CFT and Entanglement Entropy

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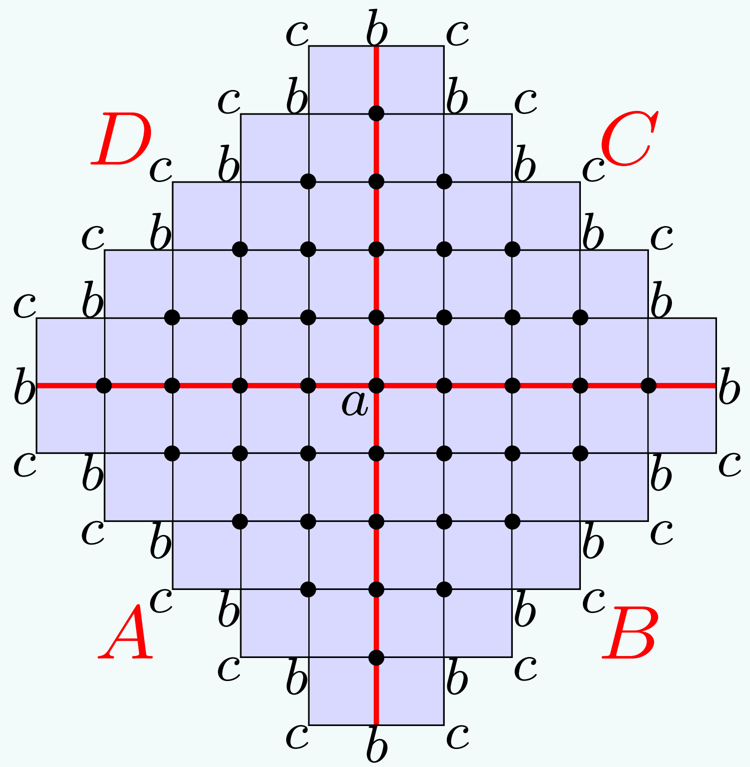

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Outline

- **Orús: Entanglement and Tensor Networks**

(Numerical Aspects)

- **Fazio: Entanglement and Quantum Phase Transitions**

(General Theory)

- **Pearce: Conformal Field Theory and Entanglement Entropy**

(Analytic Calculation)

Section 1: Fundamental Concepts

Section 2: Eight-Vertex/XYZ Chain

Section 3: Unitary RSOS Models

Section 4: Nonunitary Models

Quotes/Abstract

1919 Niels Bohr: "If quantum mechanics hasn't profoundly shocked you, you haven't understood it yet."

1947 Albert Einstein: Quantum mechanics "cannot be reconciled with the idea that physics should represent reality in time and space, free from spooky action at a distance".

1985 John Bell: "There is a way to escape the inference of superluminal speeds and spooky action at a distance. But it involves absolute determinism in the universe, the complete absence of free will."

1989 David Mermin (often attributed to Richard Feynman): "Shut up and calculate!"

Abstract

One-dimensional Hamiltonians, such as the XYZ quantum spin chain, exhibit quantum phase transitions. The universal behaviours at such critical points are described, in the continuum scaling limit, by Conformal Field Theories (CFTs). In the first instance, CFTs are characterized by a central charge c. Entanglement entropy provides a convenient means to determine the central charge either numerically or analytically through entanglement "Area Laws". In these lectures, we review the application of Yang-Baxter methods and Corner Transfer Matrices to obtain the exact entanglement entropy and central charge for the unitary XYZ quantum spin chain and its specializations. The extension of these methods to other unitary and nonunitary models will also be discussed. In particular, we consider the minimal models $\mathcal{M}(m, m')$ as the continuum scaling limit of the RSOS(m, m') lattice models.

Section 1: Fundamental Concepts

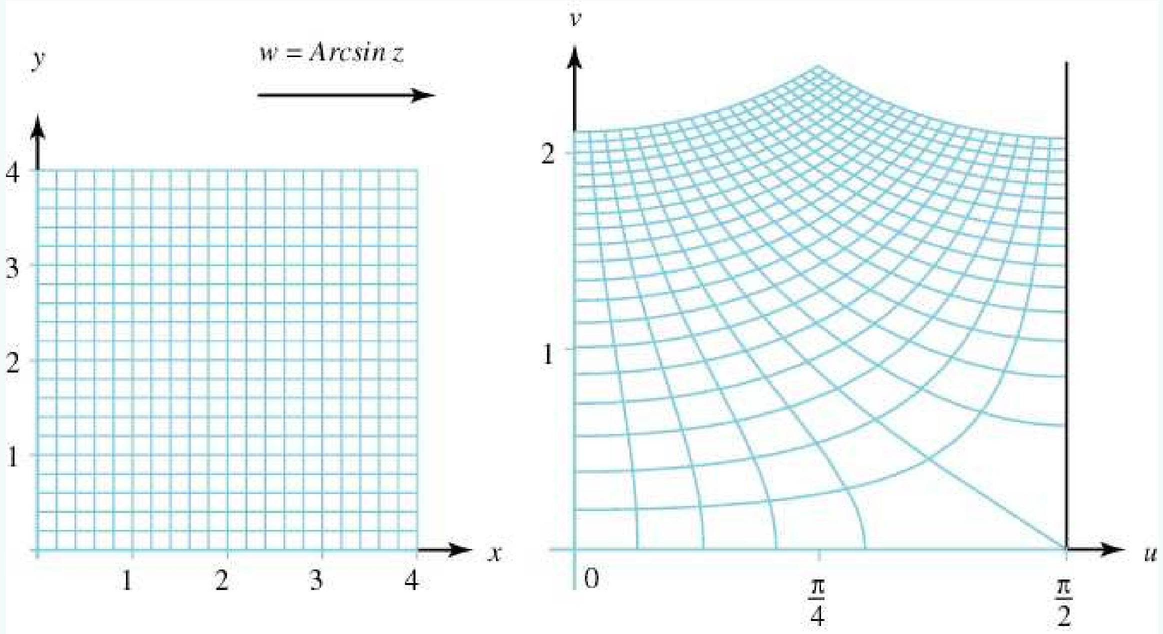
1. Scaling and Conformal Field Theory (CFT)
2. Central charge, effective central charge and conformal weights
3. von Neumann and Rényi entanglement entropies, Schmidt decomposition

John von Neumann (1903–1957)

Alfréd Rényi (1921–1970)

- von Neumann: "There's no sense in being precise when you don't even know what you're talking about".
- Rényi: "A mathematician is a device for turning coffee into theorems".

Scaling and CFT



- **Classical versus Quantum Models**: Quantum models in d-dimensions are related to classical models in $(d+1)$-dimensions. In this way, properties of 1-dimensional quantum Hamiltonians can be obtained by studying 2-dimensional classical systems (say via transfer matrices).

- **Scaling**: Critical systems (those with correlation length $\xi = \infty$) exhibit translational, rotational and scale invariance. Remarkably, they are also invariant under local scale transformations (preserving local angles). In 2-dimensions, these infinitessimal conformal transformations form an infinite-dimensional group corresponding to the group of analytic mappings of the plane.

- **Virasoro Algebra**: The Lie algebra (describing the infinitessimal conformal transformations) of the conformal symmetry group is the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \qquad m, n \in \mathbb{Z}, \quad c \in \mathbb{R}$$

- **Central Charge**: The scalar central charge c is a number characterizing the CFT.

Central Charge and Finite-Size Behaviour

- At criticality, the continuum scaling limit ($N \to \infty$, $a \to 0$, $Na = x \in \mathbb{R}$) of a 2-d lattice model (and its associated quantum Hamiltonian) is described by a CFT characterized by a set of conformal data:
  - The central charge c.
  - The conformal weights Δ.
  - Conformal characters and partitions functions.

- Traditionally (Blöte, Cardy, Nightingale 1986), the conformal data is calculated from the universal finite-size corrections to the eigenvalues of the periodic transfer matrix $\boldsymbol{T}(u)$ on the cylinder and double row transfer matrix $\boldsymbol{D}(u)$ on the strip. The spectral parameter u is essentially the spatial anisotropy.

- The universal finite-size behaviour of eigenvalues for large system size N is

$$\begin{aligned}\log T(u) &\sim -N f_{\mathrm{bulk}}(u) - \tfrac{2\pi}{N}\Big[(-\tfrac{c}{12}+\Delta+\bar{\Delta}+k+\bar{k})\sin\vartheta + (-\tfrac{c}{12}+\Delta-\bar{\Delta}+k-\bar{k})\, i\cos\vartheta\Big] \\ \tfrac{1}{2}\log D(u) &\sim -N f_{\mathrm{bulk}}(u) - f_{\mathrm{bdy}}(u) - \tfrac{\pi}{N}\Big[(-\tfrac{c}{24}+\Delta+k)\sin\vartheta\Big]\end{aligned}$$

where $\vartheta = \frac{\pi u}{\lambda}$ and $k, \bar{k}$ are non-negative integer levels. Similar formulas apply to the eigenenergies of 1-d quantum chains.

- For the 2-d critical Ising model $\mathcal{M}(3,4)$ and the related 1-d quantum transverse Ising model, $c = \frac{1}{2}$ and $\Delta, \bar{\Delta} \in \{0, \frac{1}{16}, \frac{1}{2}\} = \{\Delta_{r,s}\}$ where r, s are Kac labels. For the Minimal Models $\mathcal{M}(m, m')$

$$c = 1 - \frac{6(m'-m)^2}{mm'}, \quad \Delta_{r,s} = \frac{(m'r-ms)^2-(m'-m)^2}{4mm'}, \quad r = 1,2,\ldots,m-1, \quad s = 1,2,\ldots,m'-1$$

Unitary vs Nonunitary

- A CFT is unitary if $L_n^\dagger = L_{-n}$ with respect to a positive definite inner product on states. Theories with Hermitian Hamiltonians/transfer matrices are unitary.

