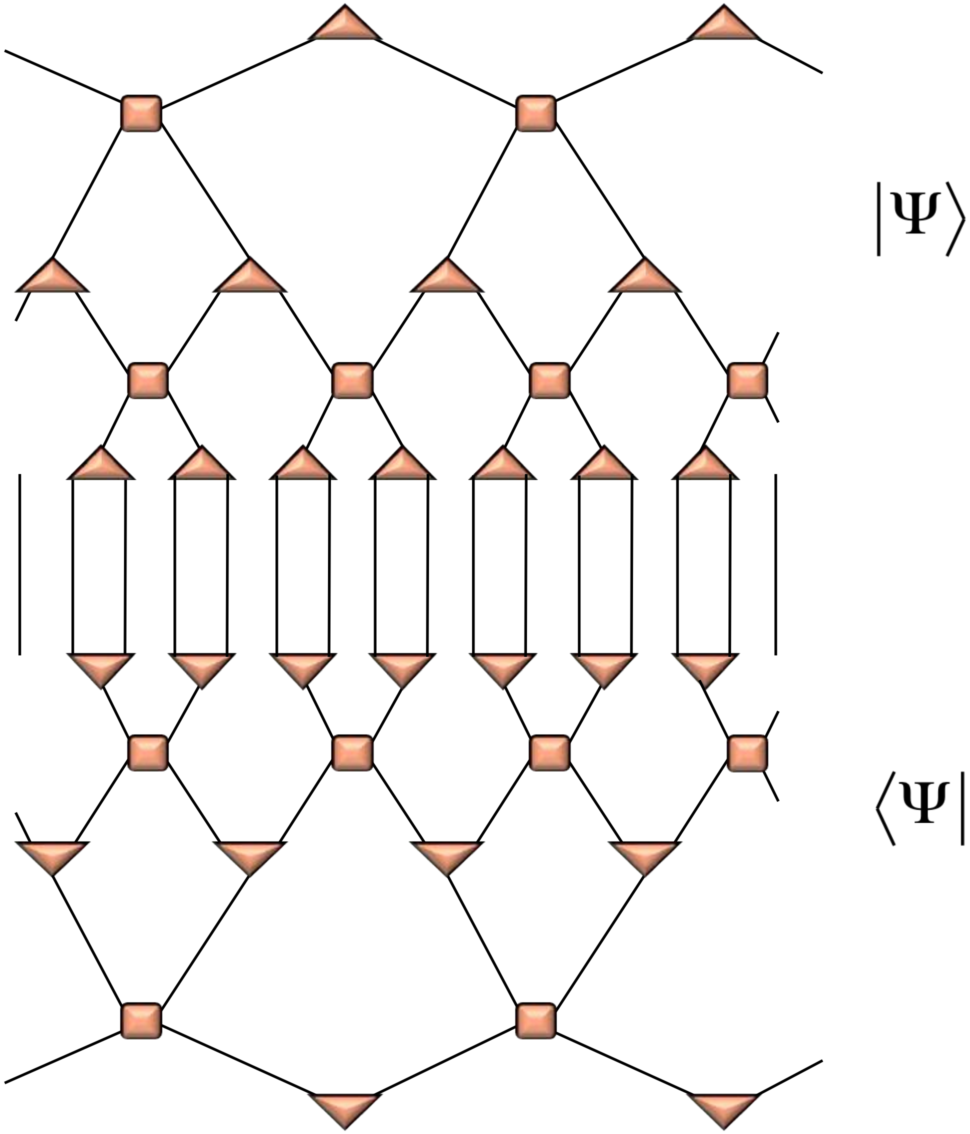

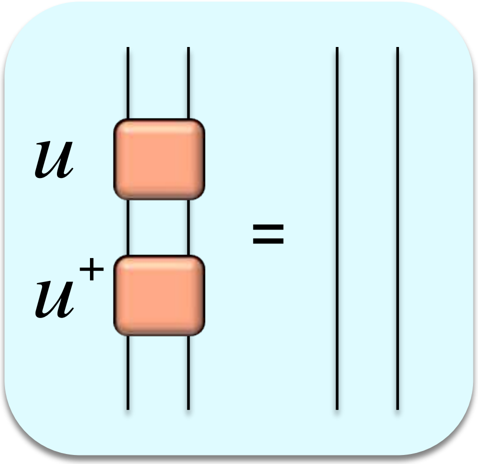

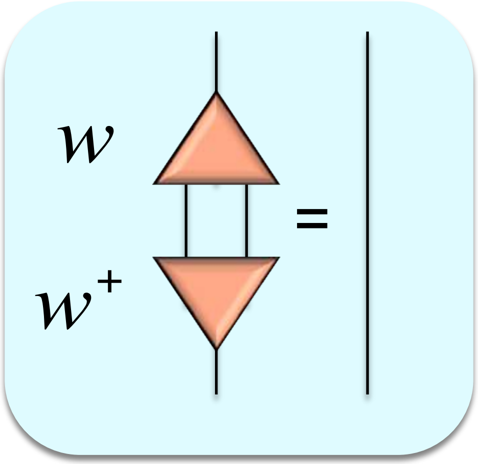

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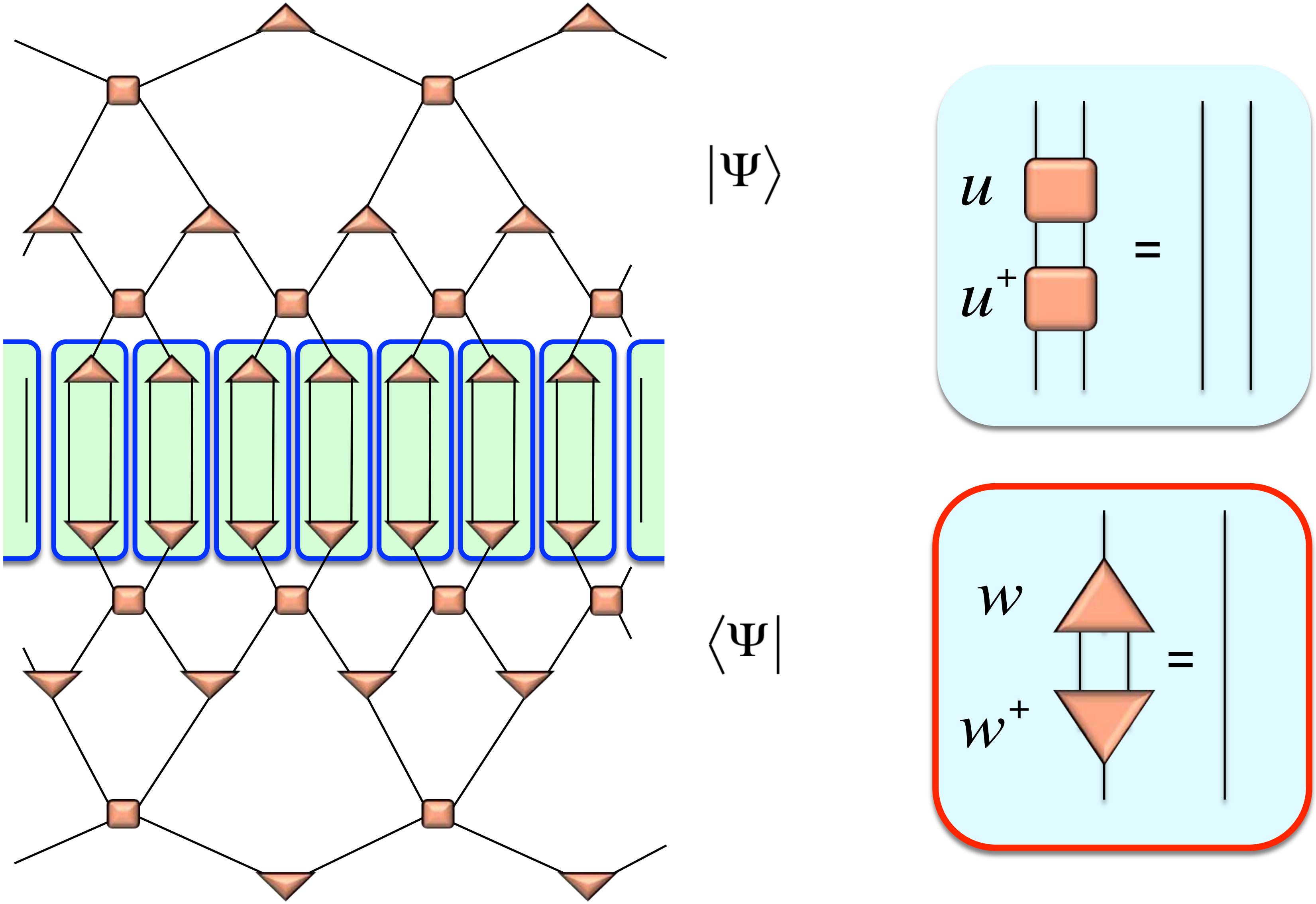

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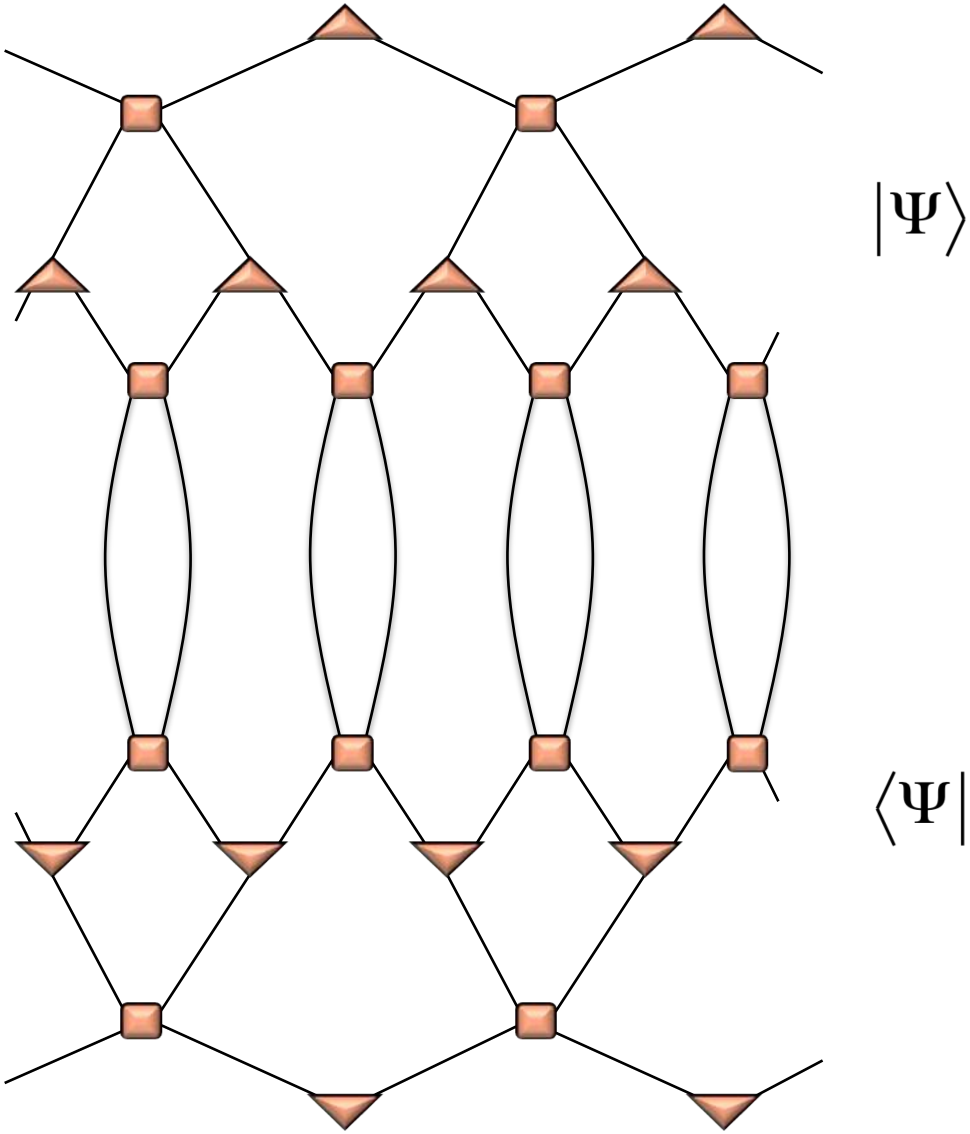

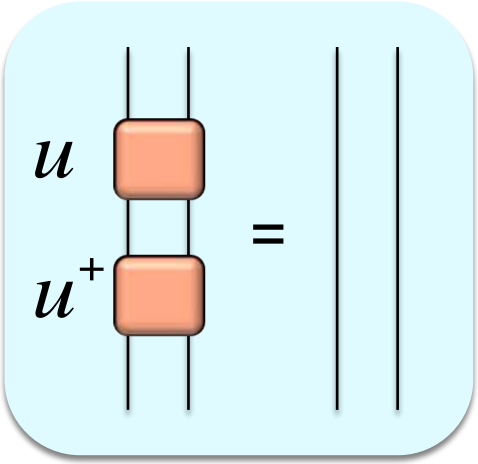

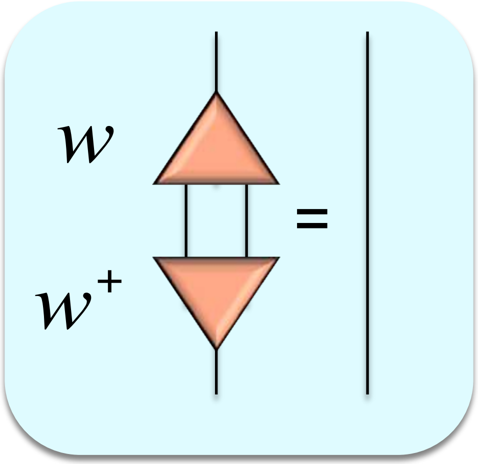

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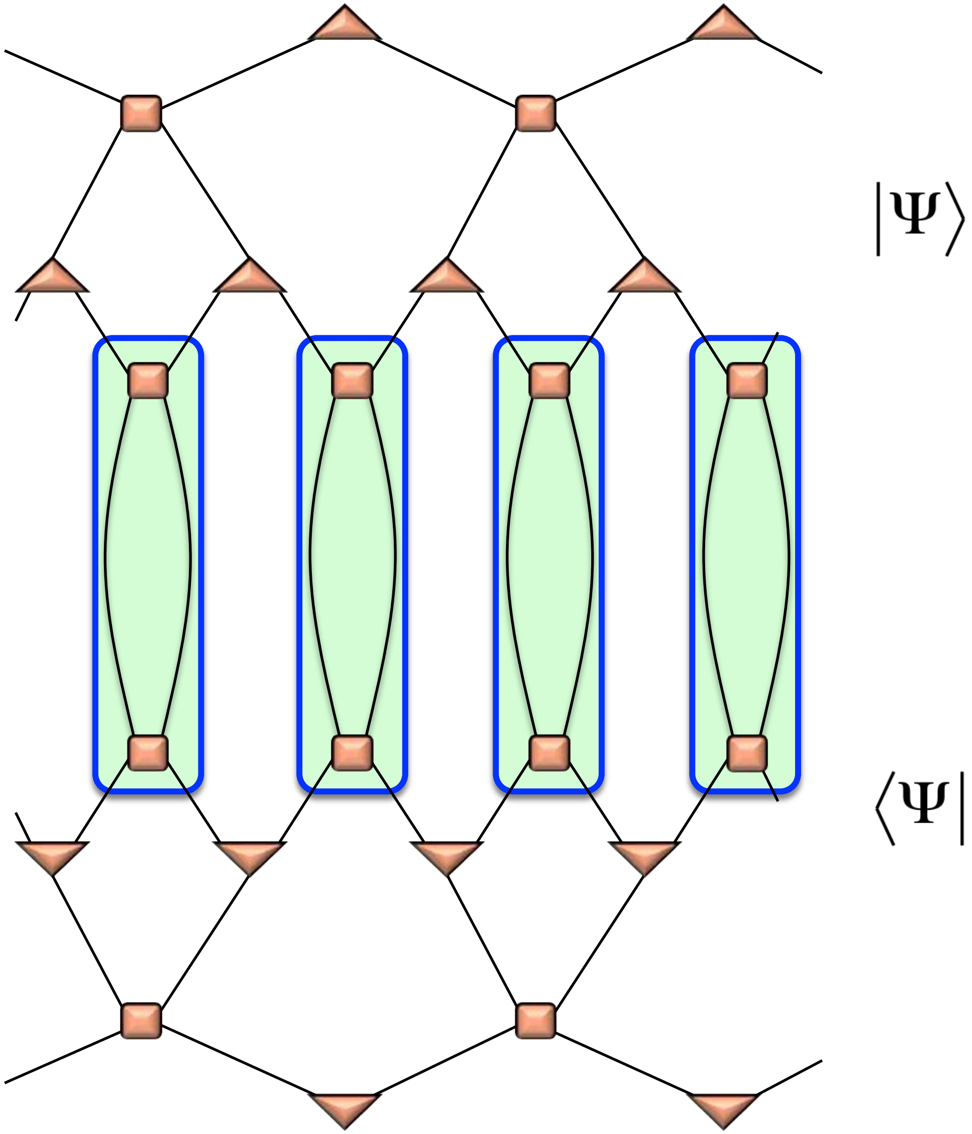

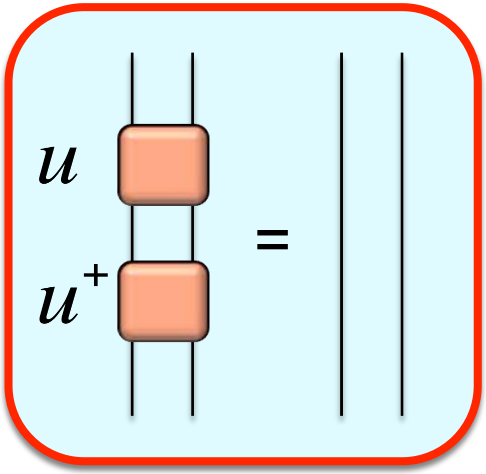

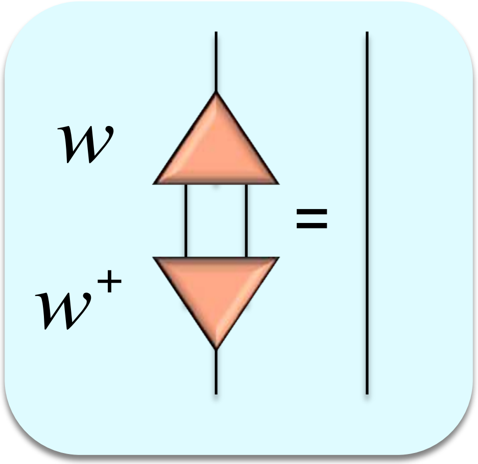

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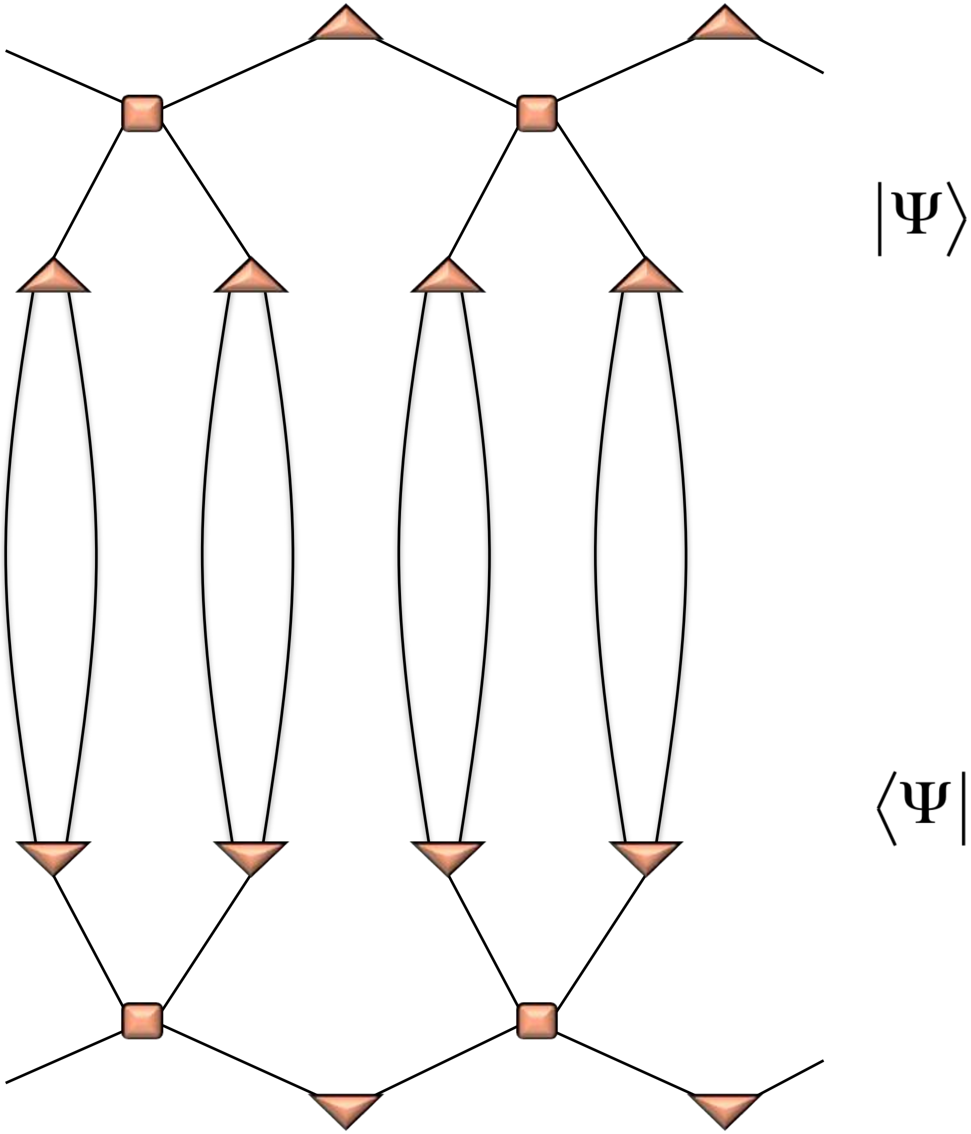

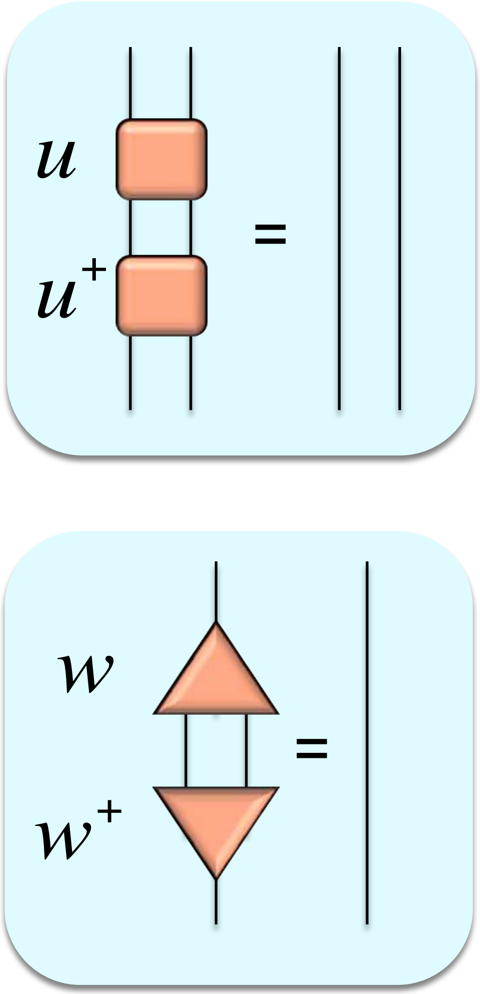

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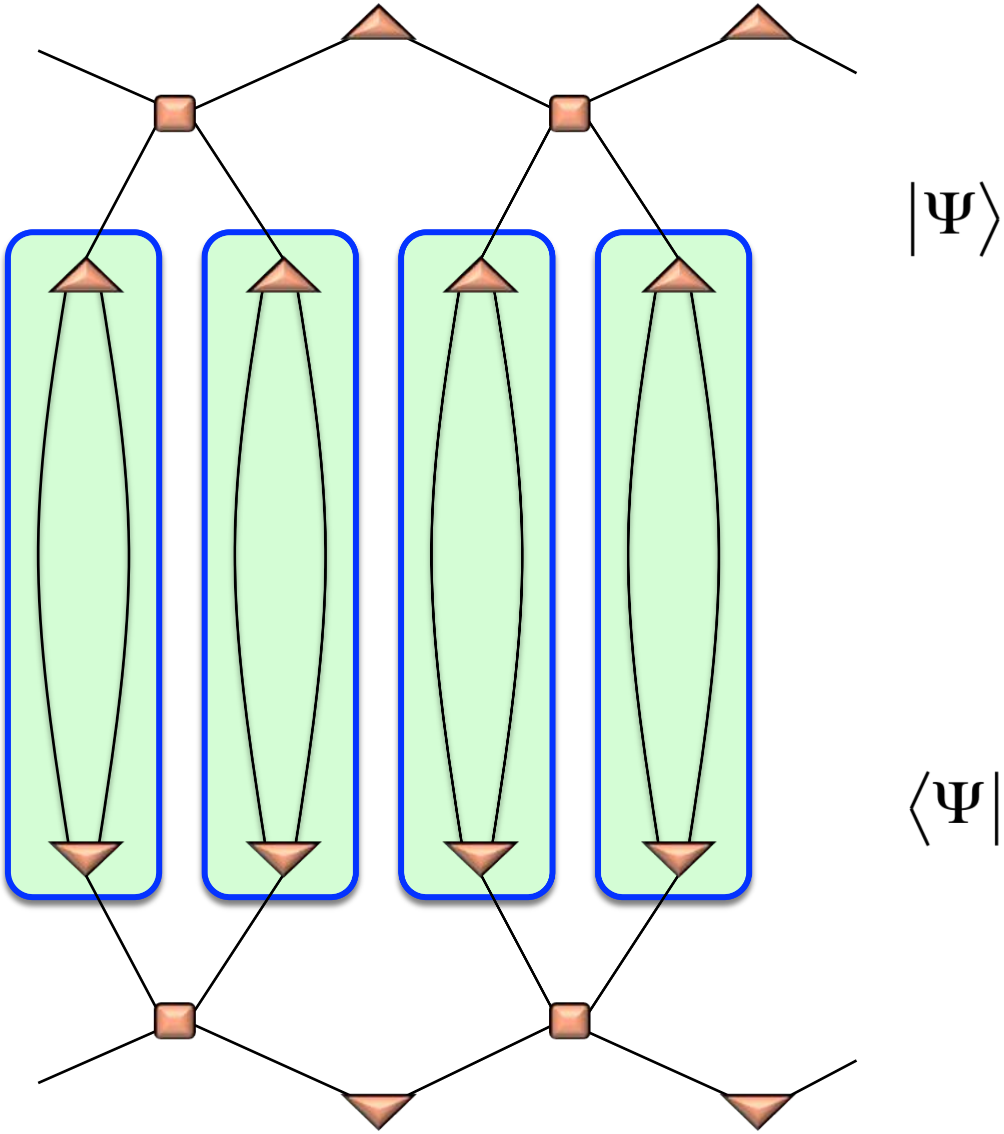

