대칭성과 위상수학, 그리고 상전이
Symmetry, Topology, and Phase Transitions

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- Prologue: Statistical Mechanics
- Symmetry and Order
- Topological Perspectives
- Berezinskii-Kosterlitz-Thouless Transition
- Dynamic Properties
- Quantum Properties
- Epilogue

Prologue: Statistical Mechanics

Matter in our daily life (incl. biological systems)

macroscopic, many constituents *many-particle system*

e.g., air in this classroom ($N \sim 10^{25}$ molecules)

microscopic description: dynamics (classical or quantum)

(micro) state $\{q_i, p_i\}$ 6N (micro) variables

Can't specify in practice! / in principle!

macroscopic description: statistical mechanics

(macro) state $\{p, T, \ldots\}$ a few macro variables

macro variables: collective degrees of freedom

external parameters + (internal) energy

Social system: individual states vs societal variables

(area, living level, technology, organization,…)

에너지, 일, 열 Energy, Work and Heat

What are these?

Energy levels E_n depends on external parameters $\{y_\alpha\}$

(mean) energy $E = \sum_n p_n E_n$ p_n: prob. for (micro) state n

Change of energy E

via change of $\{y_\alpha\}$ (i.e., of E_n) : work done $W \equiv -\Delta_y E$

via change of p_n: heat absorbed $Q \equiv \Delta E$

Energy transfer bet. two (macro) systems: work + heat

$\Delta E \equiv Q - W$ (heat absorbed - work done) by the system

엔트로피 Entropy

To a given macro state $(E,\{y_\alpha\})$

$\longleftarrow$ many micro states correspond e.g. 윷놀이

accessible states

number of accessible states $\Omega(E,\{y_\alpha\})$

$\Omega > 1 \Rightarrow$ missing information "entropy"

probability for the system in (macro) state $(E,\{y_\alpha\})$

$$p(E,\{y_\alpha\}) \propto \Omega(E,\{y_\alpha\})$$ postulate of equal a priori probability

macro state i (Ω_i small) $\rightleftarrows$ macro state f (Ω_f large)

irreversibility

e.g. 강의실 안의 공기: 앞에만 있는 상태 vs 고르게 퍼진 상태

$$\frac{p_i}{p_f} = \frac{\Omega_i}{\Omega_f} = \frac{(V/2)^N}{V^N} = 2^{-N} \sim 2^{-10^{25}} \approx 0$$

entropy $S \equiv k \log \Omega$ (Boltzmann) function of (macro) state

irreversibility: initial state $\longrightarrow$ equilibrium state (S maximum)

i.e., $S \rightarrow \max$ or $\Delta S \geq 0$ isolated system

엔트로피 S: 정보의 부족분

정보 I: 음의 엔트로피 negentropy

$$S = -I + I_0$$

heat $đQ$ absorbed via a quasi-static process:

$dS = đQ/T$ (can be negative)

Clausius' definition, but S?, holonomy, T?, very limited

temperature $\dfrac{1}{T} \equiv \dfrac{\partial S}{\partial E}$ ⇒ energy $E = E(T, \{y_\alpha\})$

1st law of thermodynamics

$Q = \Delta E + W$ definition of heat $\Longrightarrow$ energy conservation

infinitesimal change: $đQ = TdS = dE + \sum_\alpha X_\alpha dy_\alpha$

$$X_\alpha \equiv -\frac{\partial E}{\partial y_\alpha}$$ generalized force

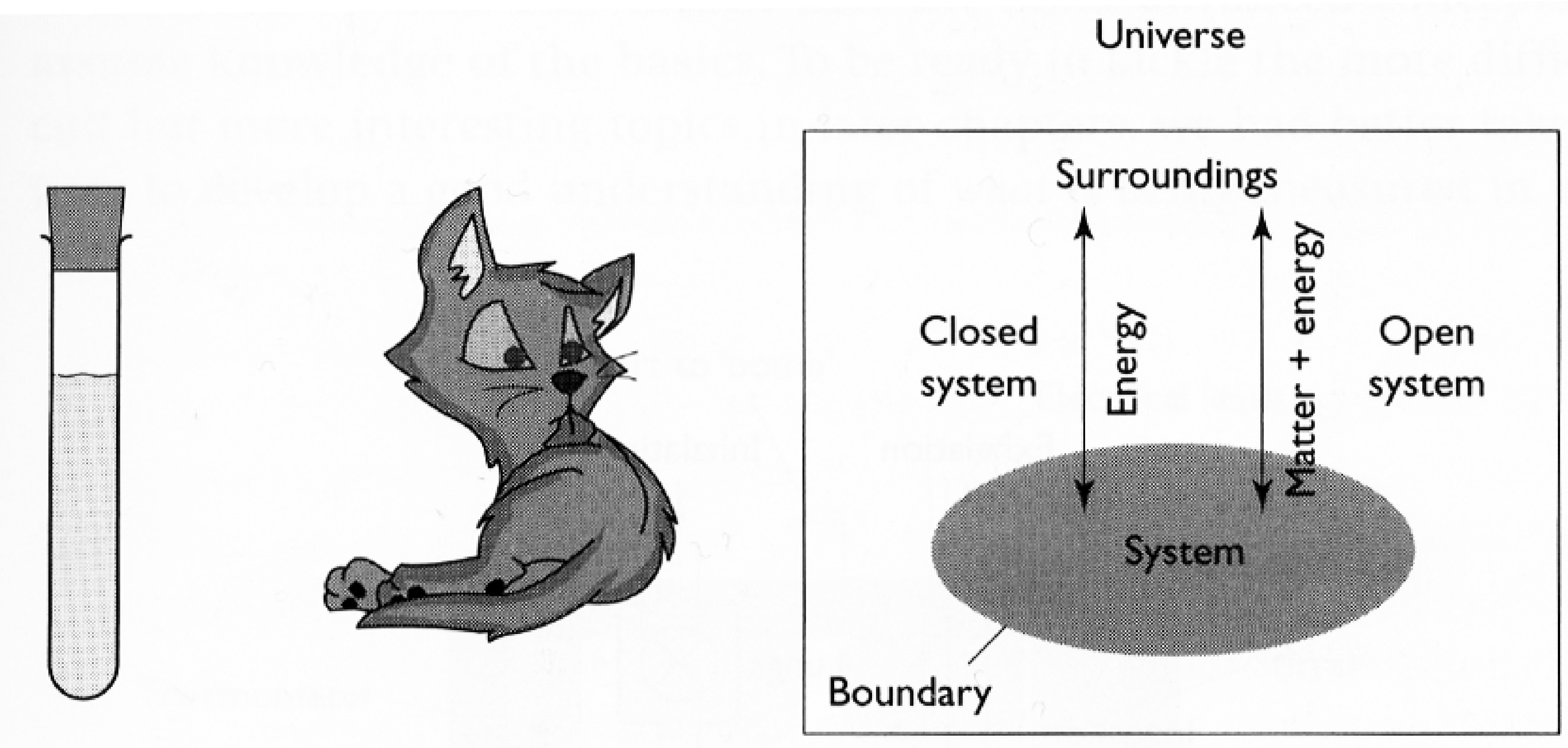
2nd law of thermodynamics

$\Delta S \geq 0$ Spontaneous change in an isolated system is non-decreasing.

More generally, $\frac{p(\Delta S)}{p(-\Delta S)} = e^{\Delta S/k}$ 요동정리 (detailed) fluctuation theorem

How can life survive the 2nd law?

외떨어지지 않음 → 닫히거나 열림



열역학 둘째 법칙 (2nd law of thermodynamics)

- 외떨어진 계: 엔트로피 $S \rightarrow \max$
- 외부와 교류하는 계 (에너지 등): ?

Thermodynamic potentials

single-component fluid: $\{y_\alpha\} = V; \quad X_\alpha = p$

$E(S, V) \quad (\Leftarrow \; dE = TdS - pdV)$

$F(T, V) = E - TS \quad (\Rightarrow \; dF = -SdT - pdV)$

$H(S, p) = E + pV \quad (\Rightarrow \; dH = TdS + Vdp)$

$G(T, p) = F + pV = E - TS + pV \quad (\Rightarrow dG = -SdT + Vdp)$

System exchanging energy (work + heat) with environment at temperature T

$$-W = \Delta E - Q \qquad \Leftarrow -Q = T\Delta S_{\text{env}} = T(\Delta S_{\text{tot}} - \Delta S)$$

$$= \Delta E - T\Delta S + T\Delta S_{\text{tot}} \qquad \Leftarrow \Delta E - T\Delta S = \Delta F$$

$$= \Delta F + T\Delta S_{\text{tot}} \qquad \Leftarrow \Delta S_{\text{tot}} \geq 0 \quad \text{2nd law}$$

$$\Rightarrow \; W \leq -\Delta F$$

2nd law of thermodynamics

- isolated system: $S \longrightarrow \max$
- system in contact with a heat reservoir: $F \longrightarrow \min$
- system in contact with a heat reservoir at constant pressure: $G \longrightarrow \min$

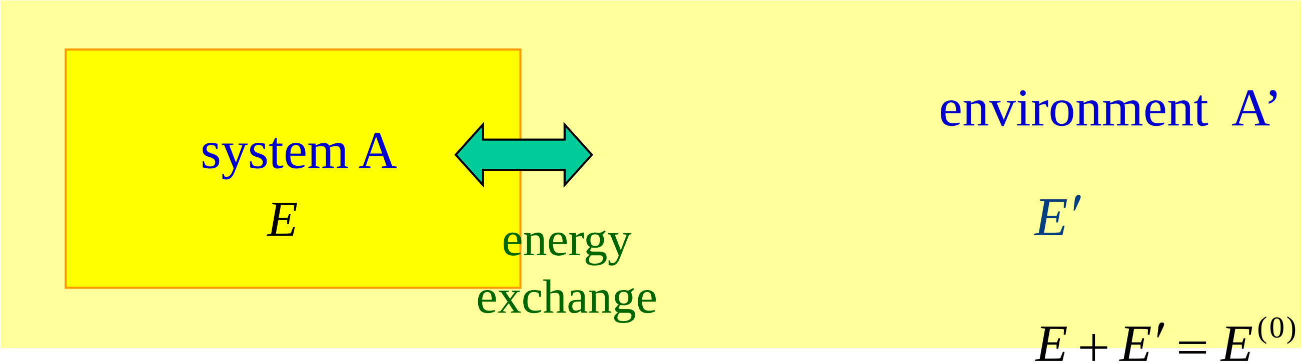
More generally, we have integral fluctuation theorems:

$$\left\langle e^{-\Delta S} \right\rangle = 1 \quad \Rightarrow \langle \Delta S \rangle \geq 0 \quad \left(\because \left\langle e^{x} \right\rangle \geq e^{\langle x \rangle} \right)$$

$$\left\langle e^{\beta (W + \Delta F)} \right\rangle = 1 \quad \text{or} \quad \left\langle e^{\beta W} \right\rangle = e^{-\beta \Delta F} \quad \Rightarrow \langle W \rangle \leq -\Delta F$$

$$W = 0 : \ \Delta F \leq 0$$

System in contact with a heat reservoir A' at temp. *T*



$$E + E' = E^{(0)}$$

Prob. for system A in state r with energy E_r

$$p_r \propto \Omega'(E') = \Omega'(E^{(0)} - E_r)$$

$$\ln \Omega'(E^{(0)} - E_r) = \ln \Omega'(E^{(0)}) - E_r \frac{\partial}{\partial E} \ln \Omega'(E) \Big|_{E^{(0)}} = C - \beta E_r \qquad \beta \equiv \frac{1}{kT}$$

$$\rightarrow \quad p_r \propto e^{-\beta E_r} \quad \text{or} \quad p_r = \frac{1}{Z} e^{-\beta E_r}$$

canonical distribution

$$Z \equiv \sum_r e^{-\beta E_r} = \mathrm{Tr}\, e^{-\beta E_r}$$

partition function

Connection to thermodynamics

free energy

$$F = E - TS = -kT \ln Z \quad \text{or} \quad \beta f \equiv \frac{F}{NkT} = -\frac{1}{N} \ln Z$$

mean (internal) energy and heat capacity

$$E \equiv \sum_r p_r E_r = \frac{1}{Z} \sum_r e^{-\beta E_r} E_r = -\frac{\partial}{\partial \beta} \ln Z \quad \text{and} \quad C \equiv \frac{\partial E}{\partial T} \quad \text{or} \quad c \equiv \frac{C}{N} = -k\beta^2 \frac{\partial^2}{\partial \beta^2} (\beta f)$$

pressure and compressibility

$$p = -\frac{\partial F}{\partial V} = -\frac{\partial f}{\partial v} \quad \text{and} \quad \kappa \equiv -\frac{1}{V} \frac{\partial V}{\partial p} = -\frac{1}{v} \left(\frac{\partial p}{\partial v} \right)^{-1} = \frac{1}{v} \left(\frac{\partial^2 f}{\partial v^2} \right)^{-1}$$

magnetization and susceptibility

$$m = -\frac{\partial f}{\partial H} \quad \text{and} \quad \chi \equiv \frac{\partial m}{\partial H} = -\left(\frac{\partial^2 f}{\partial H^2} \right)$$

$$F = F_0 - MH = F_0 - H \left\langle \sum_i s_i \right\rangle \quad \Rightarrow \quad m = \frac{1}{N} \left\langle \sum_i s_i \right\rangle, \quad \chi = \frac{1}{NkT} \sum_{i,j} \left[\langle s_i s_j \rangle - \langle s_i \rangle \langle s_j \rangle \right]$$

fluctuation-dissipation thm

Hamiltonian $H = H_0 + U$

Partition function $Z \equiv \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\, e^{-\beta H_0} e^{-\beta U}$

high-temp. limit: $\beta U \to 0$ noninteracting system

low-temp. limit: $\beta U \to \infty$ ground state

$T \to \infty$ noninteracting system (ideal) gas

↓ perturbation expansion: density exp. (no λ-exp.)

What is in between? phase transition

↑ elementary excitations: collective modes

$T \to 0$ ground state solid crystal

바닥상태 (= 진공)의 대칭성 깨짐 → 물질의 대칭성 깨짐

Symmetry and Order

Spacetime: Homogeneous and Isotropic

Symmetry of Physical Law: e.g. 뉴턴의 운동 법칙 $\mathbf{a} = \mathbf{F}/m$

Invariance under symmetry transformation

나란히 옮김translation, 돌림rotation, 시간 진행time translation

전하켤레charge conjugation, 홀짝성parity, 시간 되짚기time reversal

맞바꿈exchange/permutation

게이지gauge

응집물질condensed matter: 대칭성이 절로 깨질 수 있음
spontaneous symmetry breaking

→ 정돈(질서) order

Water and Ice: H_2O 분자들의 집단

T

water (symmetry: translation, rot) disorder entropy

T_c ← phase transition (symmetry breaking) free energy

ice (broken symmetry) order energy

0

Cooperativity among many constituents → emergent property

액체-고체, 자석, 초전도, 초기 우주, 기억 작용, DNA 풀어짐, 세포 분화, 피의 산소운반, 효소 작용, 여론 형성, 지각 작용, 도시 형성, 경기 변동과 공황, …

Order Parameter

How to specify the broken-symmetry state?

→ order parameter

$$\psi \begin{cases} = 0 & \text{sym. state (disordered)} \\ \neq 0 & \text{unsym. state (ordered)} \end{cases}$$

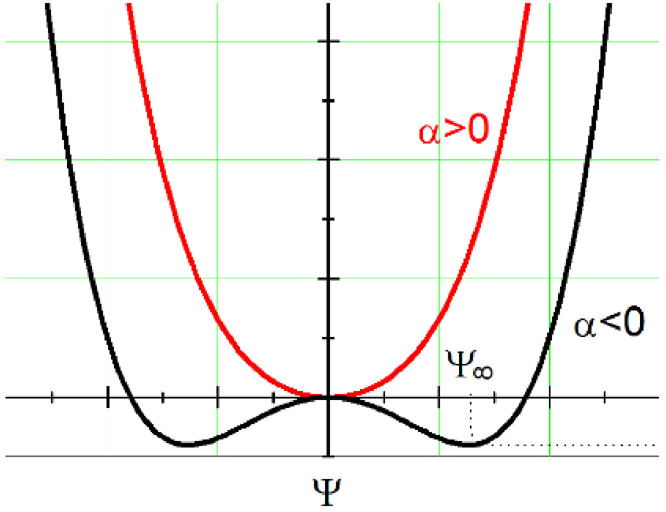
Free energy functional expanded in powers of ψ Landau theory

$$F(\psi) = F_0 + a|\psi|^2 + b|\psi|^4 + \cdots$$

$$\rightarrow F_{\min}$$

⇒ equilibrium order parameter

$$\psi = \begin{cases} 0, & a>0 \\ \sqrt{\dfrac{-a}{b}}, & a<0 \end{cases}$$



Usually ψ: continuous at transition, i.e., $\psi \rightarrow 0$ as $T \rightarrow T_c$ 2nd-order tr.