- For unitary theories, $\Delta, \bar{\Delta} \geq 0$ and $\Delta = \bar{\Delta} = 0$ for the ground state (vacuum $|0\rangle$). A theory with $\Delta_{\text{min}} < 0$ must be nonunitary. This can happen for theories with real spectra even though the Hamiltonian/transfer matrices are not Hermitian (non-Hermitian quantum mechanics).

- Some examples of nonunitary theories include the Yang-Lee theory, critical dense polymers and critical percolation. In such cases, the effective central charge is defined as

$$-\frac{c}{24} + \Delta_{\text{min}} = -\frac{c_{\text{eff}}}{24}, \qquad c_{\text{eff}} = c - 24\Delta_{\text{min}}$$

Effectively, the central charge and conformal weights are "shifted" so that $\tilde{\Delta} = \Delta - \Delta_{\text{min}} \geq 0$.

- Note that c always occurs in the combination $-\frac{c}{24} + \Delta$ in the universal finite-size corrections. So such calculations cannot distinguish between c and c_{eff}.

- As we will see, entanglement entropy provides an alternative means to analytically calculate the central charge c.

- It is an interesting open question as to whether quantum entanglement can distinguish between the central charge c and the effective central charge c_{eff} for nonunitary theories.

Reduced Density Matrices

- Consider a quantum system with a Hermitian Hamiltonian so that left- and right-eigenvectors coincide and the associated CFT is unitary. A mixed state is represented (via spectral decomposition) as a density matrix

$$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|, \qquad p_j \geq 0, \quad \sum_j p_j = 1, \quad \mathrm{Tr}\rho = 1, \quad p_j = \text{probability of state } j$$

where the one-dimensional projectors have unit trace

$$\mathrm{Tr}\,|\psi_j\rangle\langle\psi_j| = 1$$

and the eigenstates $|\psi_j\rangle$ are orthonormal. For a pure state $\rho = |\psi\rangle\langle\psi|$ and $p_j = \delta_{j,1}$.

- If A, B are two complementary subsystems of $A \cup B$, a *bipartition* of a 1-d quantum system acting on a Hilbert space H is given by

$$\mathsf{H} = \mathsf{H}_A \otimes \mathsf{H}_B, \qquad \mathsf{H} = \otimes^N \mathbb{C}^2 \text{ for spin systems}$$

- Let $|j\rangle_A, |j\rangle_B$ be orthonormal bases for $\mathsf{H}_A, \mathsf{H}_B$. Then, for the normalized ground state $|\psi\rangle \in \mathsf{H}_A \otimes \mathsf{H}_B$, the reduced density matrix is given by the partial trace

$$\rho_A = \mathrm{Tr}_B\,\rho = \sum_j \langle j|_B(|\psi\rangle\langle\psi|)|j\rangle_B, \qquad \rho = |\psi\rangle\langle\psi| = \text{projector onto state } \psi$$

where we trace out on system B. Clearly, for a normalized state, $\mathrm{Tr}_A\,\rho_A = \mathrm{Tr}\rho = 1$.

- For example, the reduced density matrix for the entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B)$$

is

$$\rho_A = \tfrac{1}{2}\mathrm{Tr}_B(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B)(\langle 0|_A \otimes \langle 1|_B - \langle 1|_A \otimes \langle 0|_B) = \tfrac{1}{2}(|0\rangle_A\langle 0|_A + |1\rangle_A\langle 1|_A)$$

Von Neumann/Rényi Entanglement Entropies

- The von Neumann entanglement entropy is defined by

$$S(\rho_A) = -\mathrm{Tr}_A[\rho_A \log \rho_A], \quad \Rightarrow \quad S(\rho_B) = -\mathrm{Tr}_B[\rho_B \log \rho_B] = S(\rho_A)$$

A pure state has entropy $S = -\lambda \log \lambda|_{\lambda=1} = 0$ and is not entangled. Often $\log_2$ is used.

- The Rényi Entanglement Entropy is a generalization of the von Neumann entropy given "by taking α independent replicas" (the von Neumann limit follows from l'Hôpital's rule)

$$S_\alpha(\rho_A) = \frac{1}{1-\alpha} \log \mathrm{Tr}_A(\rho_A)^\alpha, \qquad S(\rho_A) = \lim_{\alpha\to 1} S_\alpha(\rho_A) = -\mathrm{Tr}_A[\rho_A \log \rho_A], \qquad \alpha \geq 0,\ \alpha \neq 1$$

- Consider a 1-d critical quantum system and let A be an interval of length ℓ and B the rest of the infinite real line. For large ℓ, the von Neumann entropy of the ground state $|0\rangle$ behaves as

$$S_A \sim \frac{c}{3} \log \frac{\ell}{a} + C, \qquad \ell \to \infty$$

- Similarly, if L, R are the left-, right-halves of the real line, the von Neumann and Rényi entropies of the ground state $|0\rangle$ behave as

$$S(\rho_L) \sim \frac{c}{6} \log \frac{\xi}{a} + C, \qquad S_\alpha(\rho_L) \sim \frac{c}{12}\Big(\frac{1+\alpha}{\alpha}\Big) \log \frac{\xi}{a} + C, \qquad \xi \to \infty$$

- In these "Area Laws", c is the central charge, ξ is the correlation length, a is the lattice spacing and C is a non-universal constant. They are called "Area Laws" since, for gapped systems in 3-d, S is expected to be proportional to the *area* separating A and B. The $(1+1)$-d CFT formulas are due to Holzhey, Larsen, Wilczek 1994 and Calabrese, Cardy 2004.

Schmidt Decomposition

- Suppose a bipartite system AB is in the (normalized) state $|\psi\rangle$. Then there exist orthonormal states $|j\rangle_A$ of A and $|j\rangle_B$ of B such that $|\psi\rangle$ is given by the Schmidt decomposition

$$|\psi\rangle = \sum_j \lambda_j \, |j\rangle_A \, |j\rangle_B, \qquad \lambda_j \in [0,1], \quad \sum_j \lambda_j^2 = 1$$

The proof uses the singular value decomposition of matrices (see Wikipedia).

- The number of nonzero terms in the sum is given by $\min[\dim \mathsf{H}_A, \dim \mathsf{H}_B]$. The number of strictly positive $\lambda_j > 0$ gives the Schmidt rank. A state is entangled if the Schmidt rank is greater than one.

- In the Schmidt basis, the spectral decomposition of the reduced density matrices is

$$\rho_A = \sum_j \lambda_j^2 \, |j\rangle_A \langle j|_A, \qquad \rho_B = \sum_j \lambda_j^2 \, |j\rangle_B \langle j|_B$$

The reduced density matrices have common nonzero eigenvalues λ_j^2 so they have the same von Neumann entropy

$$S_A = S_B = -\sum_j \lambda_j^2 \log \lambda_j^2$$

- The set of Schmidt eigenvalues $\{-\log \lambda_j^2\}$ is referred to as the "entanglement spectrum". The entanglement spectrum provides more information than the entanglement entropy.

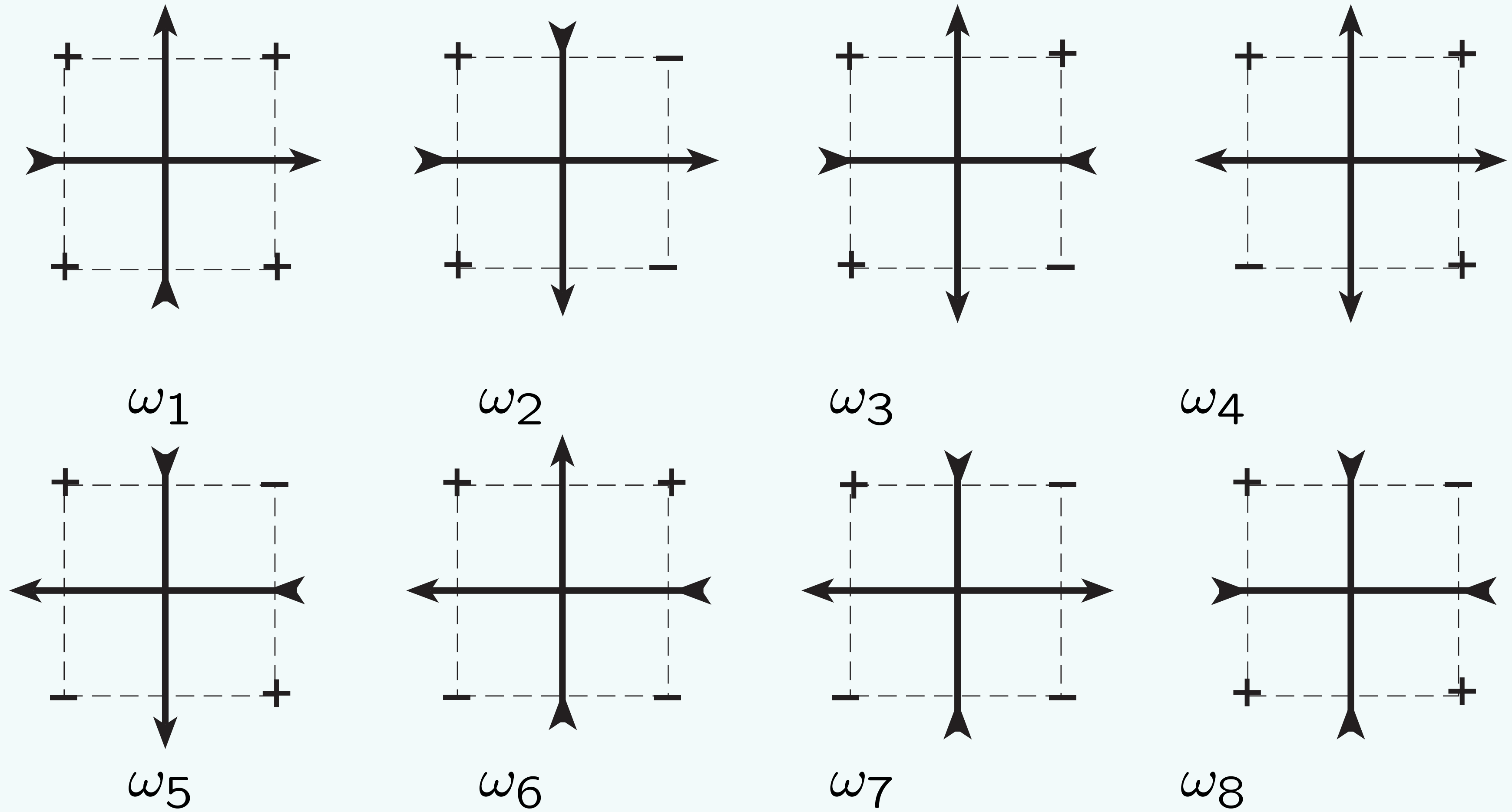
Section 2: Eight-Vertex Model/XYZ Chain

1. Definition, parametrization, integrability, XYZ chain
2. Corner Transfer Matrices (CTMs)
3. Eight-vertex entanglement entropy

Rodney James Baxter (1940–)

- R.Weston, *The entanglement entropy of solvable lattice models*, J. Stat. Mech. L03002 (2006).
- E.Ercolessi, S.Evangelisti, F.Ravanini, *Exact entanglement entropy of the XYZ model and its sine-Gordon limit*, Phys. Lett. A374 2101 (2010).