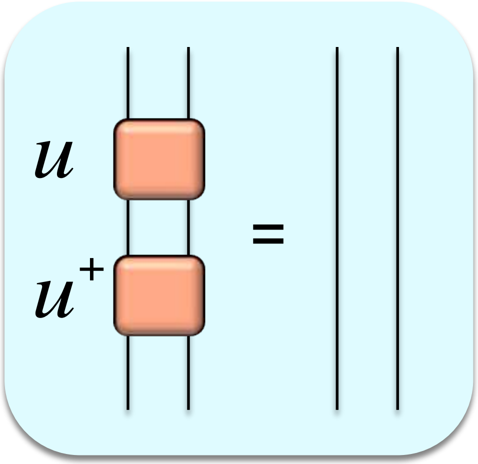

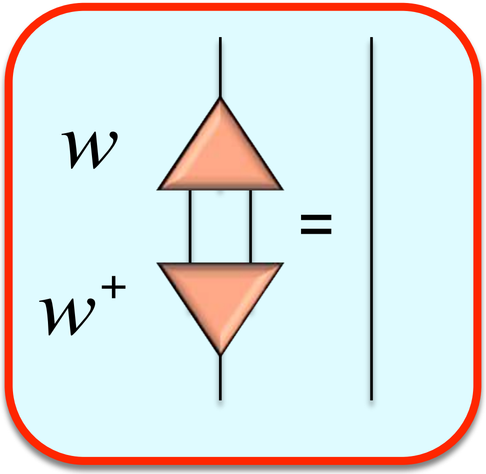

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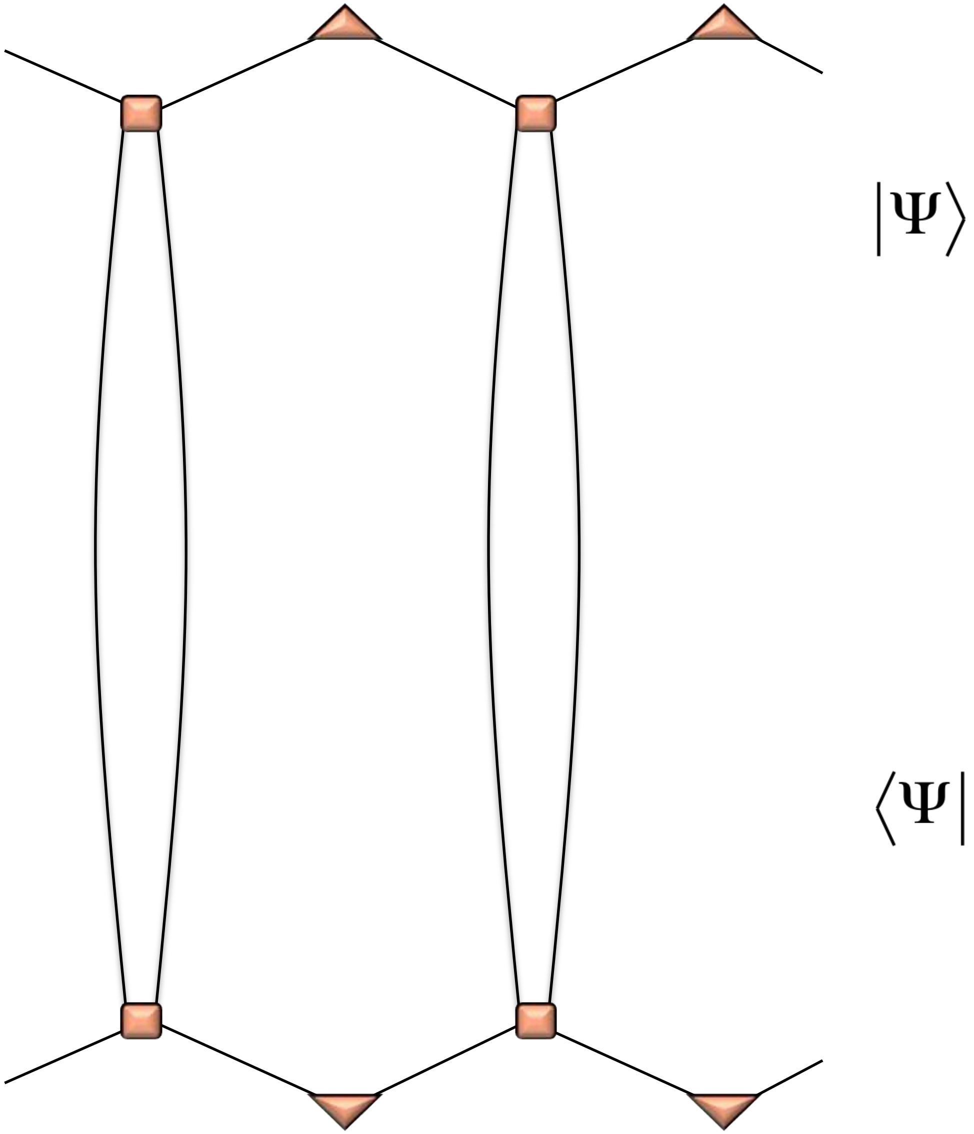

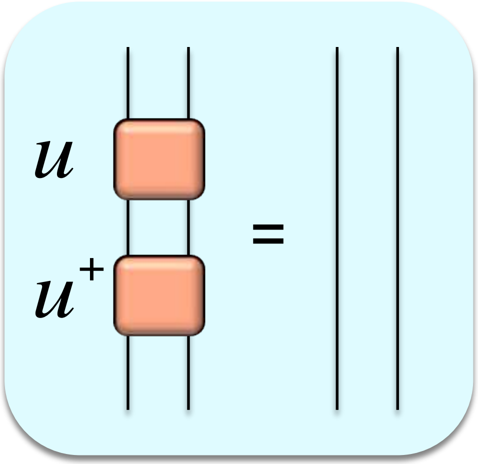

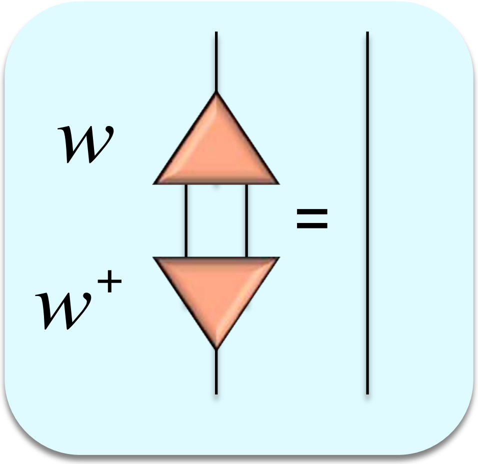

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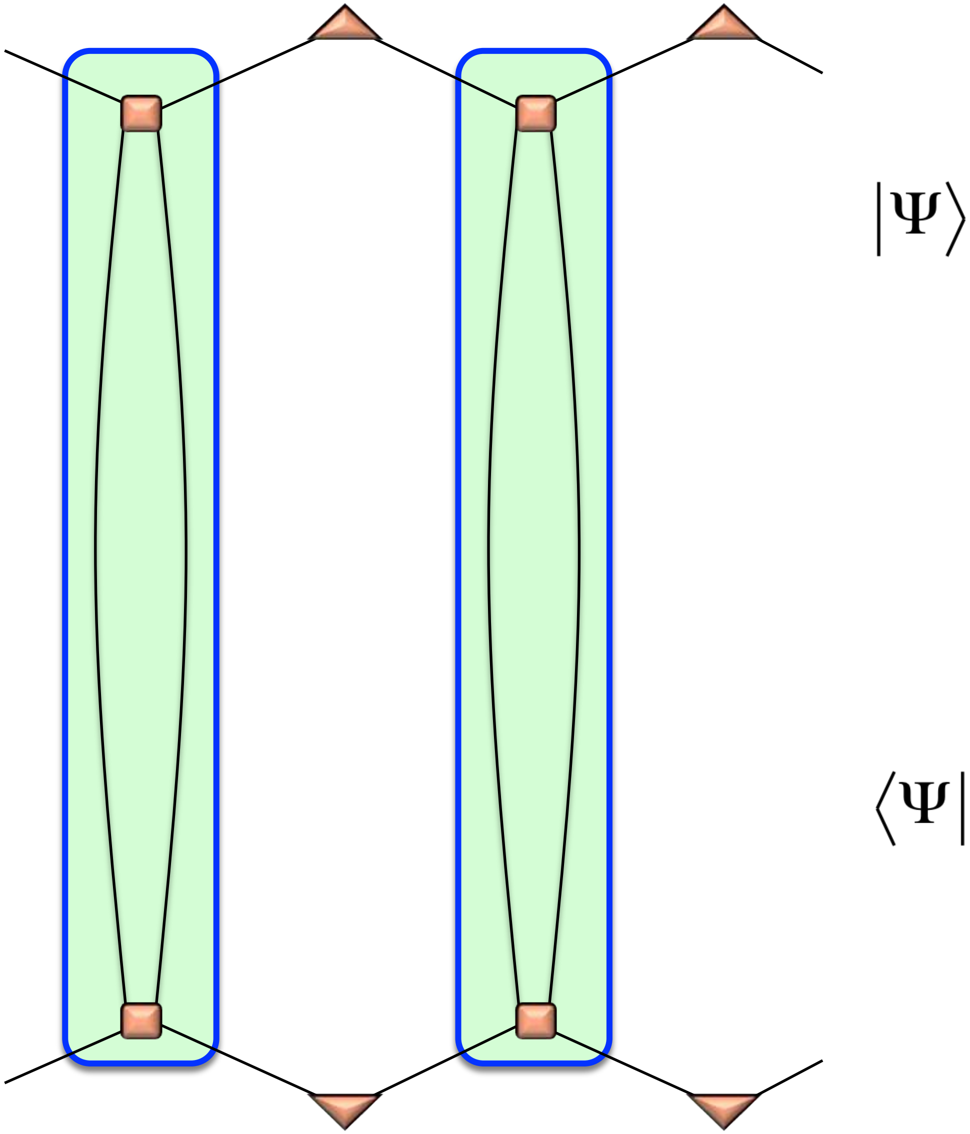

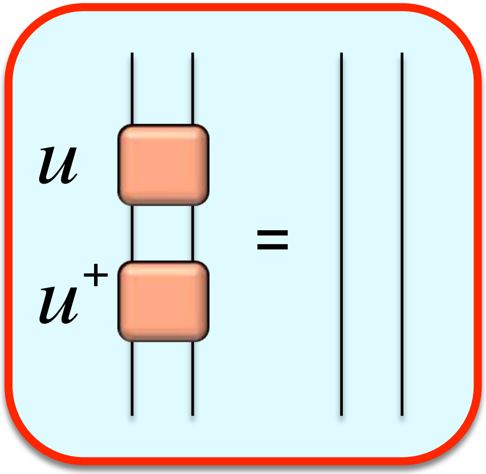

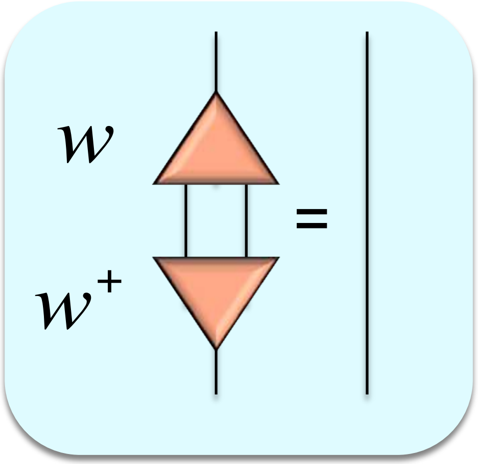

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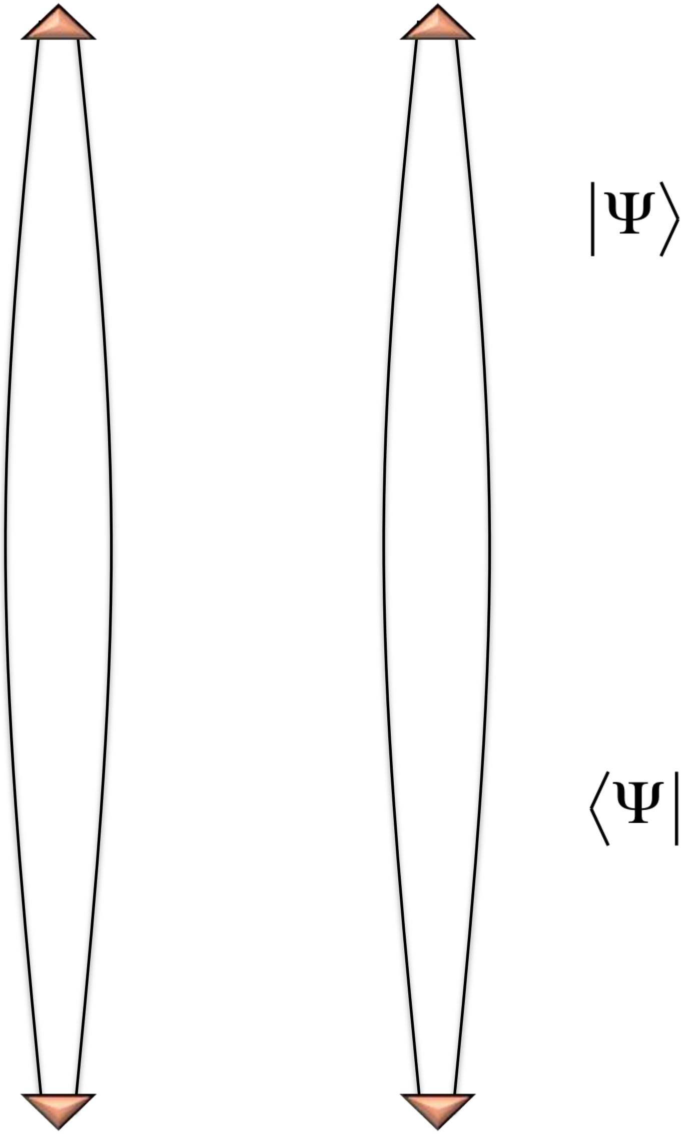

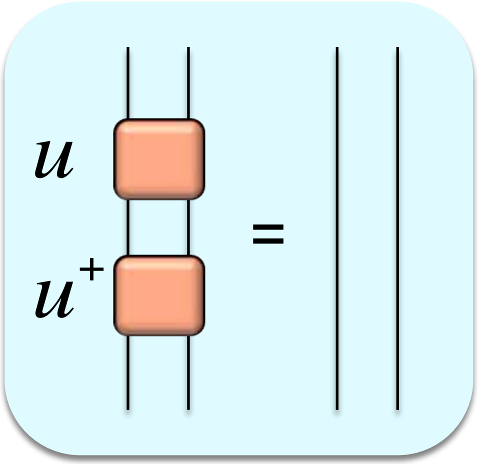

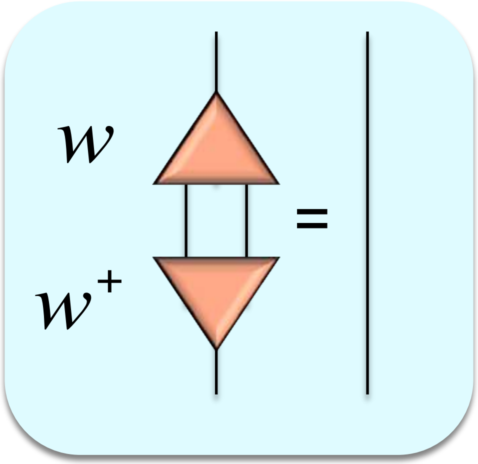

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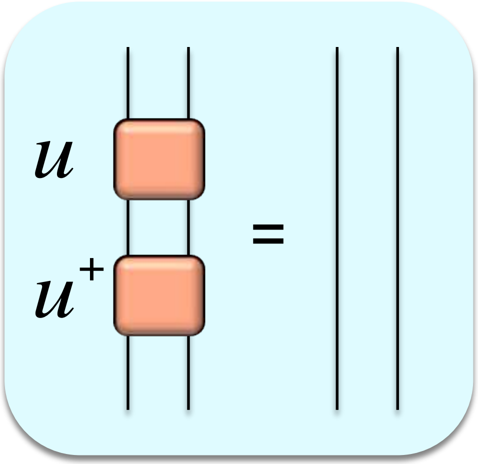
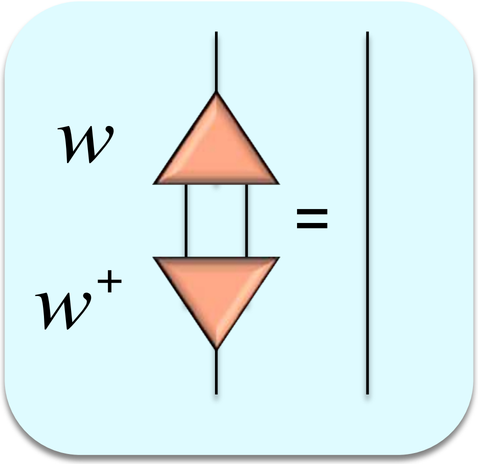
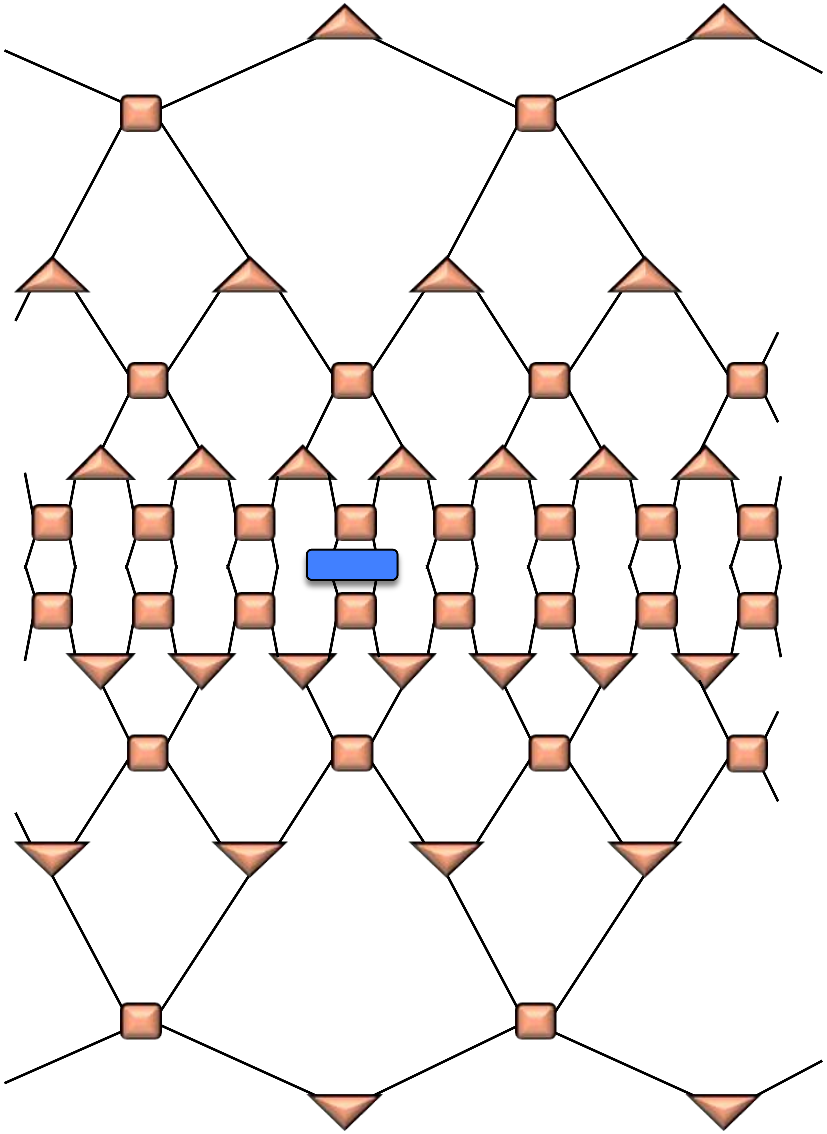