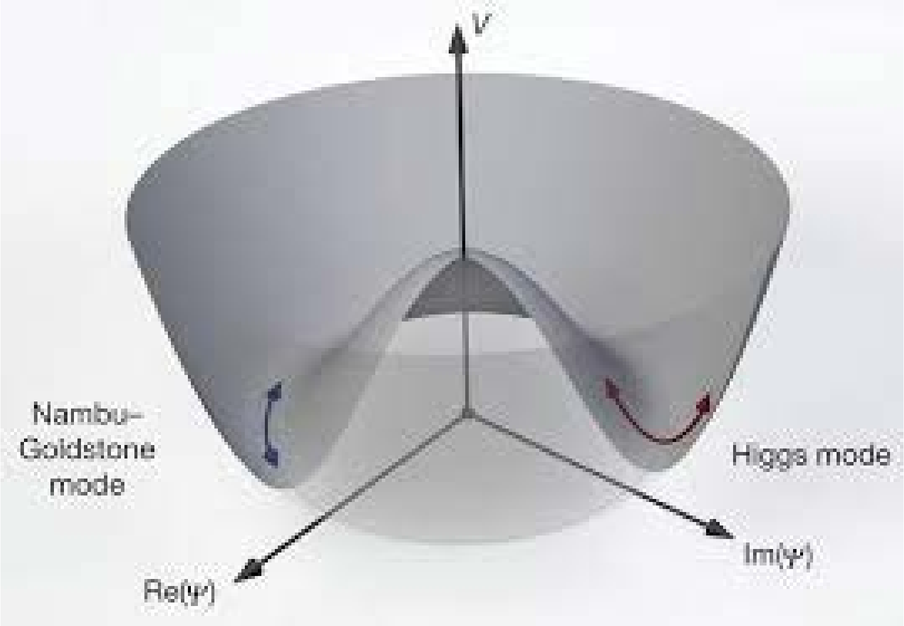
cf. 1st-order (discontinuous) transition

Discrete vs Continuous Symmetry

Discrete sym.: ψ may be a scalar (real) variable. Z_2: Ising model

Continuous sym.: ψ has components, *phase angle* → *phase field*.   U(1): XY model

$$\psi = |\psi| e^{i\phi}$$



Goldstone theorem: continuous symmetry broken → Goldstone mode (no energy gap, massless)

Mermin-Wagner theorem: Continuous symmetry may not be broken (i.e. no LRO) in $d = 2$.

Continuous sym. $U \longrightarrow \exists$ infinitesimal transformation $U_\varepsilon = 1 + i\varepsilon_i L_i$ (L_i : generators)

$$UHU^{-1} = H \Rightarrow [H, L_i] = 0,$$ L_i: const. of motion, i.e., $dL_i/dt = 0$

Suppose that L transforms ops. A into B according to

A: sym.-restoring op.
B: sym.-breaking op.

$$[L, A] = -iB,$$

where the average of B is the order parameter ψ:

$$\psi = \langle B \rangle = \mathrm{Tr}\, \rho B = i\, \mathrm{Tr}\, \rho[A, L] = i\, \mathrm{Tr}\, [\rho, L]A$$

Ordered state: $\psi \neq 0 \Rightarrow [\rho, L] \neq 0$ symmetry broken

$\rho = Z^{-1} e^{-\beta H}$ and $[H, L] = 0 \Rightarrow [\rho, L] = 0$??

$\rho \propto P e^{-\beta H}$ restricted ensemble $\longleftarrow$ ergodicity broken

Linear Response Theory

Perturbation → system H → Response

ext. field $h(\mathbf{r}, t)$ phys. quantity $\delta B(\mathbf{r}, t)$

$$H = H_0 + H', \quad H' = -Ah = -\int d^3r\, A(\mathbf{r})h(\mathbf{r},t)$$

$$\delta\langle B(\mathbf{r},t)\rangle = \int d^3r' \int_{-\infty}^{t} dt' K_{BA}(\mathbf{r},\mathbf{r}';t,t')\, h(\mathbf{r}',t')$$

$$K_{BA}(\mathbf{r},\mathbf{r}';t,t') \equiv i\langle [B(\mathbf{r},t), A(\mathbf{r}',t')]\rangle = K_{BA}(\mathbf{r}-\mathbf{r}',t-t')$$ linear response function

$$\delta\langle B(\mathbf{q},\omega)\rangle = \chi_{BA}(\mathbf{q},\omega)\, h(\mathbf{q},\omega)$$

$$\chi_{BA}(\mathbf{q},\omega) = \int_0^\infty dt \int d^3r\, e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega t)} K_{BA}(\mathbf{r},t) = \int_0^\infty dt\, e^{i\omega t} K_{BA}(\mathbf{q},t)$$ generalized suscept. (LT of K)

$$K_{BA}(\mathbf{q},t) = \frac{i}{V}\langle [B(\mathbf{q},t), A(-\mathbf{q})]\rangle$$

Goldstone theorem

Linear response function

$$K_{lA}(\mathbf{q},t) = i\left\langle [l(\mathbf{q},t), A(-\mathbf{q})]\right\rangle \qquad L = \int d^3r\, l(\mathbf{r},t)$$

$$K_{lA}(\mathbf{q}{=}0,\omega) = i\int_{-\infty}^{\infty} dt\, e^{i\omega t} \left\langle [L, A]\right\rangle = \left\langle B\right\rangle \int_{-\infty}^{\infty} dt\, e^{i\omega t} = 2\pi \left\langle B\right\rangle \delta(\omega)$$

$\Rightarrow$ Usually, $K_{lA}(\mathbf{q},\omega) = 2\pi\left\langle B\right\rangle \delta[\omega - \omega(\mathbf{q})]$ with $\lim_{\mathbf{q}\to 0}\omega(\mathbf{q}) = 0$ if no long-range int.

$\Rightarrow$ $K_{LA}(\mathbf{q},\omega) \to \infty$ as $\omega \to \omega(\mathbf{q})$ i.e., collective mode: $\omega \to 0$ as $\mathbf{q} \to 0$

ferromagnet: $\mathbf{L} = \mathbf{S} = \int d^3r\, \mathbf{s}(\mathbf{r}) = (S_x, S_y, S_z)$, i.e., $L = S_x$, $A = S_y$, $B = S_z$ $([\mathbf{S}, H] = 0)$

antiferromagnet: $\mathbf{L} = \mathbf{S}$, A and B: comp. of stag. magnetization $\tilde{\mathbf{S}} = \sum_{i\in A} \mathbf{s}_i - \sum_{i\in B} \mathbf{s}_i$ $([\tilde{\mathbf{S}}, H] \neq 0)$

superfluid: $L = N$ (number op.), A/B: phase/amplitude of field op. $\hat{\psi}(\mathbf{r})$

↘ generator of gauge sym.

Goldstone mode

1. This mode may not be obvious.
2. It depends on the dynamics of the system. → No general theory.
3. Applicable at $T = 0$ as well, where $\langle B\rangle$ represents vac. expectation value. If g.s. (vac.) breaks conti. sym. ($\langle B\rangle = \langle 0|B|0\rangle \neq 0$), there exist elementary exc. with $\omega \to 0$ as $\mathbf{q} \to 0$.

   no energy gap: massless Goldstone boson
4. Continuous sym. cannot be broken in 2D. Mermin-Wagner theorem
5. Plasma osc. due to the presence of long-range Coulomb int. (gauge field)

   local gauge sym.

$$\omega(q) = \sqrt{\omega_p^2 + c_s^2 q^2} \xrightarrow[q\to 0]{} \omega_p^2 = \sqrt{\frac{4\pi n e^2}{m}} \neq 0$$

Thus breaking of local gauge sym. $\nrightarrow$ Goldstone bosons

Instead gauge particles acquire mass. (Anderson-)Higgs [ABEGHHK] mechanism

e.g., gauge bosons in weak int. $W^{+/-}$, Z^0

Meissner effect in superconductivity: gauge particle (photon) gets massive

Examples of Goldstone Modes

1. Lattice (translation sym. broken)

   → lattice vibrations (phonons): $\omega \propto q$ time rev. sym. unbroken

2. Magnet (rotational sym. broken)

   → spin waves (magnons)

   (F) $\omega \propto q^2$ time rev. sym. broken

   (AF) $\omega \propto q$ time rev. sym. broken, but order parameter not const. of motion

3. Charge-density waves (CDW) (translational sym. broken)

   Peierls instability → periodic lattice distortion (PLD) (static → insulator)

   CD $\Delta\rho \propto e^{i(kx+\varphi)}$, energy indep. of $\varphi \Rightarrow$ long-wavelength fluc. in φ

   → phason $\varphi \propto e^{i(qx-\omega t)}$ ($\omega \to 0$ as $q \to 0$) phonon in the soliton lattice

   sliding of PLD → Fröhlich conduction

   cf. soliton condensation: commensurate-incommensurate (C-IC) transition

Topological Perspectives

Elementary particle (← *Field theory*)

symmetry, unification, charm, beauty, TOE,…

Condensed matter (← *Statistical mechanics*)

symmetry-breaking, disorder, randomness, frustration, chaos,…

Yet best precision comes from “dirty” condensed matter!

voltage standard: Josephson effect $V_{dc} \equiv \langle V \rangle = \frac{\hbar}{2e}\langle \dot{\phi} \rangle = n\frac{\hbar\omega}{2e} \quad (\sim 10^{-17})$

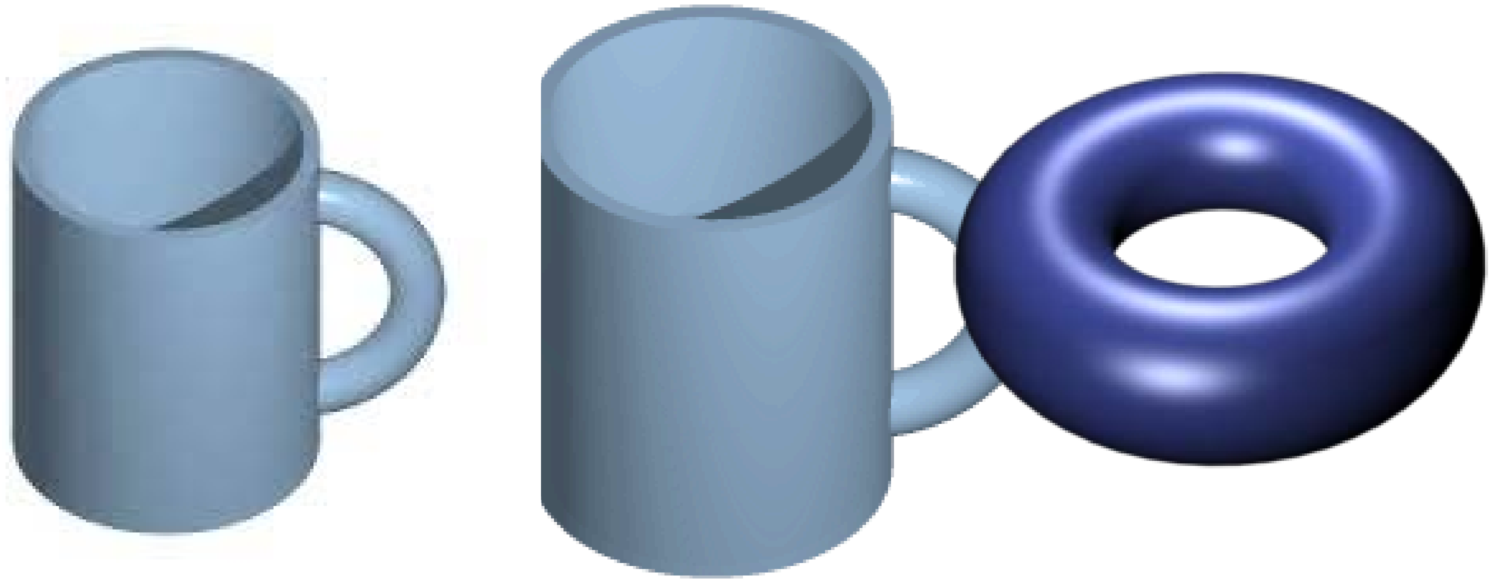
resistance standard: quantum Hall effect $\sigma_H = n\frac{e^2}{h} \quad (\sim 10^{-10})$

Why?

Quantum Numbers: Symmetry vs Topology

Symmetry vs Topology

Mug and doughnut: different symmetry but the same topology



Topological Singularities

- Defects in ordered state (⟶ order-breaking, sym.-restoring) can play important role in phase transitions (both conti. & discrete sym.)
- Finite energy gap ↔ Goldstone mode (only for conti. sym.)

Discrete symmetry

1D order parameter (e.g., Ising)

domain wall (magnetism, ferroelectricity, CDW)

Continuous symmetry

2D order parameter (XY model) $\psi = |\psi| e^{i\phi}$

vortex (supercond., superfluid He^4)

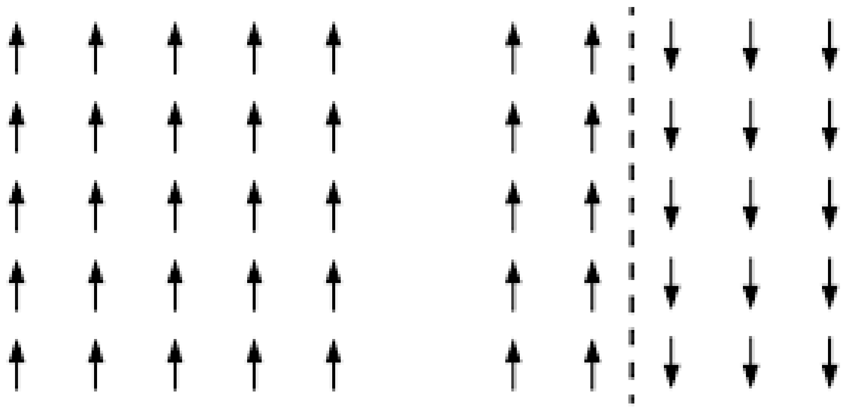
dislocation (crystal)

3D order parameter (Heisenberg model)

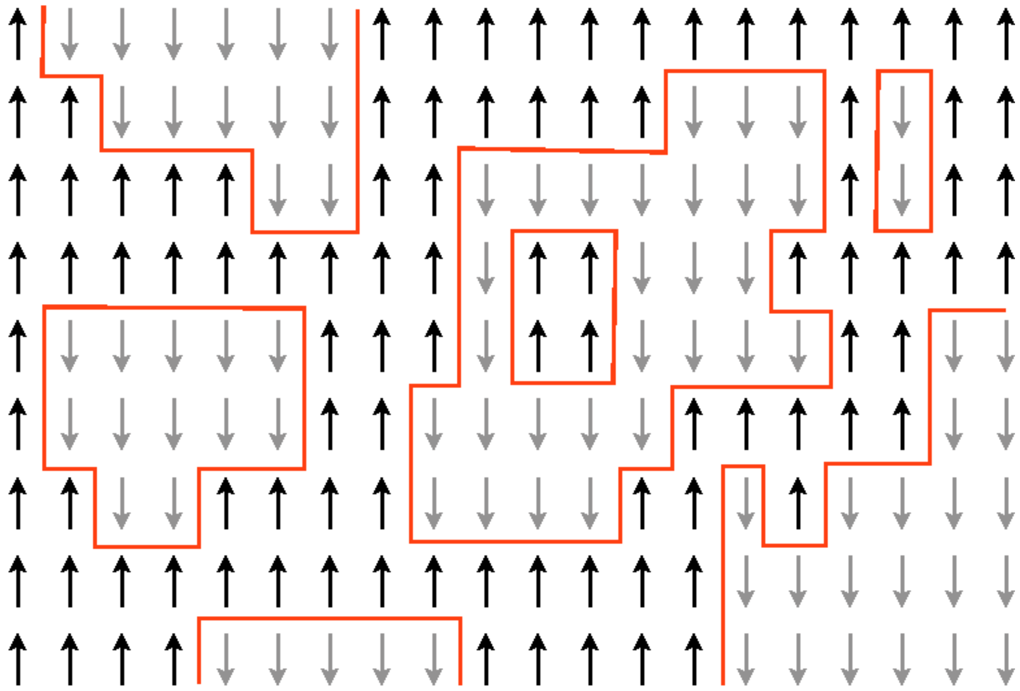
monopole, vacancy or interstitial (crystal)

domain wall

spin up/down (s = +1/−1) configuration



(a) (b)

