Foundations of Critical Phenomena

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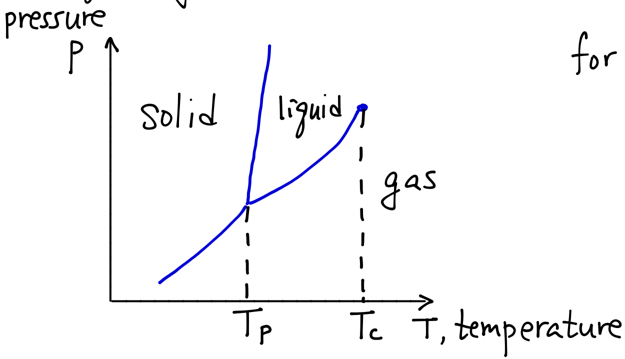
Contents

I. Phase Transitions and Critical Phenomena

phase transition : abrupt change of the system by the variation of external parameters

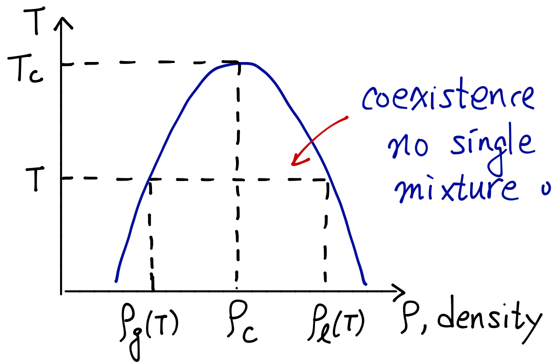
<liquid-gas transition>



for $T_p < T < T_c$

gas $\rightarrow$ liquid as $p \uparrow$

"liquid-gas transition"



coexistence region
no single phase
mixture of liquid & gas

for $T > T_c$, continuous change in ρ

for $T < T_c$, abrupt change in ρ

$$\rho_g \rightarrow \rho_\ell$$

"order parameter"

$$\Delta\rho(T) \equiv \rho_\ell(T) - \rho_g(T)$$

<magnetic transition>

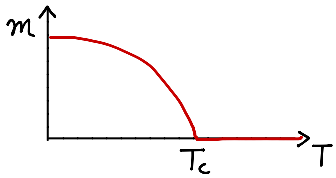
magnet $\begin{cases} \text{no magnetization (paramagnetic) for } T > T_c \\ \text{finite magnetization (ferromagnetic) for } T < T_c \end{cases}$

alignment of spins $\Rightarrow$ ferromagnetism

magnetization $\vec{m} \equiv \frac{1}{N} \sum_i \vec{S}_i$ "order parameter"

Two kinds of phase transitions

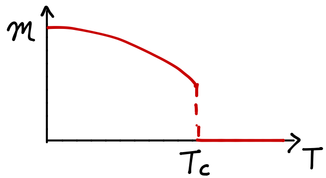
① continuous phase transition



order parameter is continuous at the transition

critical phenomena show up

② discontinuous transition



finite jump in order parameter

coexistence of two phases at the transition

In a conventional transition

order parameter : first-order derivative of free energy

ex) $m = -\dfrac{\partial f}{\partial H}$ $\quad$ m : magnetization, f : free energy density

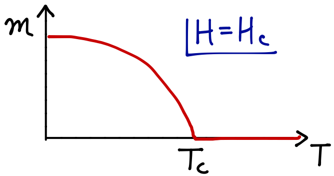
H : magnetic field

$\Rightarrow$ phase transition : singularity in free energy

Some derivatives of free energy is not continuous at transition

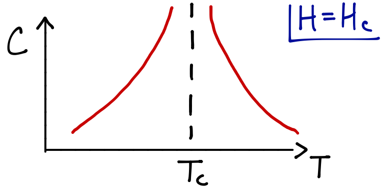
critical phenomena : at continuous transition,

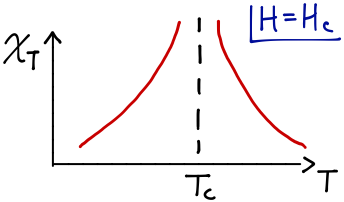
physical quantities show powe-law behaviors



$m \sim (T_c - T)^{\beta}$

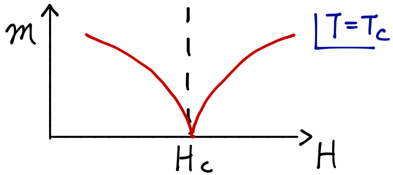
$C \sim |T - T_c|^{-\alpha}$





$\chi_T \sim |T - T_c|^{-\gamma}$

$m \sim |H - H_c|^{1/\delta}$



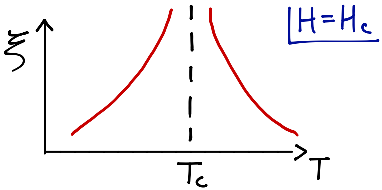
correlation length ξ

characteristic decay length for correlation function of order parameters

ex) corr. ftn. $G(\vec{r}-\vec{r}\,') = \langle \vec{S}(\vec{r}) \cdot \vec{S}(\vec{r}\,') \rangle - \langle \vec{S}(\vec{r}) \rangle \cdot \langle \vec{S}(\vec{r}\,') \rangle$

generally $G(\vec{r}) \sim e^{-r/\xi}$: exponential decay

at critical point T_c $G(r) \sim \dfrac{1}{r^{d-2+\eta}}$: power-law decay $(\xi = \infty)$



$\xi \sim |T-T_c|^{-\nu}$

critical exponents : $\alpha, \beta, \gamma, \delta, \eta, \nu$

they depend only on
- symmetries of the system
- spatial dimension
- short-rangeness of interactions

"universality class"

< mean-field theory >

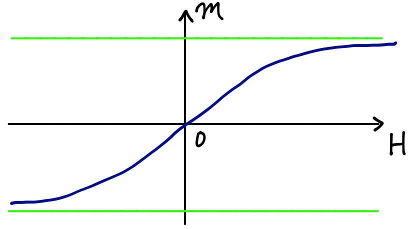
Weiss mean-field theory

nearest-neighbor Ising model

$$\mathcal{H}[S] = -H\sum_i S_i - J\sum_{\langle i,j\rangle} S_i S_j \qquad S_i = \pm 1\ ,\ \langle i,j\rangle : \text{nearest-neighbor pairs}$$

i) $J=0$: partition function $Z[H] = \sum_{\{S_i\}} \prod_i e^{-\beta H S_i} = \left[2\cosh(H/k_BT)\right]^N \qquad (\beta \equiv 1/k_BT)$

$$m = -\frac{1}{N}\frac{\partial F}{\partial H} = \frac{k_BT}{N}\frac{\partial}{\partial H}\ln Z[H] = \tanh\left(\frac{H}{k_BT}\right)$$



ii) $J \neq 0$: $\mathcal{H} \xrightarrow{\text{mean-field}} \mathcal{H}_{MF} = -\sum_i H_{MF}^{(i)} S_i$

$$\mathcal{H} = -\sum_i H_i S_i$$

n.n. of i

$$H_i = H + J\sum_j{}' S_j = H + J\sum_j{}'\langle S_j\rangle + \cancel{J\sum_j{}'(S_j - \langle S_j\rangle)}$$

neglect fluctuations from mean value

$$H_{MF}^{(i)} = H + zJm \qquad (z : \text{number of nearest neighbors},\ \ m = \langle S_i\rangle)$$

$$\therefore\ m = \tanh\left(\frac{H_{MF}^{(i)}}{k_BT}\right) = \tanh\left(\frac{H + zJm}{k_BT}\right)$$

Hamiltonian $\mathcal{H} = -H\sum_i S_i - J\sum_{\langle i,j\rangle}(m+\tilde{S}_i)(m+\tilde{S}_j)$ $\quad (\tilde{S}_i \equiv S_i - m)$

$$= -H\sum_i S_i - J\sum_{\langle i,j\rangle}\left[m^2 + m(\tilde{S}_i+\tilde{S}_j) + \cancel{\tilde{S}_i\tilde{S}_j}\right] \quad \text{neglect}$$

$$\mathcal{H}_{MF} = -\sum_i S_i \underbrace{\left(H + \sum_j{}' Jm\right)}_{H_{MF}^{(i)}} + Jm^2 \underbrace{\sum_{\langle i,j\rangle} 1}_{=\frac{1}{2}zN}$$

