Renormalization Group Transformation (RGT) of Two-dimensional Ising spins by decimation projector

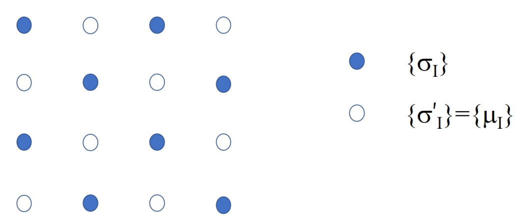
In this project we consider two-dimensional N Ising spins on a square lattice with the Hamiltonian

$$\bar{K}[S] \equiv -\beta\mathcal{H}[S] = K\sum_{\langle i,j\rangle} S_i S_j,$$

where the summation runs over the nearest-neighbor pairs $\langle i,j\rangle$. We use a projector operator of decimation defined by

$$P(\mu,\sigma) = \prod_I \delta_{\mu_I,\sigma'_I},$$

where the schematic block spins are represented in the figure.



1. Show that the RGT does not produce the Hamiltonian which matches exactly $\bar{K}'[\mu] = K'\sum_{\langle i,j\rangle} \mu_i\mu_j$,

2. Show that the Hamiltonian produced by the RGT can be represented by the following Hamiltonian

$$\bar{K}'[\mu] = K_1\sum_{\langle i,j\rangle} \mu_i\mu_j + K_2\sum_{\langle\langle i,j\rangle\rangle} \mu_i\mu_j + K_3\sum_{\square} \mu_i\mu_j\mu_k\mu_l,$$

   where the second summation runs over next-nearest pairs and the third over squares made of four spins (i,j,k,l). Express K_1, K_2, K_3 in terms of K.

3. We assume that K_3 is negligible and that we can estimate K' in the mean-field spirit:

$$K_1\sum_{\langle i,j\rangle} \mu_i\mu_j + K_2\sum_{\langle\langle i,j\rangle\rangle} \mu_i\mu_j \approx K'\sum_{\langle i,j\rangle} \mu_i\mu_j.$$

   Estimate $K'(K_1,K_2)$ in the case that all spins are aligned and find the recursion relation between K and K'.

4. Use the recursion relation to find all fixed points and critical exponents at the critical fixed point. Compare them with the exact results.