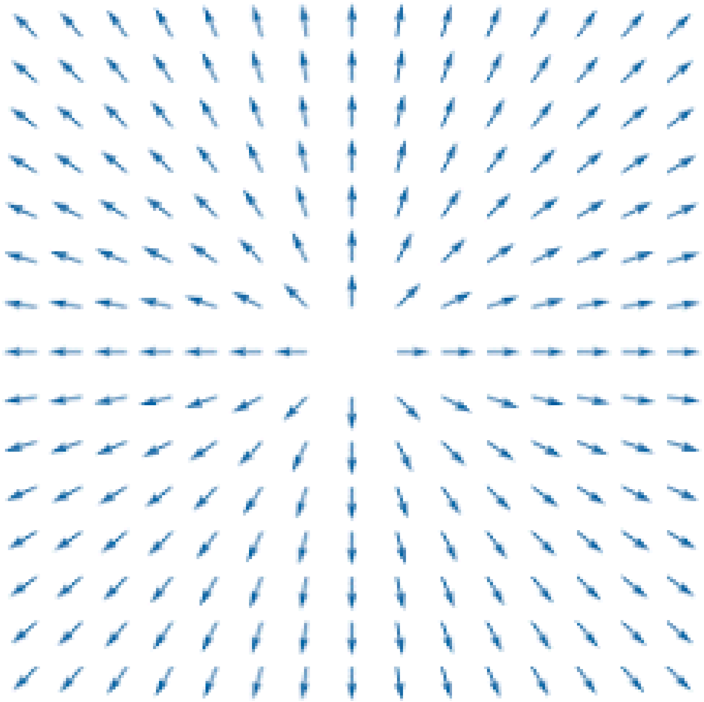


line (space dim d = 2)
surface (space dim d = 3)

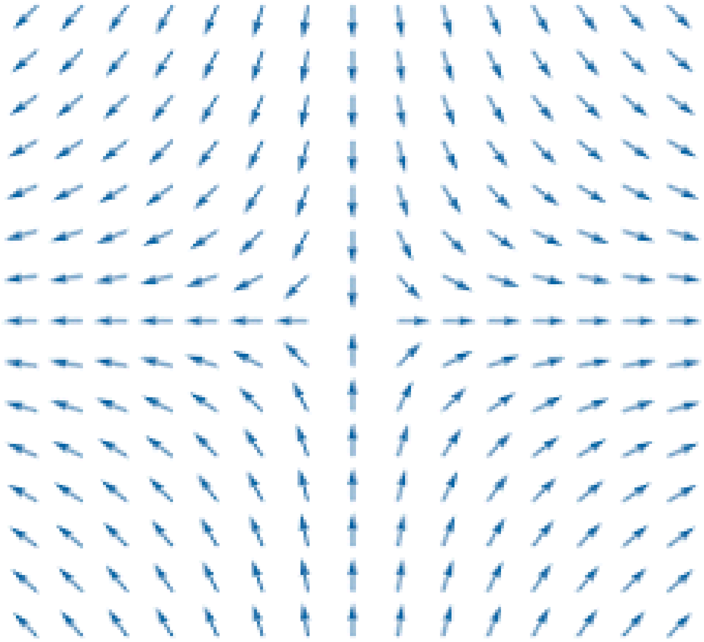
vortex

phase configuration ($0 \le \varphi < 2\pi$)

point vortex ($d = 2$)
vortex/flux line ($d = 3$)

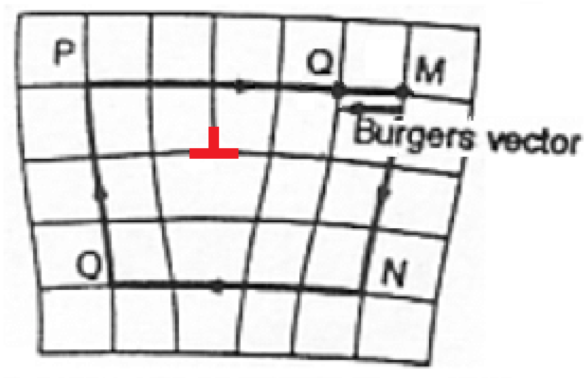


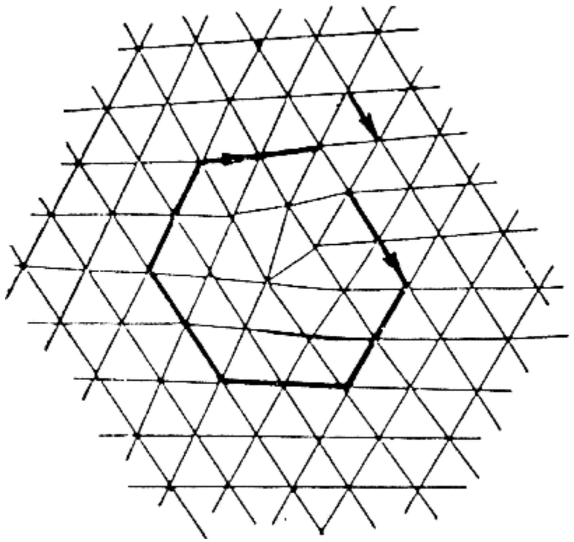
charge +1: **vortex**



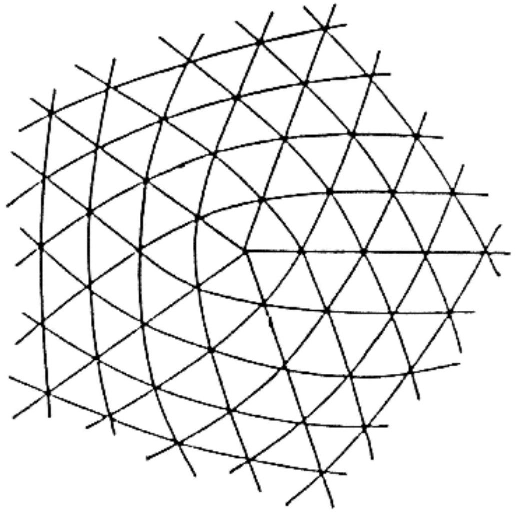
charge −1: **antivortex**

dislocation



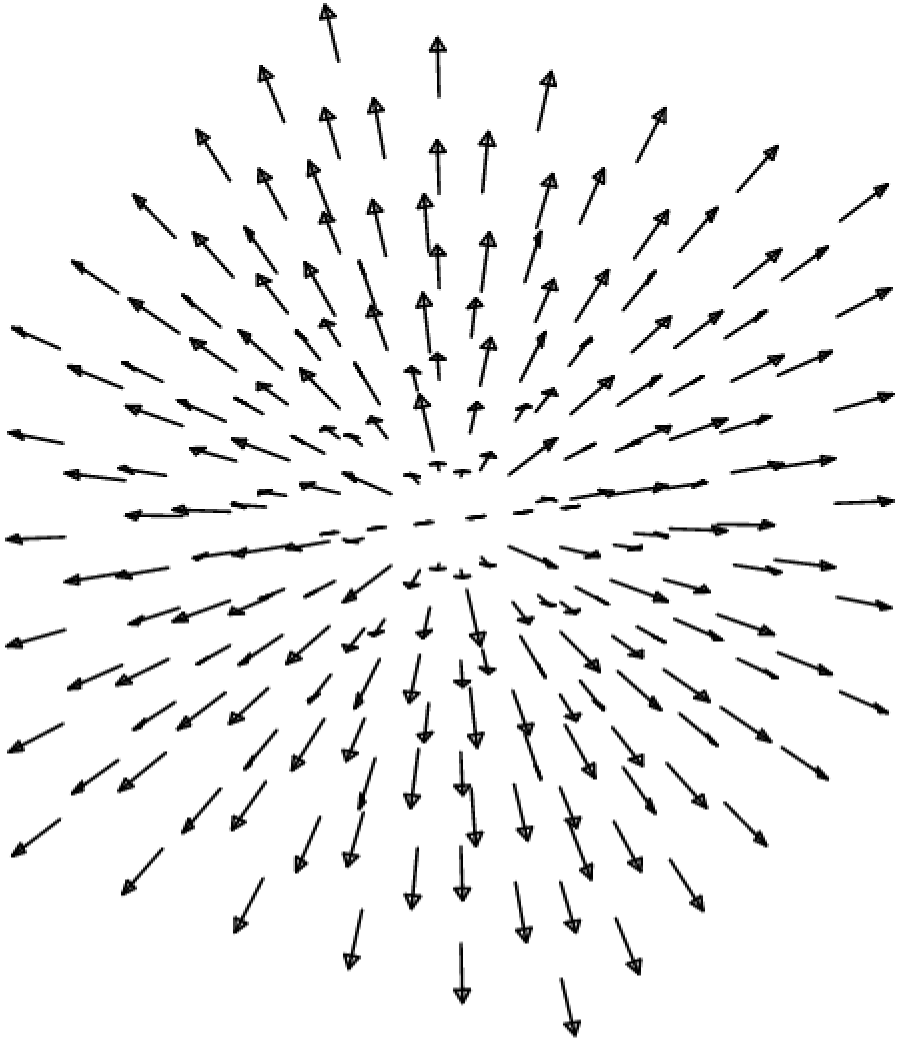


pair of disclinations



disclination

monopole



Symmetry → conserved quantity   Noether's theorem

discrete eigenvalues for the operator quantum numbers

symmetry: fragile, subject to perturbation

broken → mixing of the quantum numbers

Topology → winding numbers

topological charge (quantum numbers)

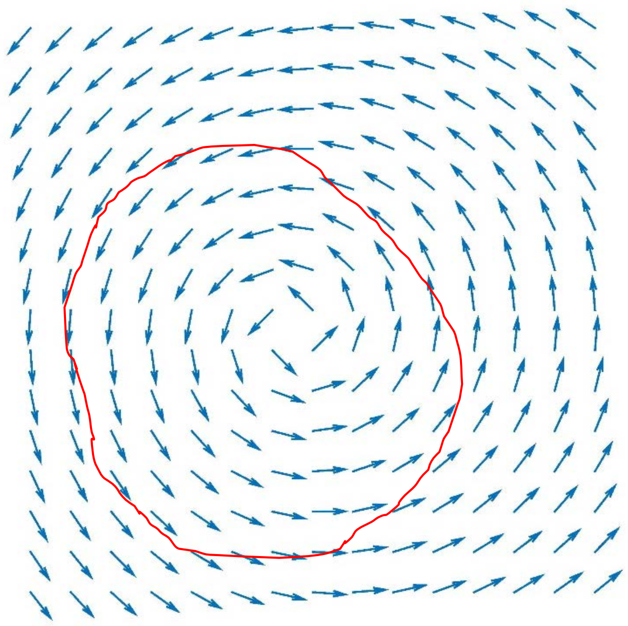
topology: robust against perturbation

→ high precision

Condensate wave function

$$\psi(\mathbf{r}) = |\psi(\mathbf{r})| e^{i\phi(\mathbf{r})}$$

$\oint \nabla\phi \cdot d\mathbf{l} = 2\pi n$ topological charge $n > 0$: vortex



Quantum Interference

- Moving charge in the presence of vector potential **A**
  - Hamiltonian

$$H = \frac{1}{2m}\left(\mathbf{p} + \frac{e}{c}\mathbf{A}\right)^2$$

AB phase acquired

$$\phi_{AB} = \frac{e}{\hbar c}\oint \mathbf{A}\cdot d\mathbf{l} = \frac{e}{\hbar c}\int \mathbf{B}\cdot d\mathbf{a} = \frac{e}{\hbar c}\Phi_{AB}$$

gauge transformation: $\mathbf{A} \longrightarrow \mathbf{A} + \nabla\Lambda \quad \Rightarrow \quad \psi \longrightarrow \psi e^{i(e/\hbar c)\Lambda}$

⇨ Bohm-Aharonov effect

- Moving magnetic moment in the presence of scalar potential A_0 (moving solenoid in the presence of charge)
  - Hamiltonian

$$H = \frac{1}{2m}\left(\mathbf{p} - \frac{1}{c}\mathbf{E}\times\boldsymbol{\mu}\right)^2$$

AC phase acquired

$$\phi_{AC} = \frac{e}{\hbar c}\oint \mathbf{A}_{AC}\cdot d\mathbf{l} \quad \left(\mathbf{A}_{AC} \equiv \frac{1}{e}\boldsymbol{\mu}\times\mathbf{E}\right)$$

⇨ Aharonov-Casher effect

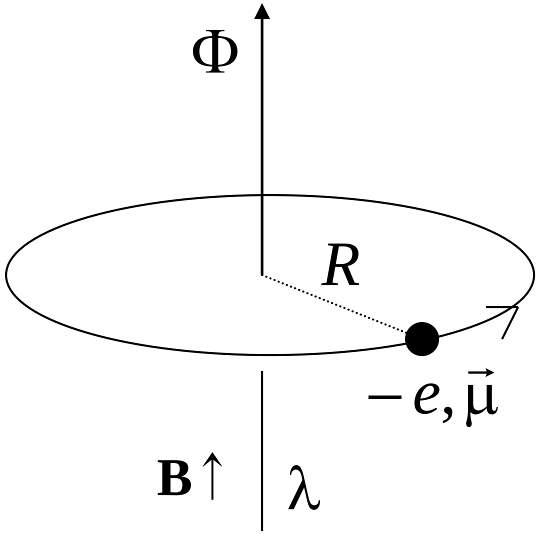
Persistent Currents

Free electrons in a metallic loop

- BA flux $\Phi_{AB} = \oint \mathbf{A}\cdot d\mathbf{l} = \int \mathbf{B}\cdot d\mathbf{a} \equiv f_{AB}\Phi_0 \qquad (\Phi_0 \equiv hc/e)$
- AC flux $\Phi_{AC} = \oint \mathbf{A}_{AC}\cdot d\mathbf{l} = \frac{\mu\sigma}{e}\int \nabla\cdot\mathbf{E}\, da \equiv \frac{\mu\sigma}{e}4\pi\lambda \equiv \sigma f_{AC}\Phi_0$

  (σ = ±1: spin state; λ: linear charge density)
- Energy level $E_{n\sigma} = \frac{\hbar^2}{2mR^2}(n+f)^2 \quad (-1/2 < f \le 1/2) \quad f \equiv f_{AB} + \sigma f_{AC}$
- Total energy $E = \sum_{n\sigma} E_{n\sigma}$
- Charge current Spin current

$$I_c = -\frac{e}{2\pi\hbar}\frac{\partial E}{\partial f_{AB}} \qquad I_s = \frac{1}{4\pi}\frac{\partial E}{\partial f_{AC}}$$



Superfluids and Superconductors

Condensate wave function $\psi(\mathbf{r}) = |\psi(\mathbf{r})| e^{i\phi(\mathbf{r})}$

Superfluid velocity $\mathbf{v}_s = \frac{\hbar}{m}\nabla\phi$

circulation $\oint \mathbf{v}_s \cdot d\mathbf{l} = \frac{\hbar}{m}\oint \nabla\phi \cdot d\mathbf{l} = n\frac{h}{m}$ winding number *n*: topological q. number