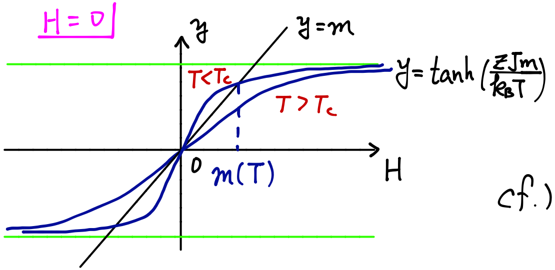
$$= -(H+zJm)\sum_i S_i + \frac{1}{2}NzJm^2$$

free energy $F = -k_BT \ln \sum_{\{S_i\}} e^{-\beta\mathcal{H}_{MF}} = \frac{1}{2}NzJm^2 - k_BTN\ln\left[2\cosh\left(\frac{H+zJm}{k_BT}\right)\right]$

minimize F : $\frac{\partial F}{\partial m} = 0 = NzJm - NzJ\,\cancel{\ln\left[2\cosh\left(\frac{H+zJm}{k_BT}\right)\right]}\ \tanh\left(\frac{H+zJm}{k_BT}\right)$

$$\Rightarrow m = \tanh\left(\frac{H+zJm}{k_BT}\right)$$

$\boxed{H=0}$



$y'(m=0)_{T=T_c} = 1 \Rightarrow \frac{zJ}{k_BT_c} = 1,\ T_c = \frac{zJ}{k_B}$

$$\begin{cases} T>T_c : m=0 \\ T<T_c : m\neq 0 \end{cases}$$

cf.) $d=1,\ z=2$: $T_{c,MF}^{1D} = \frac{2J}{k_B}$, $T_{c,\text{exact}} = 0$

$d=2,\ z=4$ (square lattice) : $T_{c,MF}^{2D} = \frac{4J}{k_B}$ $\quad T_{c,\text{exact}} = \frac{2}{\ln(\sqrt{2}+1)}\frac{J}{k_B} \simeq 2.3J/k_B$

Critical phenomena in mean-field theory

$$m = \tanh\left(h + \frac{T_c}{T} m\right)$$

$$\Rightarrow h + \frac{T_c}{T} m = \tanh^{-1}(m) = m + \frac{1}{3} m^3 + \mathcal{O}(m^5)$$

$$h = t \cdot m + \frac{1}{3} m^3 \qquad \left(t \equiv 1 - \frac{T_c}{T} : \text{reduced temperature}, \; h \equiv \frac{H}{k_B T} \right)$$

① $H = 0$, $T \to T_c^-$: $m \sim (-t)^\beta$

$$m\left(t + \frac{1}{3} m^2\right) = 0 \quad \to \quad m^2 = -3t \qquad \beta = \frac{1}{2}$$

② $T = T_c$, $H \to 0$: $m \sim |H|^{1/\delta}$

$$h \simeq \frac{1}{3} m^3 \qquad \delta = 3$$

③ $H = 0$, $T \to T_c$: $\chi_T \equiv \left(\frac{\partial m}{\partial H}\right)_T \sim |t|^{-\gamma}$

$$\frac{\partial}{\partial H} : \quad \frac{1}{k_B T} = \chi_T \left(t + m^2 \right)$$

i) $T \to T_c^+$: $m = 0$, $\chi_T \simeq \dfrac{1}{k_B (T - T_c)}$

ii) $T \to T_c^-$: $m \simeq \sqrt{-3t}$, $\chi_T \simeq \dfrac{1}{2 k_B (T_c - T)}$

$$\gamma = 1$$

④ $H = 0$, $T \to T_c$: $C_H \equiv \frac{1}{N} \frac{\partial E}{\partial T}\Big|_H \sim |t|^{-\alpha}$

$$E = \langle \mathcal{H}_{MF} \rangle = -\frac{1}{2} N z J m^2$$

i) $T > T_c$: $m = 0$, $E = 0$, $C_H = 0$

$$\alpha = 0$$

ii) $T \to T_c^-$: $m = \sqrt{-3t}$, $E \simeq \frac{3}{2} N k_B (T - T_c)$, $C_H \simeq \frac{3}{2} N k_B$

critical exponents for n.n. Ising model

	mean-field	2D Ising	3D Ising
α	0	0	0.11
β	$\frac{1}{2}$	$\frac{1}{8}$	0.33
γ	1	$\frac{7}{4}$	1.24
δ	3	15	4.82

II. Landau Theory

<phenomenological Landau theory>

Landau free energy $L[m]$ (m : order parameter)

① L should be consistent with symmetries of the system

② near the transition

L can be expanded in a power series of m

Landau free energy density

$$\mathcal{L} \equiv \frac{L}{V} = \sum_{n=0}^{\infty} a_n(K, T)\, m^n$$

③ find m_0 which minimizes $\mathcal{L}$

$m_0 = 0$: disordered phase $(T > T_c)$

$m_0 \neq 0$: ordered phase $(T < T_c)$

< Landau theory for Ising model >

near T_c $\quad \mathcal{L} = a_0 + a_1 m + a_2 m^2 + a_3 m^3 + a_4 m^4 + \cancel{\cdots}$ neglect

for $H=0$: $\quad \mathcal{L}(m) = \mathcal{L}(-m) \Rightarrow a_1 = a_3 = 0$

$a_2(T) = a_2^{(0)} + t \cdot a_2^{(1)} + \mathcal{O}(t^2) \qquad t \equiv \dfrac{T-T_c}{T}$: reduced temperature

$a_4(T) = a_4^{(0)} + \mathcal{O}(t)$

$$\frac{\partial \mathcal{L}}{\partial m} = 0 = 2a_2(t)\, m + 4 a_4 m^3 \Rightarrow m = 0 \text{ or } \sqrt{\frac{-a_2(t)}{2a_4}}$$

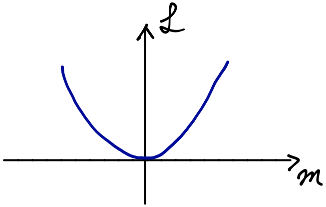
if $m = \begin{cases} 0 & t>0 \\ \text{finite} & t<0 \end{cases}, \quad \Rightarrow \quad a_2^{(0)} = 0 \quad a_4^{(0)} > 0, \quad a_2^{(1)} > 0$

for $H \neq 0$, add $-Hm$ $\qquad$ cf.) $-H\sum_i S_i$

$$\Rightarrow \mathcal{L} = -Hm + a t m^2 + \frac{1}{2} b m^4 \quad \left(a \equiv a_2^{(1)},\ b = 2a_4^{(0)} \right)$$