The norm is just the contraction of the top tensors

Expectation values

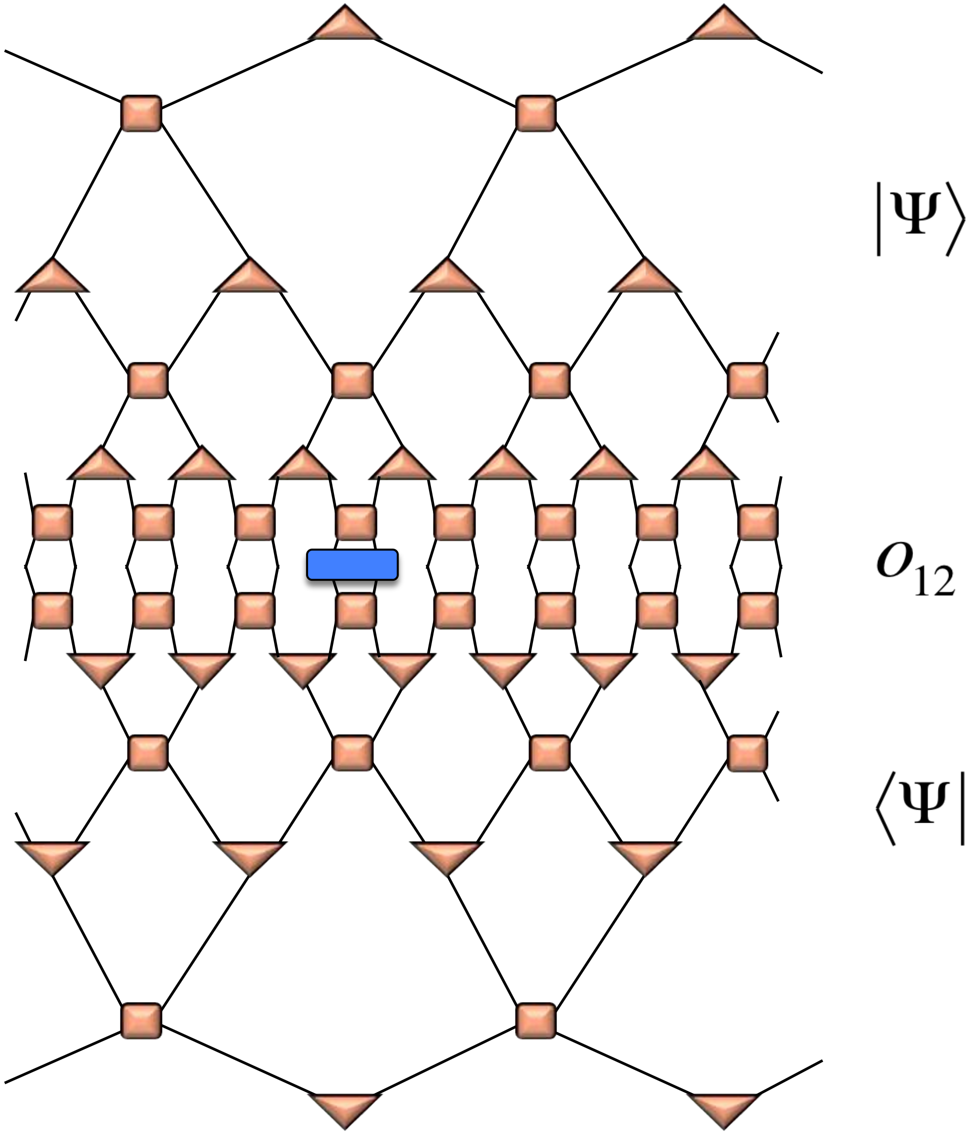


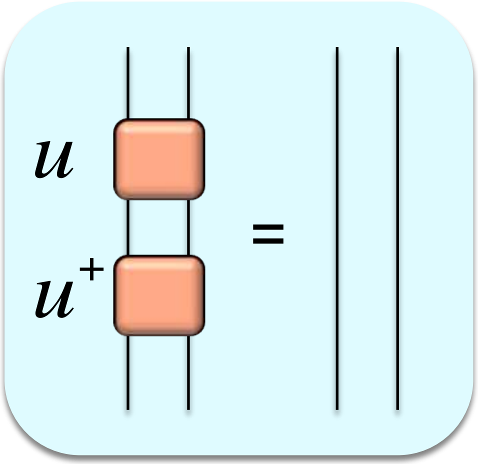
$|\Psi\rangle$

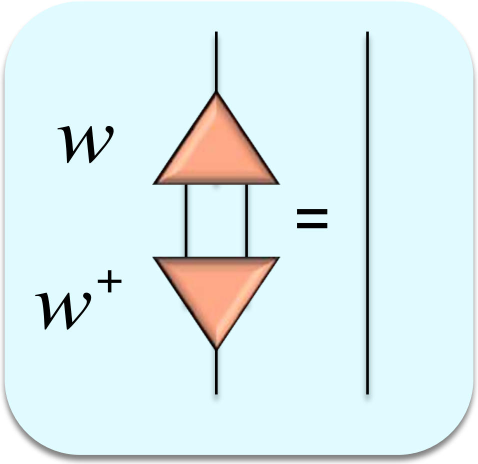
o_{12}

$\langle\Psi|$

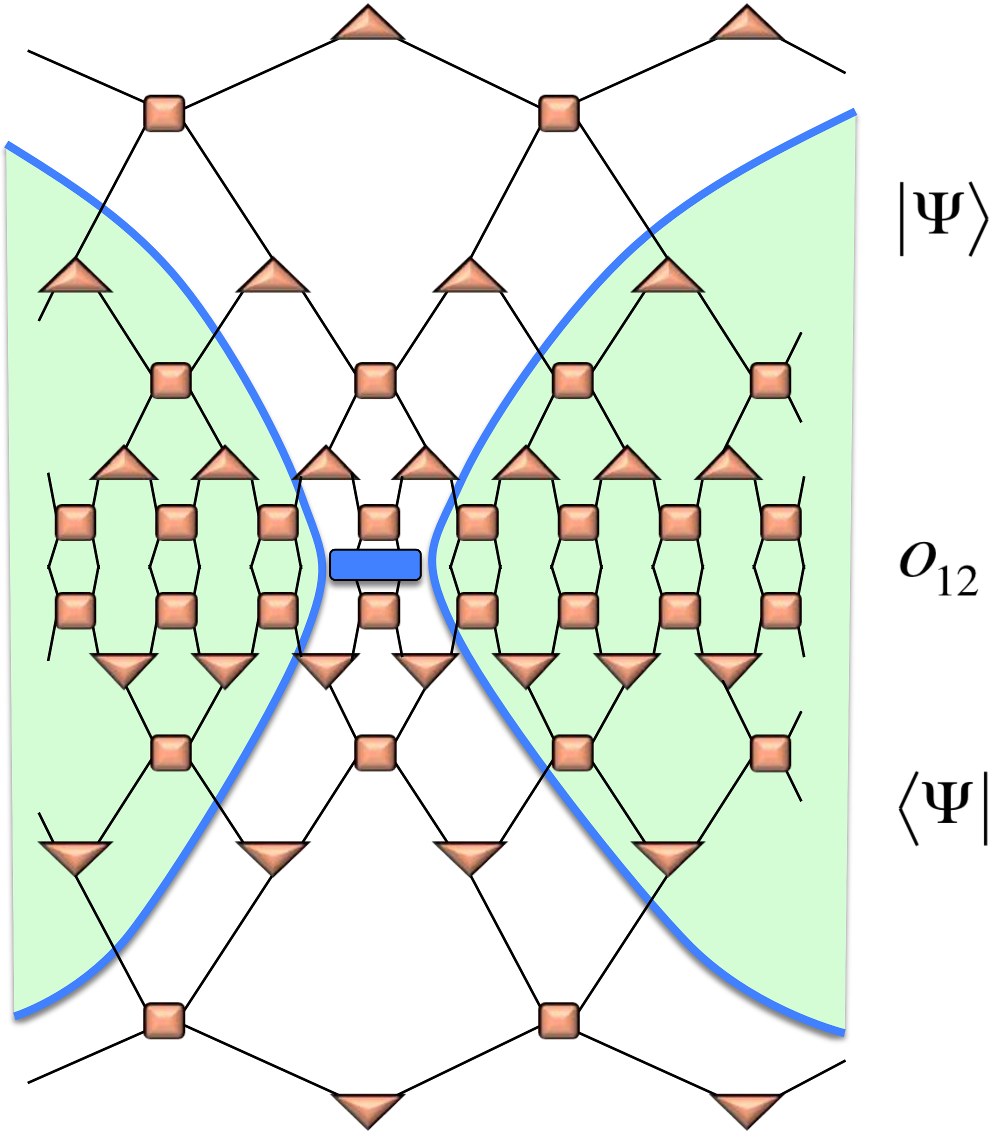
Expectation values

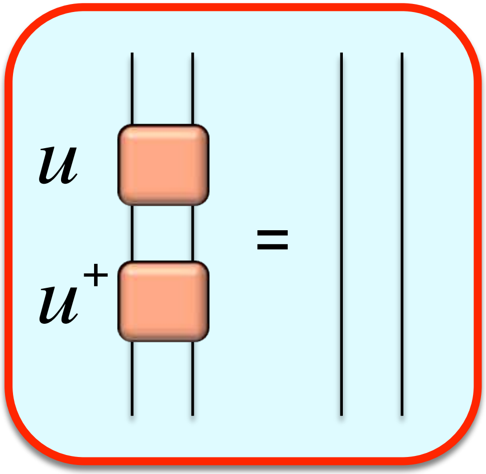


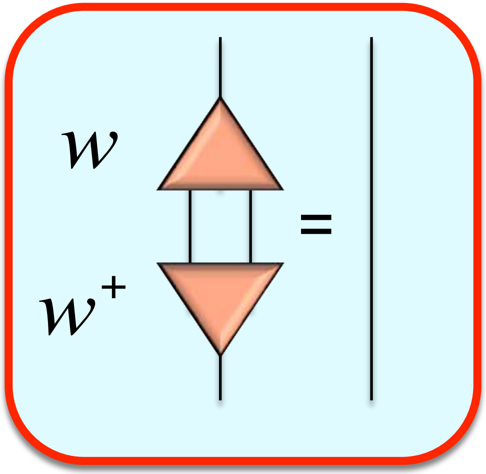




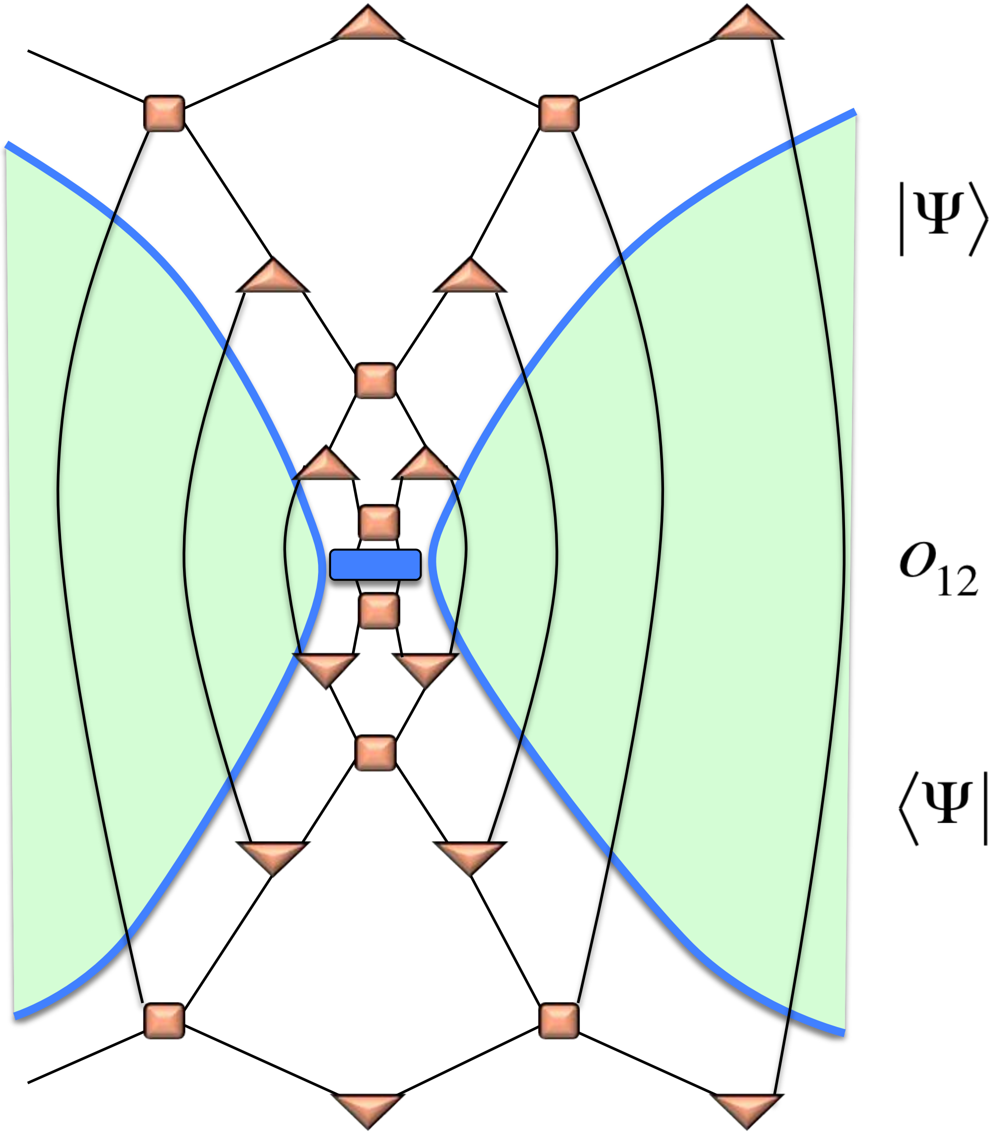
Expectation values

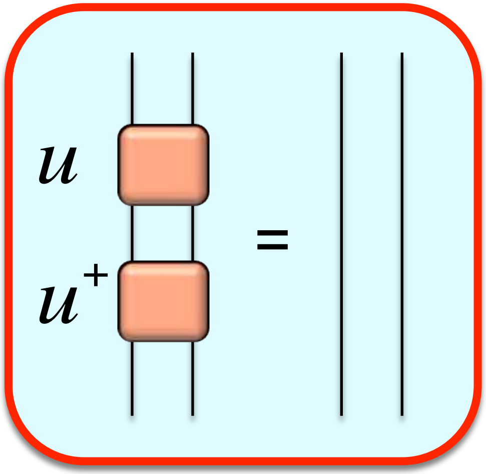


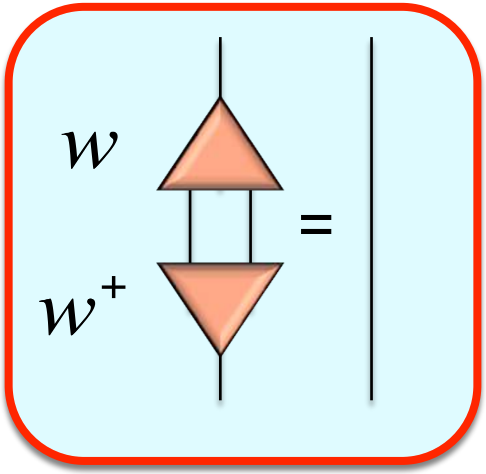




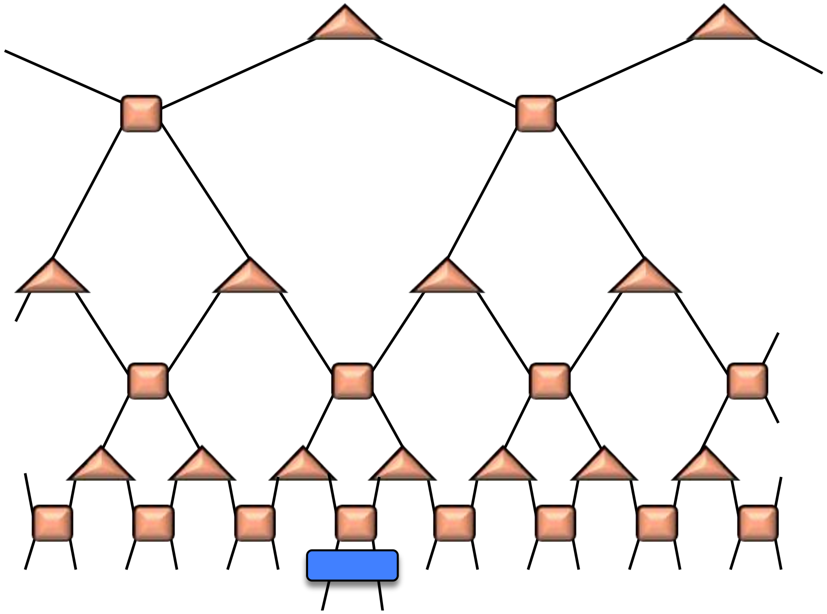
Expectation values







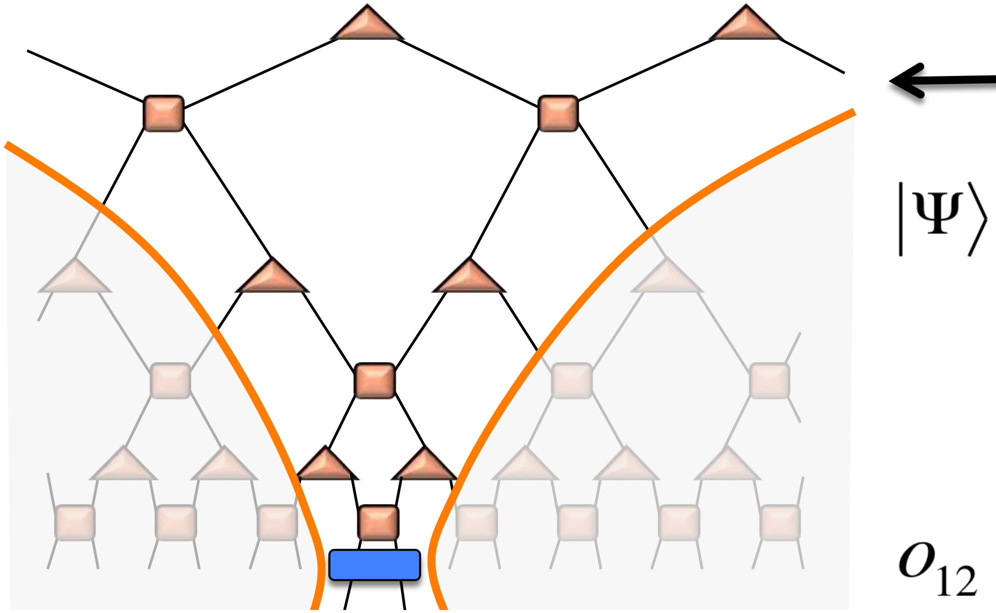
Expectation values



$|\Psi\rangle$

o_{12}

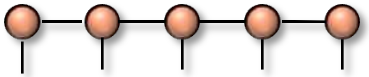
Expectation values

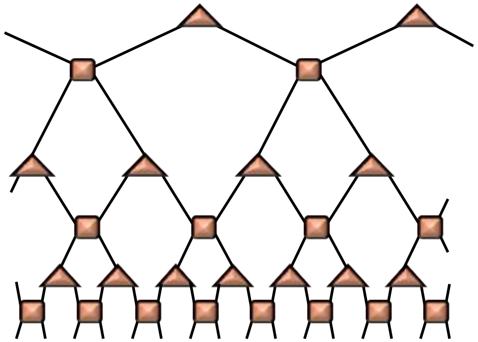


„causal cone“ with bounded width

Only tensors inside of the causal cone contribute to the expectation value

MPS vs 1d MERA





MPS in 1d

1d area law $S(L)=O(1)$

Exact contraction is efficient

Finite correlation length

to/from 1d Hamiltonians

Arbitrary tensors

MERA in 1d

Beyond 1d area law $S(L)=O(\log L)$

Exact contraction is efficient

Finite and infinite correlation lengths

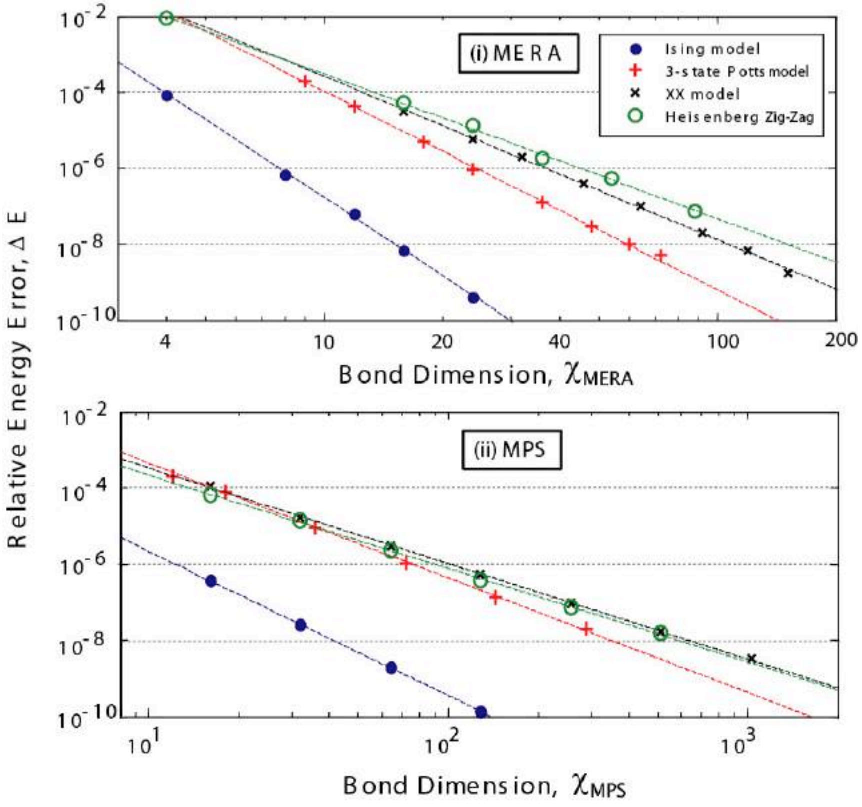
to/from 1d Hamiltonians

Constrained tensors

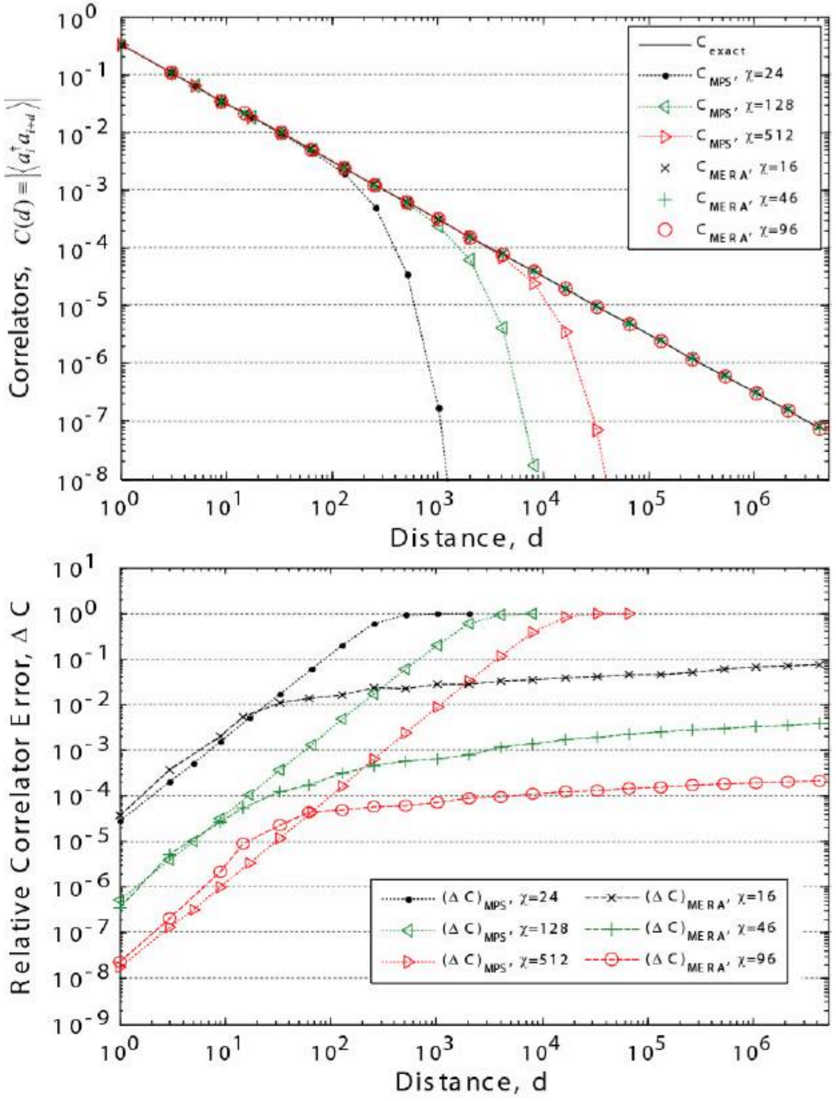
An example:
1d critical systems

G. Evenbly, G. Vidal, in "Strongly Correlated Systems. Numerical Methods", Springer, Vol. 176 (2013)

Critical Energies



Critical XX Correlators



6) Further topics

Román Orús

University of Mainz

November 2nd 2017

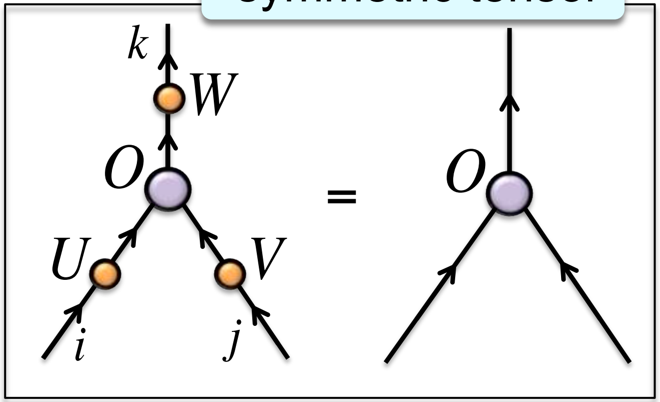
TNs with symmetries

e.g., S. Singh, R. N. C. Pfeifer, G. Vidal, PRA 82, 050301 (2010)

Symmetric tensors and Schur’s lemma

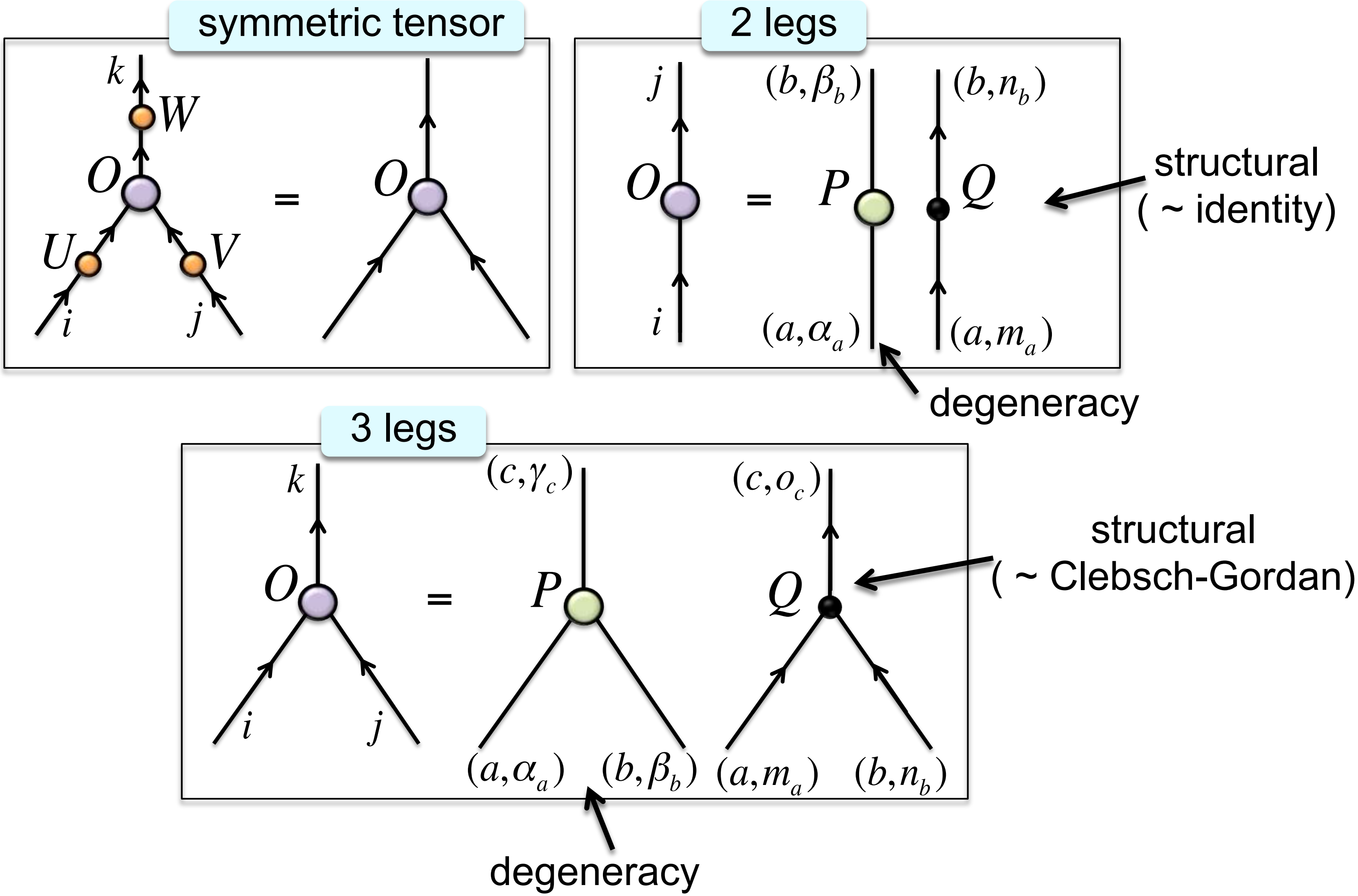
e.g., S. Singh, R. N. C. Pfeifer, G. Vidal, PRA 82, 050301 (2010)

symmetric tensor



Symmetric tensors and Schur's lemma

e.g., S. Singh, R. N. C. Pfeifer, G. Vidal, PRA 82, 050301 (2010)



symmetric tensor

2 legs

structural
(~ identity)

degeneracy

3 legs

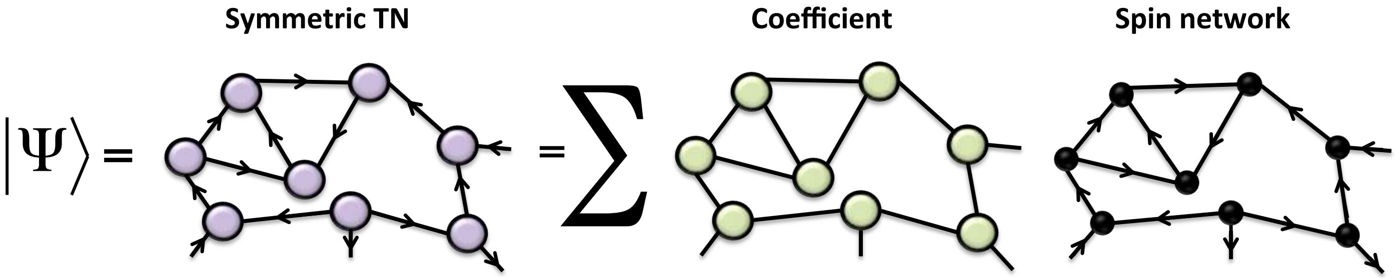
structural
(~ Clebsch-Gordan)

degeneracy

Structural part depends only on the group properties (intertwiners)

Emergent spin networks

e.g., S. Singh, R. N. C. Pfeifer, G. Vidal, PRA 82, 050301 (2010)



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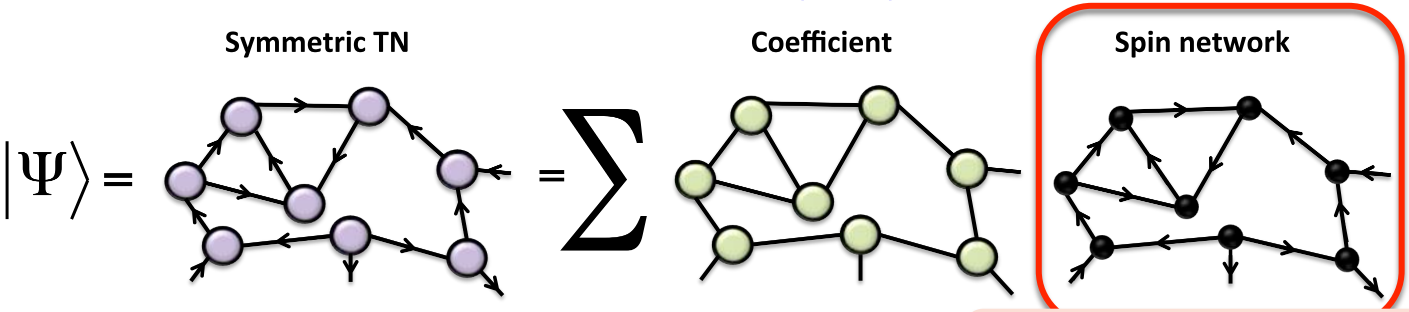
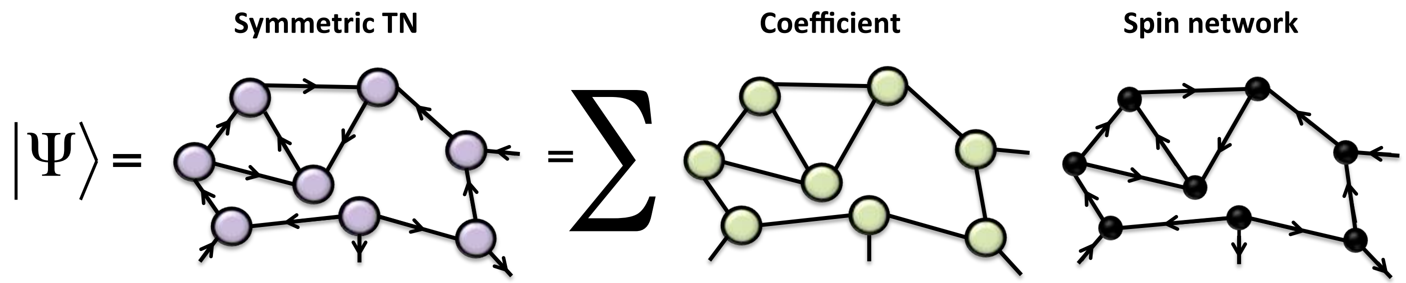
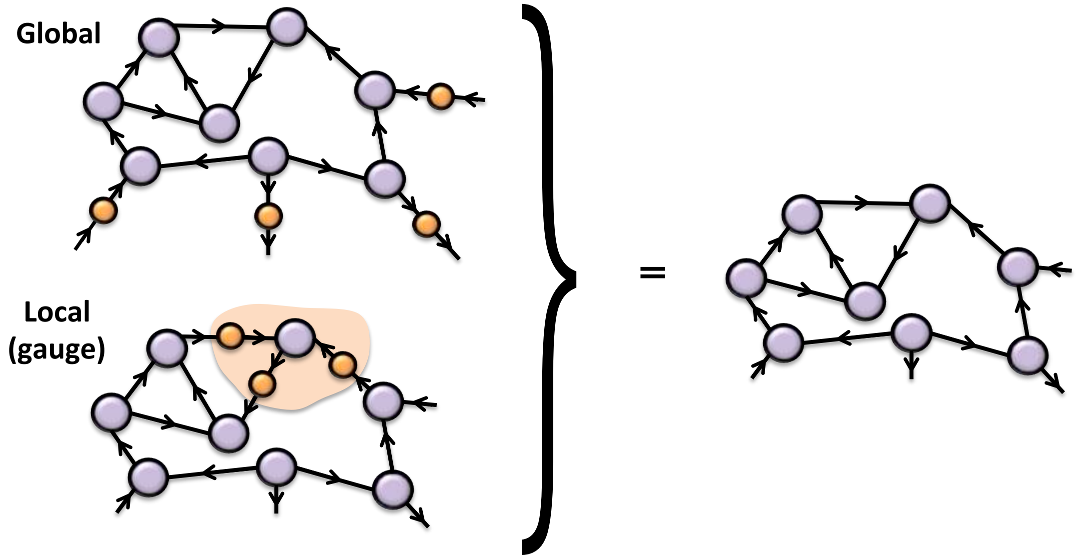


States of quantum geometry in loop quantum gravity...