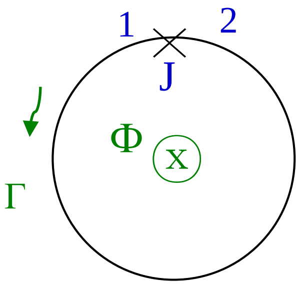
vortex quantization

Superconductor

$$\mathbf{j}_s = 2e\psi^* \mathbf{v}\psi = \frac{2e|\psi|}{m}\left(\hbar\nabla\phi - \frac{2e}{c}\mathbf{A}\right) = 0, \text{ inside superconductor} \quad \Rightarrow \quad \nabla\phi = \frac{2e}{\hbar c}\mathbf{A} = \frac{2\pi}{\Phi_0}\mathbf{A}$$

flux $\Phi = \oint \mathbf{A}\cdot d\mathbf{l} = \frac{\hbar c}{2e}\oint \nabla\phi \cdot d\mathbf{l} = 2n\pi\frac{\hbar c}{2e} \equiv n\Phi_0$ winding number *n*:

flux quantization



SQUID

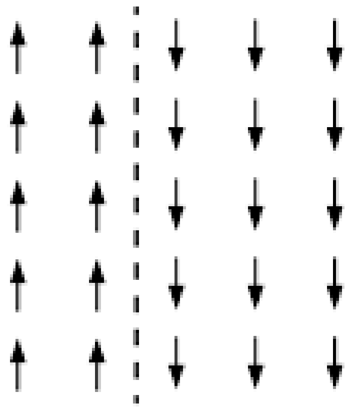
$$\int_{1,\Gamma}^{2} \nabla\phi \cdot d\mathbf{l} = \oint \nabla\phi \cdot d\mathbf{l} - \int_{2,J}^{1} \nabla\phi \cdot d\mathbf{l} = 2n\pi - \phi \quad (\phi \equiv \phi_1 - \phi_2)$$

$$\int_{1,\Gamma}^{2} \mathbf{A}\cdot d\mathbf{l} = \oint \mathbf{A}\cdot d\mathbf{l} - \int_{2,J}^{1} \mathbf{A}\cdot d\mathbf{l} = \Phi - \int_{2,J}^{1} \mathbf{A}\cdot d\mathbf{l}$$

gauge-invariant phase difference $\tilde{\phi} \equiv \phi - \frac{2\pi}{\Phi_0}\int_2^1 \mathbf{A}\cdot d\mathbf{l} = 2n\pi - 2\pi\frac{\Phi}{\Phi_0} = 2\pi(n - f)$

Phase Transitions: Topological Perspectives

Discrete symmetry: Ising model in d dimensions $\quad H = -J \sum_{\langle i,j \rangle} s_i \, s_j$



Domain wall in the system of linear size L

entropy $S = k \log \Omega = kd \log L$

($\because$ # of possible positions of d.w.: $\Omega = L^d$)

energy cost $E = 2JL^{d-1}$ ($\because$ # of spins at d.w. $\sim L^{d-1}$)

Free energy cost $\Delta F = E - TS = 2JL^{d-1} - dkT \log L$

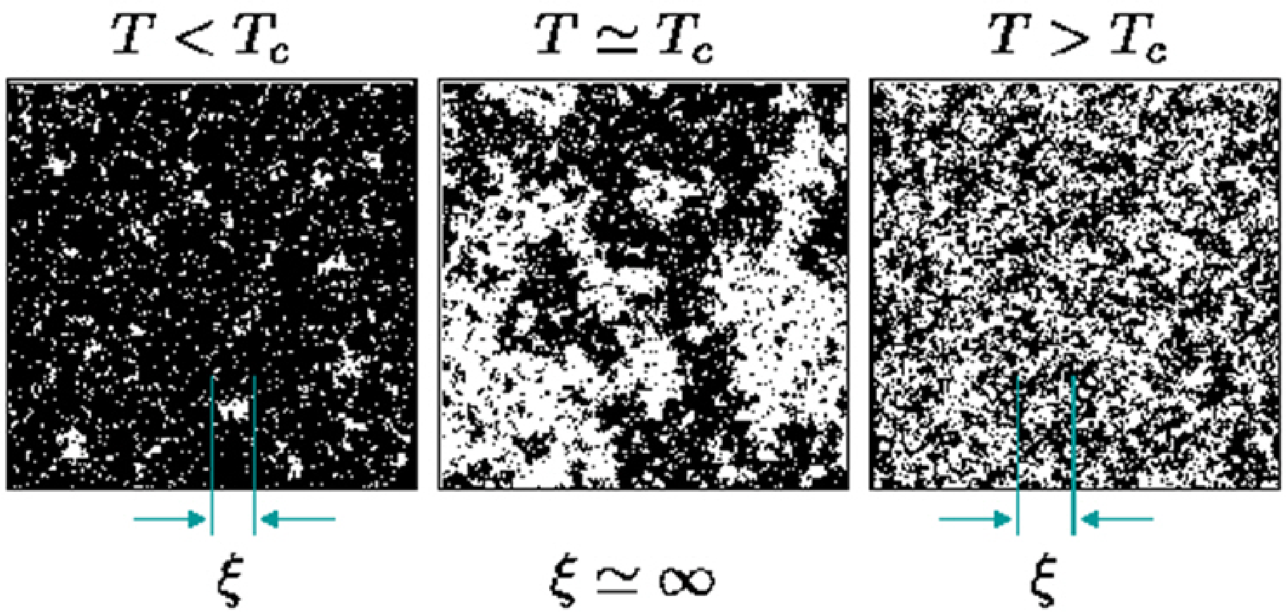
$d \leq 1$: entropy dominates, $\Delta F < 0 \Rightarrow$ many domains, no order

$d > 1$: energy dominates below some finite $T \Rightarrow$ ph. tr. into ordered state

Thus the lower critical dim. $d_l = 1$ for the Ising model.

Correlation length

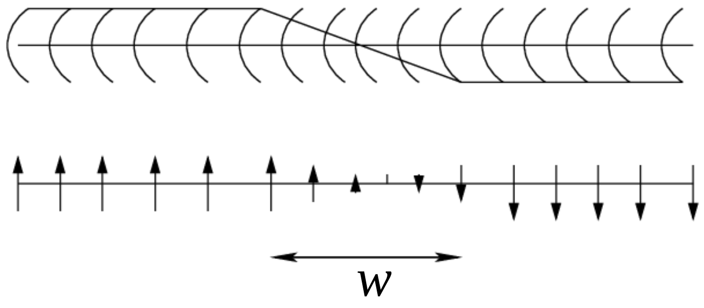
2D Ising model



Divergence of correlation length at T_c

$\rightarrow$ scale invariance

Continuous symmetry: n-vector model in d dimensions $(n \geq 2)$



$$H = -J \sum_{\langle i,j \rangle} \mathbf{s}_i \cdot \mathbf{s}_j \qquad (\mathbf{s}_i\text{: } n\text{-dim. Vector})$$

angle bet adjacent layers $\approx L\theta/w$

energy cost $E \sim JwL^{d-1}[1 - \cos(\theta/w)] \sim Jw^{-1} L^{d-1}\theta^2 \geq J L^{d-2}\theta^2$

entropy $S \leq kd \log L$

free energy cost $\Delta F \geq J L^{d-2}\theta^2 - dkT \log L$

$d > 2$: $\Delta F > 0$ below some finite T $\Rightarrow$ ph. tr. into ordered state

$d \leq 2$: ? It turns out that phase fluctuations (spin waves) destroy LRO.

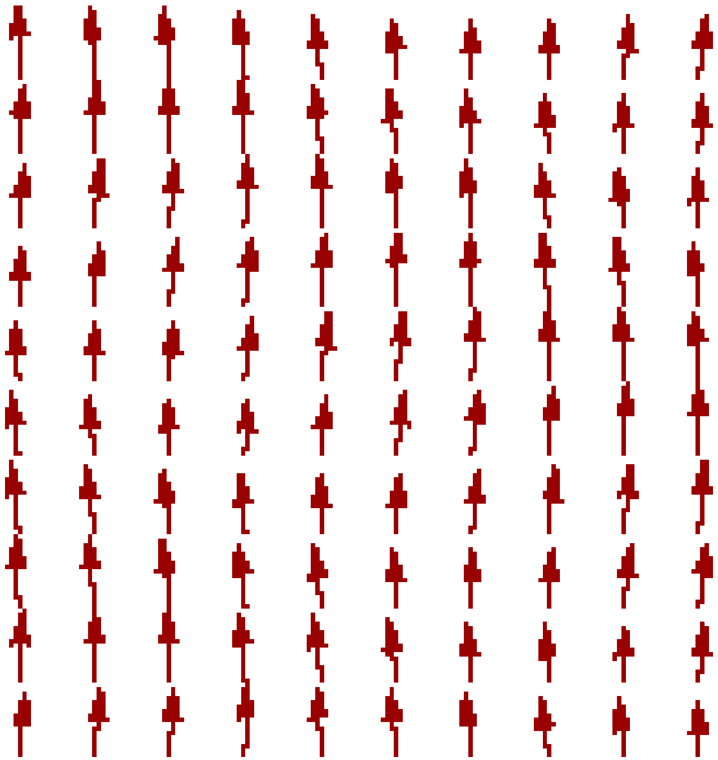
Thus the lower critical dim. $d_l = 2$ for the n-vector model $(n \geq 2)$.

Mermin-Wagner theorem: Continuous sym. cannot be broken (i.e. no LRO) in $d = 2$.

Phase fluctuations in two dimensions (d = 2) $\psi(\mathbf{r}) = |\psi(\mathbf{r})|e^{i\phi(\mathbf{r})}$

Order parameter $|\psi| = 0$: no long-range order ← no broken (continuous) symmetry

spin-wave excitations



Correlation function

$$\Gamma(\mathbf{r}) \equiv \left\langle \psi(\mathbf{r})\psi^*(0) \right\rangle = \left\langle e^{i\phi(\mathbf{r})} e^{-i\phi(0)} \right\rangle$$ superfluids/supercond

$$\left\langle \mathbf{s}(\mathbf{r}) \cdot \mathbf{s}(0) \right\rangle$$ magnets

$$\left\langle \rho_{\mathbf{G}}(\mathbf{r}) \rho_{\mathbf{G}}^*(0) \right\rangle = \left\langle e^{i\mathbf{G}\cdot[\mathbf{u}(\mathbf{r})-\mathbf{u}(0)]} \right\rangle$$ crystals (Debye-Waller factor)

$$\rho_{\mathbf{G}}(\mathbf{r}) \equiv e^{i\mathbf{G}\cdot\mathbf{R}(\mathbf{r})} \text{ with } \mathbf{R}(\mathbf{r}) = \mathbf{r} + \mathbf{u}(\mathbf{r})$$

In the limit $r \to \infty$,

$$\Gamma(r) \sim \begin{cases} \Gamma_\infty (\neq 0) \Rightarrow \text{LRO} \\ e^{-r/\xi} \quad \text{disorder} \end{cases}$$

Spin wave excitations

$\Gamma(r) \sim r^{-\eta}$ algebraic (QLRO) "critical"

⇒ algebraic (↔ δ-ftn) Bragg peaks

Examples of 2D phase fluctuations

Superfluid ^4He films

third sound, oscillating substrate

Superconducting films

type II (effective penetration depth large), ac impedance measurement

transverse magnetic field ⇒ Abrikosov flux lattice, 2D melting

Superconducting arrays

precise realization of the XY model

Liquid crystal films

Lipid monolayers floating on water

Adsorption (e.g. Xe, Kr on graphite)

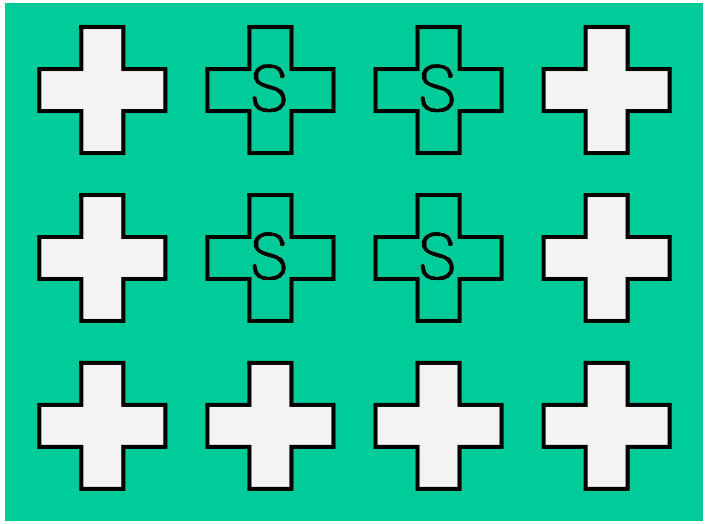
incommensurate melting (IC (floating) solid - IC fluid)

Electron systems (e.g. MnO films)

disorder-driven M-I transition

Berezinskii-Kosterlitz-Thouless Transition

2D Superconducting Arrays



superconducting islands weakly coupled by Josephson junctions

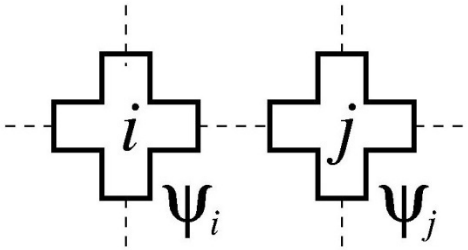
- Described by the 2D XY model
- Study of low-dim. physics
- Related to a variety of systems

e.g. superconducting networks
tight-binding electrons
high-T_c superconductors
quantum Hall system

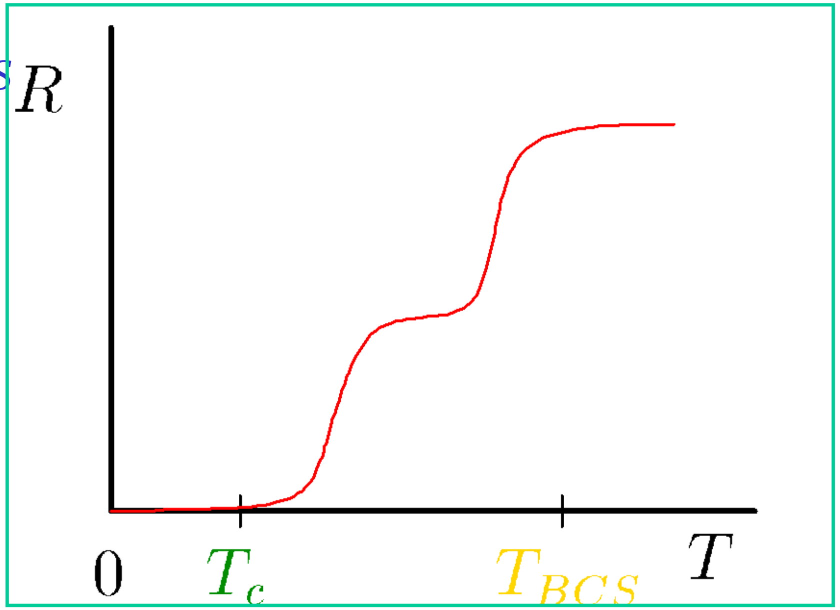
Ginzburg-Landau Description

- GL free energy

$$F = \sum_i \left(a|\psi_i|^2 + b|\psi_i|^4\right) + \sum_{\langle i,j \rangle} c|\psi_i - \psi_j|^2$$



- Two transition regions
  - $T = T_{BCS}$ $(a = 0)$
  - $T = T_c$: $(\approx c)$



$R \neq 0$

$R \rightarrow 0$

Lower transition region: Phase fluctuations only

amplitude fluctuations: negligible (only slight renormalization)

$$\psi(\mathbf{r}) = |\psi| e^{i\phi(\mathbf{r})} \text{ with } |\psi| = \text{const. or } \psi_i = |\psi| e^{i\phi_i}$$

$$\Rightarrow F = \sum_i (a|\psi_i|^2 + b|\psi_i|^4) + c\sum_{\langle i,j\rangle} |\psi_i - \psi_j|^2 = N(a+b) + c|\psi|^2 \sum_{\langle i,j\rangle} \left|e^{i(\phi_i - \phi_j)} - 1\right|^2$$

$$= F_0 - J\sum_{\langle i,j\rangle} \cos(\phi_i - \phi_j)$$

spin-wave excitations (continuum system)

$$\cos(\phi_i - \phi_j) \approx 1 - \frac{1}{2}(\phi_i - \phi_j)^2 \approx 1 - \frac{1}{2}(\nabla\phi)^2$$

$$\Rightarrow F \approx F_1 + \frac{J}{2}\int d^2r\,(\nabla\phi)^2$$

Correlation function

$$\Gamma(\mathbf{r}) \equiv \langle \psi(\mathbf{r})\psi^*(0)\rangle = \langle e^{i\phi(\mathbf{r})} e^{-i\phi(0)} \rangle \propto \frac{1}{Z}\int \prod_{\mathbf{r}} d\phi(\mathbf{r})\, e^{-\beta F} e^{i[\phi(\mathbf{r})-\phi(0)]}$$

$$\text{FT:}\quad \phi(\mathbf{r}) \equiv \frac{1}{\sqrt{V}}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}}\phi_{\mathbf{k}}, \qquad F = \frac{J}{2}\sum_{\mathbf{k}} k^2 |\phi_{\mathbf{k}}|^2$$