Eight-Vertex Model



- The eight distinct vertex configurations of the eight-vertex model showing one of the two corresponding configurations of the related Ising model. The spins and arrows are related by

$$\alpha = ab, \quad \beta = bc, \quad \gamma = cd, \quad \delta = da$$

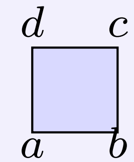
- The model is Yang-Baxter integrable in the symmetric case, $\omega_1 = \omega_5$, $\omega_2 = \omega_6$, $\omega_3 = \omega_7$, $\omega_4 = \omega_8$, when the Boltzmann weights are equal in pairs under arrow reversal.

- The partition function is

$$Z = \sum_{\text{arrow states}} \prod_{\text{faces}} W\begin{pmatrix} & \gamma & \\ \delta & & \beta \\ & \alpha & \end{pmatrix}, \qquad \alpha, \beta, \gamma, \delta = \pm 1$$

Eight-Vertex Ising Model

- The eight vertex model on the square lattice can be formulated as an Ising model with spins at the corners of the elementary faces and Boltzmann face weights

$$W\begin{pmatrix} d & c \\ a & b \end{pmatrix} = R\exp(Kac + Lbd + Mabcd) =$$ 

$$a, b, c, d = \pm 1$$

- The four independent vertex weights are related to R, K, L, M by

$$\omega_1 = \omega_5 = Re^{K+L+M}, \qquad \omega_2 = \omega_6 = Re^{-K-L+M}$$
$$\omega_3 = \omega_7 = Re^{K-L-M}, \qquad \omega_4 = \omega_8 = Re^{-K+L-M}$$

- This is not the usual rectangular Ising model since it involves four-spin interactions in addition to two-spin interactions. The case $M = 0$ corresponds to two decoupled Ising models. This mapping is one-to-two since we can arbitrarily fix one spin somewhere on the lattice.
It follows that

$$Z^2_{\text{Ising}} = 2Z_{\text{Vertex}}\Big|_{M=0}$$

- In general, the eight-vertex model is not critical ($\xi < \infty$). The critical case corresponds to the six-vertex model ($\omega_4 = \omega_8 = 0$). On the critical manifold, the correlation length ξ diverges ($\xi = \infty$).

Eight-Vertex Parametrization

- The face weights of the (symmetric) eight-vertex model can be parametrized in terms of elliptic functions

$$\omega_1 = R\,\mathrm{snh}\,\lambda, \quad \omega_2 = R\,k\,\mathrm{snh}\,\lambda\,\mathrm{snh}\,u\,\mathrm{snh}(\lambda-u), \quad \omega_3 = R\,\mathrm{snh}(\lambda-u), \quad \omega_4 = R\,\mathrm{snh}\,u$$

- Here $k = k(q)$ is the elliptic modulus, q is the elliptic nome and

$$\mathrm{snh}\,u = \mathrm{snh}(u,k) = -i\,\mathrm{sn}(iu,k)$$

$$\mathrm{sn}(u,k) = 2q^{1/4}k^{-1/2}\sin\frac{\pi u}{2I}\prod_{n=1}^{\infty}\frac{1-2q^{2n}\cos\frac{\pi u}{I}+q^{4n}}{1-2q^{2n-1}\cos\frac{\pi u}{I}+q^{4n-2}}$$

$$q = \exp(-\tfrac{\pi I'}{I}), \qquad k(q) = 4\sqrt{q}\prod_{n=1}^{\infty}\left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^4, \qquad I(q) = \frac{\pi}{2}\prod_{n=1}^{\infty}\left(\frac{1+q^{2n-1}}{1-q^{2n-1}}\,\frac{1-q^{2n}}{1+q^{2n}}\right)^2$$

- The four face weights depend on the 4 variables

$R =$ overall constant normalization $= R'/\,\mathrm{snh}\,\lambda$
$u =$ the spectral parameter
$\lambda =$ the crossing parameter $=$ constant
$q =$ the departure-from-criticality variable $(0 < q < 1) =$ constant

- The critical six-vertex manifold corresponds to setting $q = k = 0$. In this case, the (ratios of) elliptic functions reduce to trigonometric functions

$$\mathrm{snh}(u,0) \mapsto \sin u$$

Yang-Baxter Integrability

- In the basis $\{+,+\},\{+,-\},\{-,+\},\{-,-\}$, the quantum $\check{\boldsymbol{R}}$-matrix (acting from $\{\gamma,\beta\}$ to $\{\delta,\alpha\}$) adding a single face to the lattice is:

$$W\begin{pmatrix} & \gamma & \\ \delta & & \beta \\ & \alpha & \end{pmatrix}: \quad \check{\boldsymbol{R}}(u) = R\begin{pmatrix} \mathrm{snh}(\lambda-u) & 0 & 0 & k\,\mathrm{snh}\,\lambda\,\mathrm{snh}\,u\,\mathrm{snh}(\lambda-u) \\ 0 & \mathrm{snh}\,\lambda & \mathrm{snh}\,u & 0 \\ 0 & \mathrm{snh}\,u & \mathrm{snh}\,\lambda & 0 \\ k\,\mathrm{snh}\,\lambda\,\mathrm{snh}\,u\,\mathrm{snh}(\lambda-u) & 0 & 0 & \mathrm{snh}(\lambda-u) \end{pmatrix}$$

This action extends to a length N chain of spins by taking tensor products. For example,

$$\check{\boldsymbol{R}}_2(u) = \boldsymbol{I}\otimes\check{\boldsymbol{R}}(u)\otimes\boldsymbol{I}\otimes\cdots\otimes\boldsymbol{I}$$

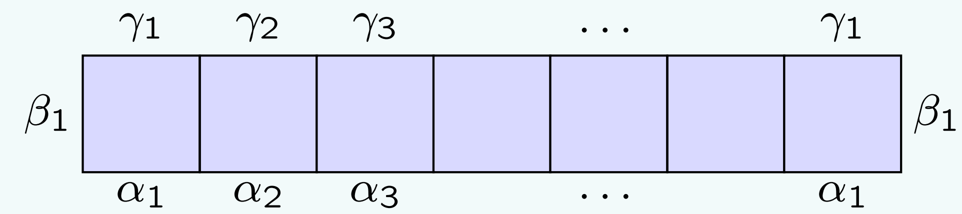
- The R-matrix satisfies an inversion relation and the Yang-Baxter Equation (YBE)

$$\check{\boldsymbol{R}}_j(u)\check{\boldsymbol{R}}_j(-u) = R^2\,\mathrm{snh}(\lambda-u)\,\mathrm{snh}(\lambda+u)\boldsymbol{I}$$

$$\check{\boldsymbol{R}}_j(u)\check{\boldsymbol{R}}_{j+1}(u+v)\check{\boldsymbol{R}}_j(v) = \check{\boldsymbol{R}}_{j+1}(v)\check{\boldsymbol{R}}_j(u+v)\check{\boldsymbol{R}}_{j+1}(u)$$

- The YBE implies commutation of the row transfer matrices

$$\langle\boldsymbol{\alpha}|\boldsymbol{T}(u)|\boldsymbol{\gamma}\rangle = \sum_{\beta_1,\beta_2,\ldots,\beta_N=\pm1}\prod_{j=1}^{N} W\left(\begin{matrix} & \gamma_j & \\ \beta_j & & \beta_{j+1} \\ & \alpha_j & \end{matrix}\middle|\, u\right) =$$

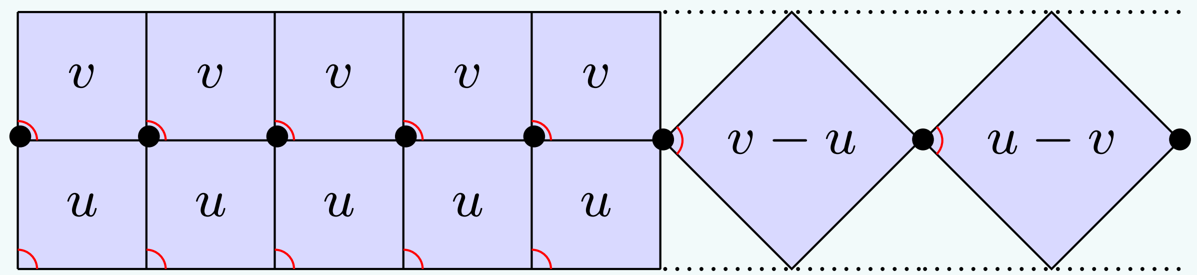


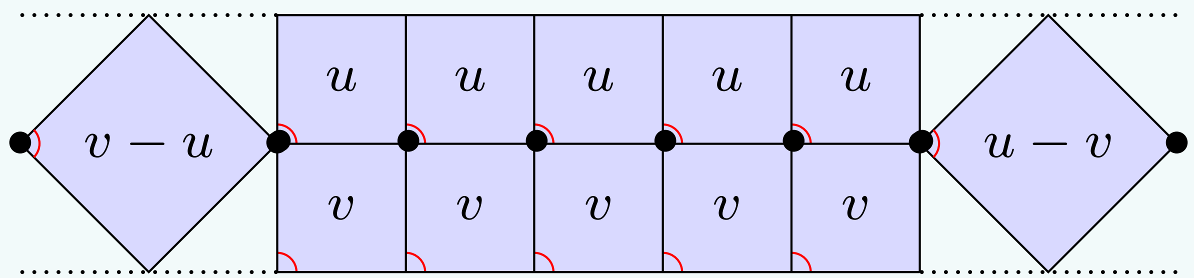
$$[\boldsymbol{T}(u),\boldsymbol{T}(v)] = 0$$