Emergent spin networks

e.g., S. Singh, R. N. C. Pfeifer, G. Vidal, PRA 82, 050301 (2010)



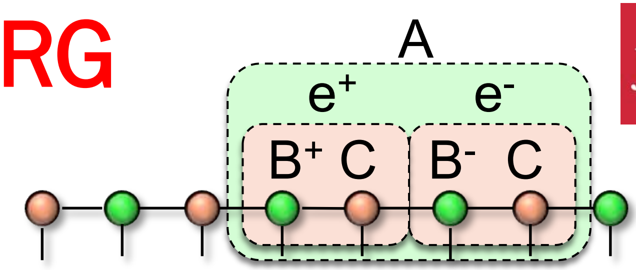


Global and gauge symmetries are handled naturally

Concerning numerics: HUGE computational savings, e.g., SU(2)-DMRG

Example: U(1)-Gauge iDMRG

Gauge-invariant iDMRG simulations



fermion → $(B_n)^{s_n}_{(q,\alpha_q)(r,\beta_r)} = (b_{n,q})^{s_n}_{\alpha_q,\beta_r}\delta_{q+(s_n+(-1)^n)/2,r},$

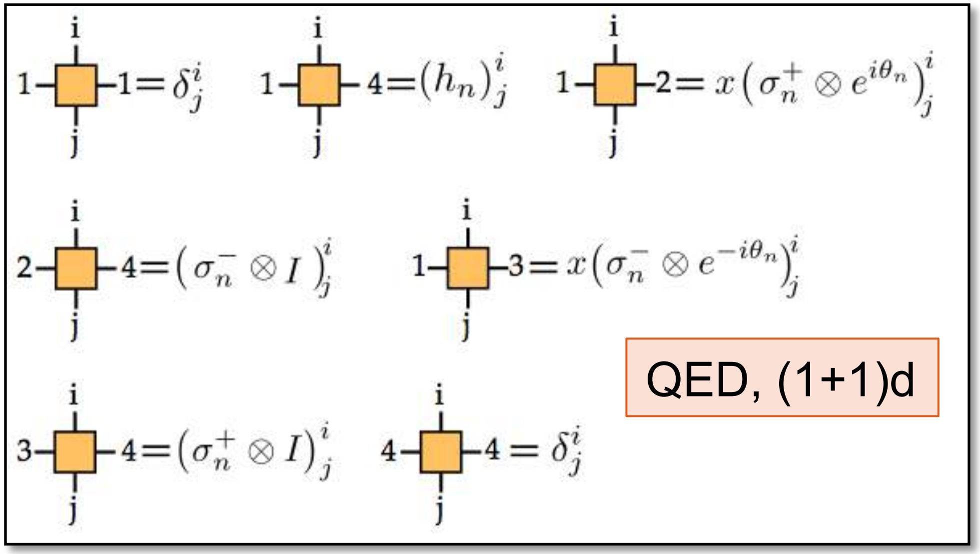
gauge boson → $(C_n)^{p_n}_{(q,\alpha_q)(r,\beta_r)} = (c_n)^{p_n}_{\alpha_q,\beta_r}\delta_{q,p_n}\delta_{r,p_n}.$

degeneracy structural

B. Buyens, K. Van Acoleyen, J. Haegeman, F. Verstraete, PoS(LATTICE2014)308.

q,r indices for U(1) gauge symmetry sector (structural)

Greek degeneracy indices



1–1 $= \delta^i_j$

1–4 $= (h_n)^i_j$

1–2 $= x\left(\sigma^+_n \otimes e^{i\theta_n}\right)^i_j$

2–4 $= \left(\sigma^-_n \otimes I\right)^i_j$

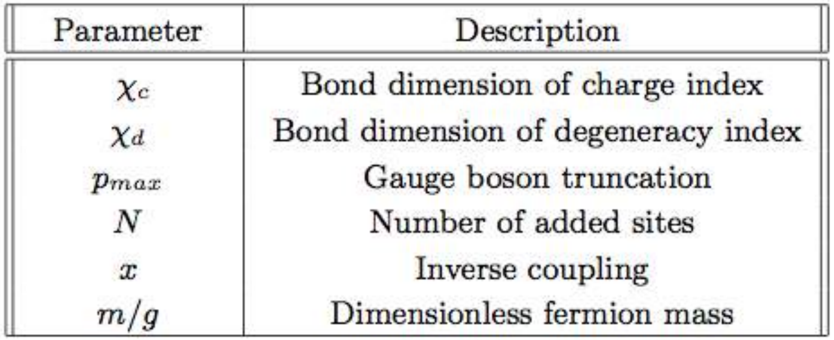
1–3 $= x\left(\sigma^-_n \otimes e^{-i\theta_n}\right)^i_j$

3–4 $= \left(\sigma^+_n \otimes I\right)^i_j$

4–4 $= \delta^i_j$

QED, (1+1)d

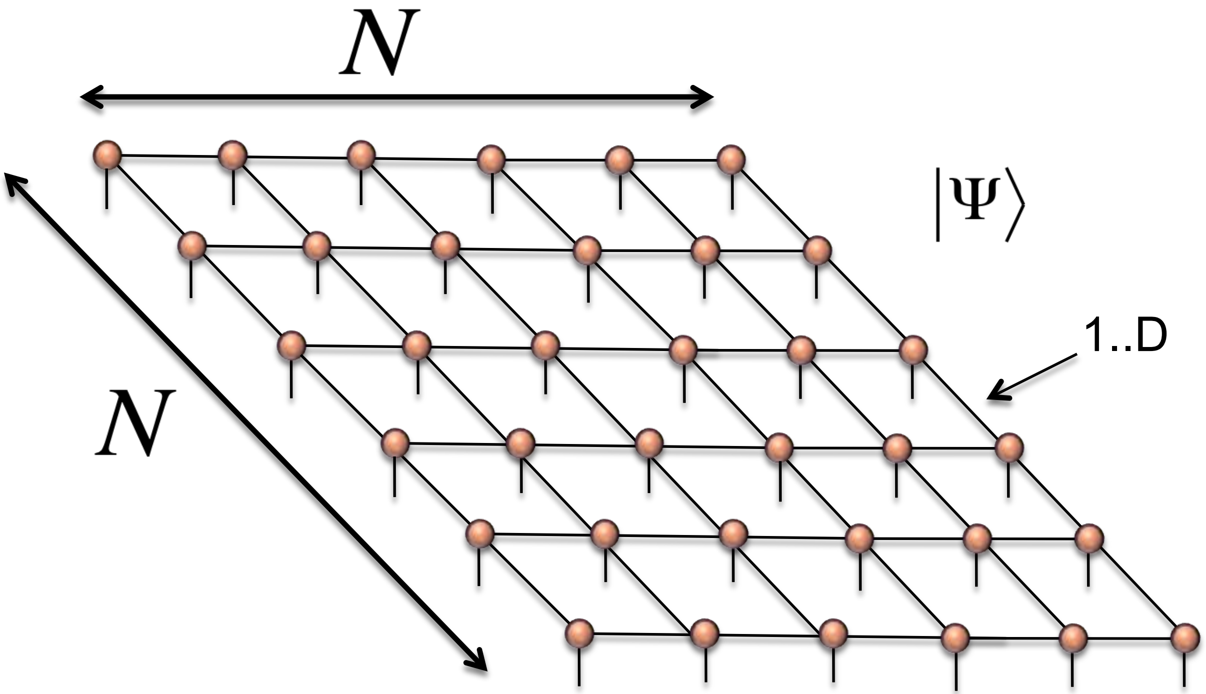
H = translation invariant MPO, bond dimension = 4

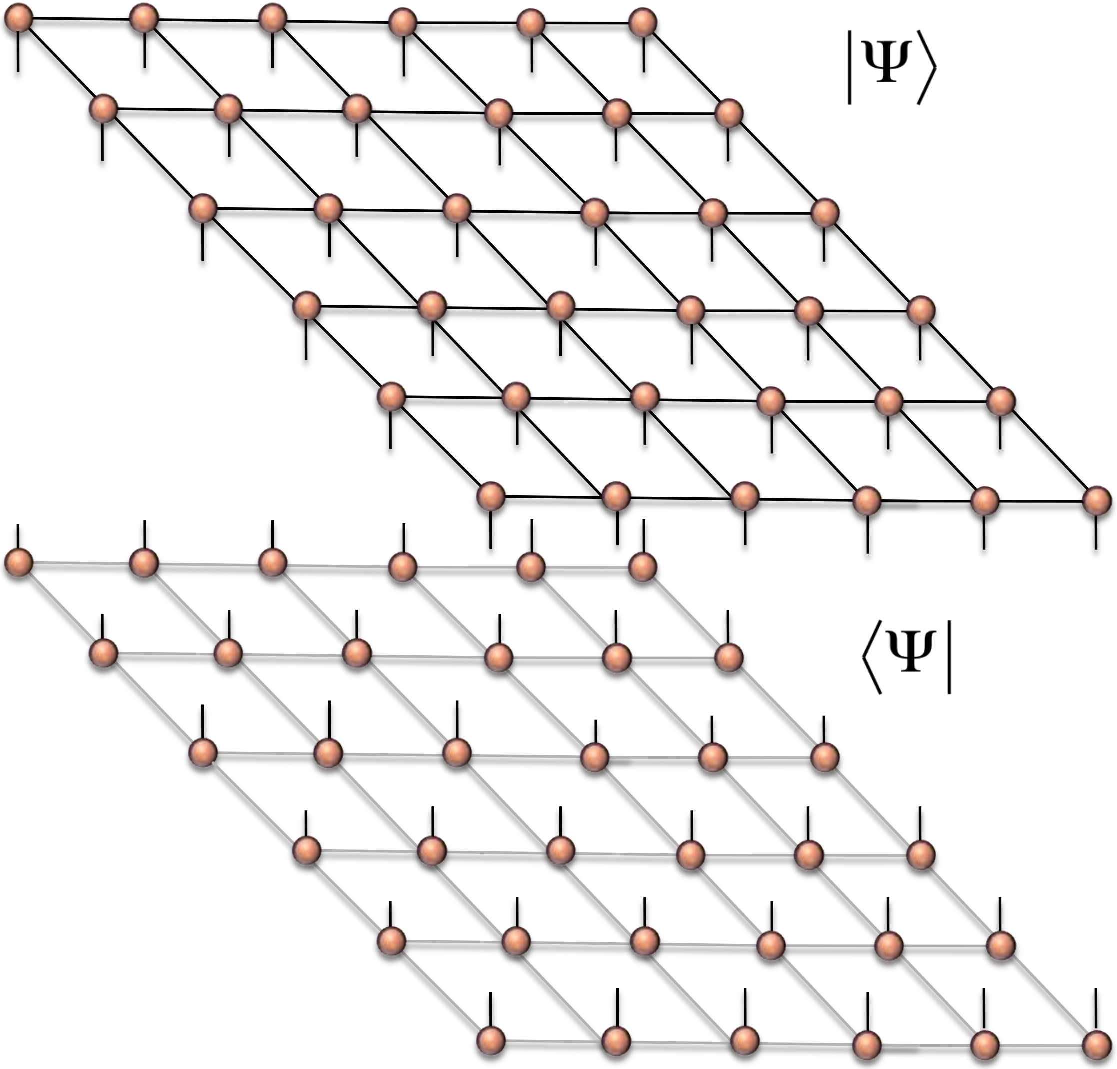


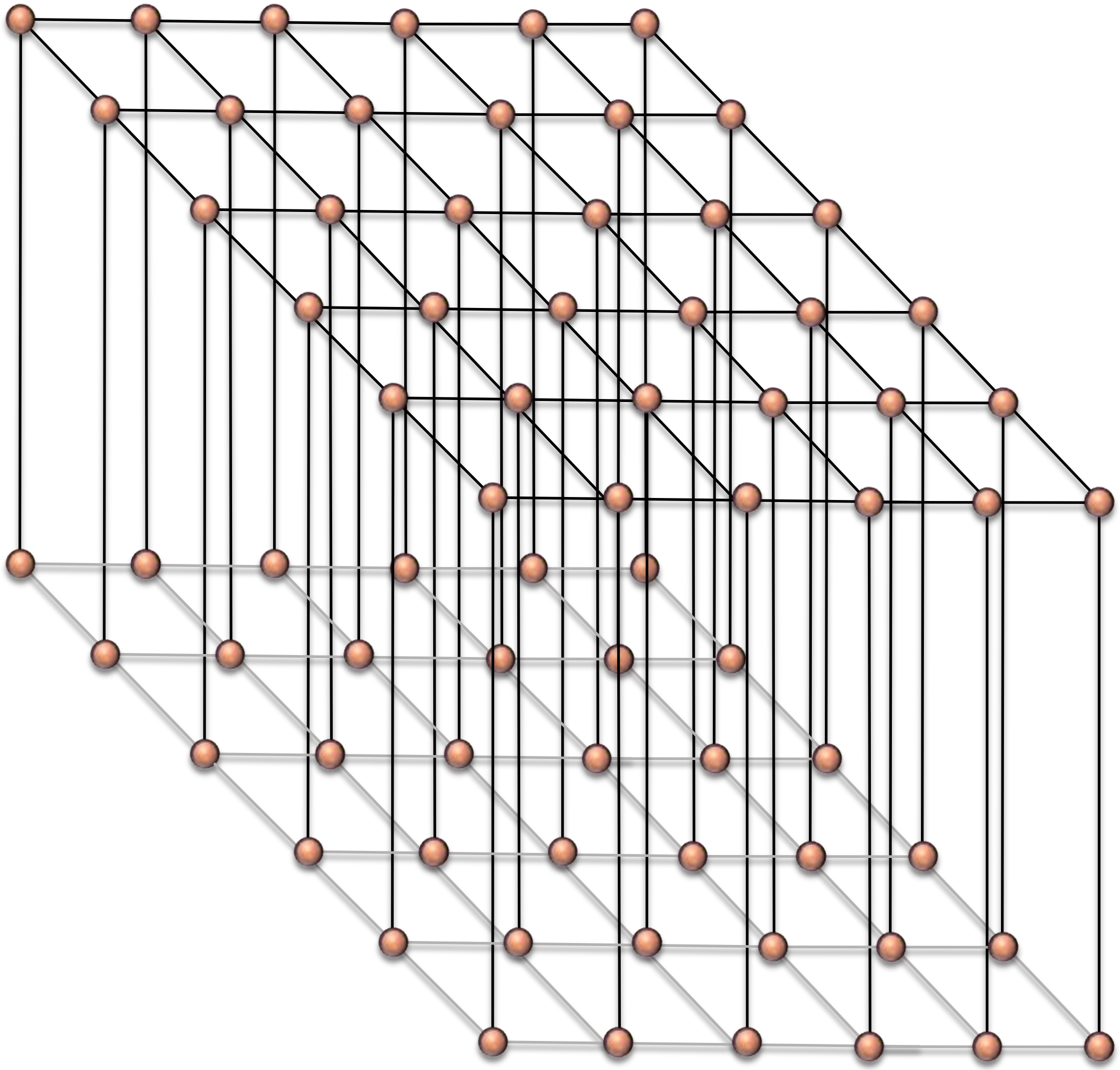
Parameter	Description
χ_c	Bond dimension of charge index
χ_d	Bond dimension of degeneracy index
p_{max}	Gauge boson truncation
N	Number of added sites
x	Inverse coupling
m/g	Dimensionless fermion mass

PEPS & Entanglement Hamiltonians

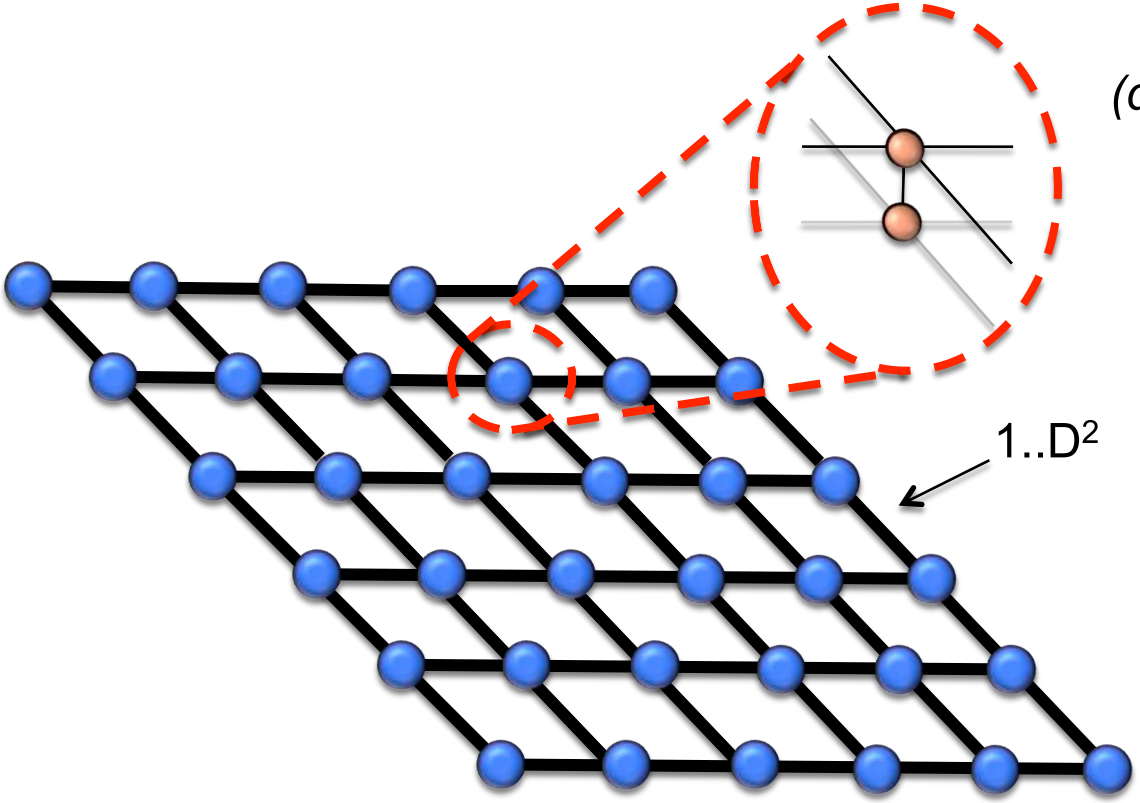
e.g. I. Cirac et al, PRB 83, 245134 (2011), N. Schuch et al, PRL 111, 090501 (2013)


N
N
$|\Psi\rangle$
1..D


$|\Psi\rangle$
$\langle\Psi|$

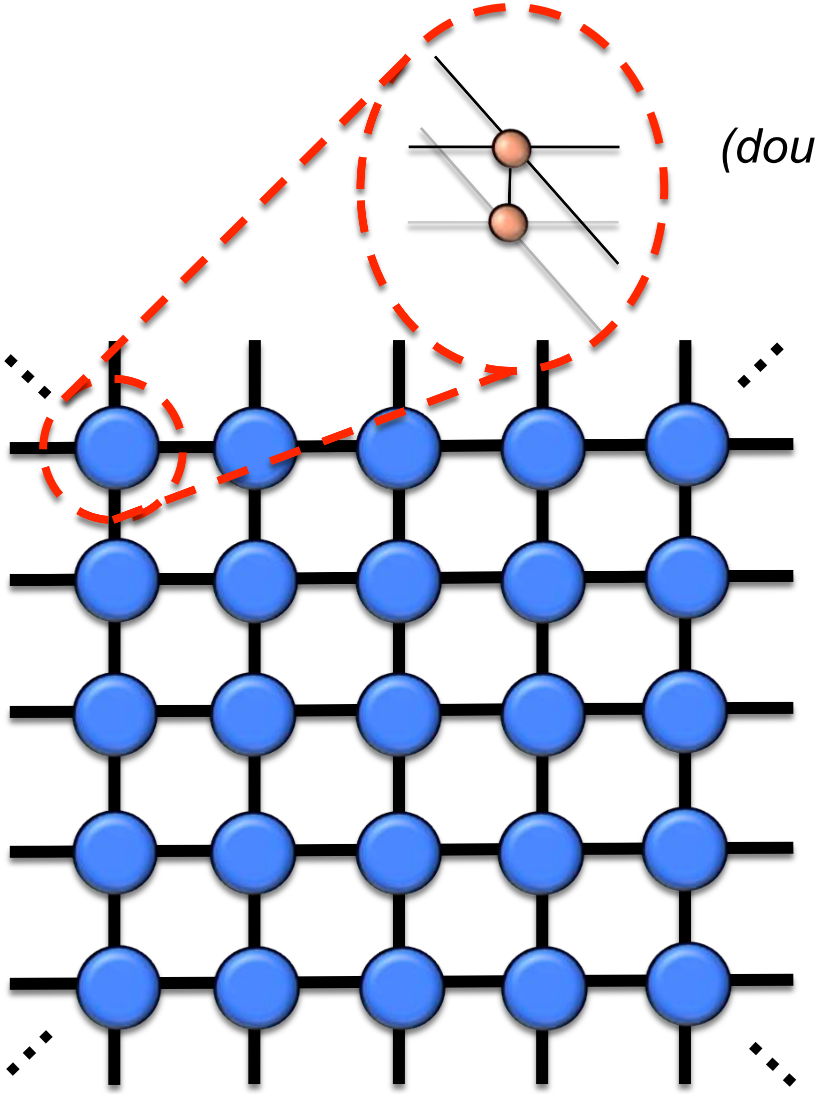


$\langle\Psi|\Psi\rangle$

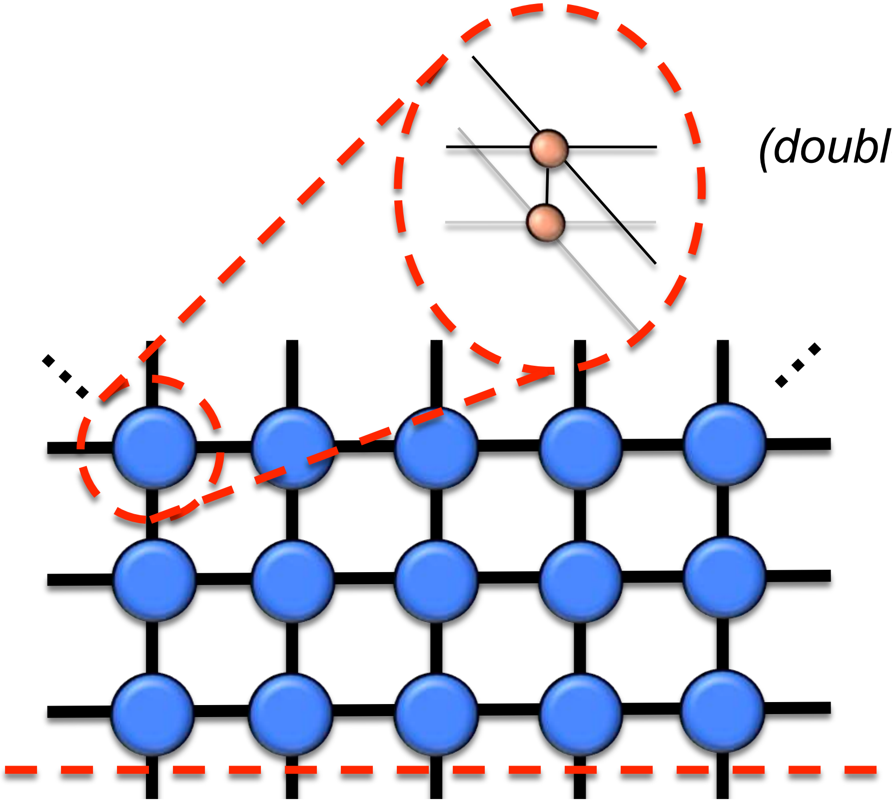


(double indices)

$$\langle\Psi|\Psi\rangle$$



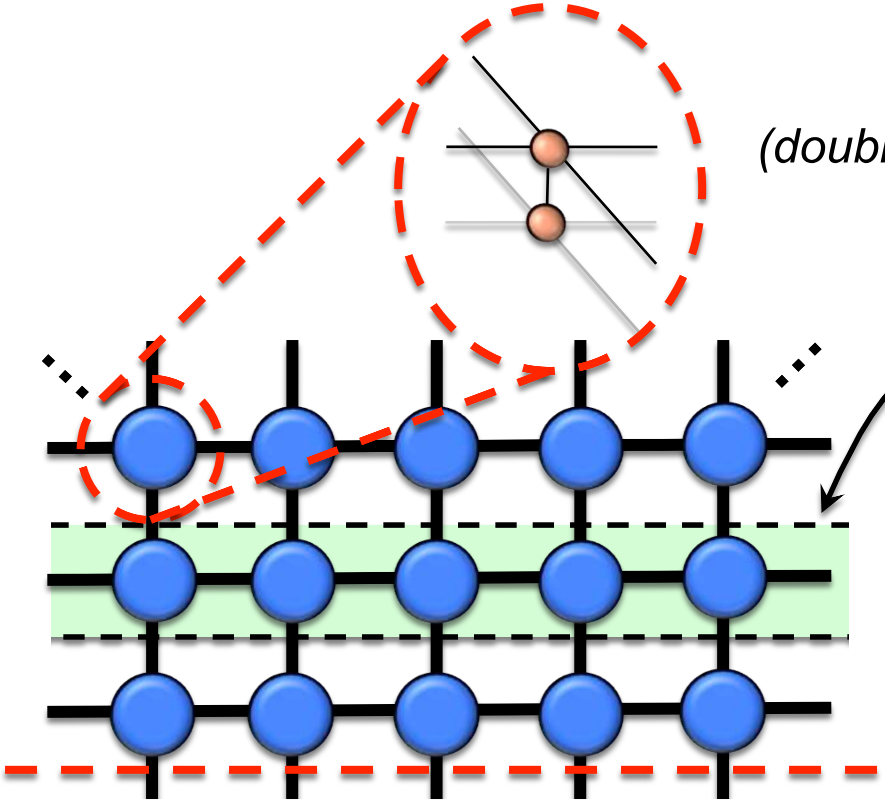
(double indices)



(double indices)

Boundary

How is physics described here?