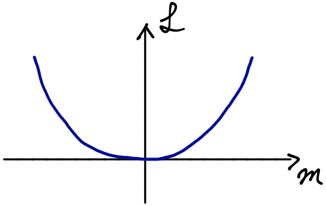
$\underline{H=0}$

$T>T_c$



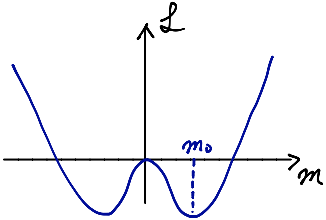
$m=0$

$T=T_c$



$m=0$

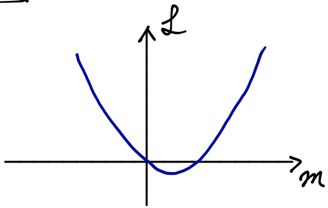
$T<T_c$



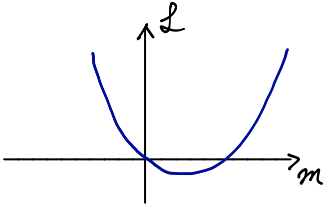
$m\neq 0$

$\underline{H>0}$

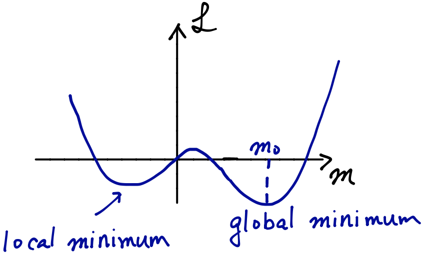
$T>T_c$



$T=T_c$

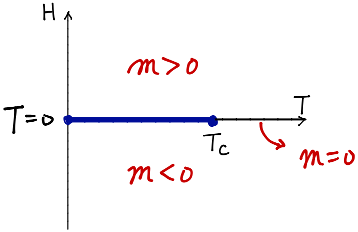


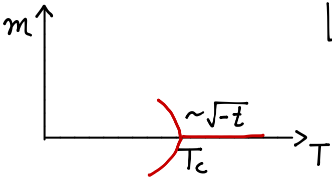
$T<T_c$



$$\frac{\partial \mathcal{L}}{\partial m} = 0 = -H + 2atm + 2bm^3$$

$$H=0 : \quad m = \begin{cases} \underset{\text{unstable}}{\cancel{0}},\ \pm\sqrt{\frac{a}{b}(-t)} & t<0 \quad (T<T_c) \\ 0 & t>0 \quad (T>T_c) \end{cases}$$





$H=0$

critical exponents

① $t<0$: $m \sim \sqrt{-t}$ $\qquad \beta = \frac{1}{2}$

② $H=0$: $\mathcal{L} = \begin{cases} 0 & (t>0) \\ -\frac{a^2}{2b}t & (t>0) \end{cases} \Rightarrow C = -T\frac{\partial^2 \mathcal{L}}{\partial T^2} = \begin{cases} 0 & t>0 \\ \frac{a^2}{b}\cdot\frac{1}{T_c} & t<0 \end{cases}$

: finite jump $\qquad \alpha = 0$

③ $t=0$: $H = 2bm^3 \Rightarrow \delta = 3$

④ $$0 = \frac{\partial}{\partial H}\left(-H + 2atm + 2bm^3\right)\Big|_t = -1 + \left(\frac{\partial m}{\partial H}\right)_T \left(2at + 6bm^2\right)$$

$$\chi_T = \left(\frac{\partial m}{\partial H}\right)_T = \frac{1}{2at^2 + 6bn^2} \quad \to m$$

$$= \begin{cases} \frac{1}{2at} & t>0 \\ -\frac{1}{4at} & t<0 \end{cases} \qquad \gamma = 1$$

if no symmetry on m

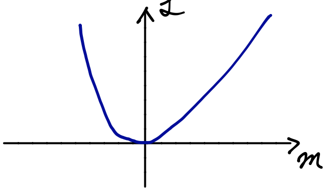
$$\mathcal{L} = -Hm + atm^2 + \frac{1}{2}bm^4 + a_3 m^3$$

$H=0$: $$\frac{\partial \mathcal{L}}{\partial m} = (2at + 3Cm + 2bm^2)m = 0$$

$$m = 0,\ -C \pm \sqrt{C^2 - \frac{at}{b}} \qquad \left(C \equiv \frac{3a_3}{4b}\right)$$

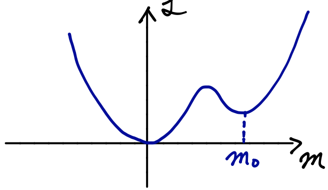
$t < t^* \equiv \frac{bc^2}{a}$: two maxima and one minimum

<u>$c<0$</u>



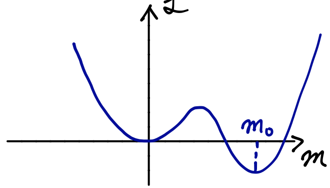
$m=0$

$t > t^*$



$m=0$

$t_1 < t < t^*$



$m=0$

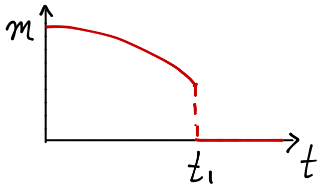
$t > t_1$

discontinuous jump at $t = t_1$: transition temperature

$$m_0 = -c + \sqrt{c^2 - at/b}$$

$$\mathcal{L}(m_0) = m_0^2\left(at^2 + \frac{4bc}{3}m_0 + \frac{1}{2}bm_0^2\right) = 0$$

$$\Rightarrow t_1 = \frac{8bc^2}{9a},\quad m_0(t_1) = -\frac{4}{3}c$$



Allow some spatial variation of m

"local order parameter" $m(\vec{r})$

Landau free energy functional

$$L[m(\vec{r})] \equiv \int d^d\vec{r} \; \mathcal{L}(m(\vec{r}))$$

$$\mathcal{L}(\underset{m}{\cancel{x}}(\vec{r})) = a\,t\,m^2 + \frac{1}{2}bm^4 - H(\vec{r})m(\vec{r}) + \frac{1}{2}\gamma\left(\vec{\nabla}m(\vec{r})\right)^2$$

order parameter

$$\langle m(\vec{r})\rangle = -\frac{\delta F}{\delta H(\vec{r})} = k_B T \frac{1}{Z}\frac{\delta Z}{\delta H(\vec{r})} \qquad (Z\text{: partition function})$$

generalized isothermal susceptibility

$$\chi_T(\vec{r},\vec{r}\,') \equiv \frac{\delta\langle m(\vec{r})\rangle}{\delta H(\vec{r}\,')} = k_B T \frac{\delta}{\delta H(\vec{r}\,')}\left[\frac{1}{Z}\frac{\delta Z}{\delta H(\vec{r})}\right] = \frac{1}{k_B T}\left[\frac{1}{Z}\frac{\delta^2 Z}{\delta H(\vec{r})\delta H(\vec{r}\,')} - \frac{1}{Z}\frac{\delta Z}{\partial H(\vec{r})}\frac{1}{Z}\frac{\delta Z}{\partial H(\vec{r}\,')}\right]$$

$$= \frac{1}{k_B T}\left[\langle m(\vec{r})m(\vec{r}\,')\rangle - \langle m(\vec{r})\rangle\langle m(\vec{r}\,')\rangle\right] \equiv \frac{1}{k_B T} G(\vec{r},\vec{r}\,') \text{ "correlation function"}$$

translationally invariant system $\quad G(\vec{r}-\vec{r}\,') = k_B T\,\chi_T(\vec{r}-\vec{r}\,')$

$$\xrightarrow{\text{Fourier transf.}} \quad \tilde{G}(\vec{k}) = k_B T\,\tilde{\chi}_T(\vec{k})$$

static susceptibility $\quad \chi_T \equiv \lim_{\vec{k}\to 0}\tilde{\chi}_T(\vec{k}) = \frac{1}{k_B T}\tilde{G}(\vec{k}=0) = \frac{1}{k_B T}\int d^d\vec{r}\; G(\vec{r})$

correlation function

$$L = \int d^d\vec r \left[\frac{1}{2}\gamma \left(\vec\nabla m(\vec r)\right)^2 + a t m^2 + \frac{1}{2} b m^4 - H(\vec r)\,\overset{m}{\cancel{\chi}}(\vec r) \right]$$

$$\frac{\delta L}{\delta \underset{m}{\cancel{\chi}}(\vec r)} = 0 \Rightarrow \quad -\gamma \vec\nabla^2 m(\vec r) + 2at\, m(\vec r) + 2b\, m(\vec r)^3 - H(\vec r) = 0$$

$$\frac{\delta}{\delta H(\vec r\,')} \otimes : \quad \left[-\gamma \vec\nabla^2 + 2at + 6b\, m(\vec r)^2\right] \chi_T(\vec r - \vec r\,') = \delta(\vec r - \vec r\,')$$

$$\frac{1}{k_B T}\left[-\gamma \vec\nabla^2 + 2at + 6b\, m(\vec r)^2\right] G(\vec r - \vec r\,') = \delta(\vec r - \vec r\,')$$

for $t > 0$, $m = 0$: $\quad (-\vec\nabla^2 + \xi_>^{-2})\, G(\vec r - \vec r\,') = \dfrac{k_B T}{\gamma}\, \delta(\vec r - \vec r\,') \qquad \xi_>(t) \equiv \left(\dfrac{\gamma}{2at}\right)^{1/2}$

for $t < 0$, $m = \pm\sqrt{-\dfrac{at}{b}}$: $\quad (-\vec\nabla^2 + \xi_<^{-2})\, G(\vec r - \vec r\,') = \dfrac{k_B T}{\gamma}\, \delta(\vec r - \vec r\,') \qquad \xi_<(t) \equiv \left(-\dfrac{\gamma}{4at}\right)^{1/2}$