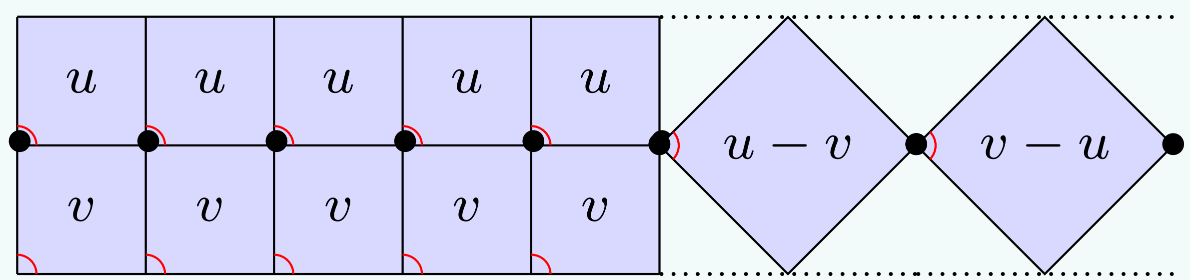
- In general, the eight-vertex transfer matrices are not Hermitian.

Commuting Periodic Row Transfer Matrices

YBE + Inversion $\Rightarrow$ $[\boldsymbol{T}(u), \boldsymbol{T}(v)] = 0$ $\Rightarrow$ Integrable

$\boldsymbol{T}(u)\boldsymbol{T}(v) \;=$ 

$=$ 

$=$ 

$= \;\boldsymbol{T}(v)\boldsymbol{T}(u)$

- Commuting (normal) transfer matrices have a common set of u-independent eigenvectors and are therefore simultaneously diagonalizable. The eigenvalue spectra can be found by solving certain functional equations in the form of T-systems, Y-systems and Baxter's T-Q equation (Bethe Ansatz).

XYZ Chain

- We regard the *crossing parameter* λ as constant, u as a variable and write the transfer matrix as $\boldsymbol{T}(u)$. Since $[\boldsymbol{T}(u), \boldsymbol{T}(v)] = 0$, $\boldsymbol{T}(u)$ is a one-parameter family of commuting transfer matrices.

- The integrable XYZ quantum spin chain belongs to the commuting family $\boldsymbol{T}(u)$. Explicitly, the logarithmic derivative of the eight vertex transfer matrix yields a Hermitian Hamiltonian

$$\frac{d}{du}\log \boldsymbol{T}(u)\Big|_{u=0} = \mathcal{H}_{\text{XYZ}} = -\sum_{j=1}^{N}(J_x\sigma_j^x\sigma_{j+1}^x + J_y\sigma_j^y\sigma_{j+1}^y + J_z\sigma_j^z\sigma_{j+1}^z), \qquad [\boldsymbol{T}(u), \mathcal{H}_{\text{XYZ}}] = 0$$

where $\sigma_j^x, \sigma_j^y, \sigma_j^z$ are Pauli matrices acting on site j and N is the length of the chain. The constants J_x, J_y and J_z allow for anisotropic interactions. If $J_x = J_y$, that is $k = q = 0$, the resulting XXZ model is critical. When $J_x = J_y = J_z = J$, the model reduces to the Heisenberg spin chain

$$\mathcal{H}_{\text{XXX}} = -J\sum_{j=1}^{N}\boldsymbol{\sigma}_j \cdot \boldsymbol{\sigma}_{j+1}$$

- Without loss of generality, we fix the ratios as

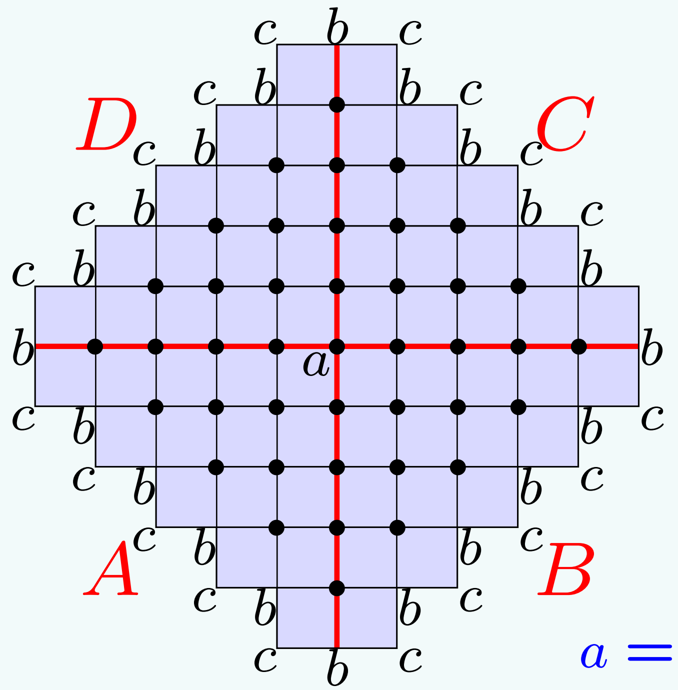
$$J_x : J_y : J_z = 1 : \Gamma : \Delta$$

In the ferromagnetic regime ($\Delta > 1$), the elliptic parametrization is

$$\Gamma = \frac{1 + k\,\text{sn}^2(i\lambda)}{1 - k\,\text{sn}^2(i\lambda)}, \qquad \Delta = -\frac{\text{cn}(i\lambda)\text{dn}(i\lambda)}{1 - k\,\text{sn}^2(i\lambda)}, \qquad 0 < k < 1, \quad 0 < \lambda < I(k')$$

where $\text{sn}(z) = \text{sn}(z,k)$, $\text{cn}(z) = \text{cn}(z,k)$, $\text{dn}(x) = \text{dn}(z,k)$ are standard Jacobi elliptic functions. $I(k')$ is the complete elliptic integral of the first kind of conjugate modulus $k' = \sqrt{1-k^2}$.

Corner Transfer Matrices



$a = \sigma_1, \quad b = c = +1$

- The one-point functions of Yang-Baxter integrable lattice models are obtained using Baxter's Corner Transfer Matrices (CTMs). The idea is to build the square lattice quadrant-by-quadrant. The partition function and one-point function of the eight-vertex Ising model are

$$Z = \mathrm{Tr}\, \boldsymbol{ABCD}, \qquad \langle\sigma_1\rangle = \frac{\mathrm{Tr}\, \boldsymbol{SABCD}}{\mathrm{Tr}\, \boldsymbol{ABCD}}, \qquad \boldsymbol{A}(u) = \boldsymbol{C}(u) = \boldsymbol{B}(\lambda - u) = \boldsymbol{D}(\lambda - u)$$

- Baxter showed the CTMs commute for $N \to \infty$ with eigenvalues that are simple exponentials! The diagonalized CTM Hamiltonian is

$$\boldsymbol{A}_d(u) = \mathrm{Tr}\, z^{2\mathcal{H}_{\mathrm{CTM}}}, \qquad \mathcal{H}_{\mathrm{CTM}} = \frac{1}{2}\sum_{j\geq 1} j(1 - \sigma_j^z \sigma_{j+2}^z), \qquad z = \exp(-\pi u/I(k))$$

Fixing $\mu_j = \sigma_j \sigma_{j+2}$ and the half-rows $\mu = \{\mu_1, \mu_2, \mu_3, \ldots\}$ gives $S_{\mu,\mu} = \sigma_1 = \mu_1\mu_3\mu_5\ldots$ and

$$\boldsymbol{S}_d = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \cdots\cdots, \qquad [A_d(u)]_{\mu,\mu} = \exp\Big[-\frac{\pi u}{4I}\sum_{j\geq 1} j(1-\mu_j)\Big]$$

Eight-Vertex Magnetization/Polarization

- The diagonalized matrices are direct products of 2×2 matrices

$$\begin{aligned}
\boldsymbol{S}_d &= \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \cdots\\
\boldsymbol{A}_d(u) &= \boldsymbol{C}_d(u) = \begin{pmatrix}1 & 0\\ 0 & s\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & s^2\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & s^3\end{pmatrix} \otimes \cdots\\
\boldsymbol{B}_d(u) &= \boldsymbol{D}_d(u) = \begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & t^2\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & t^3\end{pmatrix} \otimes \cdots\\
s &= \exp[-\pi u/2I(k)], \qquad t = \exp[-\pi(\lambda - u)/2I(k)]
\end{aligned}$$

- It follows that the *magnetization* is

$$\langle \sigma_1 \rangle = \prod_{n=1}^{\infty} \frac{1 - x^{2n-1}}{1 + x^{2n+1}} = (k')^{1/4} = \left(1 - k^2\right)^{1/8}, \qquad x = (st)^2 = e^{-\epsilon}, \quad \epsilon = \pi\lambda/I(k)$$

where $k' = k'(x)$ is the conjugate elliptic modulus of the (low-temperature) nome x. The terms with even powers of x cancel in the numerator and denominator.