(double indices)

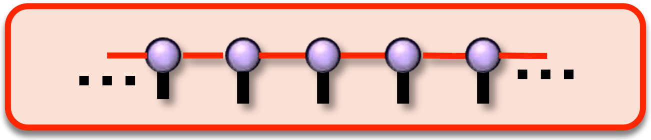
Boundary

How is physics described here?

1-dim transfer matrix:

dominant eigenvector?

Can be approximated using infinite MPS

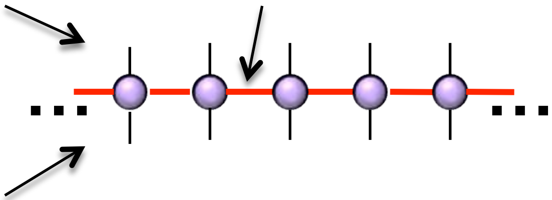


iTEBD, iDMRG, PWFRG, etc

Emergent Hamiltonians

Remember it has double indices…

Emergent Hamiltonians



Virtual indices of bra

1…D

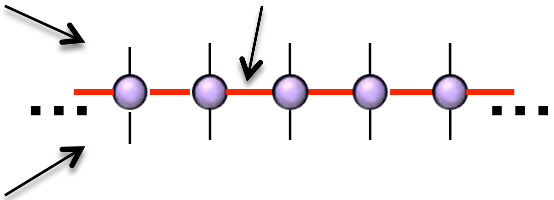
Boundary virtual index 1… χ

Virtual indices of ket

1…D

It is also hermitian and
positive by construction
(up to finite-χ *effects)*

Emergent Hamiltonians



Virtual indices of bra
1…D

Virtual indices of ket
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Boundary virtual index 1… χ

It is also hermitian and
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(up to finite-χ effects)

$$||$$

1d Entanglement Hamiltonian

$$\rho = \exp(-H_E)$$

Who is H_E ???

Emergent Hamiltonians

Virtual indices of bra 1…D

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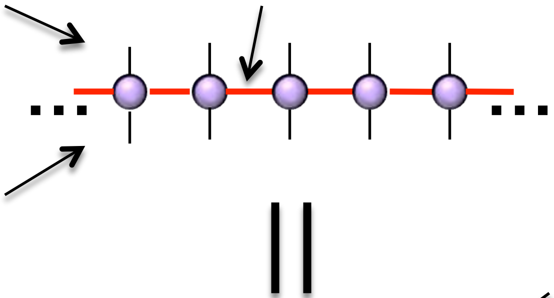
Virtual indices of ket 1…D



It is also hermitian and positive by construction (up to finite-χ effects)

1d Entanglement Hamiltonian

$$\rho = \exp(-H_E)$$

Who is H_E ???

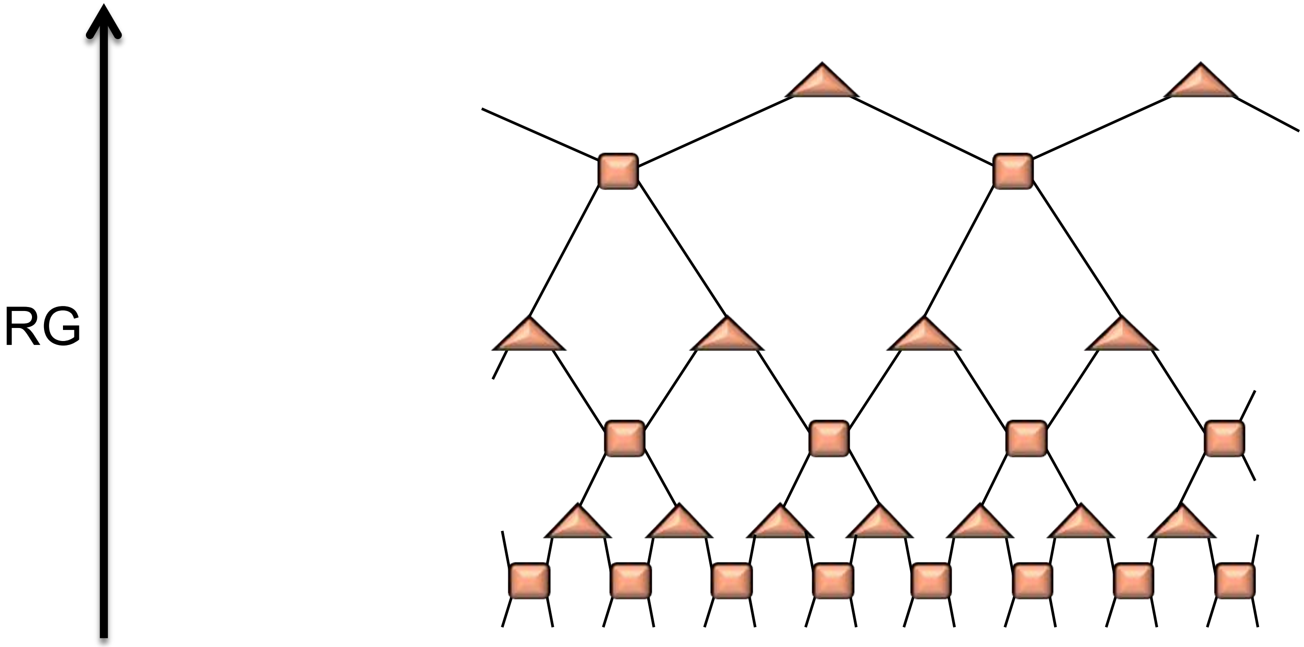
Correspondence

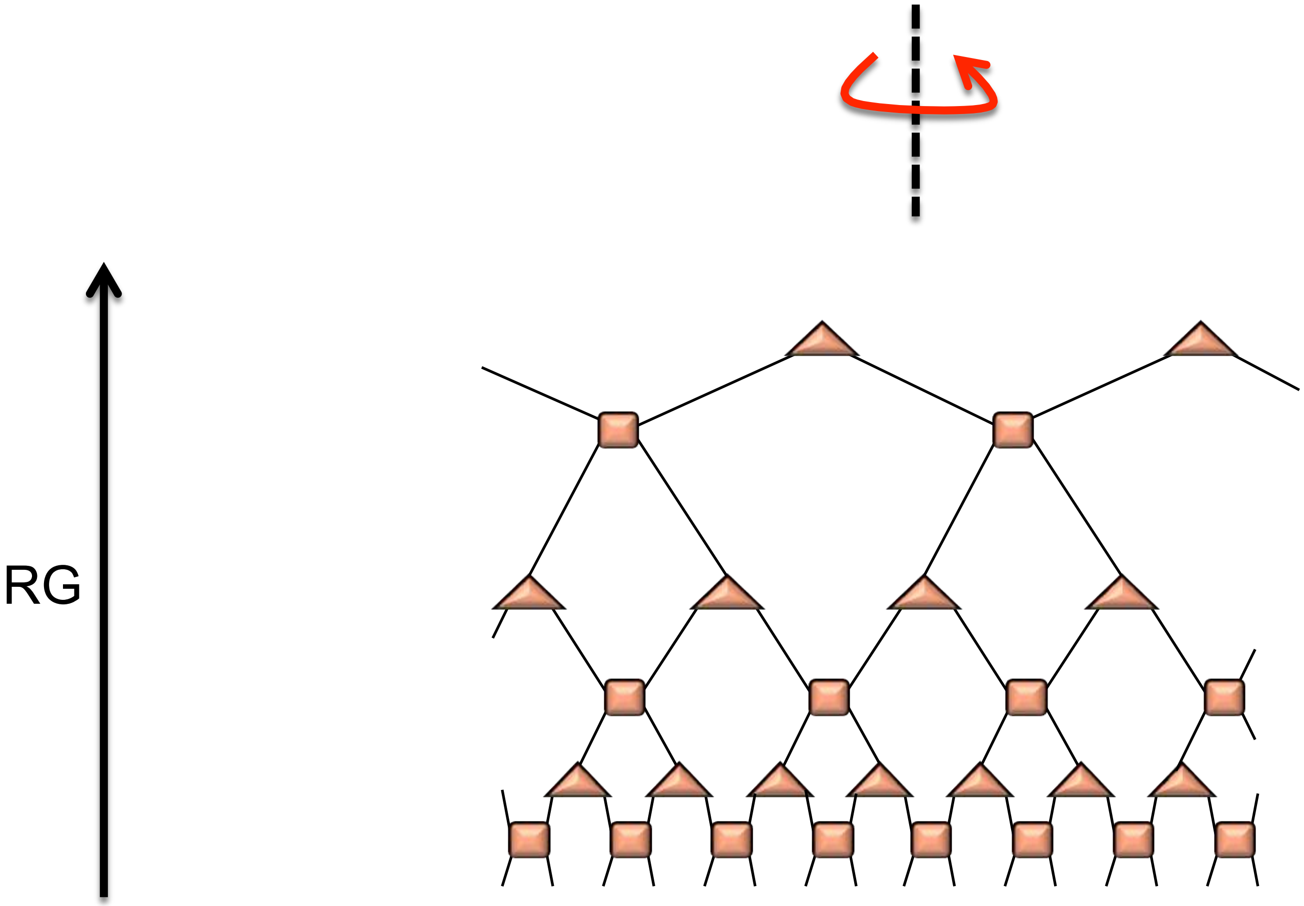
Bulk	Boundary
Gapped 2d systems, trivial phase	1d Hamiltonian, short-range
Critical 2d systems	1d Hamiltonian, long-range
Gapped 2d systems, topological order	Completely non-local (projector)
Chiral topological order, gapless	(1+1)d Conformal field theory

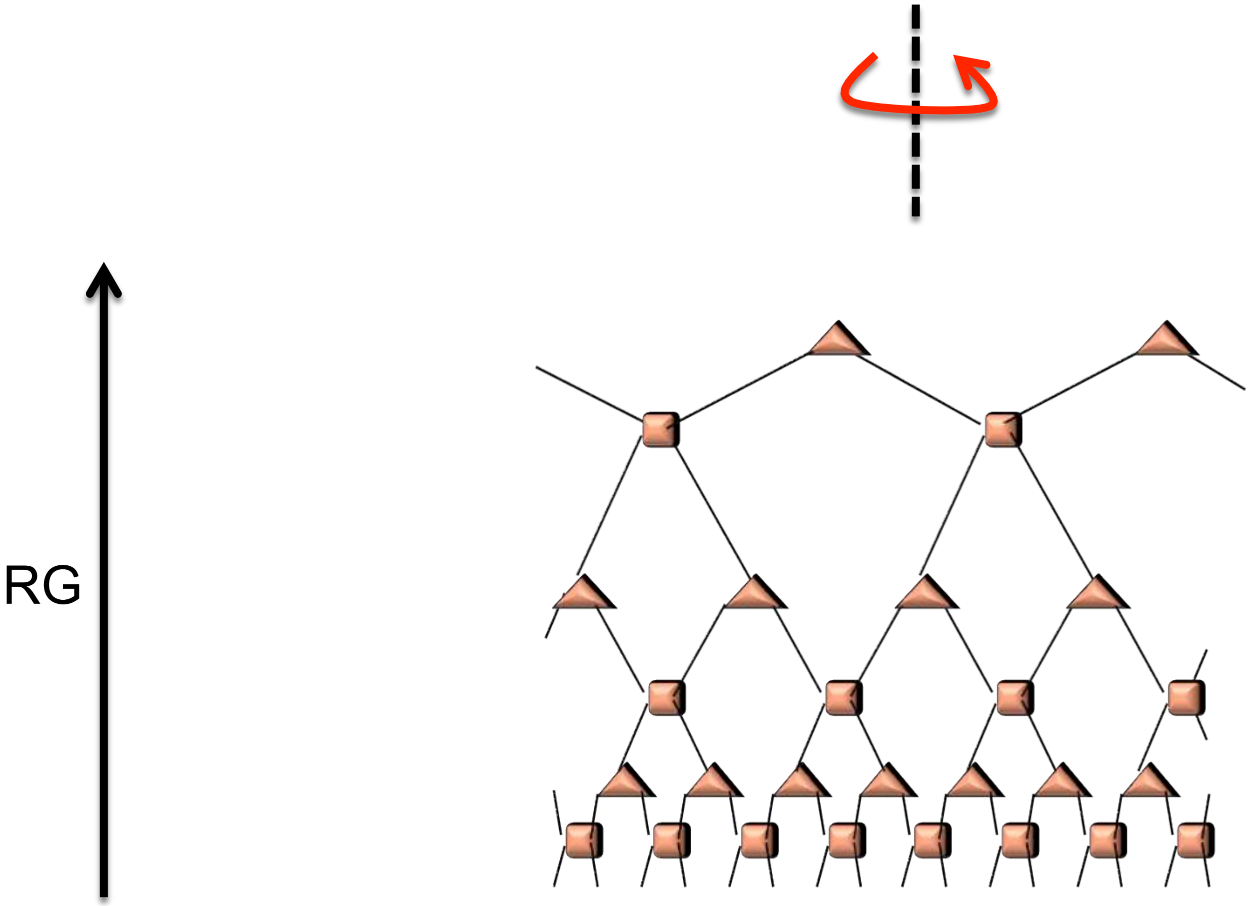
Particles and energies from Hamiltonians, and Hamiltonians from networks of entanglement + bulk-boundary correspondence