$\xi(t)$: correlation length $\sim t^{-1/2}$ $\qquad \nu = \dfrac{1}{2}$

$$(-\vec{\nabla}^2+\xi^{-2})\, G(\vec{r}-\vec{r}\,') = \frac{k_B T}{\gamma}\,\delta(\vec{r}-\vec{r}\,')$$

$$\xrightarrow{F.T.} (k^2+\xi^{-2})\,\tilde{G}(\vec{k}) = \frac{k_B T}{\gamma}$$

$$t=0\ (T=T_c):\quad \xi=\infty \quad\longrightarrow\quad \tilde{G}(\vec{k},T_c) = \frac{k_B T}{\gamma}\frac{1}{k^2}$$

$$G(\vec{r},T_c) \propto \int d^d\vec{k}\, e^{i\vec{k}\cdot\vec{r}}\,\frac{1}{k^2} \sim \frac{1}{r^{d-2}}\int d^d\vec{\rho}\, e^{i\vec{\rho}\cdot\hat{r}}\,\frac{1}{\rho^2} \qquad (\vec{\rho}\equiv\vec{k}r)$$

$$\text{at } T=T_c,\quad G(r)\sim\frac{1}{r^{d-2+\eta}} \qquad \eta=0$$

critical exponents for Ising model in Landau theory

$$\alpha=0,\quad \beta=\frac{1}{2},\quad \gamma=1,\quad \delta=3,\quad \nu=\frac{1}{2},\quad \eta=0$$

< Breakdown of Landau theory >

Ginzburg criterion

$$E_{LG} \equiv \frac{\left|\int_V d^d\vec{r}\, G(\vec{r})\right|^2}{\int_V d^d\vec{r}\ m(\vec{r})^2} \ll 1 \qquad (V \equiv \xi(T)^d)$$

$$\int_V d^d\vec{r}\ m(\vec{r})^2 \simeq m_0^2 \cdot \xi(T)^d \simeq \frac{a}{b}|t| \cdot \xi_0^d |t|^{-\frac{d}{2}} \qquad \left(\xi(T) \equiv \xi_0 |t|^{-\frac{1}{2}}\right)$$

$$\int_V d^d\vec{r}\, G(\vec{r}) \simeq k_B T_c \chi_T \simeq \frac{k_B T_c}{4a|t|}$$

$$\Rightarrow \quad E_{LG} = \frac{k_B}{4\, \Delta C\, \xi_0^d\, |t|^{(4-d)/2}} \qquad \left(\Delta C \equiv \frac{a^2}{b}\frac{1}{T_c}\right)$$

$$E_{LG} \ll 1 \quad \Leftrightarrow \quad |t|^{(4-d)/2} \gg \frac{k_B}{4 \Delta C \xi_0^d}$$

i) $d > 4$: satisfied as $t \to 0$, correct critical phenomena

ii) $d < 4$: Landau theory is not self-consistent as $t \to 0$

$d = 4$: "upper critical dimension"

beyond which critical phenomena from Landau theory are correct

<upper critical dimension>

in general

$$\int_V d^d\vec{r}\, G(\vec{r}) \sim \chi_T \sim |t|^{-\gamma}$$

$$\int_V d^d\vec{r}\, m(\vec{r})^2 \sim \xi^d m_0^2 \sim |t|^{2\beta-\nu d}$$

$$E_{LG} \ll 1 \rightarrow |t|^{-\gamma} \ll |t|^{2\beta-\nu d} \quad \text{as} \quad t \to 0$$

$$\gamma < \nu d - 2\beta \Rightarrow d > \frac{\gamma+2\beta}{\nu} \equiv d_{uc}$$

$$\left(= \frac{2-\alpha}{\cancel{\gamma}\nu}\right) \quad \text{cf.) } \alpha+2\beta+\gamma = 2.$$

III. Scaling Hypothesis and Block Spin Transformaton

Scaling Laws for critical exponents $\alpha, \beta, \gamma, \delta, \nu, \eta, \ldots$

we can derive following inequalities from thermodynamics

① $\alpha + 2\beta + \gamma \geq 2$: Rushbrooke

② $\beta + \gamma \geq \beta\delta \geq 2 - \alpha - \beta$: Griffith

③ $(2-\eta)\nu \geq \gamma$: Fisher

$$\left.\begin{array}{lll} ④ & d(\delta-1)/(\delta+1) \geq 2-\eta & : \text{Buckingham-Gunton} \\ ⑤ & d\nu \geq 2-\alpha & : \text{Josephson} \end{array}\right\} \text{hyperscaling}$$

(d : spatial dimension)

ex) Rushbrooke inequality

$$C_H = T\left(\frac{\partial S}{\partial T}\right)_H = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_H , \qquad C_m = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_m$$

$$dS = \left(\frac{\partial S}{\partial T}\right)_m dT + \left(\frac{\partial S}{\partial m}\right)_T dm = \left(\frac{\partial S}{\partial T}\right)_H dT + \left(\frac{\partial S}{\partial H}\right)_T dH$$

$$dm=0 \Rightarrow \left(\frac{\partial S}{\partial T}\right)_m = \left(\frac{\partial S}{\partial T}\right)_H + \left(\frac{\partial S}{\partial H}\right)_T \left(\frac{\partial H}{\partial T}\right)_m$$

use

$$\begin{cases} \text{Maxwell relation } \left(\frac{\partial S}{\partial H}\right)_T = \left(\frac{\partial M}{\partial T}\right)_H \\ \left(\frac{\partial H}{\partial T}\right)_m = -\dfrac{\left(\frac{\partial H}{\partial m}\right)_T}{\left(\frac{\partial T}{\partial m}\right)_H} \end{cases}$$

$$C_m = C_H - T\left(\frac{\partial M}{\partial T}\right)_H^2 \left(\frac{\partial H}{\partial m}\right)_T = C_H - \frac{(\partial M/\partial T)_H^2}{\chi_T} \geq 0$$

$$\therefore \quad C_H \geq \frac{(\partial M/\partial T)_H^2}{\chi_T}$$

as $t \to 0^-$ $(T \to T_c^-)$, $C_H \sim |t|^{-\alpha}$, $M \sim |t|^{\beta}$, $\chi_T \sim |t|^{-\gamma}$

$\Rightarrow$ l.h.s $\sim |t|^{-\alpha}$, r.h.s $\sim (|t|^{\beta-1})^2/|t|^{-\gamma} \sim |t|^{2\beta+\gamma-2}$

$$-\alpha \leq 2\beta+\gamma-2, \quad \boxed{\alpha+2\beta+\gamma \geq 2}$$

1) 2D Ising model : Onsager solution

$\alpha = 0, \quad \beta = \frac{1}{8}, \quad \gamma = \frac{7}{4}, \quad \delta = 15, \quad \nu = 1, \quad \eta = \frac{1}{4} \qquad (d=2)$

① Rushbrooke : $0 + 2 \cdot \frac{1}{8} + \frac{7}{4} = 2$ ② Griffith : $\frac{1}{8} + \frac{7}{4} = \frac{1}{8} \cdot 15 = 2 - 0 - \frac{1}{8}$