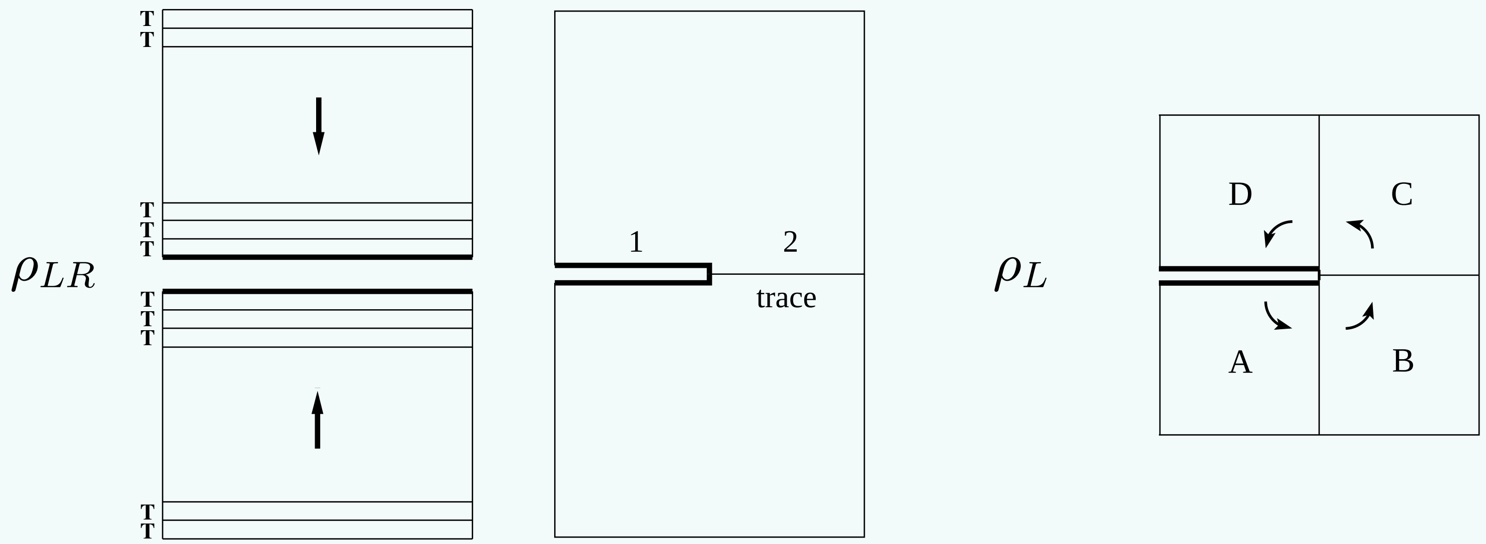
- The *polarization* of the eight-vertex model is

$$\langle \alpha \rangle = \langle \sigma_1 \sigma_2 \rangle = \prod_{n=1}^{\infty} \left(\frac{1 - x^n}{1 + x^n} \frac{1 + q^n}{1 - q^n} \right)^2$$

This cannot be obtained by a direct application of CTMs but was conjectured by Baxter and Kelland and subsequently derived by Jimbo, Miwa and Nakayashiki using difference equations.

- Note the notation here is $x = x^2_{\text{Baxter}}$ of Baxter's book.

Eight-Vertex Reduced Density Matrix



- Decompose the Hilbert space $\mathsf{H} = \mathsf{H}_L \otimes \mathsf{H}_R$ where L, R refer to the semi-infinite spin chains on the left and the right. The reduced density matrix of the ground state is

$$\rho_L = \mathrm{Tr}_R\, \rho_{LR}$$

- Now use CTMs with fixed boundary conditions. Since a trace is invariant under a change of basis/diagonalization

$$[\rho_L]_{\sigma,\sigma'} = Z^{-1}(\boldsymbol{ABCD})_{\sigma,\sigma'} = Z^{-1}(\boldsymbol{AB})^2_{\sigma,\sigma'}$$
$$(\rho_L)_d = Z^{-1}\begin{pmatrix}1 & 0\\ 0 & x\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & x^2\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\ 0 & x^3\end{pmatrix} \otimes \cdots \ldots, \qquad x = (st)^2 = e^{-\epsilon}, \quad \epsilon = \pi\lambda/I(k)$$

- The normalizing partition function is

$$Z = Z(\epsilon) = \mathrm{Tr}_L\, \boldsymbol{ABCD} = \mathrm{Tr}_L\, x^{\mathcal{H}_{\mathrm{CTM}}} = \prod_{j=1}^{\infty}(1 + x^j) = \prod_{j=1}^{\infty}(1 + e^{-j\epsilon})$$

Eight-Vertex Entanglement

- Observe that ρ_L is a function of λ and k only. Moreover, ρ_L can be written as

$$\rho_L = Z^{-1}(\boldsymbol{AB})^2 = Z^{-1}e^{-\epsilon\mathcal{N}}, \qquad Z = \mathrm{Tr}\, e^{-\epsilon\mathcal{N}}, \qquad \epsilon = \pi\lambda/I(k)$$

where $\mathcal{N}$ is an operator with integer eigenvalues.

- The von Neumann entropy S is given by

$$S = -\mathrm{Tr}_L\, \rho_L \log \rho_L = \epsilon\, Z^{-1}\mathrm{Tr}\, \mathcal{N} e^{-\epsilon\mathcal{N}} + \log Z = -\epsilon \frac{\partial \log Z}{\partial \epsilon} + \log Z$$

We thus obtain an exact analytic expression for the entanglement entropy of the XYZ model valid for general λ and k

$$S_{\mathrm{XYZ}} = \sum_{j=1}^{\infty} \frac{j\epsilon}{(1+e^{j\epsilon})} + \sum_{j=1}^{\infty} \log(1+e^{-j\epsilon})$$

- Using Euler-Maclaurin in the scaling limit $\epsilon \to 0$, this is approximated by

$$S_{\mathrm{XYZ}} \sim \frac{1}{\epsilon}\int_0^\infty \Big[\frac{y}{1+e^y} + \log(1+e^{-y})\Big]dy + C \sim \frac{\pi^2}{6\epsilon} + C, \qquad \epsilon \to 0$$

After setting $a = e^{-y}$ in the integral with $x = 1$, it becomes

$$\int_0^x \left(\frac{\log(1+a)}{a} - \frac{\log a}{1+a}\right)da = 2L_+(x) = 2L(\tfrac{x}{1+x}), \qquad 2L(\tfrac{x}{1+x})\Big|_{x=1} = 2L(\tfrac{1}{2}) = \tfrac{\pi^2}{6}$$

where $L(x)$ is the Rogers dilogarithm.

Eight-Vertex Entanglement Asymptotics

- The eight-vertex correlation length was calculated by Johnson, Krinsky, McCoy 1973

$$\xi^{-1} = -\tfrac{1}{2}\log k'(x), \qquad x = e^{-\epsilon} \to 0 \quad \text{as} \quad \xi \to \infty$$

where the elliptic moduli are

$$k(q) = 4\sqrt{q}\prod_{n=1}^{\infty}\left(\frac{1+q^{2n}}{1+q^{2n-1}}\right)^4, \qquad k'(q) = \prod_{n=1}^{\infty}\left(\frac{1-q^{2n-1}}{1+q^{2n-1}}\right)^4, \qquad k^2 + (k')^2 = 1$$

- To obtain the limiting behaviour as $\epsilon \to \infty$, we perform a conjugate modulus transformation using the fact that $k' = k(x')$

$$\xi^{-1} = -\tfrac{1}{4}\log[1 - k(x')^2] \sim \tfrac{1}{4}k(x')^2 \sim 4x', \qquad x' = \exp(-\tfrac{\pi^2}{\epsilon}) \to 0 \quad \text{as} \quad \xi \to \infty$$

to find

$$\log\xi \sim \tfrac{\pi^2}{\epsilon} + C + \cdots, \qquad \epsilon \to 0$$

- Combining this with the previous expression for S_{XYZ} gives the von Neumann entropy

$$S_{\text{XYZ}} \sim \frac{c}{6}\log\frac{\xi}{a} + C + \cdots, \qquad \xi \to \infty, \quad c = 1$$

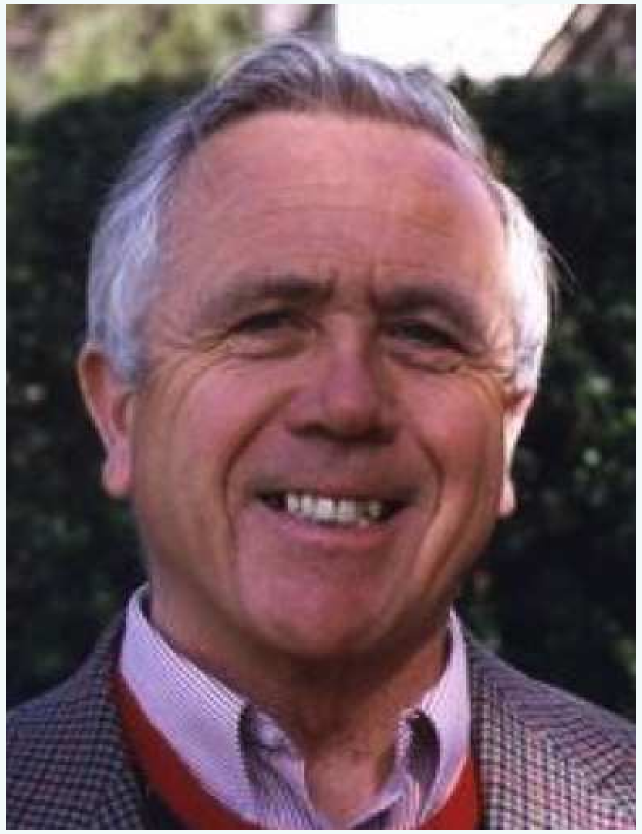
- On specializing, this result applies to the XXX, XXZ models and free fermion (Ising) models. For the Ising case ($M = 0$ at the decoupling point), $c = \frac{1}{2}$ since there are two independent Ising models on the two sublattices ($c = 1 = \frac{1}{2} + \frac{1}{2}$).