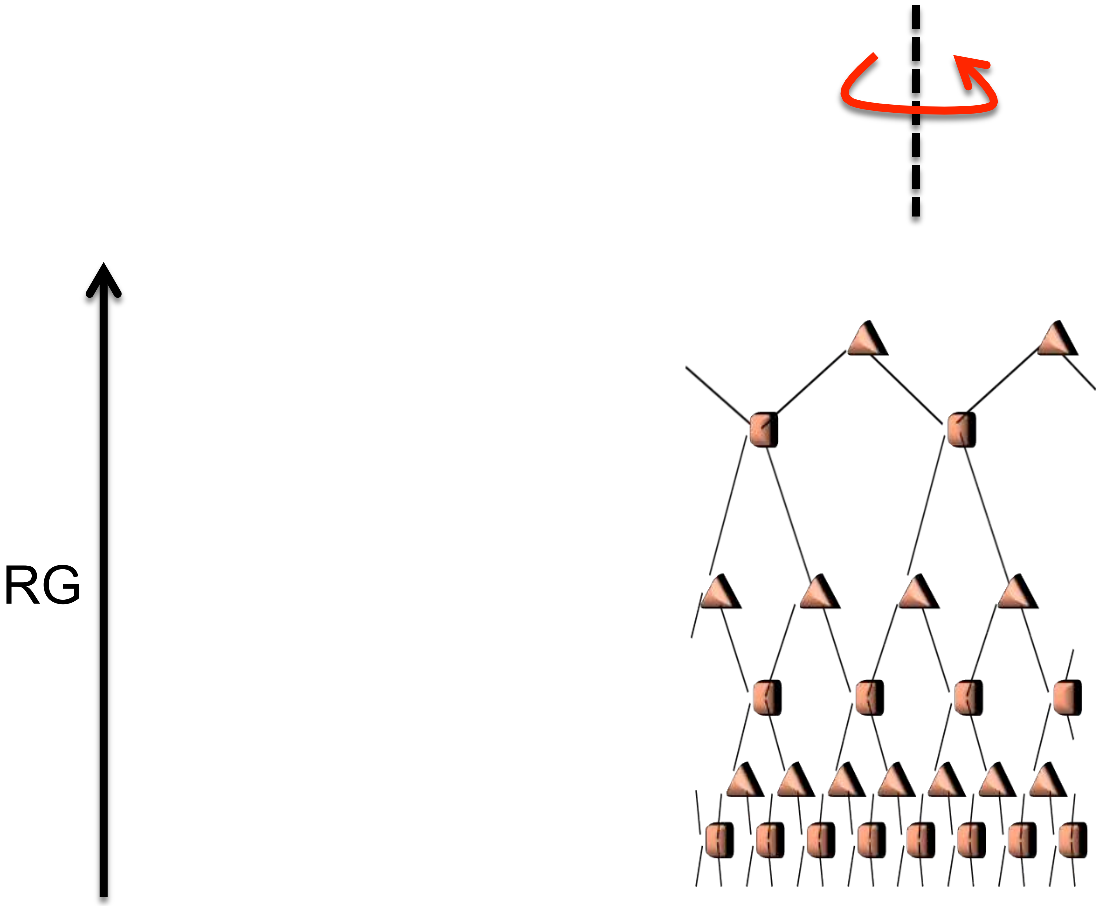
“branching” MERA

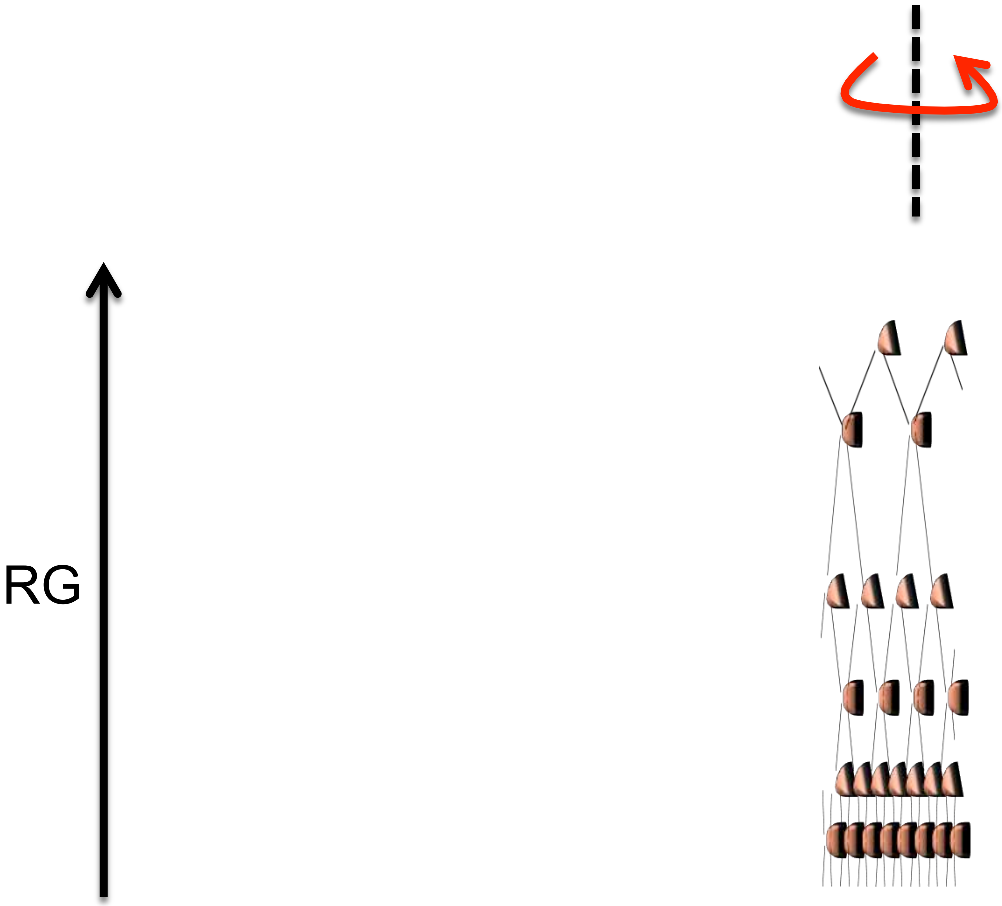
G. Evenbly, G. Vidal, arXiv:1210.1895


RG

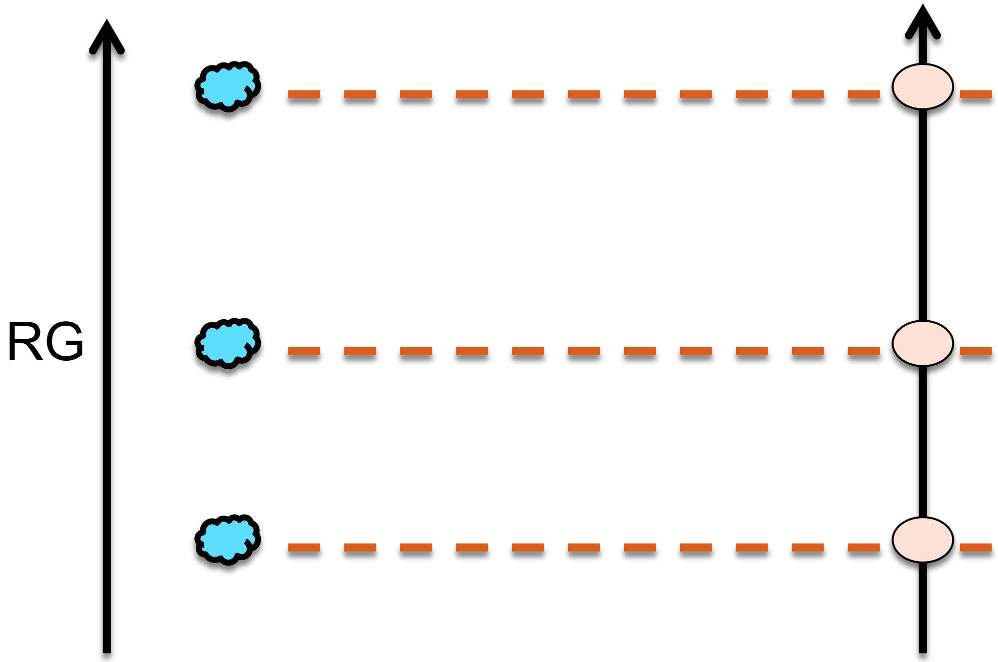

RG


RG

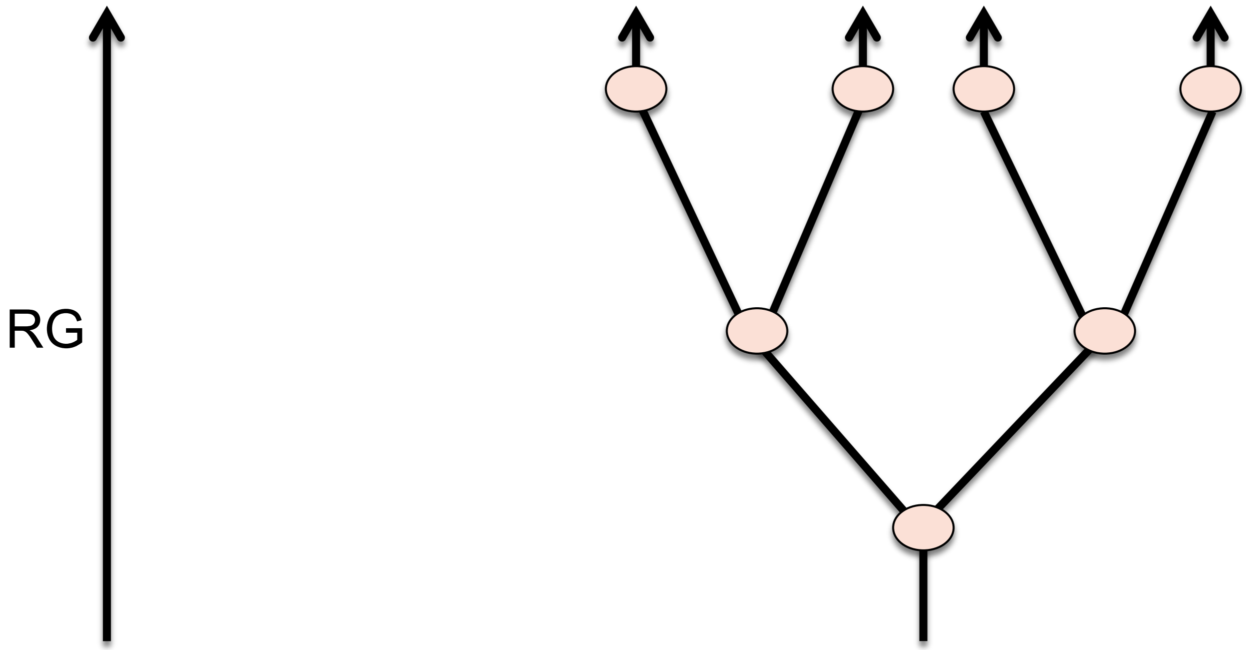

RG


RG

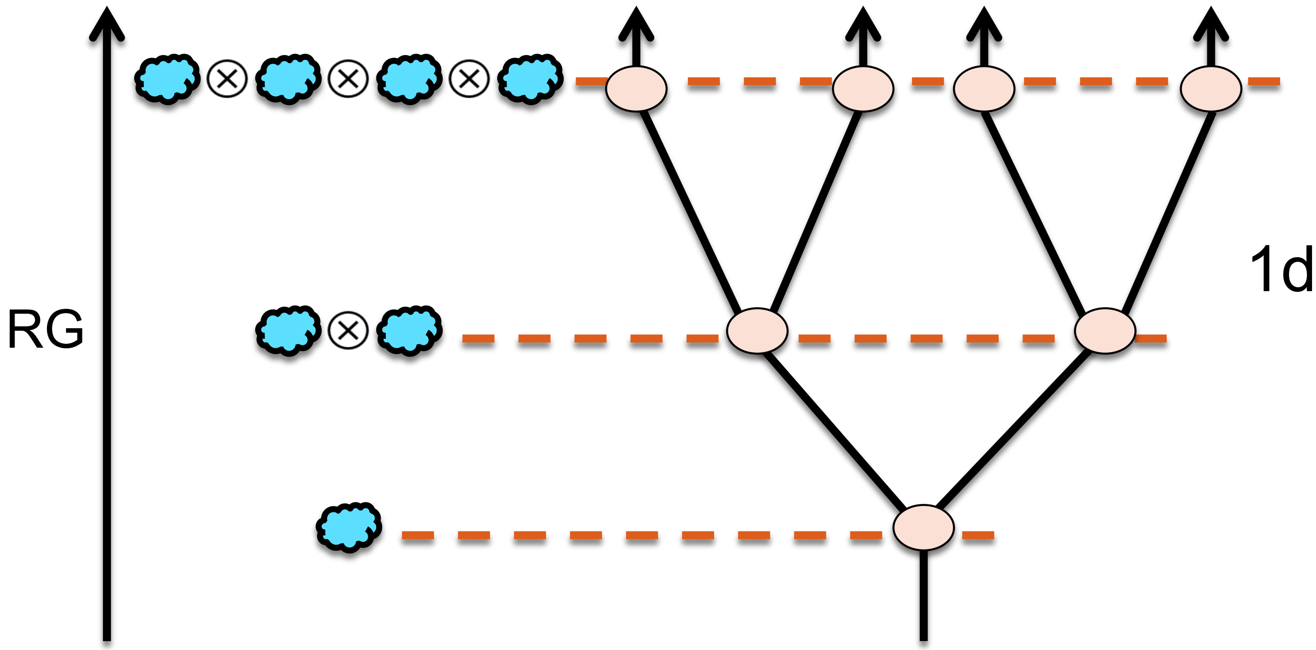
1d MERA



1d MERA



1d branching MERA

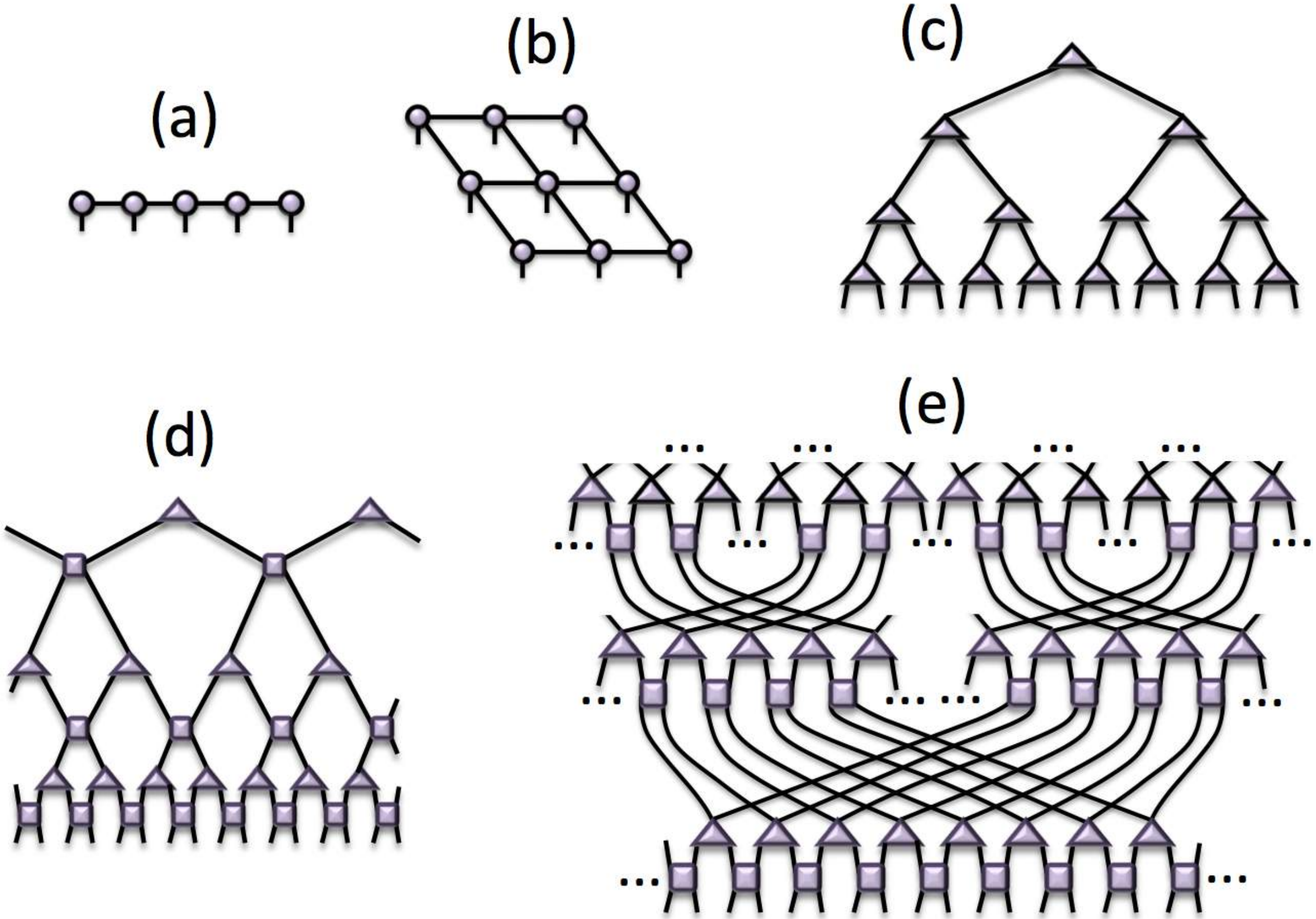


1d branching MERA

Decoupling of degrees of freedom along RG (e.g. spin-charge separation), and allows arbitrary scalings of the entanglement entropy

In 2d, ansatz for e.g., Fermi & Bose liquids, $S(L) \approx L \log L$

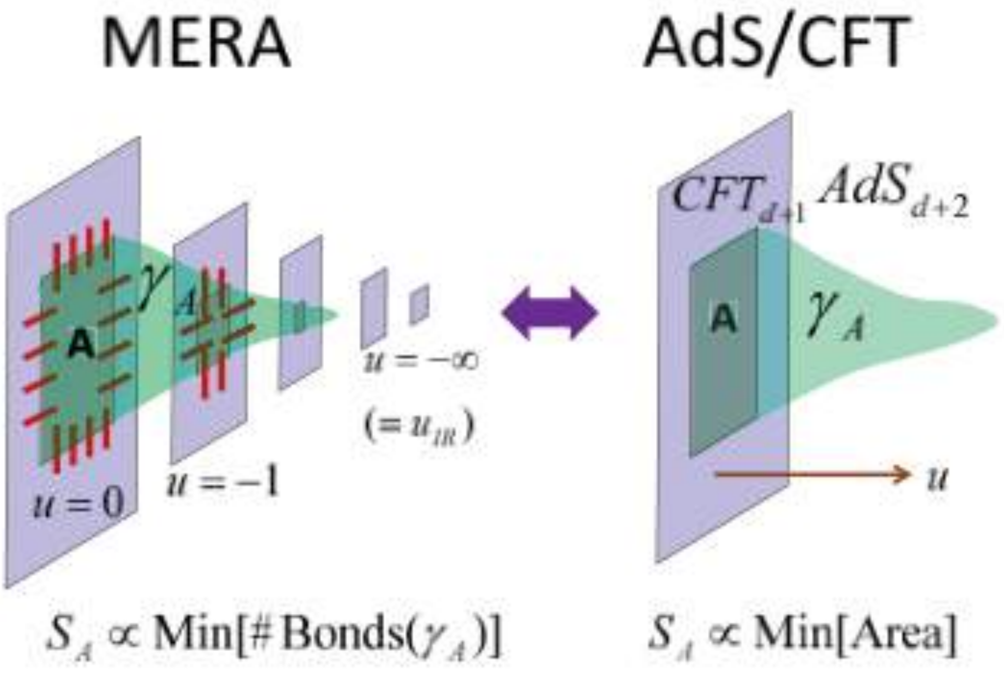
Increasing complexity...



MERA & AdS/CFT

e.g. B. Swingle, PRD 86, 065007 (2012), G. Evenbly, G. Vidal, JSTAT 145:891-918 (2011)

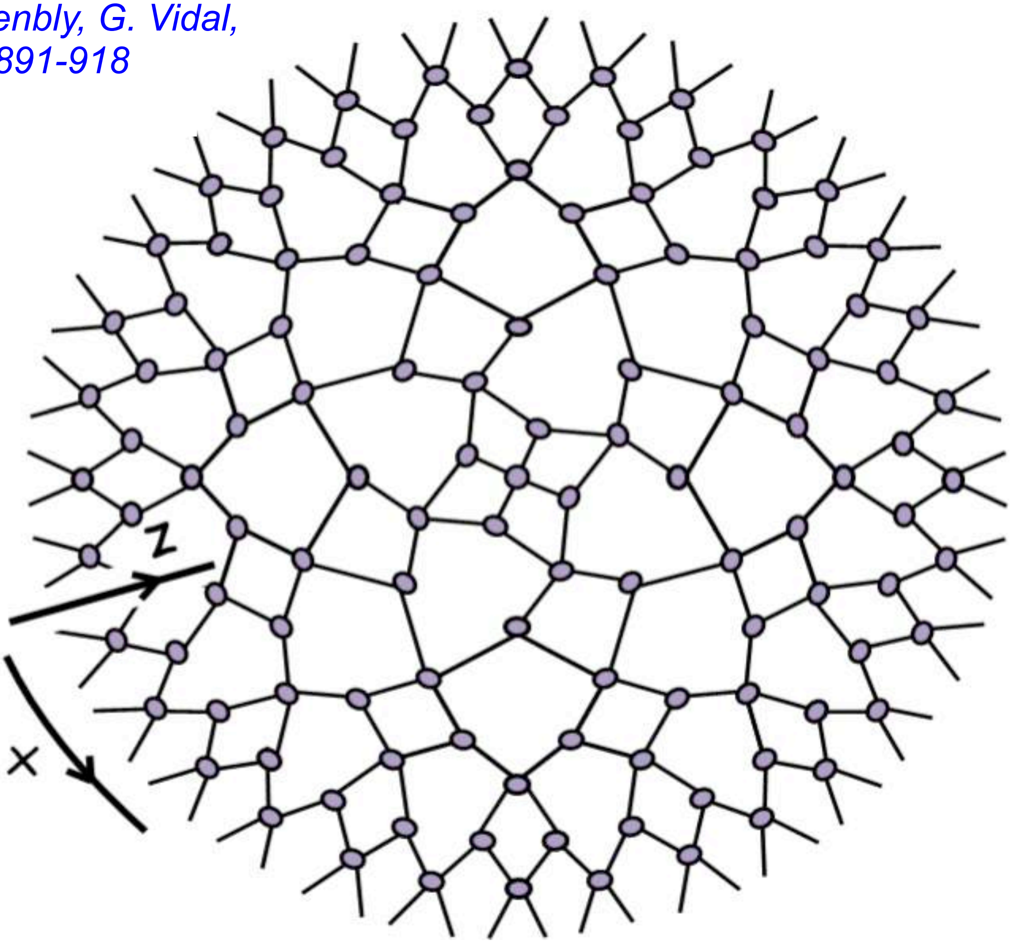
Emergent space-time



Picture from M. Nozaki, S. Ryu, T. Takayanagi, JHEP10(2012)193

MERA entropy ~ Ryu-Takayanagi prescription

Picture from G. Evenbly, G. Vidal,
(2011) JSTAT 145:891-918



(time slice)

Picture from G. Evenbly, G. Vidal,
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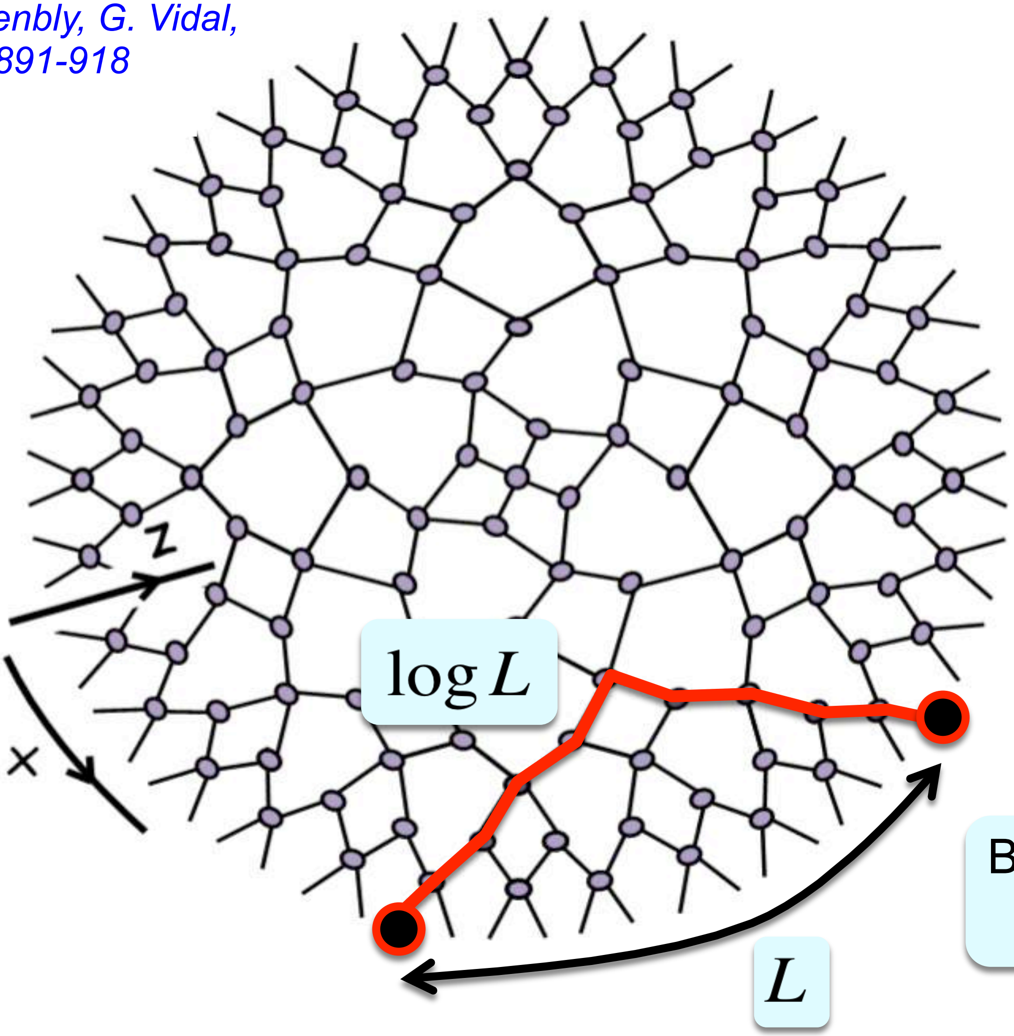
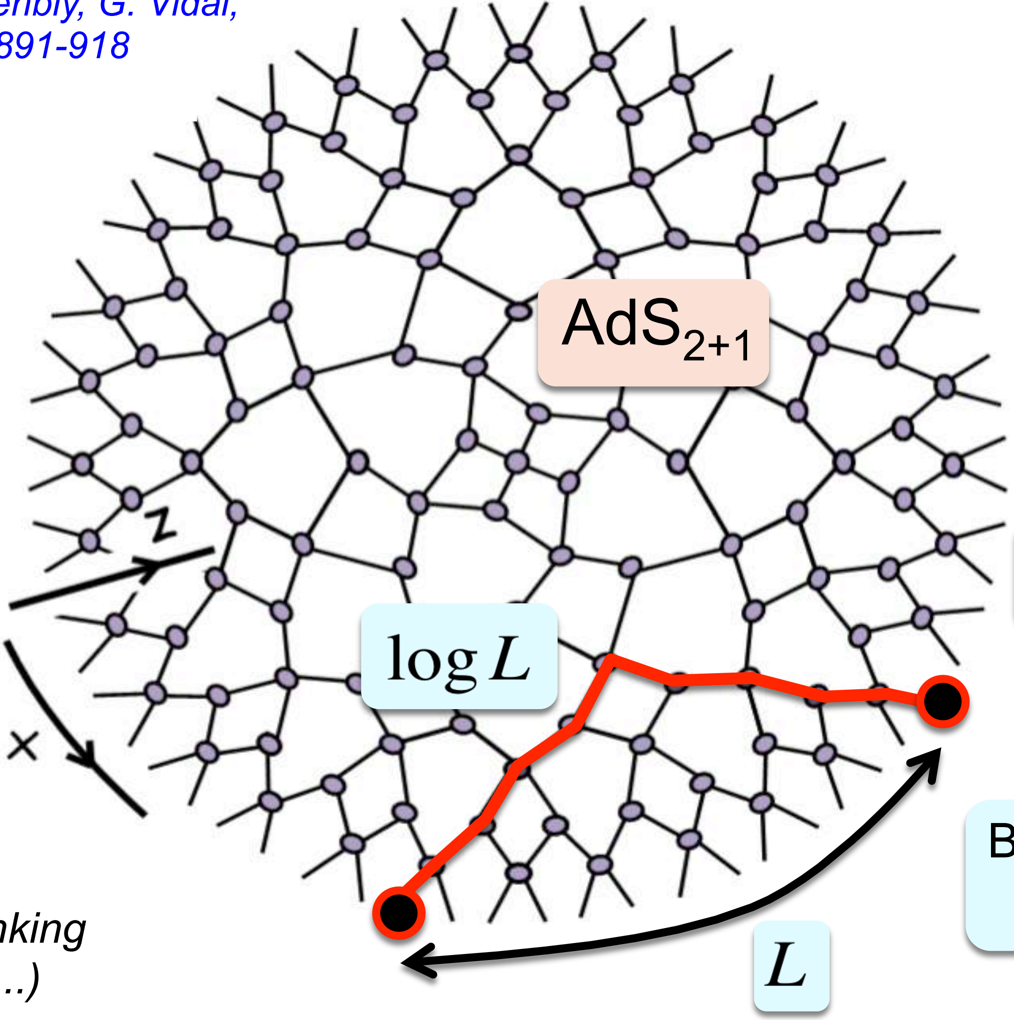


(time slice)

Bulk is a discretized AdS space

Picture from G. Evenbly, G. Vidal, (2011) JSTAT 145:891-918



(time slice)

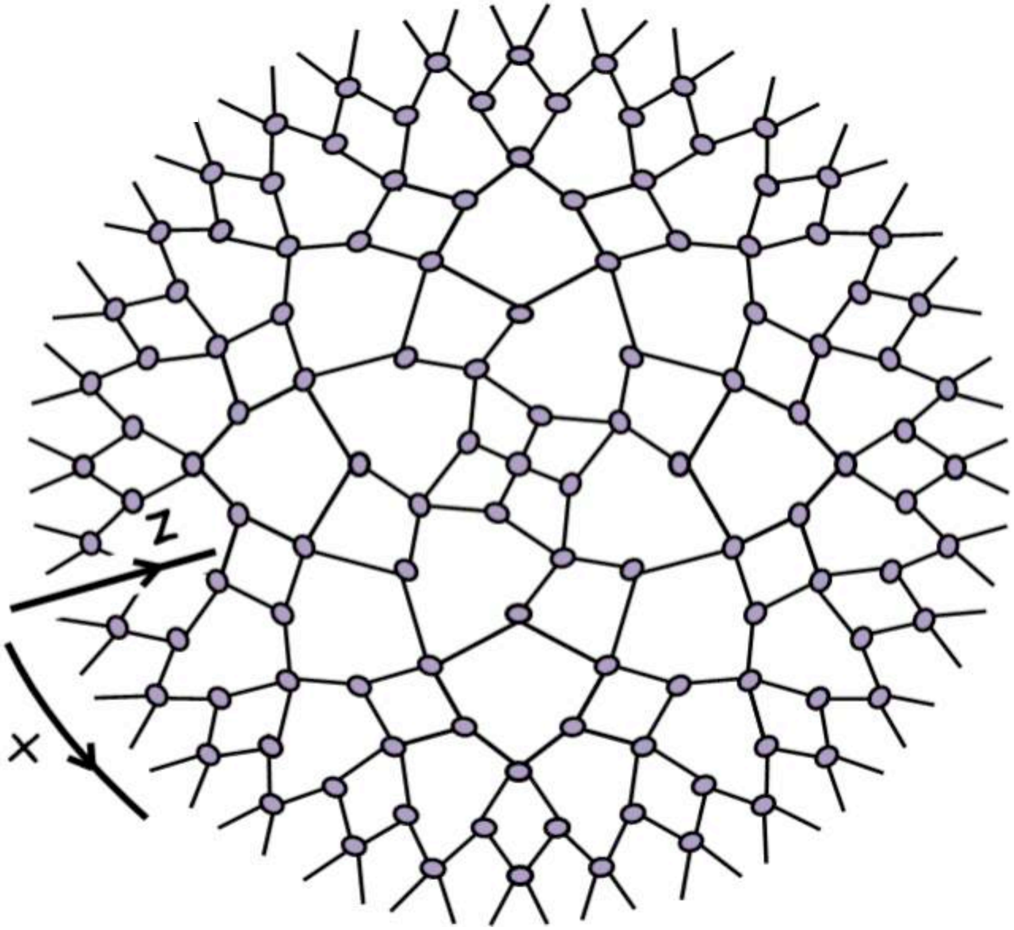
CFT_{1+1}

Bulk is a discretized AdS space

(and we were not thinking about gravity at all...)

For a scale-invariant MERA, the tensors of a critical model with a CFT limit correspond to a „gravitational“ description in a discretized AdS space:
„lattice“ realization of AdS/CFT correspondence

Let‘s now play some jazz...



(time slice)

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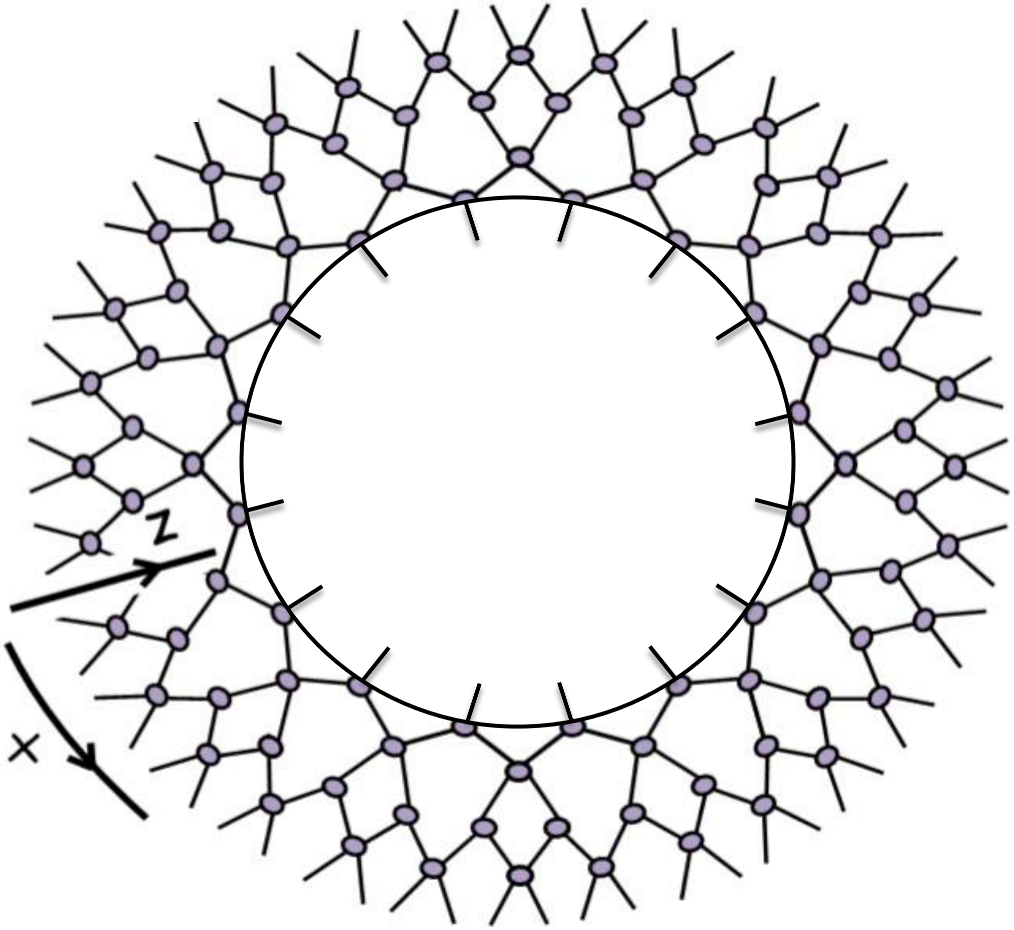
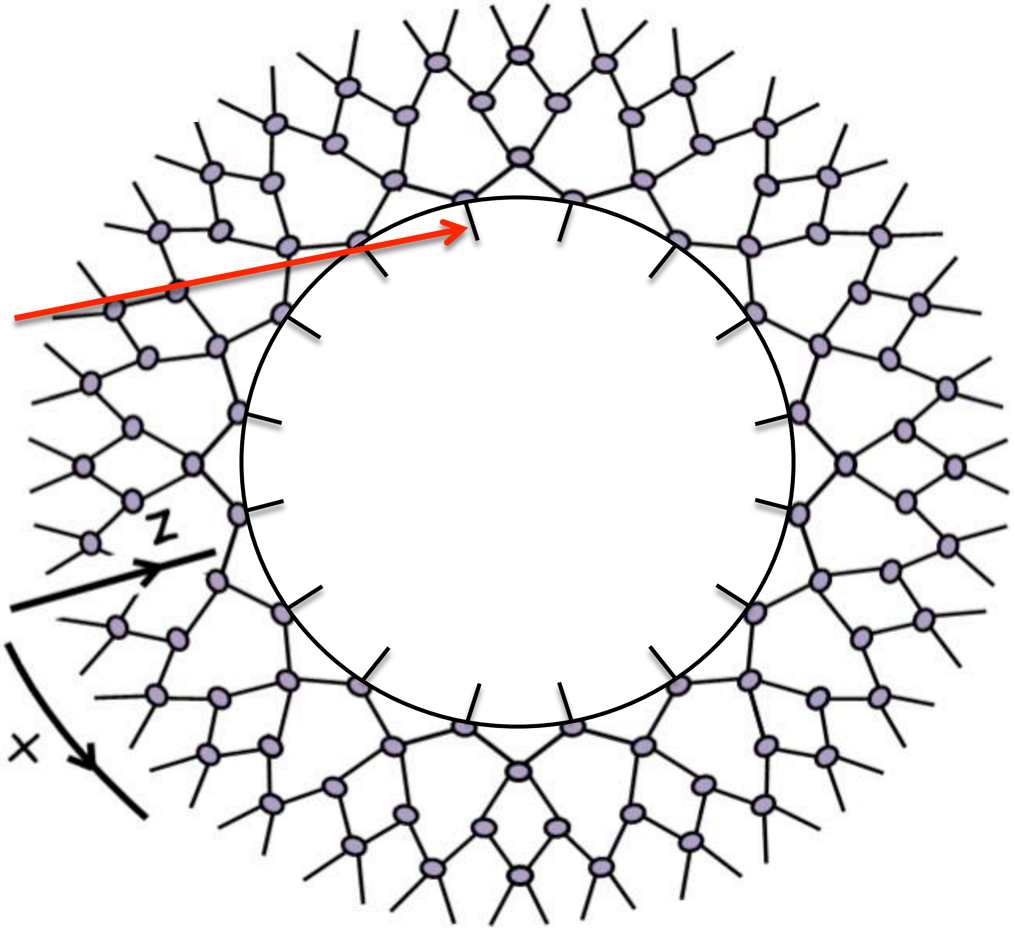


(time slice)

Finite correlation length (gapped systems) = finite number of layers

Let‘s now play
some jazz...