③ Fisher : $\left(2 - \frac{1}{4}\right) \cdot 1 = \frac{7}{4}$

④ B-G : $2 \cdot (15-1)/(15+1) = 2 - \frac{1}{4}$ ⑤ Josephson : $2 \cdot 1 = 2 - 0$

2) 3D Ising model

$\alpha \simeq 0.10, \quad \beta \approx 0.33, \quad \gamma \simeq 1.24, \quad \delta \simeq 4.8, \quad \nu \simeq 0.63, \quad \eta \approx 0.04 \qquad (d=3)$

① $0.10 + 2 \cdot 0.33 + 1.24 = 2.00$ ② $\underset{(1.57)}{0.33 + 1.24} \simeq \underset{(1.6)}{0.33 \cdot 4.8} \approx \underset{(1.57)}{2 - 0.10 - 0.33}$

③ $\underset{(1.23)}{(2 - 0.04) \cdot 0.63} \simeq 1.24$ ④ $\underset{(1.97)}{3 \cdot (4.8-1)/(4.8+1)} \simeq \underset{(1.96)}{2 - 0.04}$ ⑤ $\underset{(1.9)}{3 \cdot 0.63} \simeq \underset{(1.90)}{2 - 0.10}$

3) mean-field exponents

$\alpha = 0, \quad \beta = \frac{1}{2}, \quad \gamma = 1, \quad \delta = 3, \quad \nu = \frac{1}{2}, \quad \eta = 0 \qquad (d = ?)$

① $0 + 2 \cdot \frac{1}{2} + 1 = 2$ ② $\frac{1}{2} + 1 = \frac{1}{2} \cdot 3 = 2 - 0 - \frac{1}{2}$ ③ $(2-0) \cdot \frac{1}{2} = 1$

④ $4 \cdot (3-1)/(3+1) = 2 - 0$ ⑤ $4 \cdot \frac{1}{2} = 2 - 0$

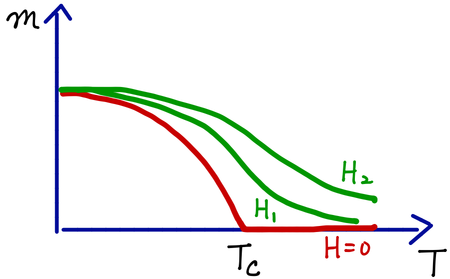
$\boxed{\text{Equality holds for all known solutions}}$

< Static Scaling Hypothesis > (B. Widom '65)

$$m(t,h) = \begin{cases} t^{\beta} F_m^{+}(h/t^{\Delta}) & \text{for } t>0 \\ (-t)^{\beta} F_m^{-}(h/(-t)^{\Delta}) & \text{for } t<0 \end{cases}$$

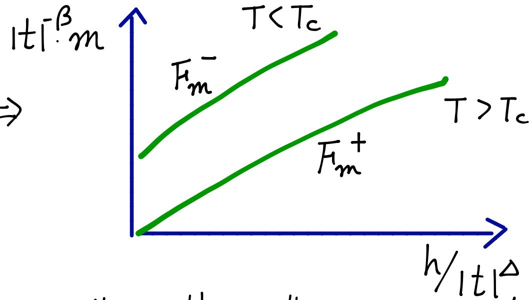
$$= |t|^{\beta} F_m^{\pm}(h/|t|^{\Delta})$$

: scaling hypothesis for order parameter



one curve for each H

$\Rightarrow$



all results collapse onto two curves

data collapse is perfect
only for correct T_c, β, Δ

<Scaling Laws from Scaling Hypothesis>

$$m(t,h) = \begin{cases} t^{\beta} F_m^{+}(h/t^{\Delta}) & \text{for } t>0 \\ (-t)^{\beta} F_m^{-}(h/(-t)^{\Delta}) & \text{for } t<0 \end{cases}$$

① time-reversal symmetry $m(t,h) = -m(t,-h)$

$\Rightarrow$ $F_m^{\pm}(x) = -F_m^{\pm}(-x)$: odd function

② limit of $h \to 0$: $m(t,0) = \begin{cases} 0 & \text{for } t>0 \\ (-t)^{\beta} \cdot \text{const} & \text{for } t<0 \end{cases}$

$\Rightarrow$ $F_m^{+}(0) = 0$, $F_m^{-}(x) \to \text{const}$ as $x \to 0$

③ susceptibility $\chi_T(H=0) = \left(\frac{\partial m}{\partial H}\right)_{H=0} \sim |t|^{\beta-\Delta} F_m^{\pm\prime}(0)$

$\Rightarrow$ $-\gamma = \beta - \Delta$ $\qquad$ $\Delta = \beta + \gamma$

④ limit of $h \to 0$ at T_c $(t=0)$: $m(0,h) \sim h^{1/\delta}$

if $F_m^{\pm}(x) \sim x^{\lambda}$ as $x \to \infty$

$t \to 0$: $m(0,h) \sim |t|^{\beta} \cdot \left(\frac{h}{|t|^{\Delta}}\right)^{\lambda} = h^{\lambda} \cdot |t|^{\beta - \lambda\Delta}$

$\Rightarrow$ $\beta - \lambda\Delta = 0$, $\lambda = 1/\delta$ $\qquad$ $\Delta = \beta \cdot \delta$

$\beta\delta = \beta + \gamma$

Griffith

< Scaling Hypothesis for Free Energy >

$$f_s(t,h) = (-t)^{2-\alpha} F_f\left(h/(-t)^{\Delta}\right) \qquad (t<0)$$

↖ singular part of free energy

① heat capacity : $C_H \sim -T\left(\frac{\partial^2 f_s}{\partial T^2}\right)_{h=0} \sim (-t)^{-\alpha}$: $F_f(0) = \text{const}$

② magnetization $m = -\frac{\partial f_s}{\partial H} \sim (-t)^{2-\alpha-\Delta} F_f'\left(\frac{h}{(-t)^{\Delta}}\right)$

$h \to 0$: $2-\alpha-\Delta = \beta$

③ susceptibility $\chi_T = \frac{\partial m}{\partial H} \sim (-t)^{2-\alpha-2\Delta} F_f''\left(\frac{h}{(-t)^{\Delta}}\right)$

$$2-\alpha-2\Delta = -\gamma$$

$\Rightarrow$ $\alpha + 2\beta + \gamma = 2$ Rushbrooke

$\beta\delta = \beta + \gamma = 2-\alpha-\beta$ Griffith

<Scaling Hypothesis for Correlation Function>

$$G(\vec{r}, t, h) = \frac{1}{r^{d-2+\eta}} F_G\Big(\underbrace{\vec{r}\cdot|t|^{\nu}}_{\vec{r}/\xi}, \frac{h}{|t|^{\Delta}}\Big)$$

$$\chi_T \sim \int G(\vec{r})\, d^d\vec{r} \sim \xi^{-(d-2+\eta)} \cdot \xi^{d} \sim \xi^{2-\eta} \sim |t|^{-\nu(2-\eta)}$$

$\gamma = \nu(2-\eta)$: Fisher

dimensional analysis $[f_s] = L^{-d}$

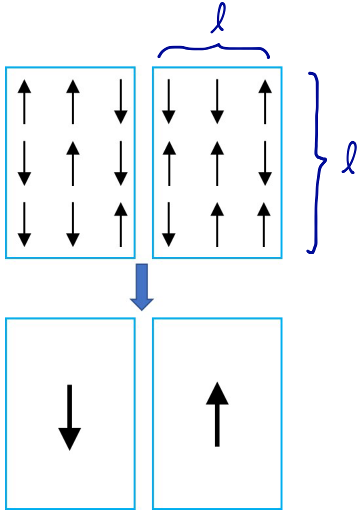
$$\Rightarrow \frac{f_s}{k_B T} \sim \xi^{-d} \sim |t|^{\nu d} \qquad \therefore\ \nu d = 2-\alpha \quad \text{Josephson}$$

$$\frac{\delta-1}{\delta+1} \underset{\substack{\uparrow \\ \text{Griffith}}}{=} \frac{\gamma/\beta}{(2-\alpha)/\beta} = \frac{\gamma}{2-\alpha} \Rightarrow d\,\frac{\delta-1}{\delta+1} = d\,\frac{\gamma}{2-\alpha} \underset{\substack{\uparrow \\ \text{Fisher}}}{=} \frac{d\nu}{2-\alpha} \underset{\substack{\uparrow \\ \text{Josephson}}}{\cdot} (2-\eta) = 2-\eta \quad : \text{B-G}$$