Section 3: Unitary RSOS Models

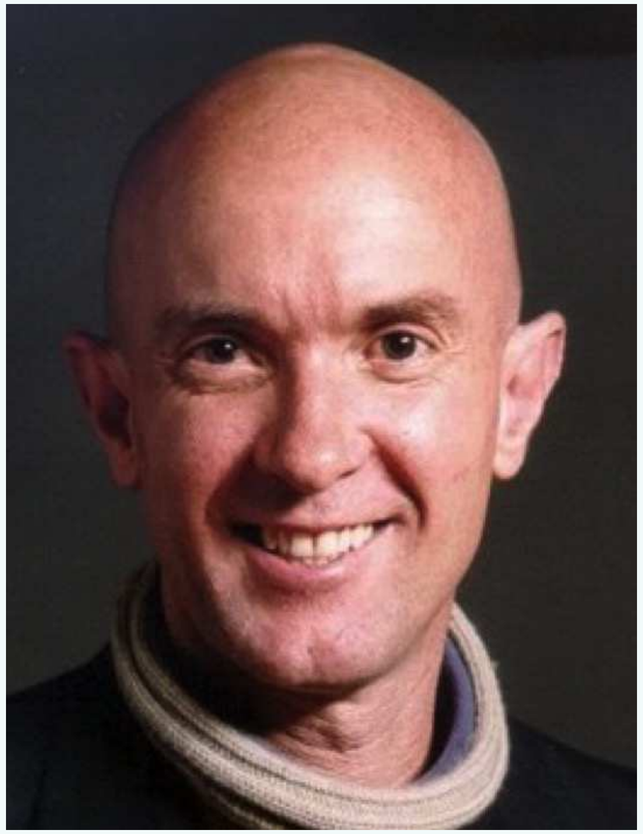
1. Definition of RSOS(m, m') models
2. Reduced density matrices and Rényi entropy
3. Correlation length and asymptotics

George Andrews

Rodney J. Baxter



Peter J. Forrester



- A.DeLuca, F.Franchini, *Approaching the RSOS critical points through entanglement: One model for many universalities*, Phys. Rev. B87, 045118 (2013).

Minimal Models RSOS$(m,m')/\mathcal{M}(m,m')$

- The Restricted Solid-on-Solid (RSOS) lattice models (Andrews, Baxter, Forrester 1984/1985) are defined on a square lattice with heights $a = 1, 2, \ldots, m'-1$. Nearest neighbour heights differ by ± 1 so the heights live on the $A_{m'-1}$ Dynkin diagram. The nonzero Boltzmann weights are

$$\begin{aligned}
W\left(\begin{matrix} a\pm 1 & a \\ a & a\mp 1 \end{matrix}\,\middle|\, u\right) &= \frac{\vartheta_1(\lambda-u)}{\vartheta_1(\lambda)} \\
W\left(\begin{matrix} a & a\pm 1 \\ a\mp 1 & a \end{matrix}\,\middle|\, u\right) &= \frac{g_{a\mp 1}}{g_{a\pm 1}}\,\frac{\vartheta_1((a\pm 1)\lambda)}{\vartheta_1(a\lambda)}\,\frac{\vartheta_1(u)}{\vartheta_1(\lambda)} \\
W\left(\begin{matrix} a & a\pm 1 \\ a\pm 1 & a \end{matrix}\,\middle|\, u\right) &= \frac{\vartheta_1(a\lambda\pm u)}{\vartheta_1(a\lambda)}
\end{aligned}$$

- Here $\vartheta_1(u) = \vartheta_1(u,p)$ is a standard elliptic theta function

$$\vartheta_1(u,p) = 2p^{1/4}\sin u \prod_{n=1}^{\infty}(1-2p^{2n}\cos 2u + p^{4n})(1-p^{2n}), \qquad 0 < p < 1$$

u is the spectral parameter and the elliptic nome $p = e^{-\epsilon}$ is a temperature-like variable, with p^2 measuring the departure from criticality. The inessential gauge factors g_a can be set to 1. The crossing parameter λ is

$$\lambda = \frac{(m'-m)\pi}{m'}, \qquad 2 \le m < m', \qquad m, m' \text{ coprime}$$

- Integrability derives from the fact that these local face weights satisfy the Yang-Baxter equation. The quantum Hamiltonians are given in BEPR2015. The continuum scaling limit is associated with the minimal model $\mathcal{M}(m,m')$. The unitary cases are given by $m' = m+1$.

Unitary RSOS$(m, m+1)$ Reduced Density Matrix

- Using commuting Corner Transfer Matrices, the partition function and Local Height Probabilities (LHPs) of the RSOS models are

$$Z = \text{Tr}\, \boldsymbol{ABCD}, \qquad P_a = \langle \delta_{l_1,a} \rangle = \frac{\text{Tr}\, \boldsymbol{S}_a \boldsymbol{ABCD}}{\text{Tr}\, \boldsymbol{ABCD}}, \qquad \boldsymbol{A}(u) = \boldsymbol{C}(u) = \boldsymbol{B}(\lambda - u) = \boldsymbol{D}(\lambda - u)$$

- It follows that

$$\rho_L = \text{Tr}_R\, \rho_{LR}, \qquad [\rho_L]_{l,l'} = Z^{-1}(\boldsymbol{ABCD})_{l,l'} = Z^{-1}(\boldsymbol{AB})^2_{l,l'}$$

where $l = \{l_1, l_2, \ldots, l_{N+1}, l_{N+2}\}$ and $l_1 = a$. Fixing $l_{N+1} = b$, $l_{N+2} = c = b \pm 1$ determines the $2(m-1)$ ground state boundary conditions in which the bulk heights alternate between b and c.

- After diagonalization, the normalizing partition function is

$$Z = \text{Tr}\, \boldsymbol{ABCD} = \sum_l E(x^{l_1}, x^{m'})\, x^{\phi[l]}, \quad \phi[l] = [\mathcal{H}_{\text{CTM}}]_{l,l} = \tfrac{1}{2} \sum_{j=1}^{N} j|l_j - l_{j+2}|$$

The low-temperature nome x is related to the conjugate nome p' of p by

$$x = \exp(-\tfrac{4\pi^2}{m'\epsilon}), \qquad x^{m'} = \exp(-\tfrac{4\pi^2}{\epsilon}) = p', \qquad p = e^{-\epsilon}, \qquad m' = m+1$$

At criticality

$$p \to 0, \qquad \epsilon \to \infty, \quad x \to 1, \qquad p' \to 1$$

Unitary RSOS$(m, m+1)$ Rényi Entropies

- The RSOS$(m, m+1)$ Rényi entropy for α replicas is given by

$$S_\alpha(\rho_L) = \frac{1}{1-\alpha} \log \mathrm{Tr}_L (\rho_L)^\alpha = \frac{1}{1-\alpha}\Big[\log Z_\alpha - \alpha \log Z_1\Big]$$

where

$$Z_\alpha^{(N)} = \mathrm{Tr}\,(\boldsymbol{ABCD})^\alpha = \sum_{a=1}^{m} \Big[E(x^a, x^{m'})\Big]^\alpha X_N(a,b,c;x^{2\alpha}), \qquad X_N(a,b,c;q) = \sum_{l_2,\ldots,l_N} q^{\phi[l]}$$

The elliptic function $E(z,q)$ is

$$E(z,q) = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(k-1)/2} z^k = \prod_{n=1}^{\infty} (1-q^{n-1}z)(1-q^n z^{-1})(1-q^n)$$

- From Andrews, Baxter, Forrester 1984 it follows that in the thermodynamic limit

$$Z_\alpha = \lim_{N\to\infty} Z_\alpha^{(N)} = x^{\frac{\alpha bc}{2}} (q)_\infty^{-1} \sum_{a=1}^{m} \Big[E(x^a, x^{m'})\Big]^\alpha \Gamma(a,d;q)$$

where $d = \frac{b+c-1}{2}$, $q = x^{2\alpha}$, $(q)_\infty = \prod_{n=1}^{\infty} (1-q^n)$ and

$$\Gamma(a,d;q) = q^{\frac{a(a-1)}{4}} \Big\{ q^{-\frac{ad}{2}} E(-q^{m(m'-a)+m'd}, q^{2mm'}) - q^{\frac{ad}{2}} E(-q^{m(m'+a)+m'd}, q^{2mm'}) \Big\}$$

Unitary RSOS$(m, m+1)$ Asymptotics

- Fixing the central height a and moving to the conjugate nome gives

$$Z_\alpha^{(a)} = \frac{\vartheta_3\Big(\frac{\pi d}{2m}-\frac{\pi a}{2m'}, p^{\frac{1}{8\alpha m}}\Big) - \vartheta_3\Big(\frac{\pi d}{2m}+\frac{\pi a}{2m'}, p^{\frac{1}{8\alpha m}}\Big)}{p^{\frac{m'}{48\alpha}}\sqrt{2mm'}\ \vartheta_4\Big(\frac{im'\epsilon}{8\alpha}, p^{\frac{3m'}{4\alpha}}\Big)}, \qquad p = e^{-\epsilon}$$

where $p \to 0$ at criticality and

$$\vartheta_3(u,t) = \prod_{n=1}^{\infty}(1+2t^{2n-1}\cos 2u + t^{4n-2})(1-t^{2n}) = 1 + 2\sum_{n=1}^{\infty} t^{n^2}\cos 2nu$$
$$\vartheta_4(u,t) = \prod_{n=1}^{\infty}(1-2t^{2n-1}\cos 2u + t^{4n-2})(1-t^{2n}) = 1 + 2\sum_{n=1}^{\infty} (-1)^n t^{n^2}\cos 2nu$$

- We use the known result for the correlation length (O'Brien, Pearce 1997)

$$\xi^{-1} = -\log k'(p^\nu) \sim 8\,p^\nu, \qquad p^\nu \sim \xi^{-1}, \qquad \nu = \frac{m'}{4} = \text{correlation length exponent}$$

- Expanding everything for small p and collecting together gives

$$\log Z_\alpha^{(a)} = -\frac{c_m}{12\alpha}\log p^\nu + C'_{adm'} + \cdots, \qquad c_m = 1 - \frac{6}{m(m+1)}$$

where the boundary entropy of Affleck-Ludwig 1991 is

$$C'_{adm'} = \log\Big(\frac{4}{\sqrt{2mm'}}\sin\frac{\pi d}{m}\sin\frac{\pi a}{m'}\Big)$$

- The leading term of the Renyi entropy is thus of the expected form

$$S_\alpha^{(a)} = \frac{1}{1-\alpha}\Big[\log Z_\alpha^{(a)} - \alpha\log Z_1^{(a)}\Big] \sim \frac{c_m}{12}\Big(\frac{1+\alpha}{\alpha}\Big)\log\xi + C'_{adm'} + \cdots, \qquad c = c_m = 1 - \frac{6}{m(m+1)}$$

Section 4: Nonunitary Models

1. $U_q[sl(2)]$ open XXZ spin chain
2. Temperley-Lieb algebra, Markov trace
3. $N = 2$ and numerical entanglement entropies for large N

Romain Couvreur

Jesper L. Jacobsen

Hubert Saleur

- R.Couvreur, J.L.Jacobsen, H.Saleur, *Entanglement in nonunitary quantum critical spin chains*, Phys. Rev. Lett. 119, 040601 (2017).