(time slice)

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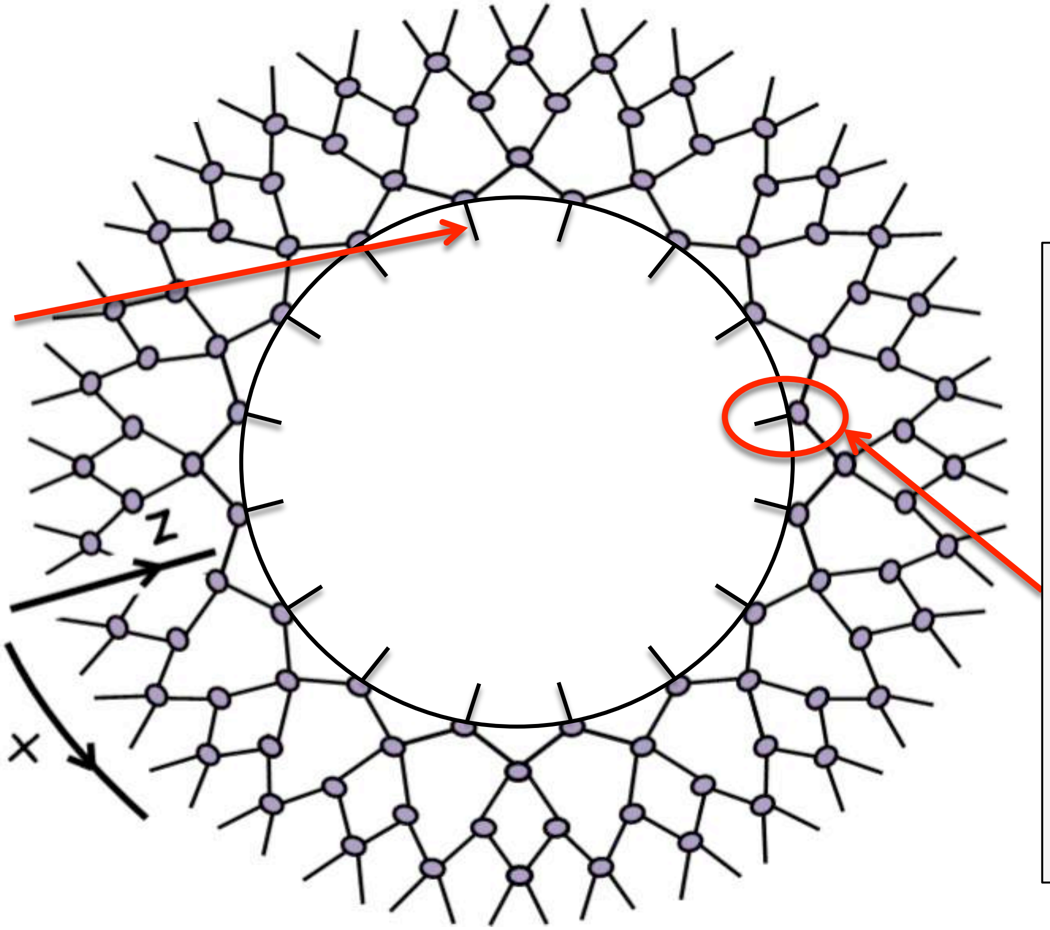


Product state = trivial fixed point

(time slice)

If arbitrary, then we can have non-trivial thermal states.

If isommetry, then all information is encoded in the network of correlations and

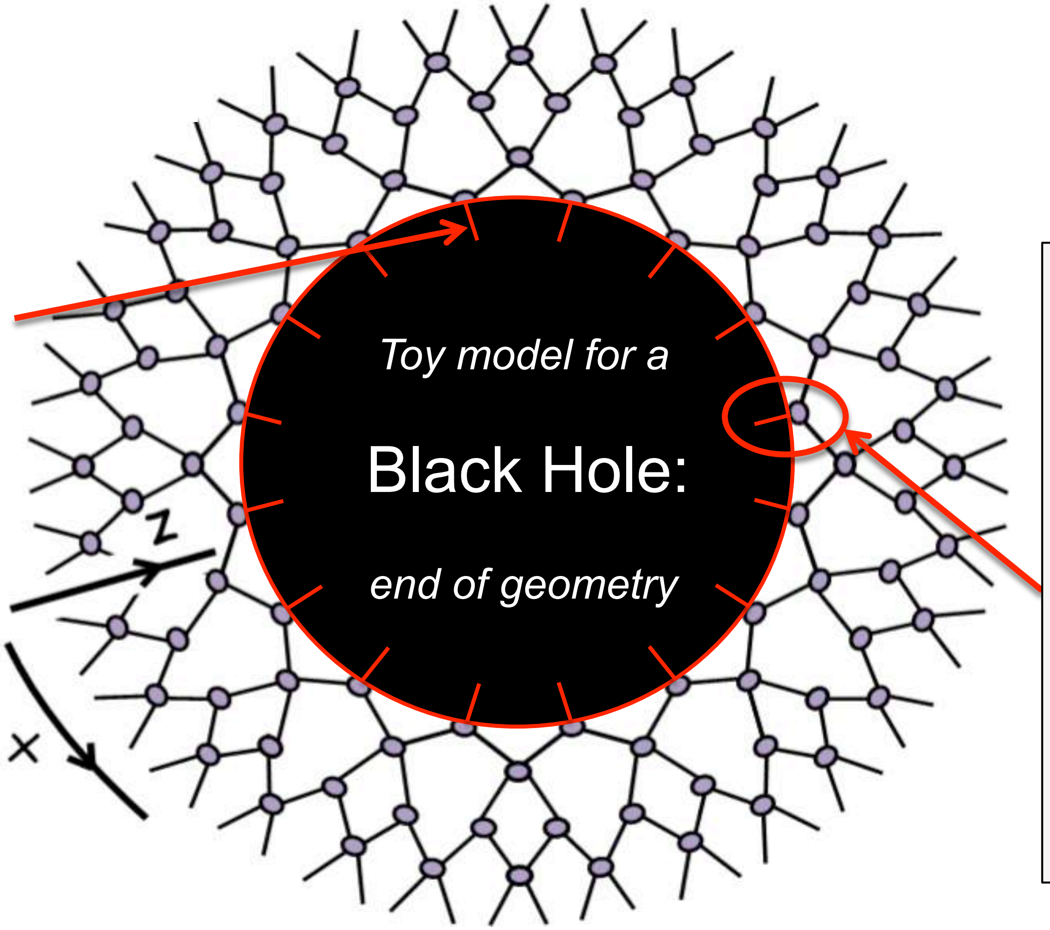
$$\rho_{in} = I$$

Finite correlation length (gapped systems) = finite number of layers

$$\left.\begin{aligned}\rho_{in} &= tr_{out}\left(|\Psi\rangle\langle\Psi|\right)\\ \rho_{out} &= tr_{in}\left(|\Psi\rangle\langle\Psi|\right)\end{aligned}\right\}$$

Same ***thermal*** *spectrum (entanglement Hamiltonian)*
finite temperature, scale invariance broken

Let‘s now play some jazz…



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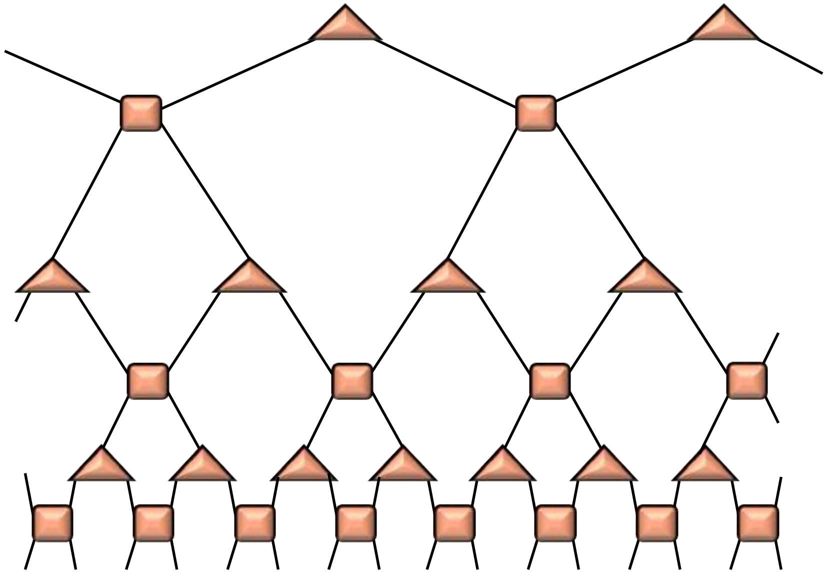
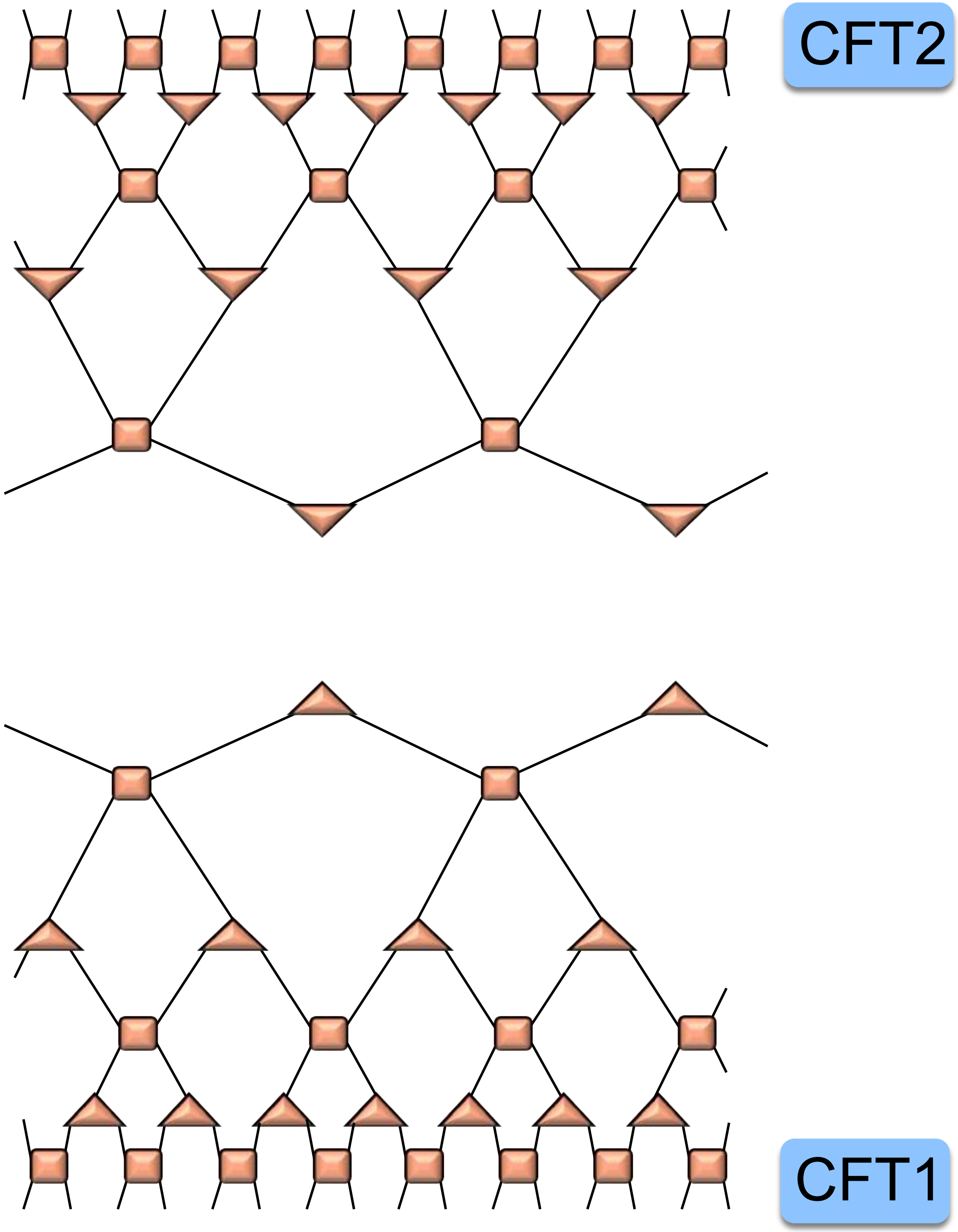
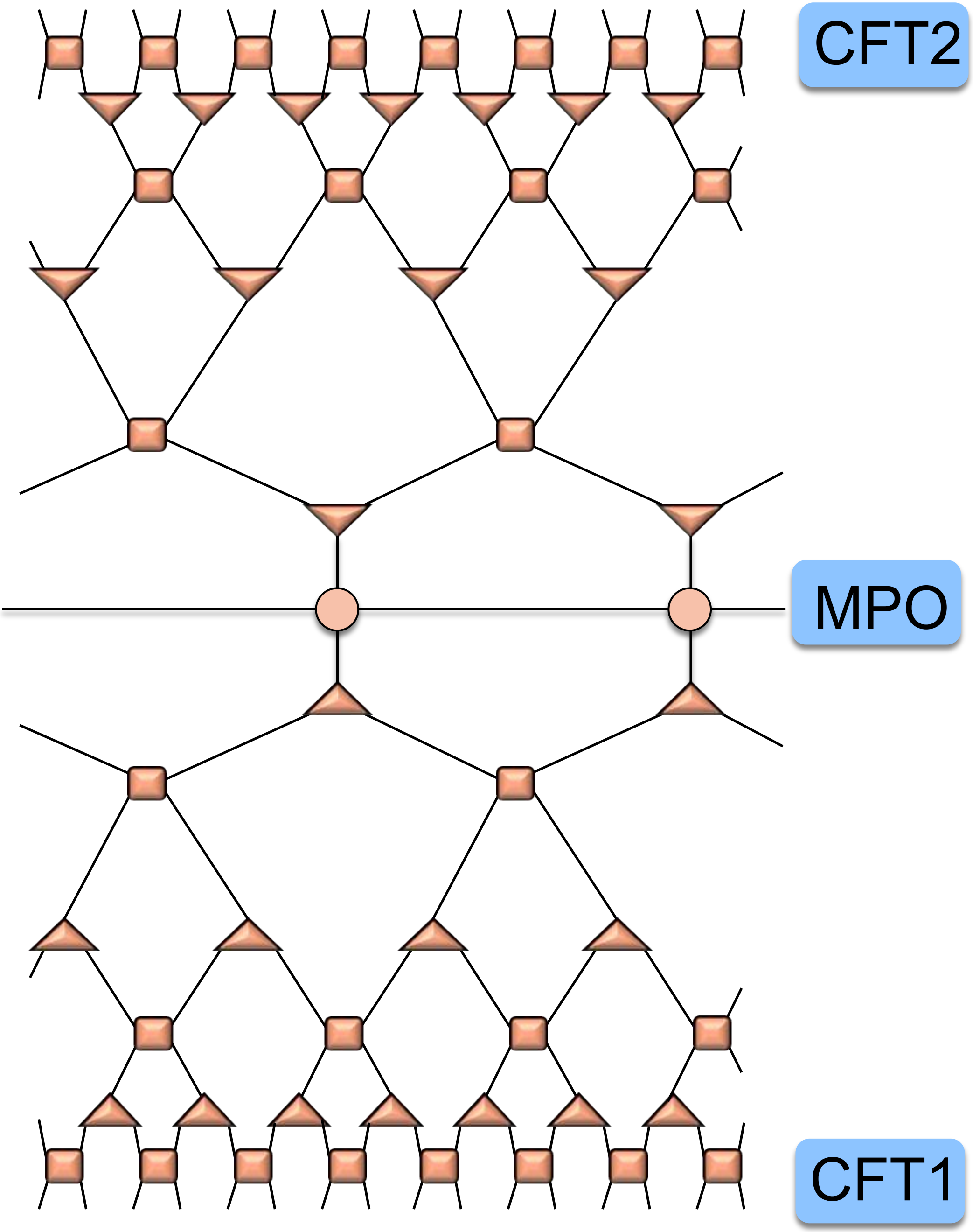
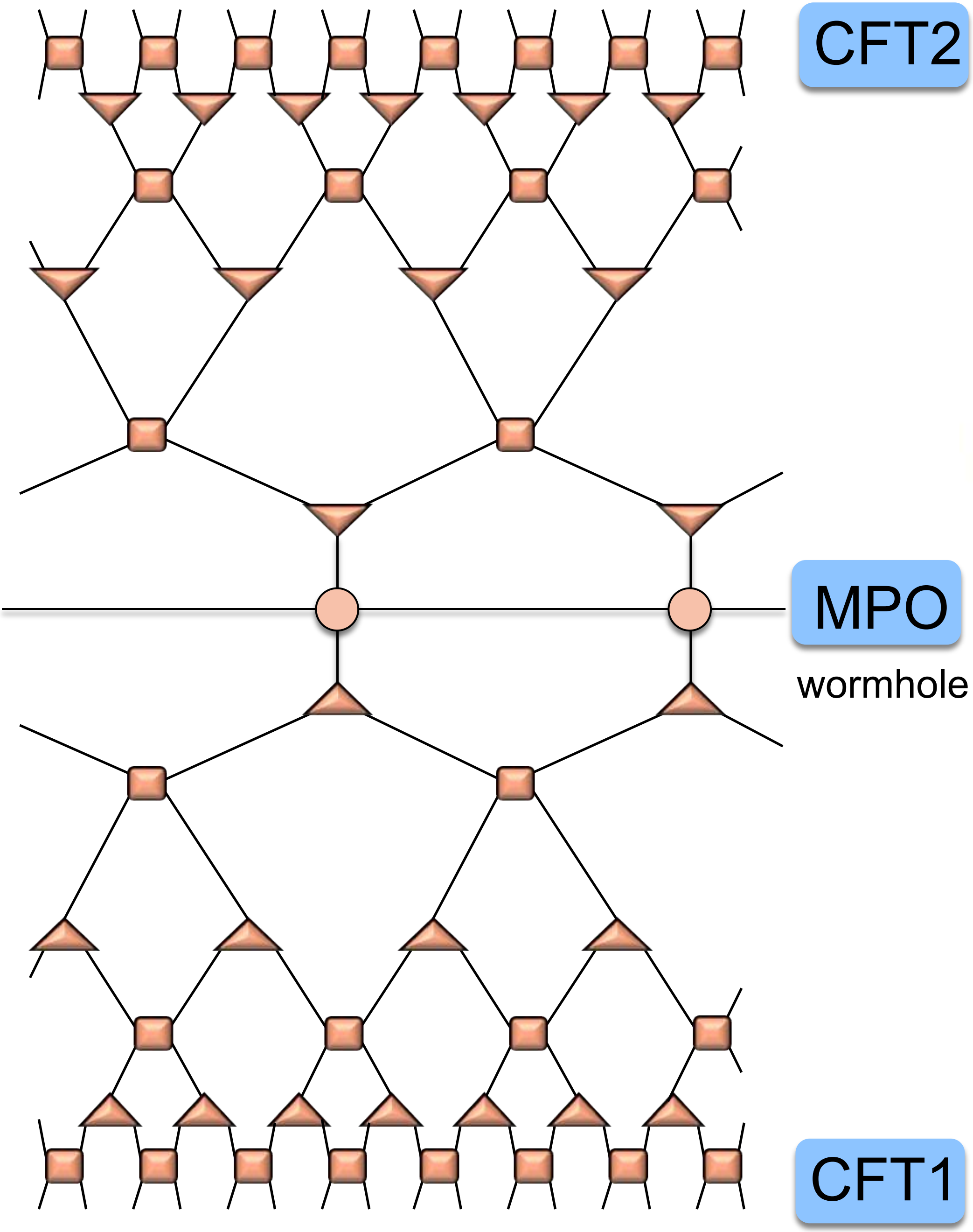


CFT1


CFT2
CFT1


CFT2
MPO
CFT1

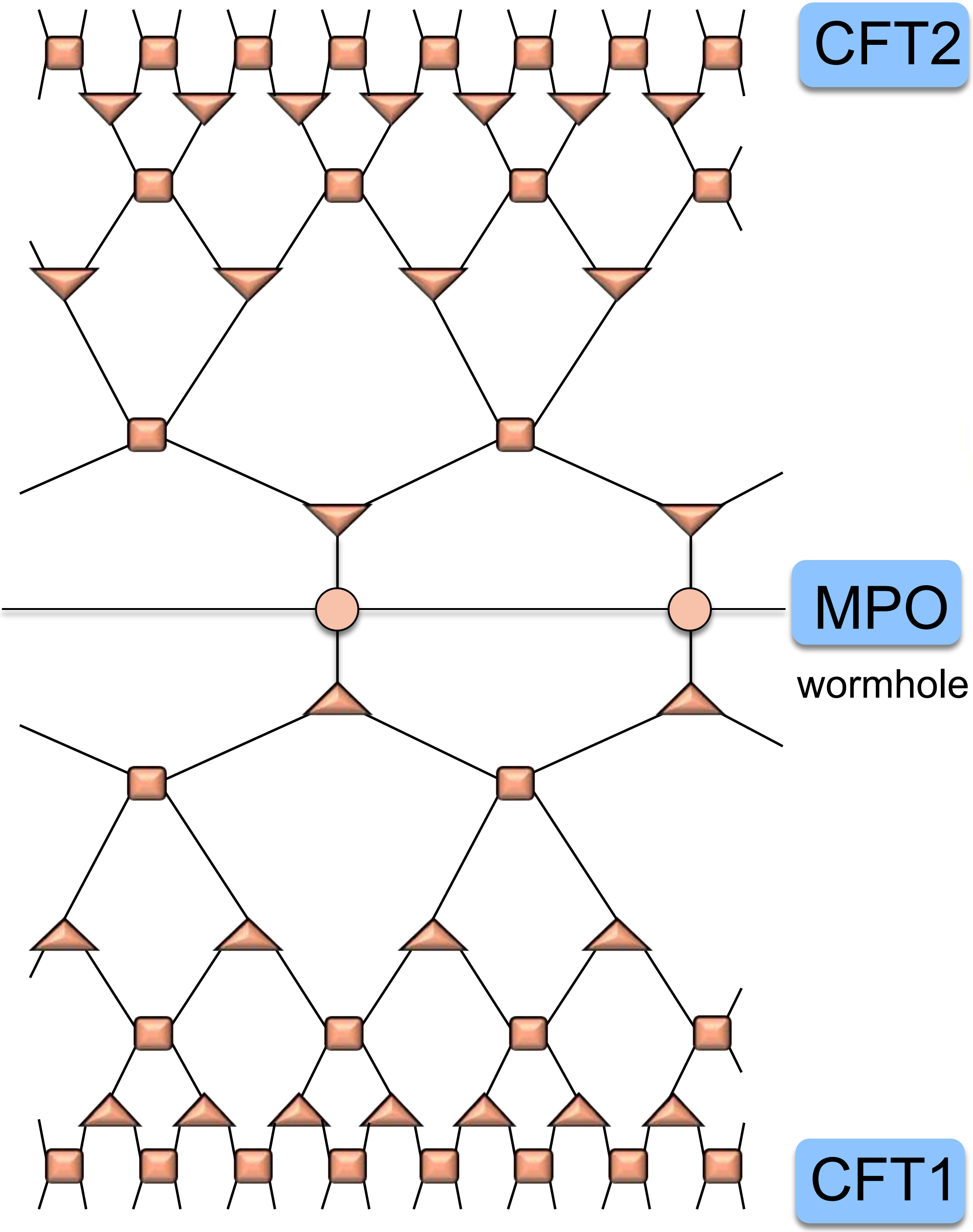


e.g., T. Hartman, J. Maldacena, JHEP05(2013)014

Thermofield double state

Eternal AdS black-hole

$$|TFD\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta E_n/2} |n\rangle_1 |n\rangle_2$$



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Thermofield double state

Eternal AdS black-hole

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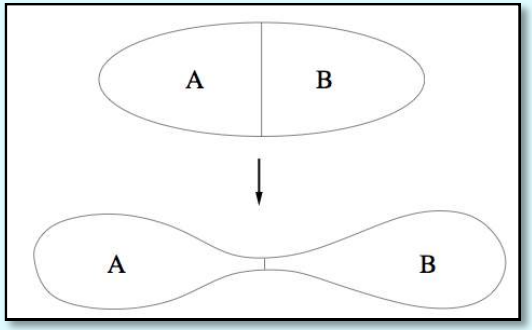
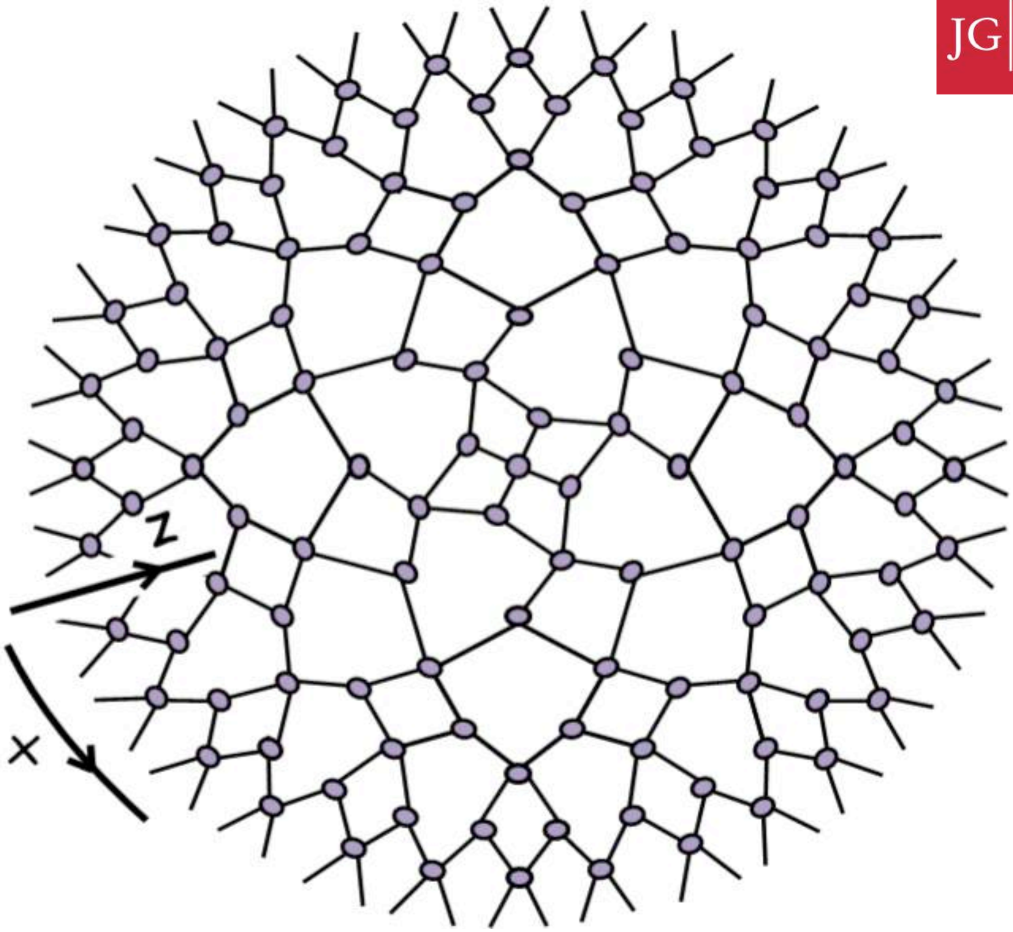
Entanglement connects upper and lower spacetimes