<block spins>

argument for scaling laws (L.P. Kadanoff '66)

block spin transformation



$\{S_i\}$

$$-\beta \mathcal{H}_\Omega = K \sum_{\langle i,j \rangle} S_i S_j + h \sum_i S_i$$

$(K \equiv \beta J,\ h \equiv \beta H)$

$$S_I = f(\{i \in I, S_i\})$$

ex) majority rule, $S_I = \mathrm{sgn}\left(\sum_{i \in I} S_i\right)$

$\{S_I\}$

$$-\beta \mathcal{H}_\ell = K_\ell \sum_{\langle I,J \rangle} S_I S_J + h_\ell \sum_I S_I$$

$(K_1 = K,\ h_1 = h)$

Assumption I : interaction types remain the same by block spin transformation

$\{S_i\}$: N spins $\rightarrow$ $\{S_I\}$: N/ℓ^d (block) spins

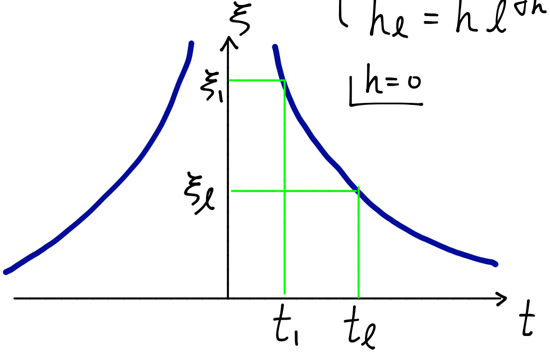
correlation length: $\xi = \xi_1 \cdot a = \xi_\ell \cdot \ell a \Rightarrow \xi_\ell = \frac{\xi_1}{\ell} < \xi_1$

external field $h \sum_i S_i \approx h_\ell \sum_I S_I$ or $h \sum_{i \in I} S_i \approx h_\ell S_I$

$$\Rightarrow h \cdot m \, \ell^d \approx h_\ell$$

free energy density $f_s(t_\ell, h_\ell) \approx f(t,h) \cdot \ell^d$

Assumption II: $\begin{cases} t_\ell = t\,\ell^{y_t} \\ h_\ell = h\,\ell^{y_h} \end{cases}$



block spin transformation $\xi_\ell < \xi_1$

$$\Rightarrow |t_\ell| > |t|$$
$$|h_\ell| > |h|$$

$$\Rightarrow y_t > 0, \; y_h > 0$$

free energy density $f_s(t,h) = \ell^{-d} f_s(t\ell^{y_t}, h\ell^{y_h})$

set $\ell = |t|^{-1/y_t} \Rightarrow f_s(t,h) = |t|^{d/y_t} f_s(\pm 1, h|t|^{-y_h/y_t})$

recall scaling hypothesis $f_s(t,h) = |t|^{2-\alpha} F_f^{\pm}(h/|t|^{\Delta})$

Kadanoff's argument $\Rightarrow$ scaling laws

if $2-\alpha = d/y_t$, $\Delta = + y_h/y_t$, $f_s(\pm 1, x) = F_f^{\pm}(x)$

correlation function

$G(\vec{r}_\ell, t_\ell, h_\ell) \equiv \langle S_I S_J \rangle - \langle S_I \rangle \langle S_J \rangle$ $\qquad \vec{r}_\ell = \vec{r}/\ell$

$$S_I \sim \frac{h}{h_\ell} \sum_{i \in I} S_i \sim \ell^{-y_h} \sum_{i \in I} S_i$$

$$\Rightarrow G(\vec{r}_\ell, t_\ell, h_\ell) \approx \ell^{-2y_h} \underbrace{\sum_{i \in I}}_{\ell^d} \underbrace{\sum_{j \in J}}_{\ell^d} \underbrace{(\langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle)}_{G(\vec{r}, t)} \approx \ell^{2(d-y_h)} G(\vec{r}, t)$$

$$G(\vec{r}/\ell, t\ell^{y_t}, h\ell^{y_h}) = \ell^{2(d-y_h)} G(\vec{r}, t, h)$$

$$G(\vec{r},t,h) = \ell^{-2(d-y_h)}\, G(\vec{r}/\ell,\, t\ell^{y_t},\, h\ell^{y_h})$$

set $\ell = |t|^{-1/y_t}$

$$G(\vec{r},t,h) = |t|^{2(d-y_h)/y_t}\, G(\vec{r}\,|t|^{1/y_t},\, \pm 1,\, h|t|^{-y_h/y_t})$$

$$= r^{2(y_h-d)}\left(r|t|^{1/y_t}\right)^{2(d-y_h)} G(\vec{r}\,|t|^{1/y_t},\, \pm 1,\, h|t|^{-y_h/y_t})$$

$$\equiv \frac{1}{r^{2(d-y_h)}}\, \mathcal{F}_G^{\pm}(r|t|^{1/y_t},\, h|t|^{-y_h/y_t})$$

recall $\quad G(\vec{r},t,h) = \dfrac{1}{r^{d-2+\eta}}\, \mathcal{F}_G^{\pm}\left(\vec{r}\cdot|t|^{\nu},\, \dfrac{h}{|t|^{\Delta}}\right)$

they are consistent if $\quad \nu = 1/y_t,\quad \Delta = y_h/y_t,\quad 2(d-y_h) = d-2+\eta$

remarks for Kadanoff's argument

1) it does not give informations for the values of y_t and y_h
the form of scaling functions

2) only two independent exponents y_t and y_h

$$\alpha = 2 - d/y_t, \quad \beta = (d - y_h)/y_t, \quad \gamma = -(d - 2y_h)/y_t$$

$$\delta = \frac{y_h}{d - y_h}, \quad \nu = 1/y_t, \quad \eta = d - 2y_h + 2$$

yields all sca~~i~~ling laws

3) "coarse graining",
elimination of short-range fluctuations
$\Rightarrow$ renormalization of coupling constants $(t, h) \to (t_\ell, h_\ell)$

IV. Renormalization Group Transformation: Basics

<Renormalization Group Transformation>

general Hamiltonian $\quad \bar{K} \equiv -\beta\mathcal{H} = \sum_n K_n \mathcal{O}_n[S] \quad (K \equiv \{K_1, K_2, \cdots\})$

interaction operators

coupling constants

renormalization group transformation (RGT)

$$[K'] \equiv R_\ell [K] \quad (\ell > 1) \qquad R_{\ell_1 \ell_2}[K] = R_{\ell_2}[R_{\ell_1}[K]]$$

partition function $\quad Z_N[K] = \mathrm{Tr}\, e^{\bar{K}(K, \{S_i\})}$

"free energy density" $g[K] = \frac{1}{N} \ln Z_N[K]$

$$e^{\bar{K}(K', \{S_I'\})} = \mathrm{Tr}'_{\{S_i\}}\, e^{\bar{K}(K, \{S_i\})} = \mathrm{Tr}_{\{S_i\}}\, P(\{S_i\}, \{S_I'\})\, e^{\bar{K}(K, \{S_i\})}$$

projection operator

ex) majority rule $\quad P(\{S_i\}, \{S_I'\}) = \prod_I \delta\left(S_I' - \mathrm{sgn}\left(\sum_{i \in I} S_i\right)\right)$

properties of projection operator

i) $P(\{S_i\},\{S_I'\}) \geq 0 \;\Rightarrow\; e^{\bar{K}[K',\{S_I'\})} \geq 0$

ii) $P(\{S_i\},\{S_I'\})$ should reflect symmetries of system

iii) $\sum_{\{S_I'\}} P(\{S_i\},\{S_I'\}) = 1$

$$\Rightarrow \; Z_{N'}[K'] = \mathrm{Tr}_{\{S_I'\}}\, e^{\bar{K}[K',\{S_I'\}]} = \mathrm{Tr}_{\{S_i\}} \underbrace{\mathrm{Tr}_{\{S_I'\}} P(\{S_i\},\{S_I'\})}_{=1} e^{\bar{K}(K,\{S_i\})}$$

$$= Z_N[K]$$

$$g[K'] = \frac{1}{N'} \ln Z_{N'}[K'] = \frac{1}{N/\ell^d} \ln Z_N[K] = \ell^d g[K]$$

$$K^{(n)} = R_\ell[K^{(n-1)}] = \cdots = R_\ell^{(n)}[K]$$

$K \equiv K^{(0)} \to K^{(1)} \to K^{(2)} \to K^{(\cancel{4}\,3)} \to \cdots$: a flow in parameter space