$U_q[sl(2)]$ Open XXZ Spin Chain

- Consider the critical open XXZ spin chain

$$H = -\sum_{j=1}^{N-1} e_j, \qquad e_j = -\frac{1}{2}\left[\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \frac{q+q^{-1}}{2}(\sigma_j^z\sigma_{j+1}^z - 1) + h_j\right]$$

where $q = e^{i\lambda}$, $|q| = 1$ and $\sigma_j^{x,y,z}$ are Pauli matrices. The case $h_j = 0$ describes the critical XYZ chain.

- Let us instead add terms with

$$h_j = \frac{q-q^{-1}}{2}(\sigma_j^z - \sigma_{j+1}^z)$$

This breaks Hermiticity but gives the $U_q[sl(2)]$ quantum group invariant open XXZ spin chain. H is not Hermitian but its eigenvalues are real (Morin-Duchesne Et Al 2016). Significantly, its left- and right-eigenstates differ. Notice that the h_j terms act as a gauge and cancel out in the periodic system.

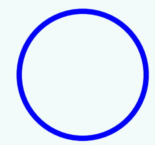
- The e_j now give a representation of the Temperley-Lieb (TL) algebra (Temperley, Lieb 1971)

$$e_j^2 = (q+q^{-1})e_j = 2\cos\lambda\, e_j, \qquad e_j e_{j\pm1} e_j = e_j, \qquad [e_j, e_k] = 0, \quad |j-k| > 1$$

These relations admit a nice diagrammatic interpretation in terms of "monoids".

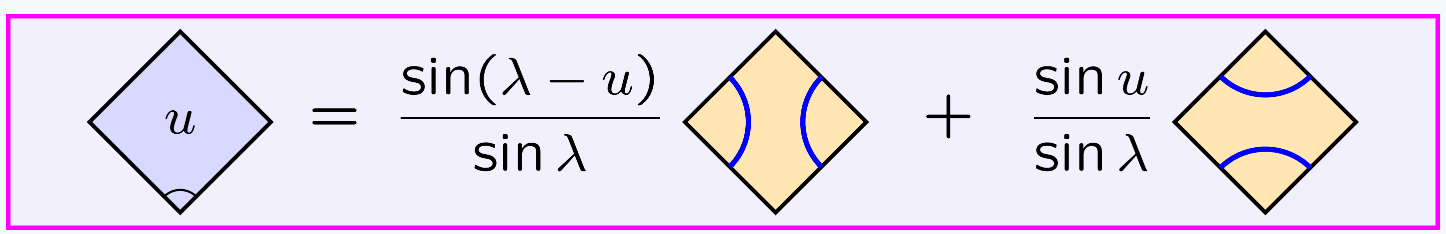
Temperley-Lieb Algebra

- The Temperley-Lieb algebra admits a planar diagrammatic representation consisting of "monoid" diagrams (physically this is a loop gas)

 $= \beta = 2\cos\lambda =$ loop fugacity

- The monoids satisfy

TL ⇒ YBE: The YBE is satisfied if the face weights are of the form



- There are three known types of TL representations yielding exactly solvable models:

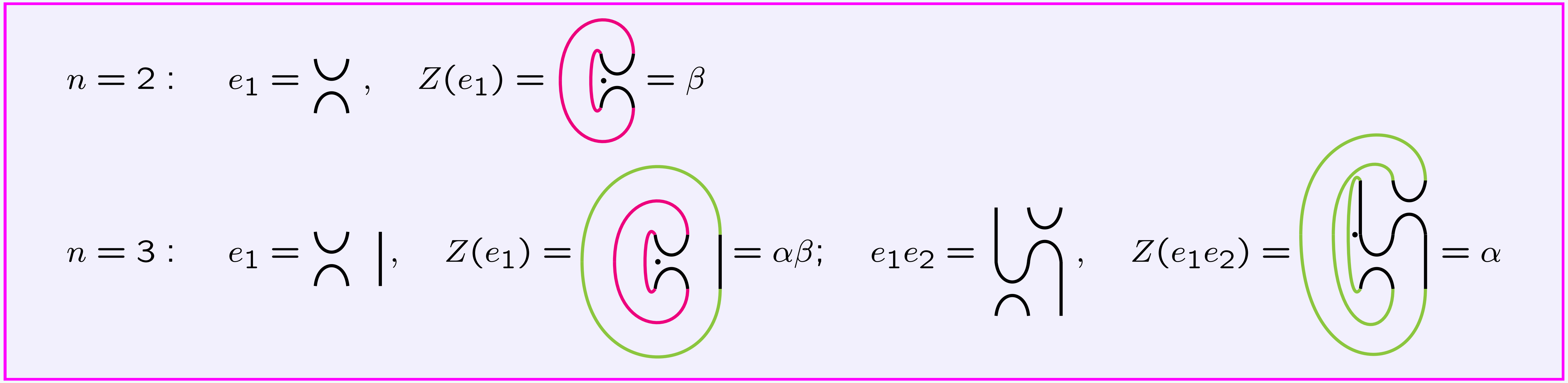
(i) quantum spin (vertex models)

(ii) height or Restricted Solid-On-Solid (RSOS)

(iii) loop (polymers/percolation)

Diagrammatic Annulus Partition Functions

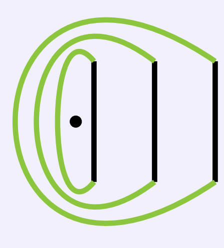
- In the Temperley-Lieb algebra, the annulus/cylinder partition function $Z(w)$ is given by closing the word w onto a link. Each non-contractible (green) loop around the annulus has weight α, each local contractible (pink) loop has weight β



- More generally, for n strings and $j = 1, \dots, n-1$

$$Z(I) = \alpha^n, \qquad Z(e_1) = \alpha^{n-2}\beta, \qquad Z(e_j) = \alpha^{n-2}\beta$$

Each strand in the identity gives a non-contractible loop as seen for $n = 3$

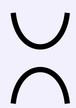
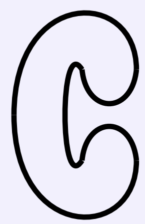
$Z(I) =$  $= \alpha^3$

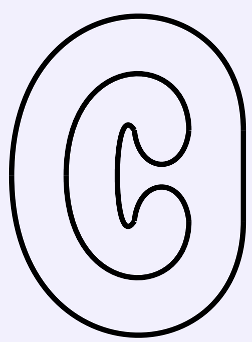
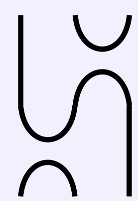
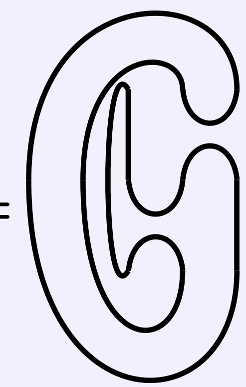
Diagrammatic Definition of Markov Trace

- The Markov trace (VFR Jones 1983), for n strings, is defined by

$$\mathrm{tr}\, w = \frac{Z(w)}{Z(I)}\bigg|_{\alpha=\beta}, \qquad \mathrm{tr}\, w = \beta^{\#\mathrm{loops}-n}$$

- It follows that

$n = 2:$ $\quad e_1 =$  , $\quad \mathrm{tr}\, e_1 =$  $= \beta^{1-2} = \beta^{-1}, \qquad \mathrm{tr}\, e_j = \beta^{-1}$

$n = 3:$ $\quad \mathrm{tr}\, e_1 =$  $= \beta^{2-3} = \beta^{-1}\,;$ $\quad e_1 e_2 =$  , $\quad \mathrm{tr}\, e_1 e_2 =$  $= \beta^{-2}$

Note that, since contractible and non-contractible loops have the same weight, the dot has been removed.

- Similarly, it is seen diagrammatically that

$$\mathrm{tr}(e_1 e_2 \ldots e_k) = \beta^{1-(k+1)} = \beta^{-k}, \qquad \mathrm{tr}(e_j^k) = \beta^{k-2}$$

- The Markov trace is the natural "trace" within the Temperley-Lieb algebra.

Naïve $N=2$ XXZ Entanglement Entropy

- Restrict $\lambda \in [0, \pi/2]$ (other values are related by a duality). Consider $N=2$ and take subsystem A (B) as the left (right) spin. The Hamiltonian is

$$H = -e_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q^{-1} & 1 & 0 \\ 0 & 1 & -q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- The lowest energy is $E^{(0)} = -(q+q^{-1}) < 0$. The other eigenenergy is $E^{(1)} = 0$. The right ground state is

$$H|0\rangle = E^{(0)}|0\rangle, \qquad |0\rangle = \frac{1}{\sqrt{2}}(q^{-1/2}|\uparrow\downarrow\rangle - q^{1/2}|\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}(0, q^{-1/2}, -q^{1/2}, 0)^T$$

- Using the convention that complex numbers are conjugated when calculating the bra associated with a given ket gives $\langle 0|0\rangle = 1$. In the basis $\uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$, the (normalized) density matrix is

$$\rho = |0\rangle\langle 0| = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -q^{-1} & 0 \\ 0 & -q & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathrm{Tr}\,\rho = 1$$

- The reduced density matrix and von Neumann entropy are

$$\rho_A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad S_A = -\mathrm{Tr}_A\,\rho_A \log \rho_A = \log 2$$

- But this does not take into account the proper left- and right-eigenvectors!