⇒ spin-wave theory

$$\Gamma(\mathbf{r}) = \langle e^{iA}\rangle = e^{-\frac{1}{2}\langle A^2\rangle} \quad (\because \text{Wick's theorem for Gaussian action}) \quad A \equiv \phi(\mathbf{r}) - \phi(0)$$

$$\frac{1}{2}\langle A^2\rangle = \frac{1}{V}\sum_{\mathbf{k}}(1-\cos\mathbf{k}\cdot\mathbf{r})\langle |\phi_{\mathbf{k}}|^2\rangle \quad \leftarrow \quad \langle |\phi_{\mathbf{k}}|^2\rangle = \frac{k_B T}{k^2 J}$$

$$= \frac{1}{KV}\sum_{\mathbf{k}}\frac{1-\cos\mathbf{k}\cdot\mathbf{r}}{k^2} = \int \frac{d^2k}{(2\pi)^2 K}\frac{1-\cos\mathbf{k}\cdot\mathbf{r}}{k^2} = \frac{1}{2\pi K}\int_{1/r}^{\Lambda}\frac{dk}{k} + \text{const.} \quad (\text{cutoff } \Lambda \approx a_0)$$

$$= \frac{1}{2\pi K}\log\left(\frac{r}{a_0}\right)$$

$$\Gamma(r) \sim e^{-\frac{1}{2\pi K}\log\left(\frac{r}{a_0}\right)} \sim r^{-\eta(T)} \quad \text{with} \quad \eta(T) = \frac{1}{2\pi K} = \frac{k_B T}{2\pi J}, \quad \text{algebraic decay (QLRO)}$$

Susceptibility

$$\chi \sim \int d^2 r\, \Gamma(r) \sim \int d^2 r\, r^{-\eta} \sim \int dr\, r^{1-\eta} \to \begin{cases} \text{finite} & \text{for } \eta > 2 \\ \infty & \text{for } \eta < 2 \end{cases}$$

Phase transition at $\eta = 2$ or $k_B T_c = 4\pi J$?

$\begin{cases} \text{spin-wave excitations} \to \text{QLRO (at low } T) \\ \text{another type of excitation: vortex excitations} \end{cases}$

$$\oint \nabla \phi \cdot \mathrm{d}\mathbf{l} = 2\pi n$$

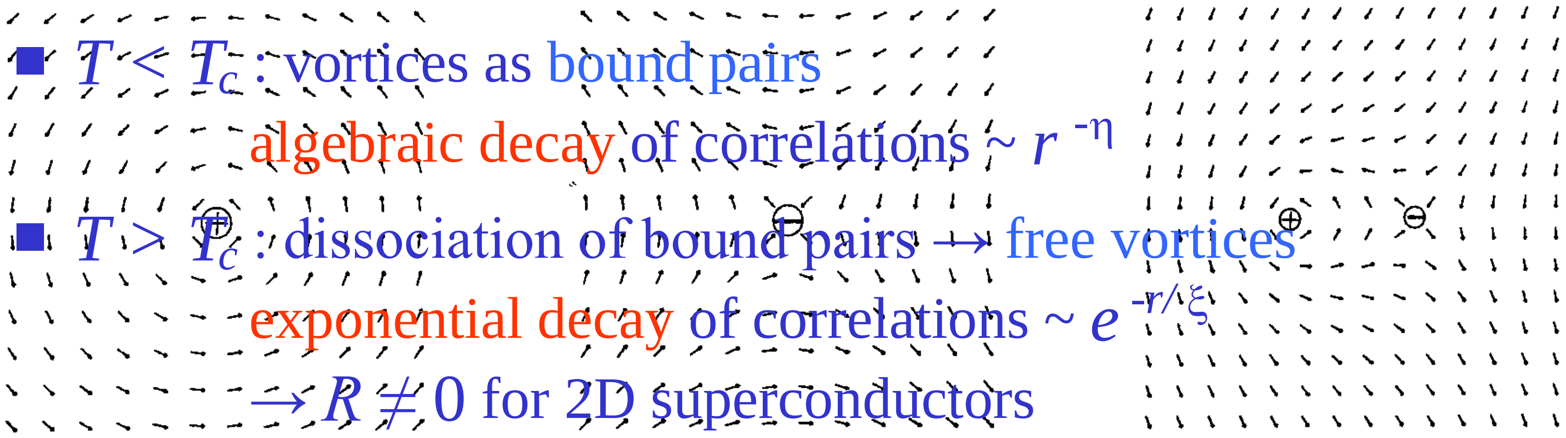
→ disorder (at relatively high T)?

2D XY Model

$$H = -J \sum_{\langle i,j \rangle} \mathbf{s}_i \cdot \mathbf{s}_j \quad \leftarrow \mathbf{s}_i = (s_{ix}, s_{iy}) = |s_i| (\cos\phi_i, \sin\phi_i) \quad \text{XY spin}$$

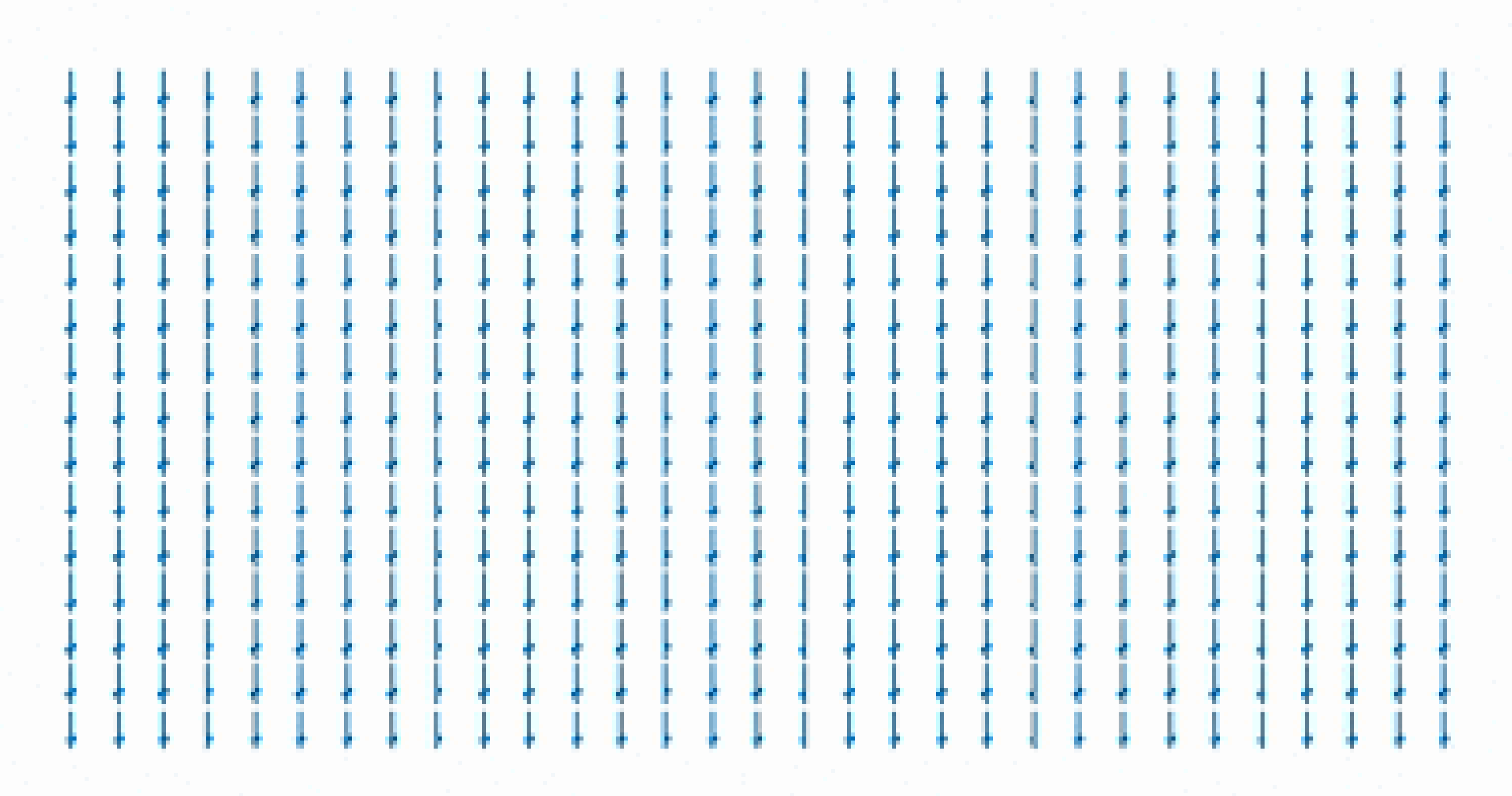
$$= -E_J \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j) \quad \text{2D XY model}$$

- Excitations
  - spin wave : Goldstone mode
  - vortex : topological defect

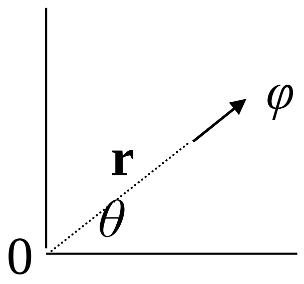


(a) (b) (c)

- $T < T_c$: vortices as bound pairs
  algebraic decay of correlations $\sim r^{-\eta}$
- $T > T_c$: dissociation of bound pairs $\rightarrow$ free vortices
  exponential decay of correlations $\sim e^{-r/\xi}$
  $\rightarrow R \neq 0$ for 2D superconductors
- BKT transition at $T = T_c$



vortex at $\mathbf{r} = 0$: $\boldsymbol{\phi}(\mathbf{r}) = \theta(\mathbf{r})$



$$\nabla\phi = \frac{1}{r}\hat{\boldsymbol{\theta}} \quad [\nabla\times\nabla\phi = 2\pi\delta(\mathbf{r})\hat{\mathbf{z}}]$$

$$c\,|\nabla\psi|^2 = c\,|\psi|^2\,(\nabla\phi)^2 = \frac{1}{2}E_J\frac{1}{r^2}$$

energy of a single vortex

$$E_{1v} = \frac{1}{2}E_J\int d^2r\,(\nabla\phi)^2 = \frac{1}{2}E_J\int d^2r\,\frac{1}{r^2} = \pi E_J\ln\frac{R}{\xi}\xrightarrow{R\to\infty}\infty$$

free energy change asso. with free vortex formation

$$\Delta F = \Delta E - T\Delta S = \pi E_J\ln\frac{R}{\xi} - k_BT\ln\Omega = (\pi E_J - 2k_BT)\ln\frac{R}{\xi}$$

$T < T_c \equiv \pi E_J/2k$: bound pairs → algebraic decay

$T > T_c$: free vortices → exponential decay

$T = T_c$: ionization of vortices → BKT transition

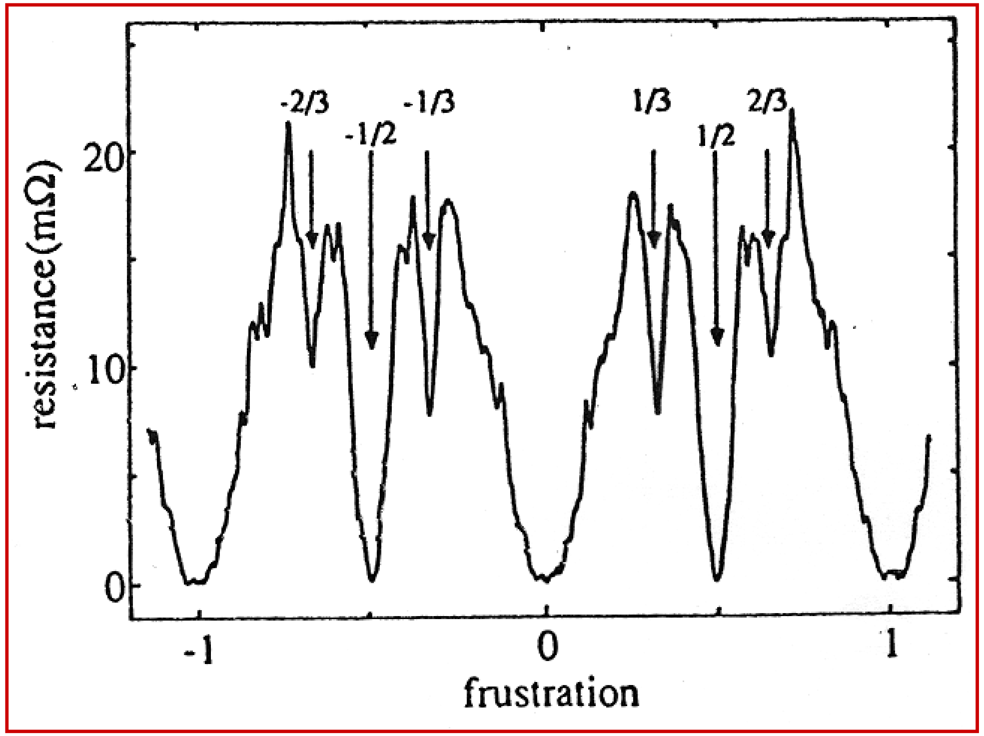
topological (no sym. breaking)

Effects of Magnetic Field

Field induced vortices (repulsive) (+ thermally excited ones)

→ tend to form regular flux lattice at $T = 0$

Competition between flux lattice and underlying array periodicity



→ commensurate

-incommensurate effects

as magnetic field is varied

⇒ frustrated XY model

Frustrated XY Model

$$H = -E_J \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j - A_{ij})$$

$$A_{ij} \equiv \frac{2e}{\hbar c} \int_i^j \mathbf{A} \cdot d\mathbf{l}, \quad \sum_P A_{ij} = 2\pi\Phi/\Phi_0 \equiv 2\pi f$$

(f : gauge-invariant frustration)

- Only frustration effects (↔ spin-glass) enter in a controllable way (← magnetic field)

complex systems

- Discrete symmetry Z_q in addition to continuous U(1) symmetry → possibility of LRO in 2D (vortex + domain wall)
- Duality transformation → Coulomb gas of (fractional) charges

$$H = 2\pi^2 E_J \sum_{\langle \mathbf{R},\mathbf{R}' \rangle} [n_{\mathbf{R}} - f] G(\mathbf{R},\mathbf{R}') [n_{\mathbf{R}'} - f]$$