< Fixed Points >

fixed point (FP) of RG transformation

$$K^* = R_\ell [K^*]$$

for any $K' = R_\ell[K]$, $\quad \xi[K'] = \xi[K]/\ell$

at FP K^* $\quad \xi[K^*] = \xi[K^*]/\ell \Rightarrow \xi(K^*) = 0, \quad \infty$

trivial FP $\qquad$ critical FP

basin of attraction : a set of initial conditions which flow to a given FP

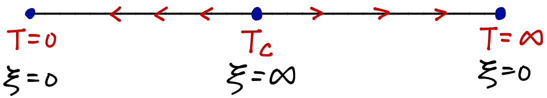
critical manifold : basin of attraction of critical FP

for a point K in critical manifold

$$\xi[K] = \ell\, \xi[K^{(1)}] = \cdots = \ell^n \xi[K^{(n)}]$$

$n \to \infty$: $K^{(n)} \to K^*$, $\xi[K^*] = \infty \rightarrow \xi[K] = \infty$

ex) nearest neighbor Ising model $(d \geq 2)$



three FPs

<Local behavior of RG flows near a fixed point>

$$K_n = K_n^* + \delta K_n \qquad (n = 1, 2, 3, \cdots, D)$$

D: dimension of parameter space

near a fixed point $\mathcal{H} = \mathcal{H}^* + \delta \mathcal{H}$

RGT : $K' = R_\ell [K]$

$$K_n' \equiv K_n^* + \delta K_n' = K_n'[K_1^* + \delta K_1, K_2^* + \delta K_2, \cdots]$$

$$= K_n^* + \sum_m \frac{\partial K_n'}{\partial K_m}\bigg|_{K^*} \delta K_m + \cancel{\mathcal{O}[(\delta K)^2]}$$

neglect

$$\delta K_n' = \sum_m M_{nm}\, \delta K_m \qquad \left(M_{nm} \equiv \frac{\partial K_n'}{\partial K_m}\bigg|_{K^*} \right)$$

linearized RGT near a given FP

assume $\mathbb{M}$ is a symmetric matrix (not in general)

$\Rightarrow$ eigenvectors $\vec{e}^{(\sigma)}$, eigenvalues $\Lambda_\ell^{(\sigma)}$

$$\mathbb{M}^{(\ell)} \vec{e}^{(\sigma)} = \Lambda_\ell^{(\sigma)} \vec{e}^{(\sigma)}$$

$$\mathbb{M}^{(\ell)} \vec{e}^{\,(\sigma)} = \Lambda_\ell^{(\sigma)} \vec{e}^{\,(\sigma)}$$

$$R_\ell R_{\ell'} = R_{\ell\ell'} \Rightarrow \mathbb{M}^{(\ell)}\mathbb{M}^{(\ell')} = \mathbb{M}^{(\ell\ell')} \Rightarrow \Lambda_\ell^{(\sigma)}\Lambda_{\ell'}^{(\sigma)} = \Lambda_{\ell\ell'}^{(\sigma)} \quad \cdots \textcircled{1}$$

from $\Lambda_1^{(\sigma)} = 1$, we can show $\Lambda_\ell^{(\sigma)} = \ell^{y_\sigma}$

(pf.) $\frac{d}{d\ell'} \textcircled{1} \Rightarrow \Lambda_\ell^{(\sigma)}\Lambda_{\ell'}^{(\sigma)\prime} = \ell\, \Lambda_{\ell\ell'}^{(\sigma)\prime}$

set $\ell' = 1 \longrightarrow \Lambda_1^{(\sigma)\prime}\Lambda_\ell^{(\sigma)} = \ell\, \Lambda_\ell^{(\sigma)\prime}$, $\quad \frac{d}{d\ell} \ln \Lambda_\ell^{(\sigma)} = \frac{1}{\ell} \Lambda_1^{(\sigma)\prime}$

$\Lambda_1^{(\sigma)} = 1 \rightarrow \ln \Lambda_\ell^{(\sigma)} = \Lambda_1^{(\sigma)\prime} \ln \ell \rightarrow \Lambda_\ell^{(\sigma)} = \ell^{\Lambda_1^{(\sigma)\prime}} \equiv \ell^{y_\sigma}$

expand $\delta\vec{K}$ in terms of $\{\vec{e}_\sigma\}$

$$\delta\vec{K} = \sum_\sigma a^{(\sigma)} \vec{e}^{\,(\sigma)} \qquad\qquad a^{(\sigma)} = \vec{e}^{\,(\sigma)} \cdot \delta\vec{K}$$

linearized RGT

$$\delta\vec{K}' = \mathbb{M}\,\delta\vec{K} = \sum_\sigma a^{(\sigma)} \mathbb{M}\, \vec{e}^{\,(\sigma)} = \sum_\sigma a^{(\sigma)} \Lambda^{(\sigma)} \vec{e}^{\,(\sigma)} \equiv \sum_\sigma a^{(\sigma)\prime} \vec{e}_\sigma$$

$$a^{(\sigma)\prime} = \Lambda^{(\sigma)} a^{(\sigma)}$$

$$a^{(\sigma)\prime} = \Lambda_\ell^{(\sigma)} a^{(\sigma)}$$

i) $\Lambda_\ell^{(\sigma)} > 1$: $y_\sigma > 0$ $\longrightarrow$ $|a^{(\sigma)}| \uparrow$ as $\ell \uparrow$ "relevant"

ii) $\Lambda_\ell^{(\sigma)} < 1$: $y_\sigma < 0$ $\longrightarrow$ $|a^{(\sigma)}| \downarrow$ as $\ell \cancel{\downarrow} \uparrow$ "irrelevant"

iii) $\Lambda^{(\sigma)} = 1$: $y_\sigma = 0$ $\longrightarrow$ $a^{(\sigma)}$ does not change "marginal"

< Types of Fixed Points >

c : codimension of basin of attraction of FP = number of ~~irrelevant~~ relevant directions

ξ : correlation length

1) sink ($c=0$, $\xi=0$) stable bulk phase

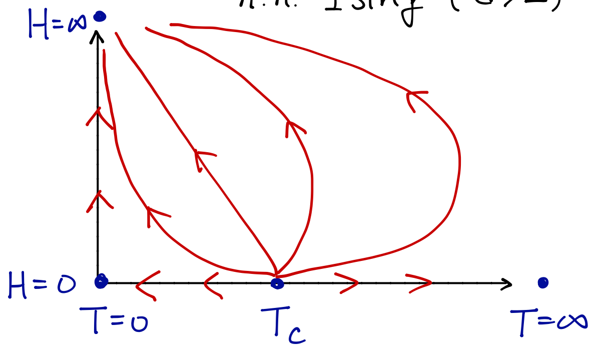
2) discontinuity FP ($c=1$, $\xi=0$) coexistence plane
phase boundary, discontinuous transition

3) continuity FP ($c=1$, $\xi=0$) bulk phase
phase boundary, continuous variation

4) critical FP ($c=2$, $\xi=\infty$) critical manifold

5) triple point ($c=2$, $\xi=0$),
multicritical FP ($c>2$, $\xi=\infty$)
multiple coexistence FP ($c>2$, $\xi=0$)

n.n. Ising ($d>2$)