Modified $N = 2$ XXZ Entanglement Entropy

- Since H is not Hermitian, it is more correct to work with left- and right-eigenstates

$$H|E_\mathsf{R}\rangle = E|E_\mathsf{R}\rangle, \qquad \langle E_\mathsf{L}|H = E\langle E_\mathsf{L}|, \qquad (\text{or } H^\dagger|E_\mathsf{L}\rangle = E|E_\mathsf{L}\rangle, \text{ since } E \in \mathbb{R})$$

- Restricting to the sector $S^z = 0$, the right eigenstates with energies $E^{(0)}, E^{(1)}$ are

$$\begin{aligned} |0_\mathsf{R}\rangle &= \tfrac{1}{\sqrt{q+q^{-1}}}\left(q^{-1/2}|\uparrow\downarrow\rangle - q^{1/2}|\downarrow\uparrow\rangle\right) \\ |1_\mathsf{R}\rangle &= \tfrac{1}{\sqrt{q+q^{-1}}}\left(q^{1/2}|\uparrow\downarrow\rangle + q^{-1/2}|\downarrow\uparrow\rangle\right) \end{aligned}$$

The left eigenstates $|0_\mathsf{L}\rangle$, $|1_\mathsf{L}\rangle$ are obtained by $q \to q^{-1}$. The normalizations are $\langle i_\mathsf{L}|i_\mathsf{R}\rangle = 1$, and $\langle i_\mathsf{L}|j_\mathsf{R}\rangle = 0$ for $i \neq j$.

- Since $\langle 0_\mathsf{R}|1_\mathsf{R}\rangle \neq 0$ we use left- and right-eigenstates to build a projector onto the ground state

$$\tilde\rho = |0_\mathsf{R}\rangle\langle 0_\mathsf{L}| = \frac{1}{q+q^{-1}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \tilde\rho_A = \mathrm{Tr}_B\left(q^{-2\sigma_B^z}\tilde\rho\right) = \tfrac{1}{q+q^{-1}}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad \sigma^z = \tfrac{1}{2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$$

Observe that $\tilde\rho_A$ is normalized for the modified Markov trace (note the opposite power of q)

$$\mathrm{Tr}_A\left(q^{2\sigma_A^z}\,\tilde\rho_A\right) = \mathrm{tr}_A\,\tilde\rho_A = 1$$

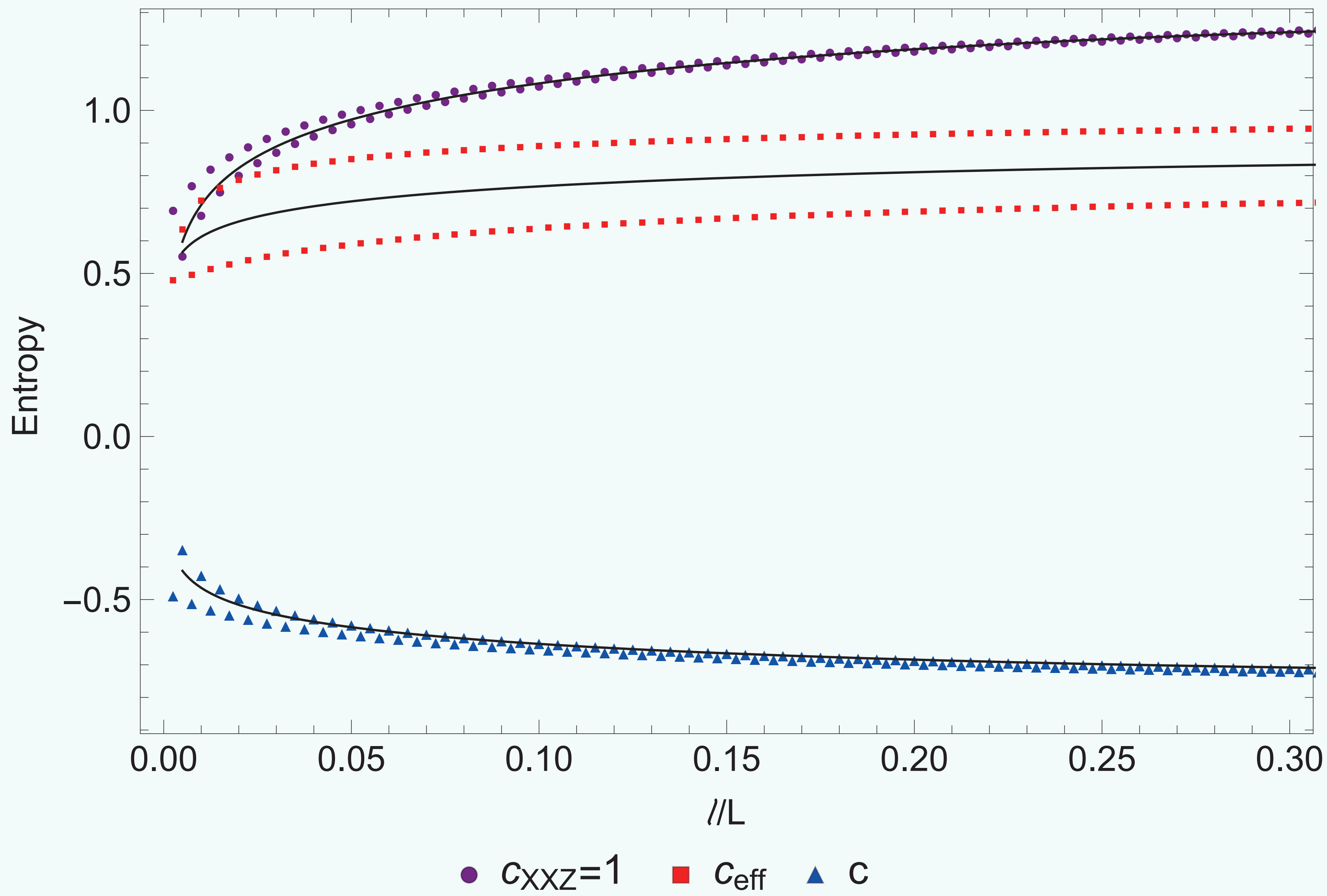
- For such nonunitary systems, the entanglement entropy is defined as the Markov trace

$$\tilde S_A = -\mathrm{tr}_A\,\tilde\rho_A \log\tilde\rho_A = -\mathrm{Tr}_A\left(q^{2\sigma_A^z}\,\tilde\rho_A \log\tilde\rho_A\right) = \log(q+q^{-1})$$

This result is more appealing since it depends on q through the combination $q+q^{-1}$ which is the quantum dimension of the spin-1/2 representation of $U_q[sl(2)]$. It also satisfies $\tilde S_A = \tilde S_B$.

Numerical Entanglement for $\mathcal{M}(3,5)$

[Couvreur, Jacobsen, Saleur 2016]



$$\Delta_{r,s} = \begin{pmatrix} \frac{3}{4} & 0 \\ \frac{1}{5} & -\frac{1}{20} \\ -\frac{1}{20} & \frac{1}{5} \\ 0 & \frac{3}{4} \end{pmatrix}$$

- Numerical entanglement entropy for the nonunitary minimal model $\mathcal{M}(3,5)$, versus $\frac{1}{\ell}$, for ℓ up to 400 sites and open boundary conditions. Purple dots show the usual entanglement entropy with the unmodified trace. Red squares show the $\alpha = 2$ Rényi entropy, with the Markov trace giving weight $n_1 = 2\cos\frac{\pi}{5}$ to non-contractible loops; this scales with $c_{\text{eff}} = \frac{3}{5}$. Blue triangles show $\tilde{S}^{(2)}$, with the Markov trace and $n_1 = 2\cos\frac{2\pi}{5}$; the scaling then involves the true central charge $c = -\frac{3}{5}$.

Summary of Exact Entanglement Results

- 2006 Weston: Higher-spin XXZ (level k = spin-$\frac{k}{2}$)

$$c_k = \frac{3k}{k+2} = \text{WZW central charges}$$

- 2013 De Luca, Franchini: Unitary minimal models $\mathcal{M}(m,m+1)$/RSOS$(m,m+1)$ models

$$c = 1 - \frac{6}{m(m+1)}$$

- 2016 Bianchini, Ravanini: Nonunitary minimal models $\mathcal{M}(m,m')$/RSOS(m,m') with $m' \neq m+1$

$$c \text{ is replaced by } c_{\text{eff}} \text{ !!}$$

These authors repeat the calculation of RSOS$(m,m+1)$ for RSOS(m,m') without modification.

- 2017 Couvreur, Jacobsen, Saleur:

Modifications are needed for nonunitary theories to see c rather than c_{eff} !!

The required modifications are as follows:
(a) The projector onto the ground state should be $|0_R\rangle\langle 0_L|$ and not $|0\rangle\langle 0|$.
(b) The trace should be replaced by the Markov trace.

- For RSOS models, the Markov trace is given by

$$\text{tr}A = \sum_{s\geq 1} [s]_q \text{Tr}_s A, \qquad [s]_q = \frac{q^s - q^{-s}}{q - q^{-1}}, \qquad q = e^{i\lambda}$$

where the quantum number $s = a$ is the Kac label and $[s]_q$ is the quantum dimension.

Open Research Problems

Calculate analytically the entanglement entropy and central charges c of the nonunitary RSOS models $\mathcal{M}(m,m')$ and logarithmic minimal models $\mathcal{LM}(p,p')$!

- The logarithmic minimal models include critical dense polymers $\mathcal{LM}(1,2)$ and critical bond percolation $\mathcal{LM}(2,3)$.

Is it possible to use entanglement entropy to analytically calculate the conformal weights $\Delta_{r,s}$?