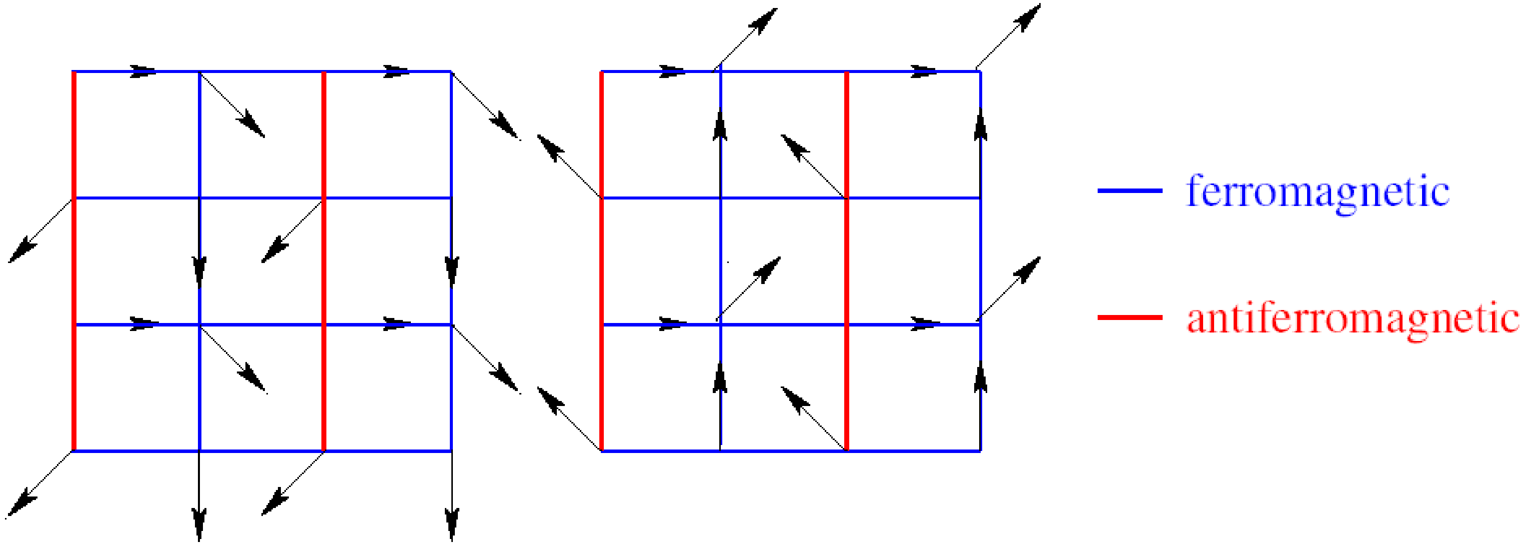
Symmetry depends on *f* in a highly discontinuous fashion

f = 0 (unfrustrated XY model): U(1), BKT transition (← RG analysis, Kosterlitz)

$T < T_c$: critical, power-law decay of phase correlation

f = ½ (fully frustrated): U(1)×Z_2

ground state: doubly degenerate (discrete) → Z_2 (Ising)



Chirality $q(\mathbf{R}, t) = \text{sgn} \sum_{\text{P}} \sin\left(\phi_i - \phi_j - A_{ij}\right) = \pm 1$

f = irrational (irrationally frustrated): glass transition? ($T_c \to 0$ as $L \to \infty$)

unit cell itself is infinite $\to$ intrinsic finite-size effects

successive orderings (corresponding to rational approx. of f) at larger length scales

f = random (gauge glass, random A_{ij}): frustration + randomness

(quasi-)glass transition at finite temperature $T_c = 0.21 \pm 0.03$

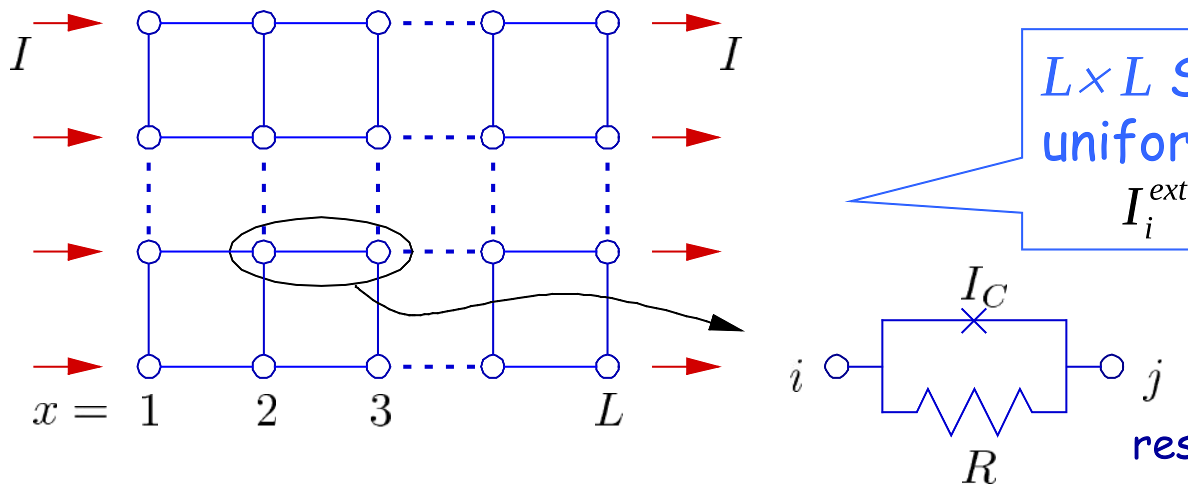
$T < T_c$: algebraic glass order

glass order parameter $$q \equiv \left[\left| \left\langle e^{i\phi_i} \right\rangle \right|^2 \right] = 0$$

correlation function of glass order parameter

$$G_{ij} \equiv \left[\left| \left\langle e^{i(\phi_i - \phi_j)} \right\rangle \right|^2 \right] \sim r^{-\eta}$$

Dynamical Properties



$L \times L$ SQ array
uniform applied currents
$I_i^{ext} = I(\delta_{x,1} - \delta_{x,L})$

resistively shunted junction (RSJ)

current conservation → equations of motion

$$\sum_j{}' \left[\frac{\hbar}{2eR} \frac{d}{dt}\left(\phi_i - \phi_j - A_{ij}\right) + I_C \sin\left(\phi_i - \phi_j - A_{ij}\right) + \eta_{ij} \right] = I_i^{ext}$$

noise current $\langle \eta_{ij}(t)\eta_{kl}(t') \rangle = \frac{2kT}{R} \delta(t-t')(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$

$I = 0$: relaxation toward equilibrium

$I = I_d$: *IV* characteristics, current-induced unbinding, coherence resonance

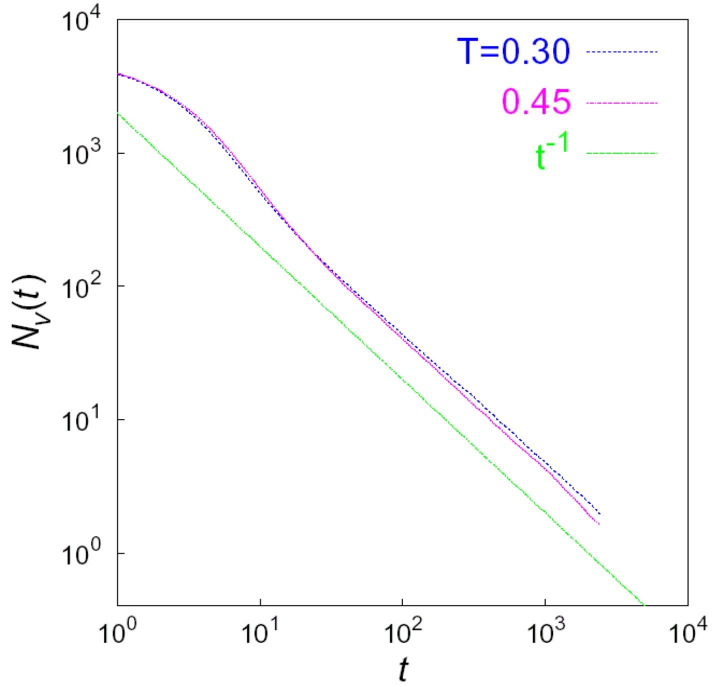
$I = I_a \cos \Omega t$: dynamics transitions, SR

$I = I_d + I_a \cos \Omega t$: mode locking, melting transition

Relaxation to Equilibrium

$f = 0$

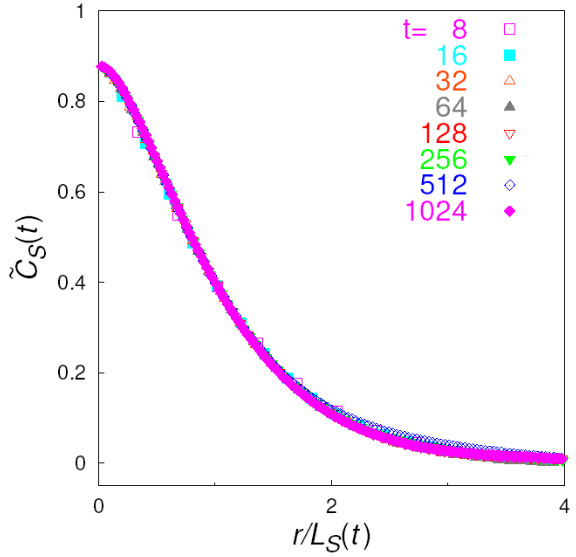
number of vortices



$$N_v \sim t^{-1} \quad (\leftarrow \gamma = \text{finite})$$

vortex separation $\xi \sim t^{1/2} \rightarrow z = 2$

phase correlation function



$$\tilde{C}_S(t) \equiv r^{\eta} C_S(t)$$

$$\eta = 0.083\,(\neq 0:\ \text{QLRO})$$

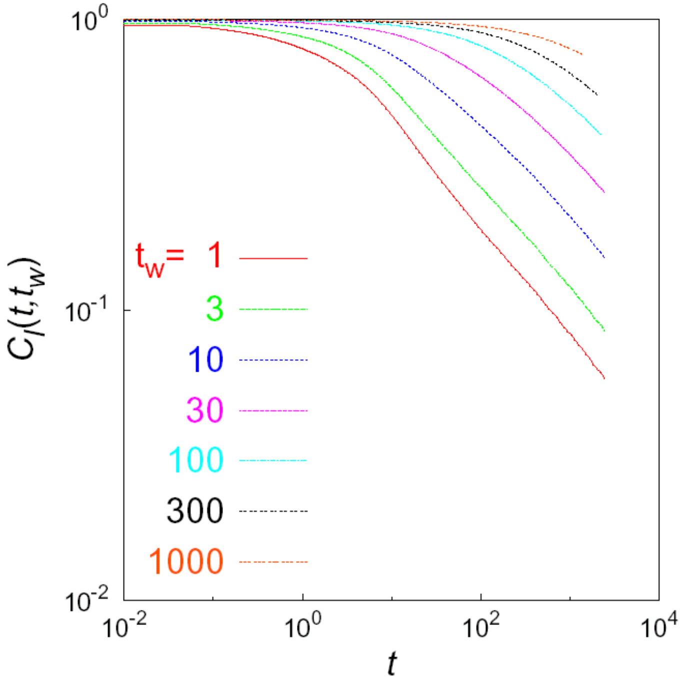
$$(\approx \eta_{eq} \approx T/2\pi + T^2/4\pi)$$

$f = 1/2$

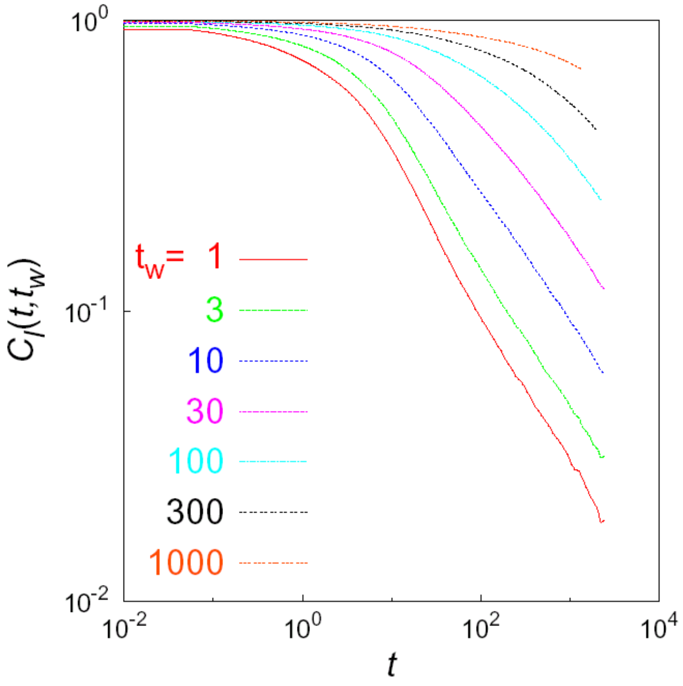
chirality autocorrelation function

$$C_I(t,t_w) \equiv \frac{1}{L^2}\sum_{\mathbf{R}}\langle q_{\mathbf{R}}(t+t_w)q_{\mathbf{R}}(t_w)]\rangle \qquad q(\mathbf{R},t) \equiv \mathrm{sgn}\sum_P \sin(\phi_i - \phi_j - A_{ij})$$

$T = 0.15$



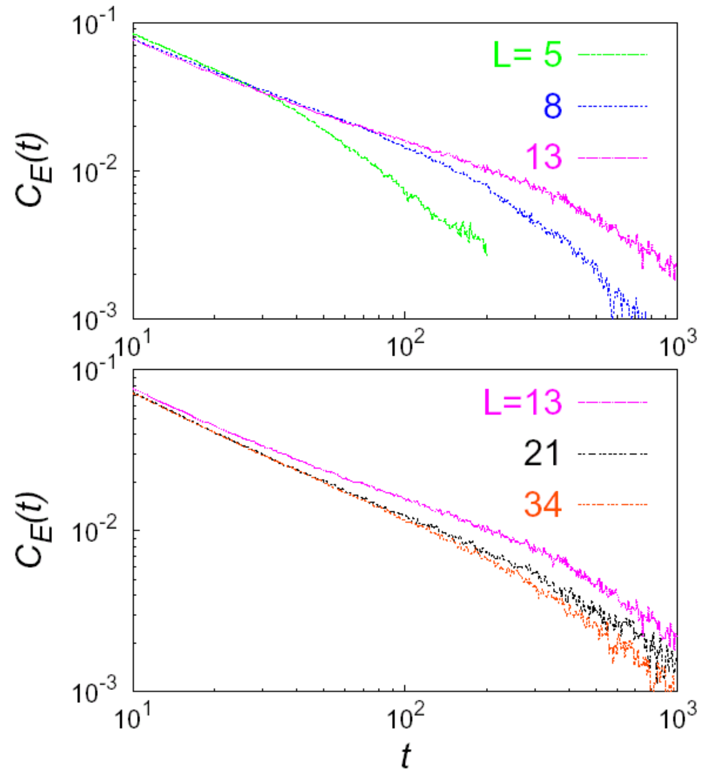
$T = 0.30$



aging

$f = \frac{\sqrt{5}-1}{2}$: irrational golden number

$T = 0.18$



$T < T_C(L)$

-algebraic slower as L↑

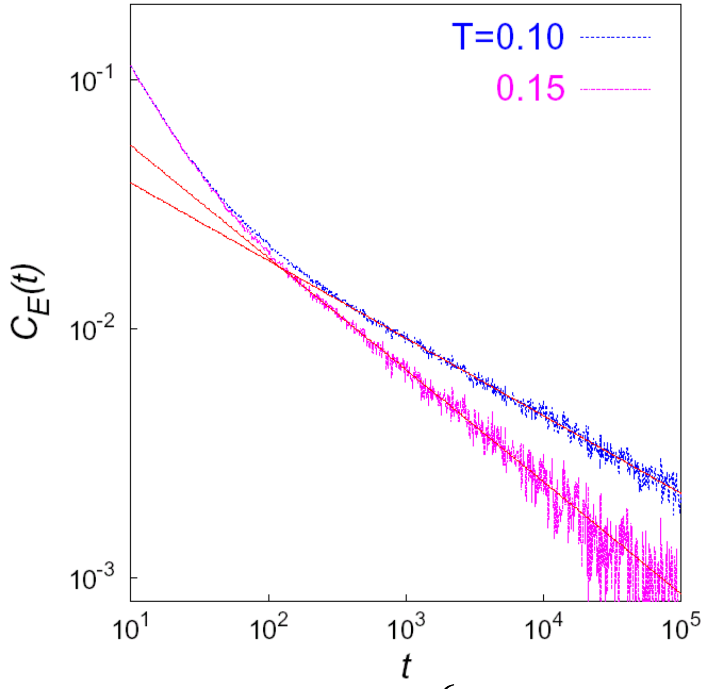
$T > T_C(L)$

-exponential faster as L↑

intrinsic finite-size effects

f = random: gauge glass

long-time regime: algebraic



$$C_E(t) \sim t^{-\alpha(T)}, \quad \alpha \approx \begin{cases} 0.31, & T = 0.10 \\ 0.45, & T = 0.15 \end{cases}$$

dep. on T → algebraic glass order

DC Driving

IV characteristics ($f = 0$)

$V \propto I^{a}$ (as $I \to 0$)

$T > T_c$: $a = 1$ (Ohmic)