M. Van Raamsdonk, arXiv:0907.2939

ER=EPR, Maldacena & Susskind

MERA



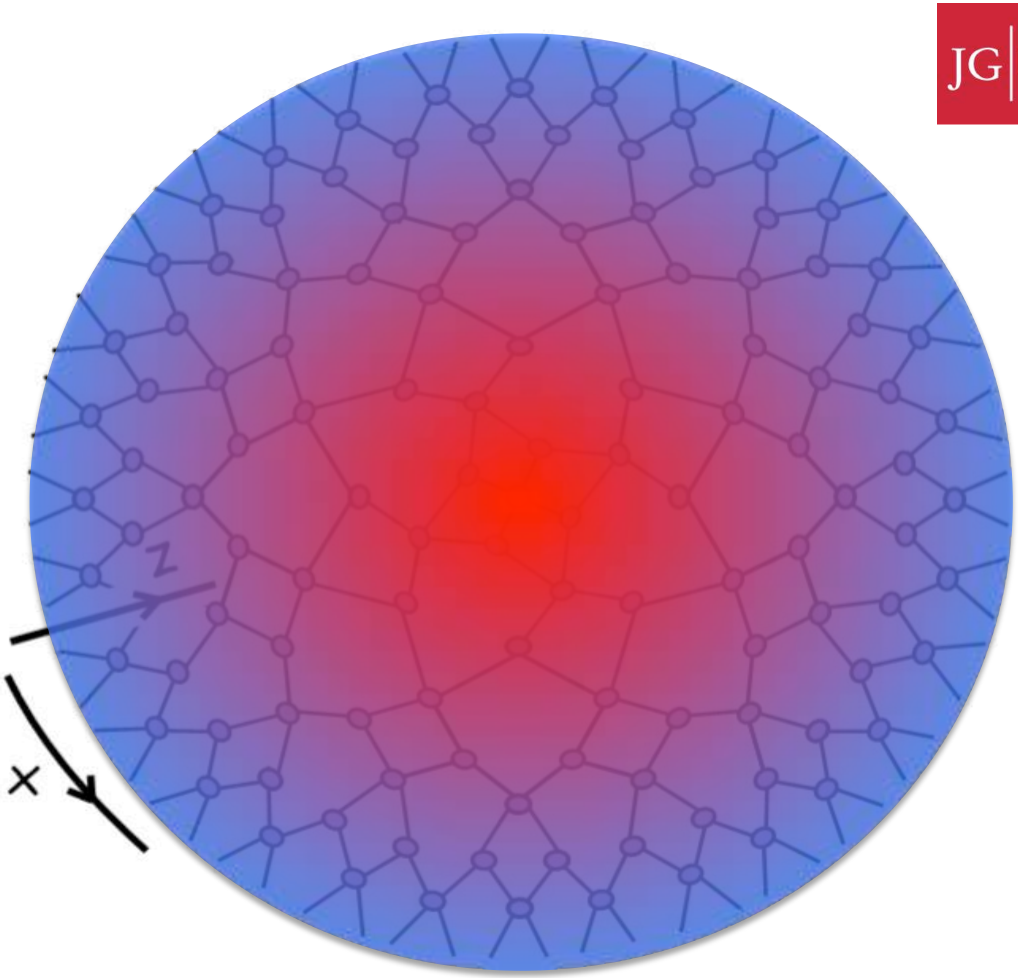
cMERA

(continuum)

$$\left|\psi\right\rangle = Pe^{-i\int_{u2}^{u1}\left(K(u)+L\right)du}\left|\Omega\right\rangle$$

J. Haegeman et al,
Phys. Rev. Lett. 110, 100402 (2013)

$K(u)$	Disentangler generator
L	Isommetry generator



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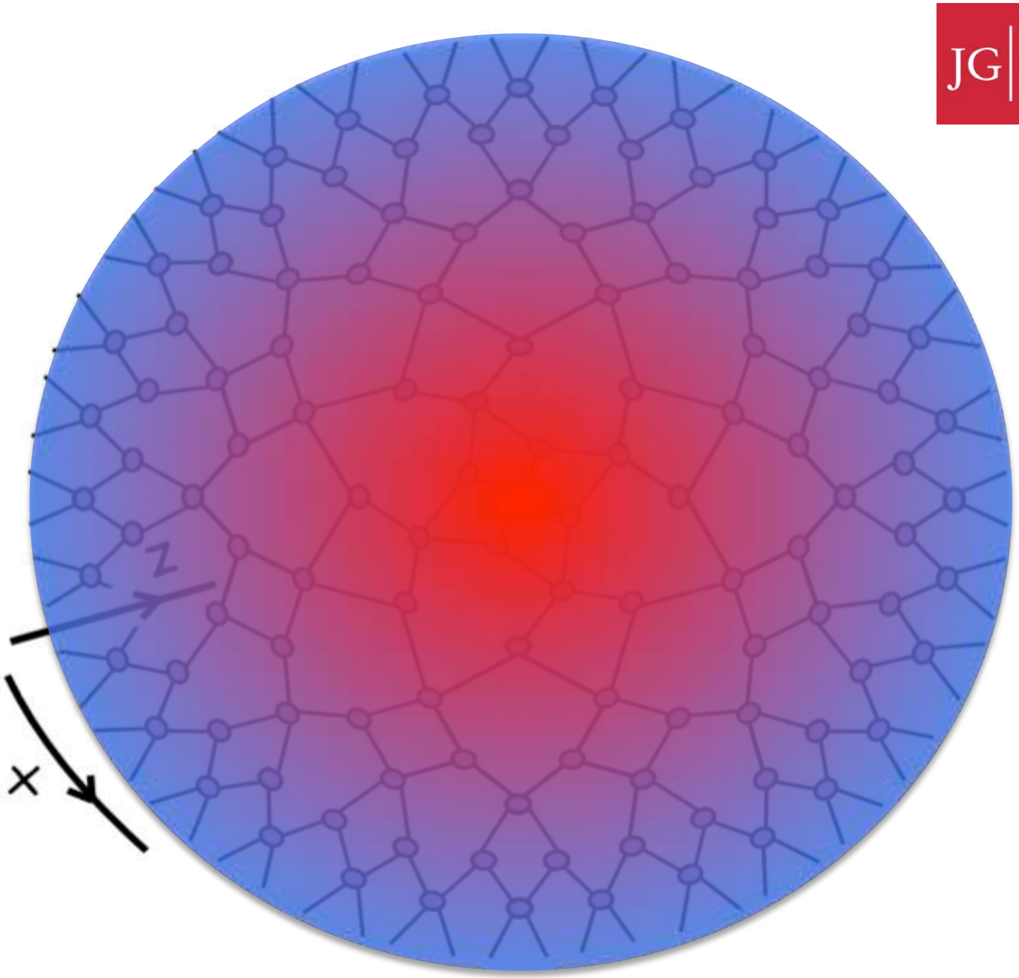

$$g_{uu}(u)du^2 = \mathcal{N}^{-1}\left(1-\left|\langle\Psi(u)|e^{iL\cdot du}|\Psi(u+du)\rangle\right|^2\right)$$

Measures the density of strength of disentanglers.
Compatible with AdS metric

M. Nozaki, S. Ryu, T. Takayanagi, JHEP10(2012)193

curvature ~ change of entanglement at every length scale

Strong subadditivity is trivial

$$S(A\cup B)+S(B\cup C)\geq S(A\cup B\cup C)+S(B)$$

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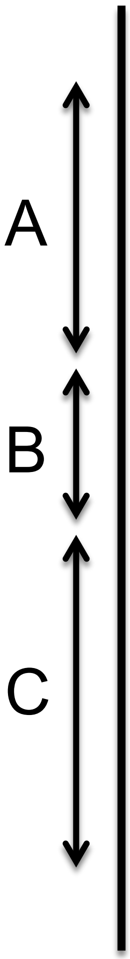
$$S(A \cup B) + S(B \cup C) \geq S(A \cup B \cup C) + S(B)$$



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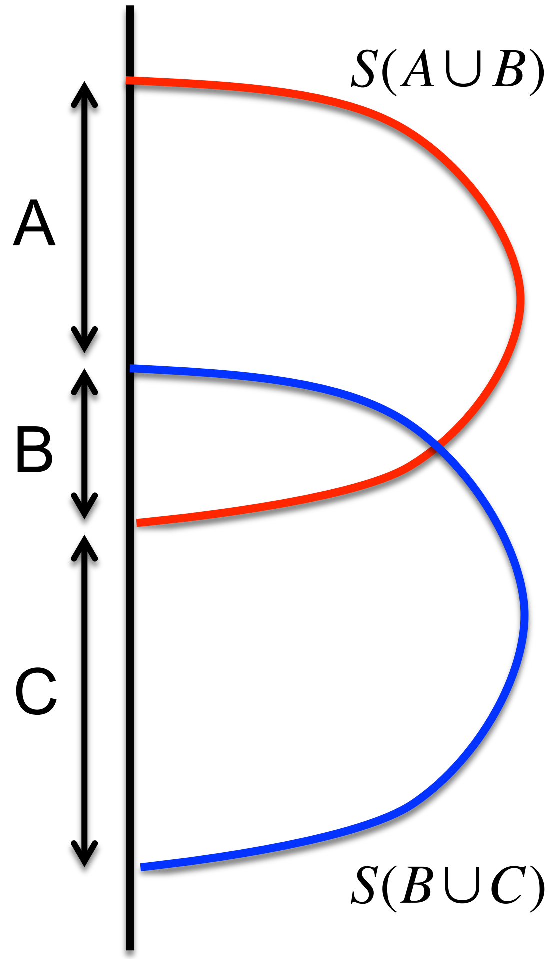

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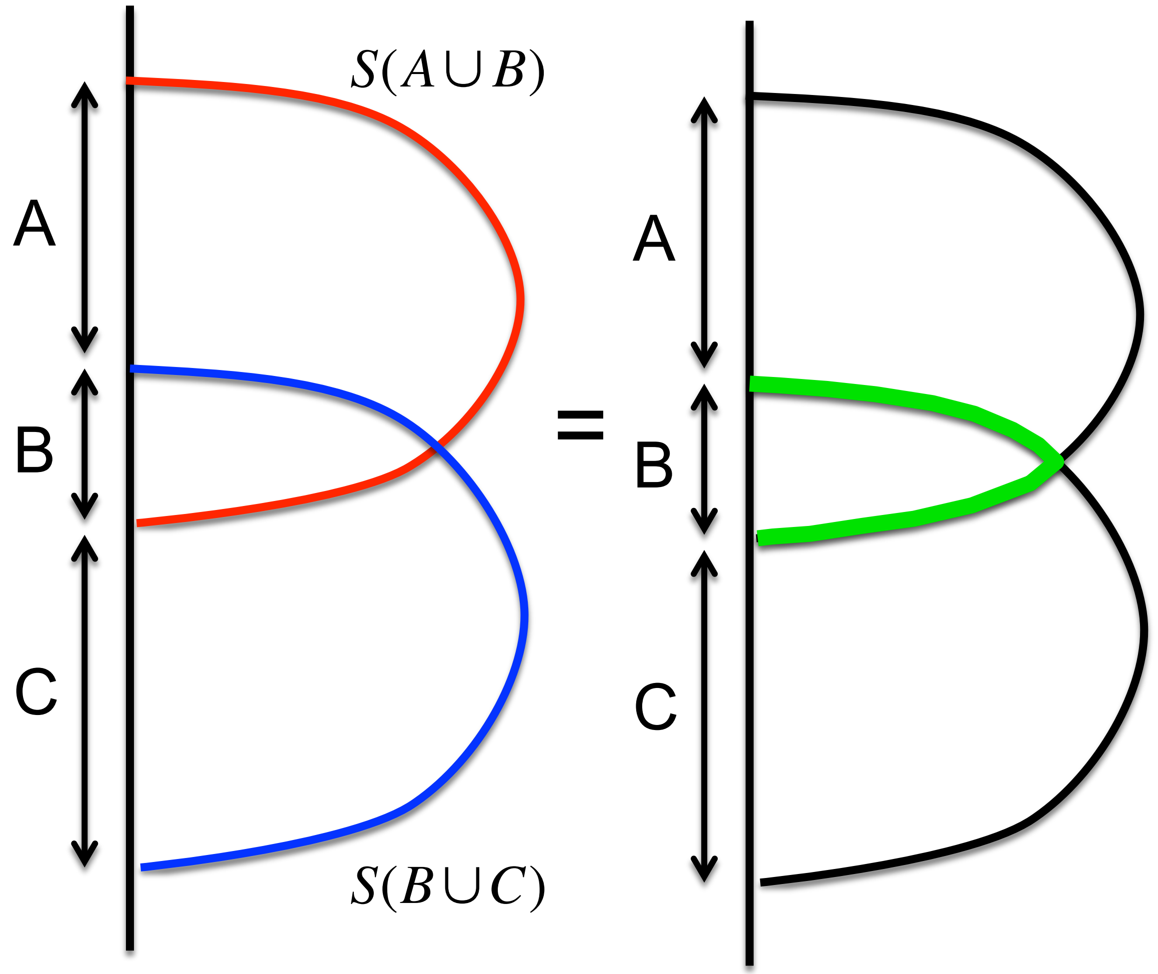

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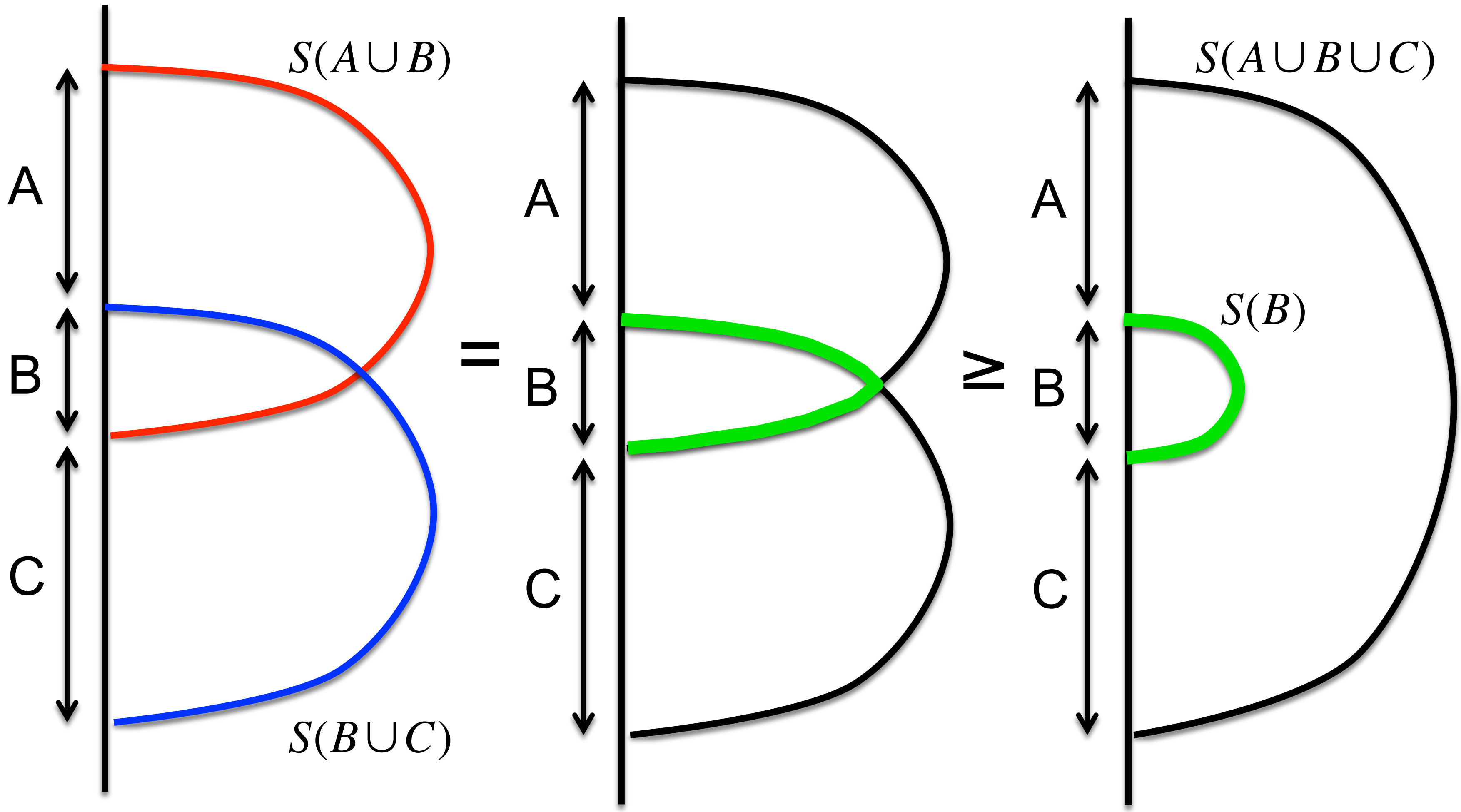
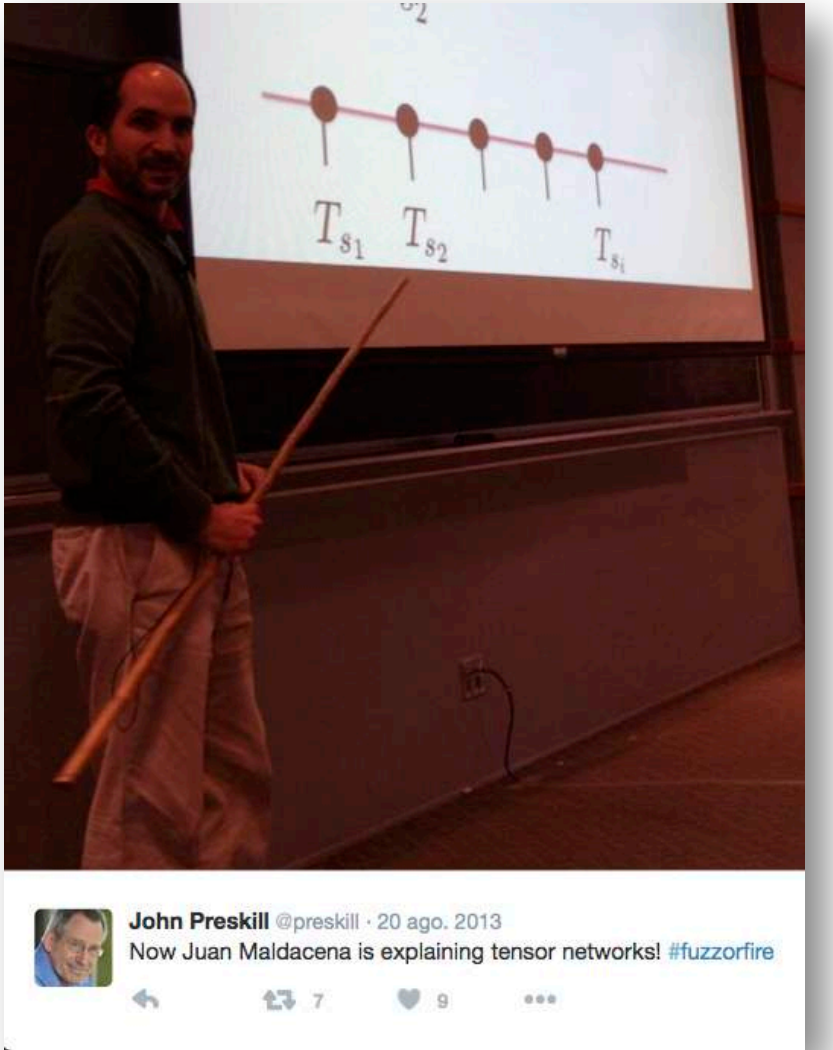
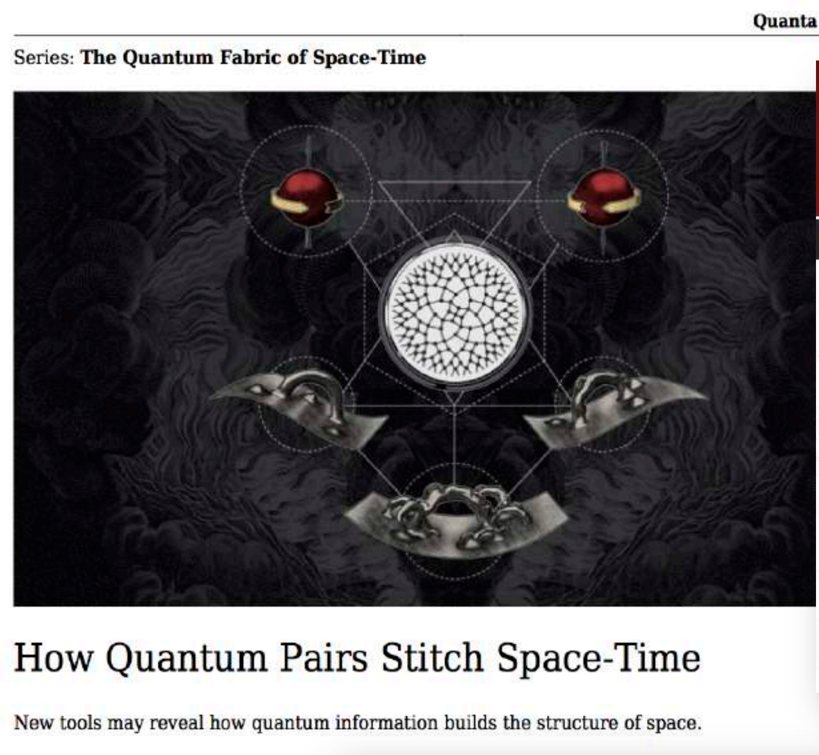


Getting popular even on Twitter…



John Preskill @preskill · 20 ago. 2013
Now Juan Maldacena is explaining tensor networks! #fuzzorfire

… and the media …



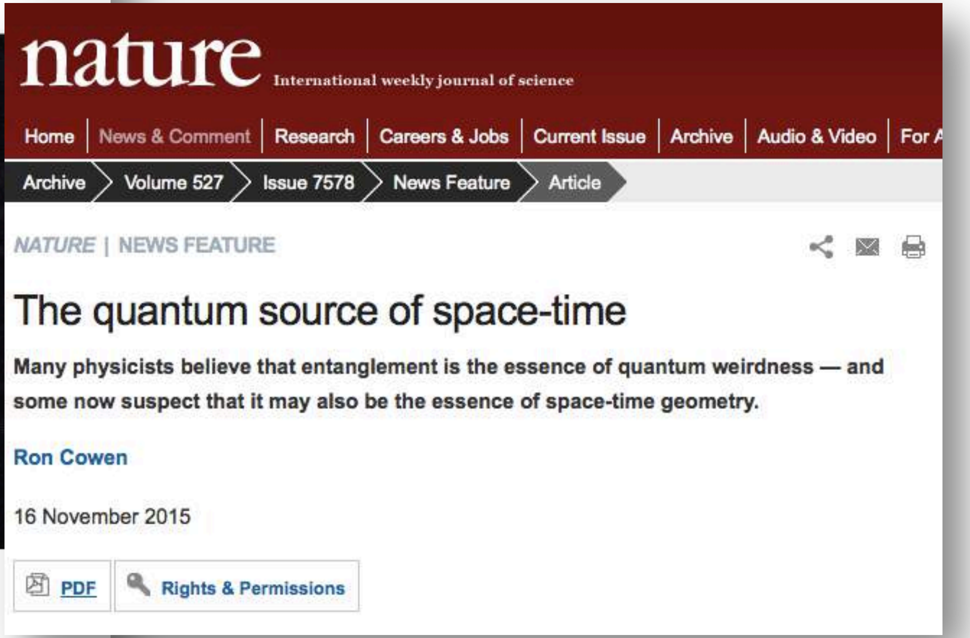
Quanta Magazine

Series: **The Quantum Fabric of Space-Time**

How Quantum Pairs Stitch Space-Time

New tools may reveal how quantum information builds the structure of space.

By Jennifer Ouellette



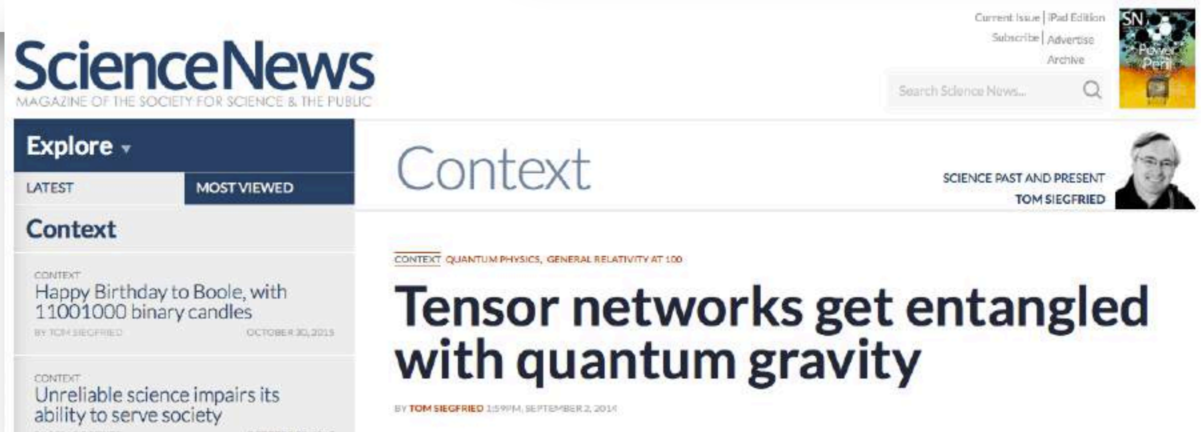
NATURE | NEWS FEATURE

The quantum source of space-time

Many physicists believe that entanglement is the essence of quantum weirdness — and some now suspect that it may also be the essence of space-time geometry.

Ron Cowen

16 November 2015



Context

CONTEXT
Happy Birthday to Boole, with 11001000 binary candles
BY TOM SIEGFRIED OCTOBER 30, 2015

CONTEXT
Unreliable science impairs its ability to serve society
BY TOM SIEGFRIED OCTOBER 26, 2015

CONTEXT QUANTUM PHYSICS, GENERAL RELATIVITY AT 100

Tensor networks get entangled with quantum gravity

BY TOM SIEGFRIED 1:59PM, SEPTEMBER 2, 2014

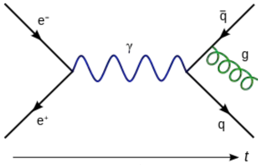
Explosion in recent years

Entanglement and Tensor Networks

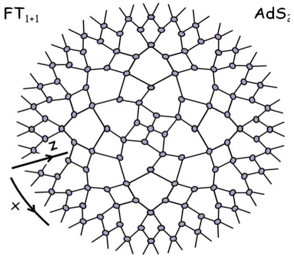
Explosion in recent years

Entanglement and Tensor Networks

HEP and lattice gauge theories



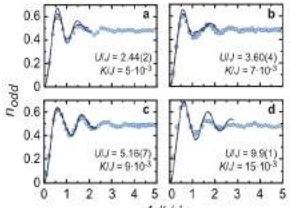
Quantum gravity, string theory and AdS/CFT



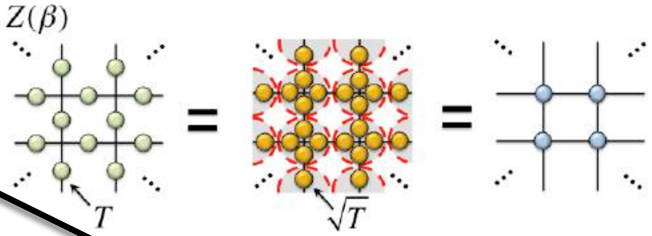
Quantum information and computation

Strongly correlated systems

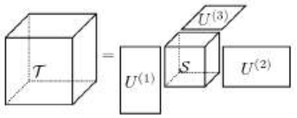
Quantum simulations



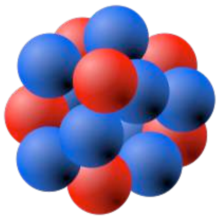
Classical statistical mechanics



Numerical tensor calculus

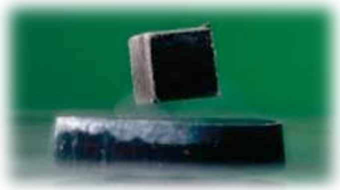


Nuclear physics



Quantum chemistry

Materials science



AI, Deep learning & Linguistics