$T \to T_c^-$: $a = 3$ ($a = z + 1$)

Effects of driving

Langevin equation of motion

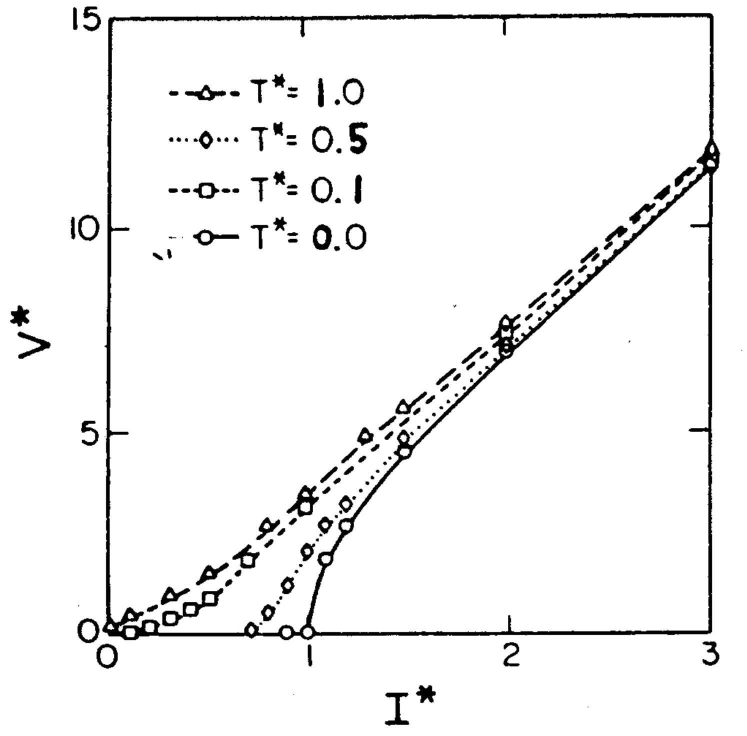
$\to$ Fokker-Planck equation for $P(\{\phi_i\},t)$

stationary sol. $P(\{\phi_i\}) \propto \exp(-\beta H[\phi])$

$\to$ effective Hamiltonian $H[\phi]$ (washboard pot.)

$$T_c(I) \approx \left[1 - \left(\frac{I}{I_c}\right)^2\right]^{1/4} T_c(I=0)$$

current-induced unbinding



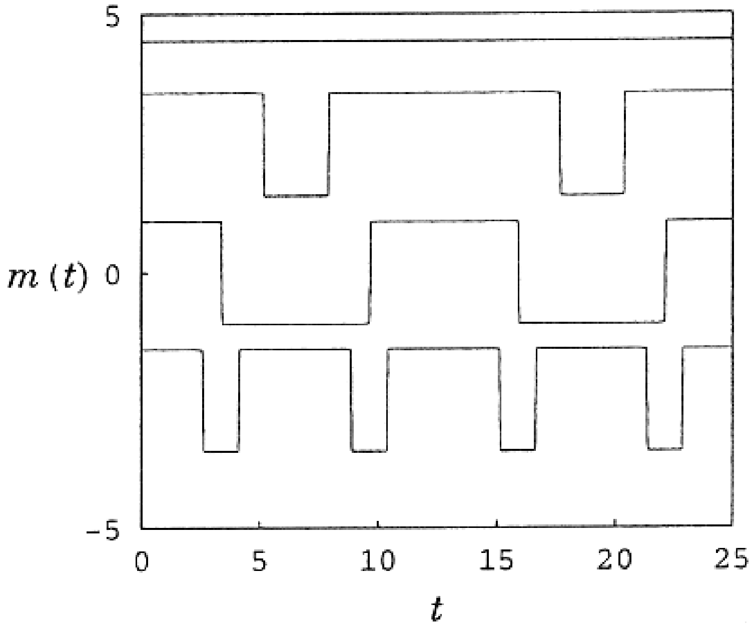
AC Driving

Dynamic Transitions: f = ½

Chirality $q(\mathbf{R},t) \equiv \operatorname{sgn} \sum_P \sin(\phi_i - \phi_j - A_{ij})$

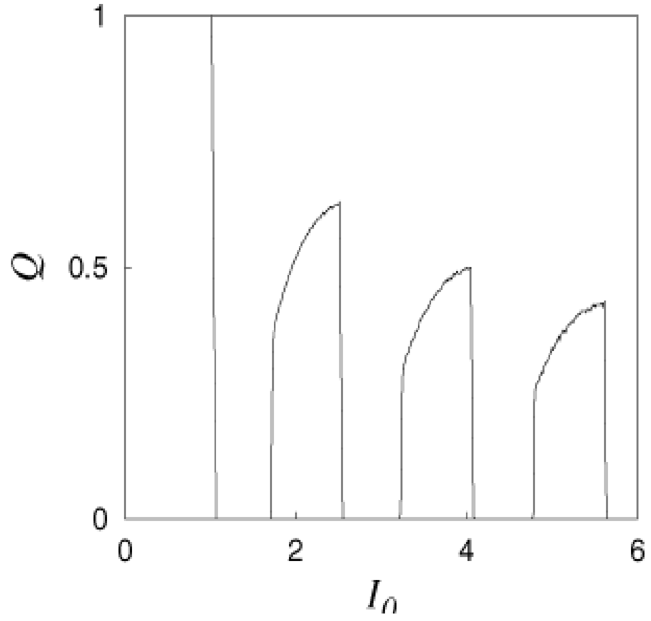
Staggered magnetization $m(t) \equiv L^{-2} \sum_{\mathbf{R}} (-1)^{x_i + y_i} q(\mathbf{R},t)$

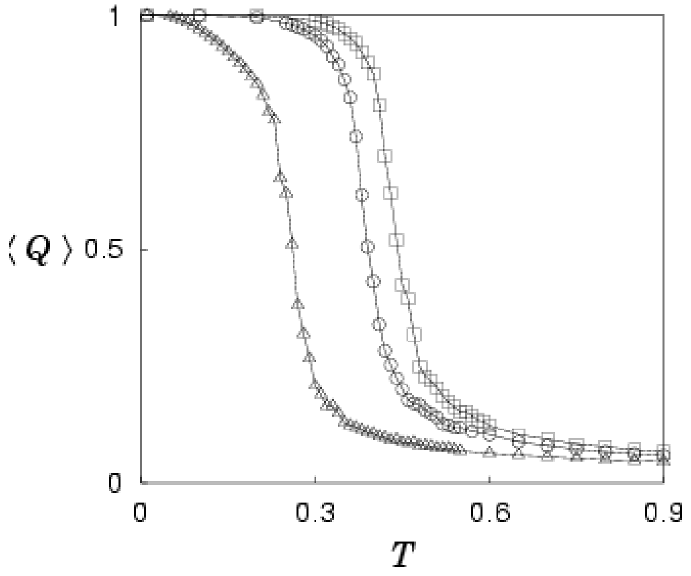
Dynamic order parameter $Q \equiv (\Omega/2\pi) \left| \oint dt \, m(t) \right|$



I_0 = 0.98, 1.03, 1.50, 2.0 from above

$\Omega/2\pi$ = 0.08 (T = 0)





Dynamic order parameter
vs temperature

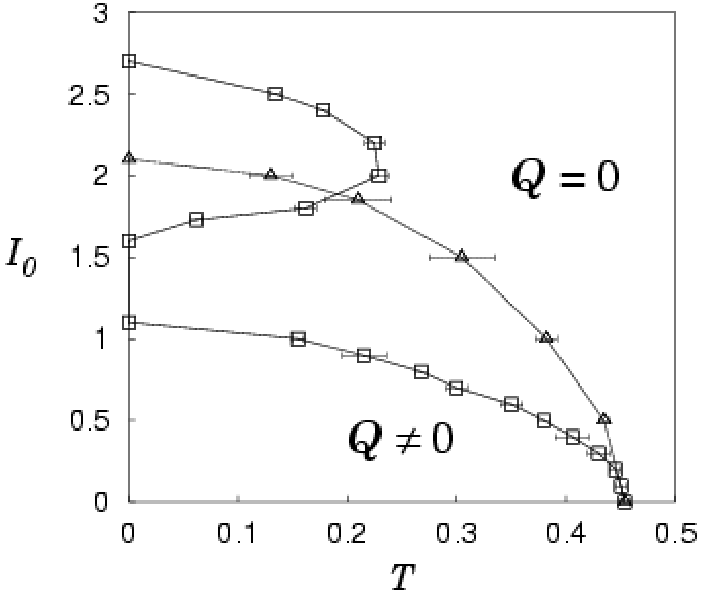
I_0 = 0.3 (□), 0.5(○), 0.8(Δ)

T_c estimated by Binder's cumulant

$$U_L = 1 - \frac{\langle Q^4 \rangle}{3\langle Q^2 \rangle^2}$$

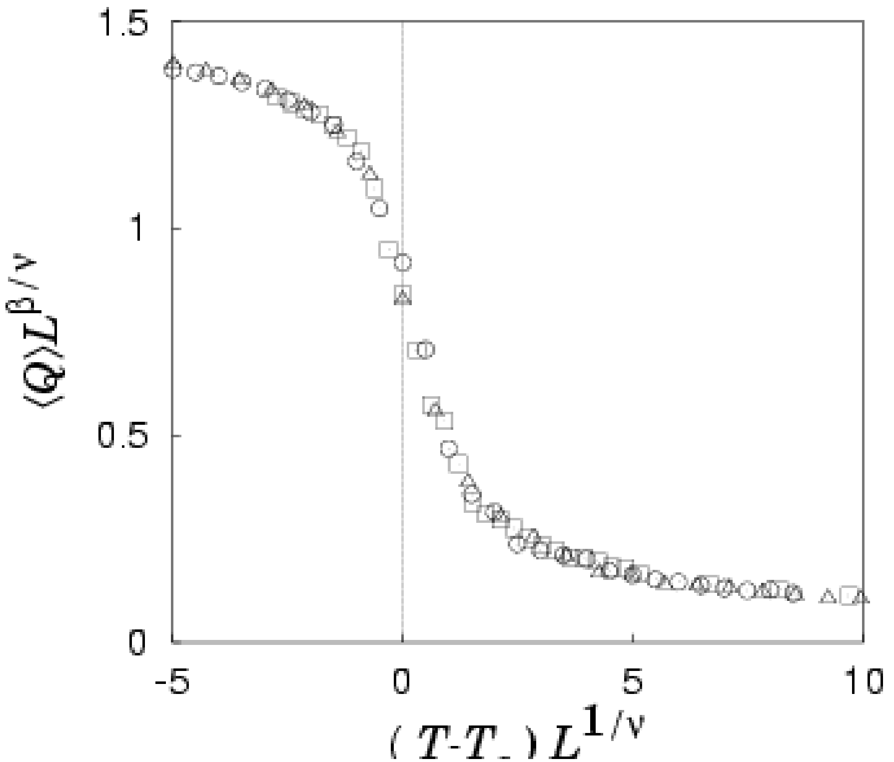
→ Phase diagram

$\Omega/2\pi = 0.08$(□), 0.16(Δ)



Scaling Relation

$$\langle Q \rangle = L^{-\beta/\nu} f((T - T_c)L^{1/\nu})$$



$I_0 = 0.3$

$\beta = 0.09; \quad \nu = 0.82$

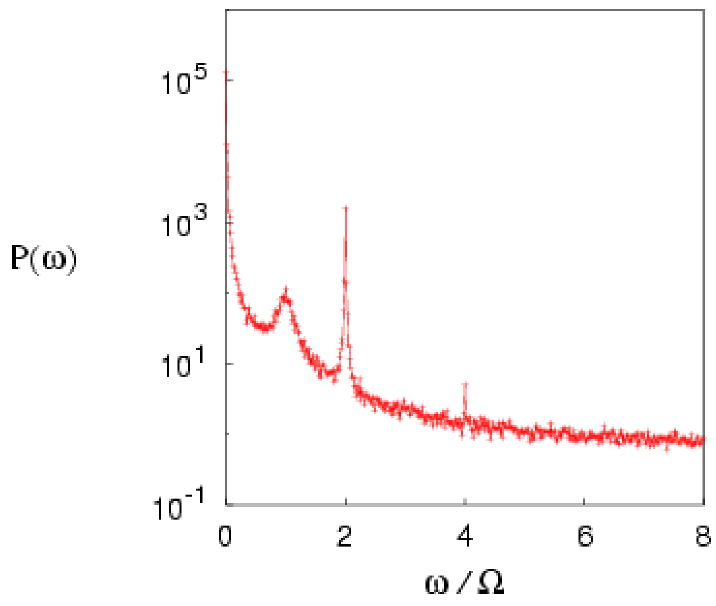
Same universality class as the equilibrium Z_2 transition in the FFXY model

Power spectrum

- $I_0 = 0.8;\ \Omega/2\pi = 0.08$

  $(Q > 0 \text{ at } T = 0)$

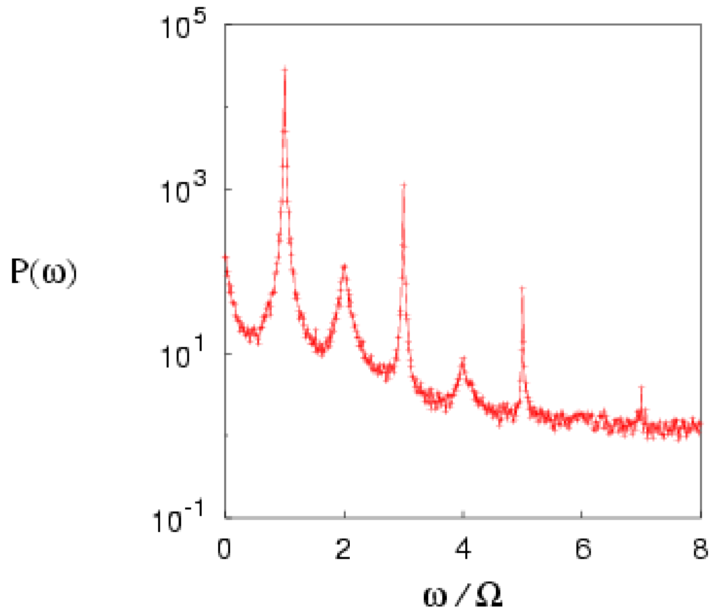
  ✓ Sharp peak at even harmonics
  ✓ Broad peak at odd harmonics



- $I_0 = 1.2;\ \Omega/2\pi = 0.08$

  $(Q = 0 \text{ at } T = 0)$

  ✓ Sharp peak at odd harmonics
  ✓ Broad peak at even harmonics

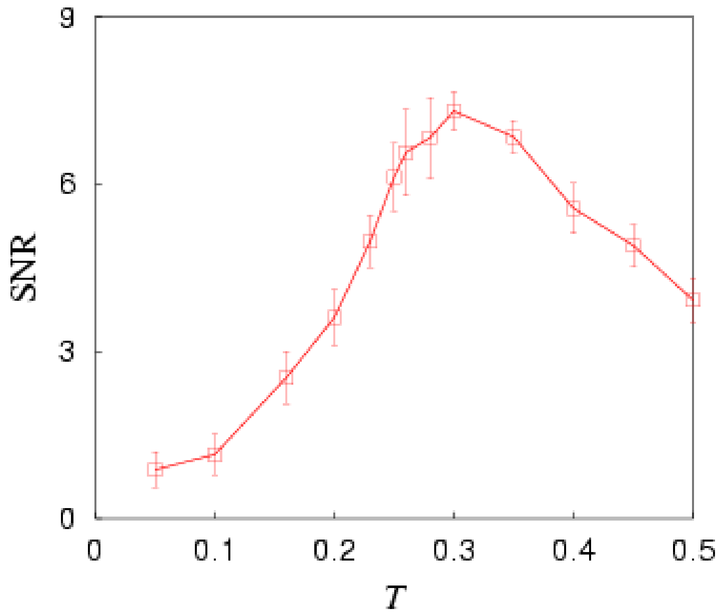


Stochastic Resonance (SR)

$$\mathrm{SNR} \equiv 10\log_{10}\left[\frac{S}{N}\right]$$

signal S : power spectrum peak at Ω
N : background noise level

- $I_0 = 0.8$; $\Omega/2\pi = 0.08$: $Q > 0$ (no osc.) at $T = 0$

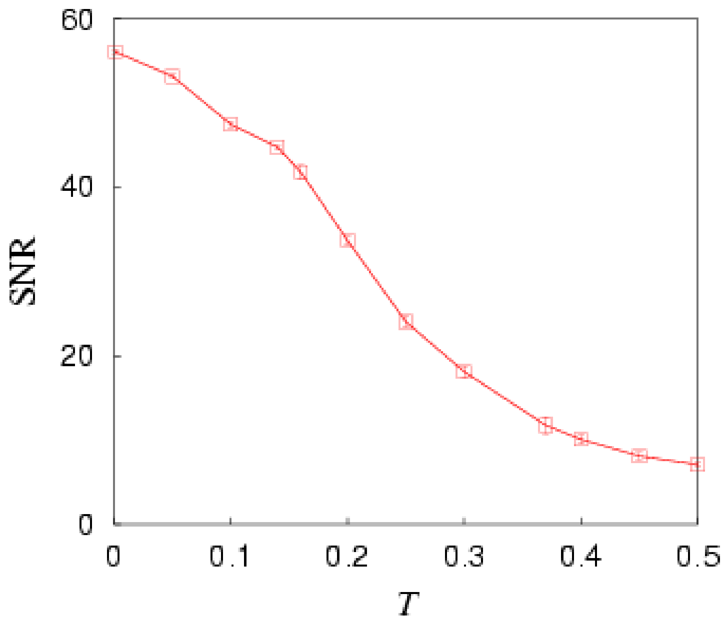


✓ SR phenomena
peak only at $T > T_c$
(⇔ double peaks around T_c)

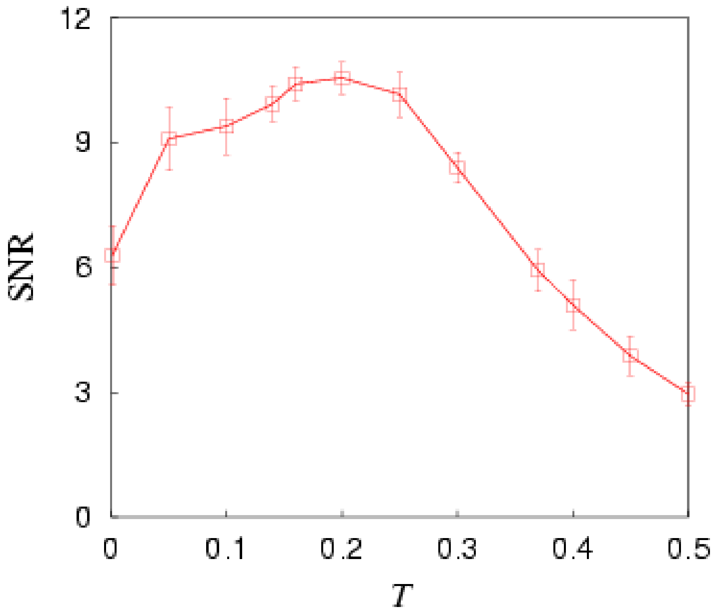
☜ $\tau \to \infty$ at $T < T_c$

- $I_0 = 1.2$; $\Omega/2\pi = 0.08$: $Q = 0$ (osc. with freq. Ω) at $T = 0$

First harmonics Ω



Second harmonics 2Ω

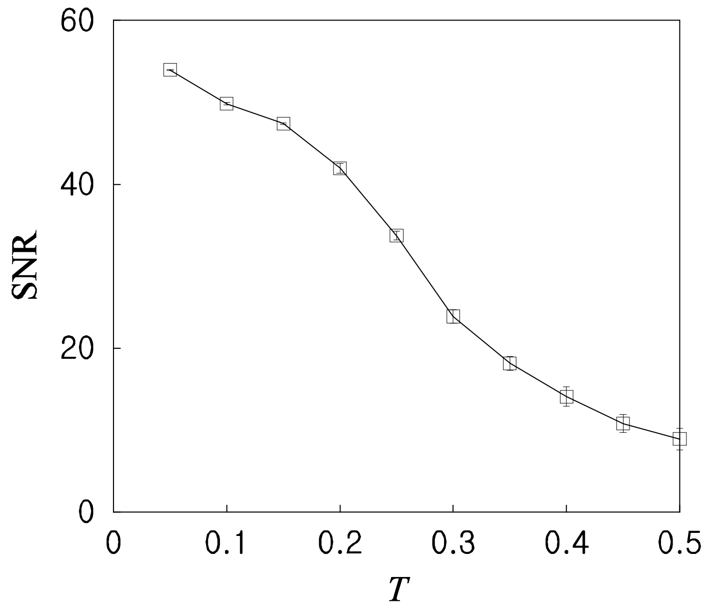


✓ No SR at 1st harmonics

✓ SR at 2nd harmonics

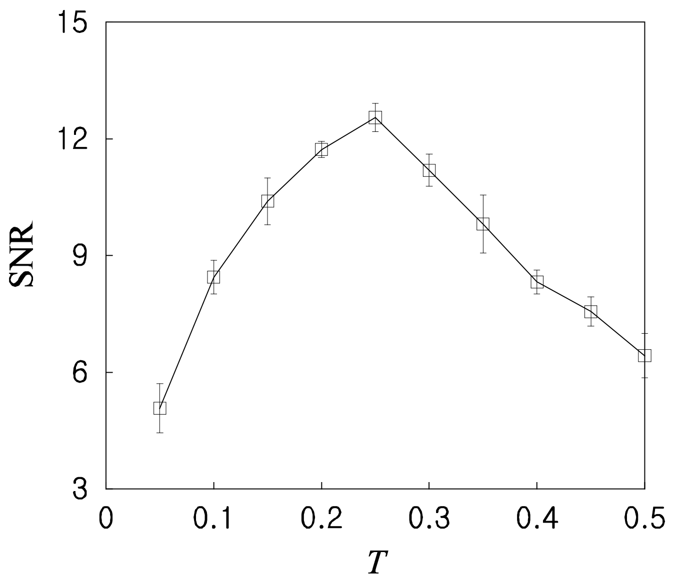
- $I_0 = 2.0$; $\Omega/2\pi = 0.08$: $Q > 0$ (osc. with freq. 2Ω) at $T = 0$

Second harmonics 2Ω



✓ No SR at 2nd harmonics

Third harmonics 3Ω



✓ SR at 3rd harmonics

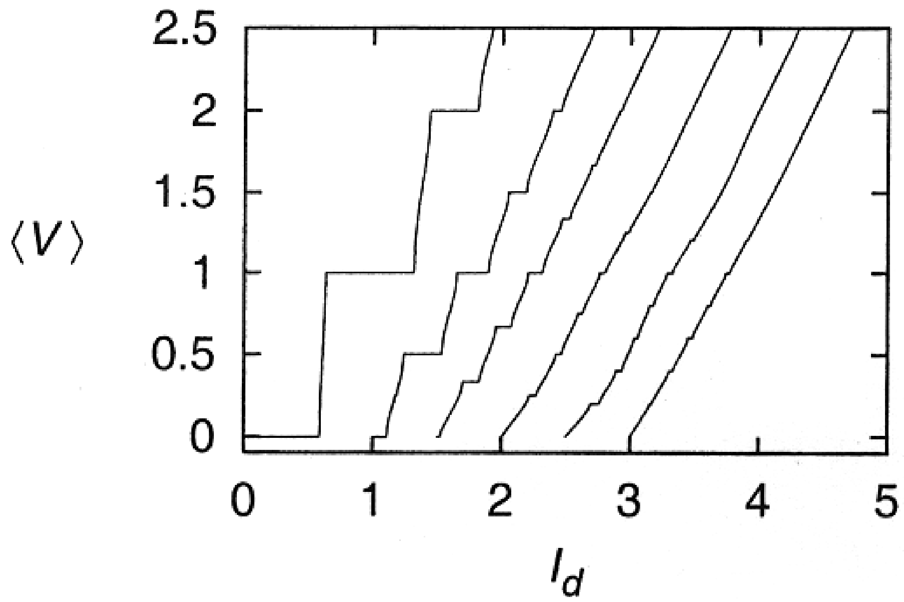
- SR also present in dc driven system (← vortex motion)

  → coherence resonance (CR)

Mode Locking, Melting, and Transitions

ac+dc driving $I = I_d + I_a \cos \Omega t$ at $T = 0$

$\rightarrow$ voltage quantization: giant Shapiro steps (GSS)



$$f = 0: \quad \langle V \rangle = n \frac{L\hbar\Omega}{2e} \quad \text{IGSS}$$

$$f = r/s: \langle V \rangle = \frac{n}{s} \frac{L\hbar\Omega}{2e} \quad \text{FGSS}$$

cf. devil's staircase

- mode locking $\leftarrow$ topological invariance
- chaos

Quantum Properties

Macroscopic Quantum Phenomena

$$H = 2e^2 \sum_{\langle i,j \rangle} [n_i - q] C_{ij}^{-1} \left[n_j - q \right] - E_J \sum_{\langle i,j \rangle} \cos\left(\phi_i - \phi_j - A_{ij} \right)$$

- Cooper pairs on the i^{th} grain

charge $2en_i$ (charging energy) $\leftrightarrow$ phase ϕ_i (Josephson energy)

canonical quantization $[n_i, \phi_j] = -i\delta_{ij} \Rightarrow$ MQP

- Magnetic field **B** $\rightarrow$ gauge field $A_{ij} \rightarrow$ magnetic frustration f
- Gate voltage (electric field) $V \rightarrow$ gauge charge $Q \rightarrow$ charge frustration $q = Q/2e$

thermal fluctuations $\leftrightarrow$ quantum fluctuations
magnetic frustration $\leftrightarrow$ charge frustration

l Quantum phase transition (S-I) at $T = 0$ ← quantum fluctuations

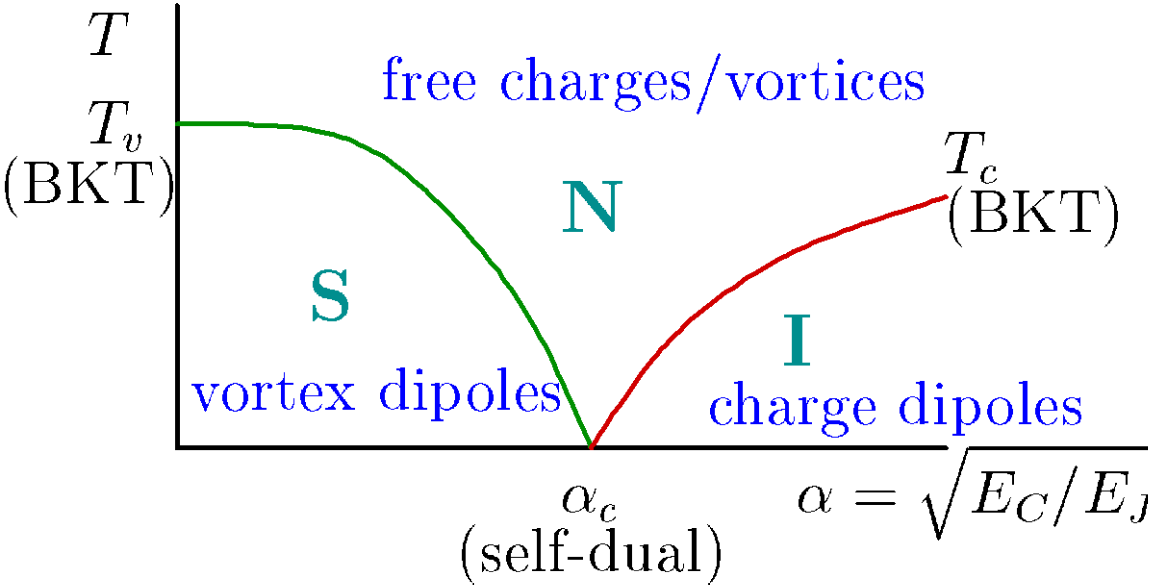
$E_C\ (\equiv 4e^2/C) \ll E_J$: phase ordering → superconductor

$E_C \gg E_J$: charge ordering → (Mott) insulator

$T > 0\ (f = q = 0)$: BKT transition

vortex unbinding (S-N) at T_v

charge unbinding (S-I) at T_c



Topological Quantization

- voltage → charge motion → current
  current → vortex motion → voltage

- $E_C \ll E_J$: ac driving → voltage quantization

$$f = \frac{r}{s}: \quad \langle V \rangle = \frac{n}{s}\frac{L\hbar\Omega}{2e}$$

Giant Shapiro steps (GSS)

- $E_C \gg E_J$: ac driving → current quantization (Bloch osc.)

$$q = \frac{r}{s}: \quad \langle I \rangle = \frac{n}{s}\frac{L2e\Omega}{2\pi}$$

Giant inverse Shapiro steps (GISS)

- $E_C \sim E_J$: E_C/E_J provides K.E. of vortices/charges, destroying lattice structure

  → quantum fluid → conductance quantization

  quantum vortex: boson with hard core (∵ Berry's phase)

  → fermion with gauge field (via Jordan-Wigner transformation)

$$\Rightarrow \quad \sigma_{xy} = m\frac{4e^2}{h} \quad (m = \text{even})$$

Quantum Hall effect (QHE)

Non-simply Connected Geometry

- Interference effects

  charge moving in magnetic field: Aharonov-Bohm effect

  $\rightarrow$ persistent current $$I = \frac{2e}{h}\frac{\partial E}{\partial f}$$

  vortex moving in gauge charge field: Aharonov-Casher effect

  $\rightarrow$ persistent voltage $$V = \frac{1}{2e}\frac{\partial E}{\partial q}$$

- Coupled Array
  - charge transport via excitons
    (pairs of excess and deficit Cooper pairs)
  - interesting phase transition and transport properties

Vortex Current

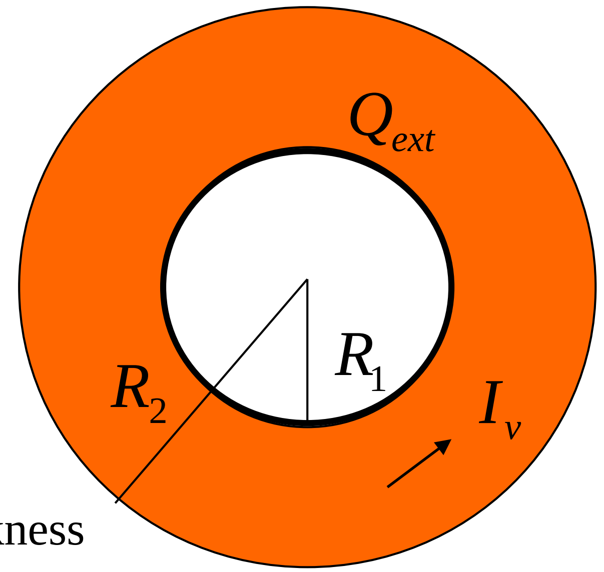
Annulus-shaped array with induced charge $Q_{ext} \equiv -2eq$ on the inner boundary

- N vortices ➡ Laughlin-type wave function
- persistent vortex current $I_v = \dfrac{c}{2e}\dfrac{\partial E}{\partial q}$
- persistent voltage $V = -\dfrac{I_v}{c}$

$\Rightarrow$ spontaneous voltage

$$V_s = \frac{NE_c}{\pi^2 e}\left[1 + \frac{aNE_c}{4\pi^2 e^2 \ln(R_2/R_1)}\right]^{-1}$$

a: thickness



Epilogue

Symmetry, Topology, and Phase Transitions

Paradigmatic system: XY model and Related Systems → rich physics

Equilibrium phase transitions

BKT transition, Ising-type transition, double transitions, 1st-order transition, $T = 0$ glass transition, algebraic glass transition

Nobel Prize in Physics 2016

½ David J. Thouless, ½ J. Michael Kosterlitz and F. Duncan M. Haldane

theoretical discoveries of topological phase transitions and topological phases of matter

Dynamic relaxation and responses

anomalous relaxation and coarsening, roughening transition, aging, nonlinear *IV* relations and mode locking, current-induced unbinding, dynamic transition, SR

Macroscopic/mesoscopic quantum phenomena

quantum fluctuations and dissipation, quantum phase transition, charge-vortex duality, quantum vortex, persistent current and voltage, topological quantization, exciton