< Case of one relevant variable>

temperature T or $K \equiv J/k_B T$

RGT $\quad T' = R_\ell(T)$, fixed point $\quad T^* = R_\ell(T^*)$

near T^* $\quad T' - T^* = R_\ell(T) - R_\ell(T^*) \simeq \Lambda_\ell (T - T^*) + \mathcal{O}(T-T^*)^2$

$$\Lambda_\ell \equiv \frac{\partial R_\ell}{\partial T} = \ell^{y_t}$$

reduced temperature $\quad t \equiv \dfrac{T - T^*}{T^*}$

$t' = t \cdot \ell^{y_t}$, $\quad t^{(n)} = t\,(\ell^{y_t})^n$ after n iterations

correlation length

$$\xi(t) = \ell\,\xi(t') = \cdots = \ell^n\,\xi(t \cdot \ell^{n y_t})$$

set $\quad t \cdot \ell^{n y_t} \equiv b \quad$ or $\quad \ell = \left(\dfrac{b}{t}\right)^{1/(n y_t)}$

$$\xi(t) = \left(\frac{b}{t}\right)^{1/y_t} \xi(b) \sim t^{-1/y_t} \Rightarrow \nu = 1/y_t$$

free energy $\quad f(t) = \ell^{-d} f(t') = \cdots = \ell^{-nd} f(t \cdot \ell^{n y_t})$

$$\Rightarrow f(t) = \left(\frac{t}{b}\right)^{d/y_t} f(b) \sim t^{d/y_t} \Rightarrow d/y_t = 2 - \alpha$$

$$d\nu = 2 - \alpha$$

< Case of two relevant variables with diagonal RGT>

$$\begin{cases} T' = R_\ell^T(T, H) \\ H' = R_\ell^H(T, H) \end{cases}$$

fixed point $\begin{cases} T^* = R_\ell^T(T^*, H^*) \\ H^* = R_\ell^H(T^*, H^*) \end{cases}$

near (T^*, H^*) $\quad \Delta T = T - T^*, \quad \Delta H = H - H^*$

$$\begin{pmatrix} \Delta T' \\ \Delta H' \end{pmatrix} = \mathbb{M} \begin{pmatrix} \Delta T \\ \Delta H \end{pmatrix}$$

$$\mathbb{M} = \begin{pmatrix} \partial R_\ell^T/\partial T & \partial R_\ell^T/\partial H \\ \partial R_\ell^H/\partial T & \partial R_\ell^H/\partial H \end{pmatrix}_{(T^*, H^*)} \equiv \begin{pmatrix} \Lambda_\ell^t & 0 \\ 0 & \Lambda_\ell^h \end{pmatrix} = \begin{pmatrix} \ell^{y_t} & 0 \\ 0 & \ell^{y_h} \end{pmatrix}$$

"diagonal RGT"

correlation length $\quad \xi(t,h) = \ell^{n}\,\xi(\ell^{n y_t}\cdot t,\ \ell^{n y_h}\cdot h)$

for $h=0$, $\xi(t,0) = \ell^{n}\,\xi(\ell^{n y_t}\cdot t,\ 0) \underset{\uparrow\ \ell^{n y_t}\cdot t \equiv b}{=} (b/t)^{1/y_t}\,\xi(b,0) \sim t^{-1/y_t}$

$\nu = 1/y_t$

for $t=0$, $\xi(0,h) = \ell^{n}\,\xi(0,\ \ell^{n y_h} h) \underset{\uparrow\ \ell^{n y_h} h = b}{=} (b/h)^{1/y_h}\,\xi(0,b) \sim h^{-1/y_h}$

free energy density

$$f(t,h) = \ell^{-nd}\, f(\ell^{n y_t} t,\ \ell^{n y_h} h)$$

set $\ell^{n y_t}\cdot t = b \rightarrow \ell = (b/t)^{1/(n y_t)}$

$$f(t,h) = t^{d/y_t}\left[\, b^{-d} f(b,\ h/t^{y_h/y_t})\right]$$

consistent with scaling hypothesis by $\begin{cases} 2-\alpha = d/y_t = d\nu \\ \Delta = y_h/y_t \end{cases}$

RGT $\Rightarrow \begin{cases} \text{① derivation of scaling law} \\ \text{② quantitative calculation of critical exponents} \end{cases}$

In the presence of irrelevant variables

$$\underbrace{t,\ h}_{\text{relevant}} \qquad \underbrace{K_3, K_4, \cdots}_{\text{irrelevant}}$$

$\Lambda_\ell^t, \Lambda_\ell^h > 1 \qquad \Lambda_\ell^{(3)}, \Lambda_\ell^{(4)}, \cdots < 1$

$y_t, y_h > 0 \qquad y_3, y_4, \cdots < 0$

$$f(t,h,K_3,\cdots) = \ell^{-nd} f(\ell^{n y_t} t,\ \ell^{n y_h} h,\ K_3 \ell^{n y_3}, \cdots)$$

$$= t^{d/y_t} b^{-d} f(b,\ h t^{-y_h/y_t},\ K_3 t^{-y_3/y_t}, \cdots)$$

$t \to 0$

$$f(t,h,K_3,\cdots) = t^{d/y_t} b^{-d} f(b,\ h t^{-y_h/y_t},\ 0, \cdots)$$

irrelevant variables are not important

when f is analytic at $K_3 = 0$

dangerous irrelevant variable

free energy is singular at $K_3 = 0$

$$f(t,h,K_3) \sim K_3^{-\mu} F(t,h) \quad \text{as } K_3 \to 0$$

it can affect scaling laws

< non-diagonal RGT >

in general, $\mathbb{M}$ is not symmetric

right eigenvectors $\mathbb{M}\vec{e}_R = \Lambda_R \vec{e}_R$, left eigenvectors $\mathbb{M}^T\vec{e}_L = \Lambda_L \vec{e}_L$

i) $\Lambda_R^{(\sigma)} = \Lambda_L^{(\sigma)}$ $\quad$ ($\because$ $\det(\mathbb{M} - \Lambda \mathbb{1}) = \det(\mathbb{M}^T - \Lambda \mathbb{1})$)

but $\vec{e}_L^{(\sigma)} \neq \vec{e}_R^{(\sigma)}$

ii) orthogonality $\quad$ $\vec{e}_L^{(\sigma')T} \cdot \vec{e}_R^{(\sigma)} = 0$ for $\Lambda^{(\sigma)} \neq \Lambda^{(\sigma')}$

($\because$ $\vec{e}_L^{(\sigma')T}\, \mathbb{M}\, \vec{e}_R^{(\sigma)} = \Lambda^{(\sigma)}\, \vec{e}_L^{(\sigma')T} \cdot \vec{e}_R^{(\sigma)} = \Lambda^{(\sigma')}\, \vec{e}_L^{(\sigma')T} \cdot \vec{e}_R^{(\sigma)}$

$(\Lambda^{(\sigma)} - \Lambda^{(\sigma')})\, \vec{e}_L^{(\sigma')T} \cdot \vec{e}_R^{(\sigma)} = 0$

if $\Lambda^{(\sigma)} \neq \Lambda^{(\sigma')}$, $\vec{e}_L^{(\sigma')T} \cdot \vec{e}_R^{(\sigma)} = 0$)

$$\delta\vec{K} = \sum_\sigma a^{(\sigma)} \vec{e}_R^{(\sigma)}, \quad a^{(\sigma)} = \vec{e}_L^{(\sigma)T} \cdot \delta\vec{K}$$

Then the remaining formulation is the same as diagonal RGT

< RG in a differential form >

$$\frac{d\vec{K}_\ell}{d\ell} \equiv \lim_{\varepsilon \to 0} \frac{\vec{K}_{\ell+\varepsilon} - \vec{K}_\ell}{\varepsilon} \equiv \vec{B}[\vec{K}_\ell]$$ coupled differential eq.

fixed points $\vec{B}(K^*) = 0$

initial conditions at $\ell = 1$: coupling constants of physical systems

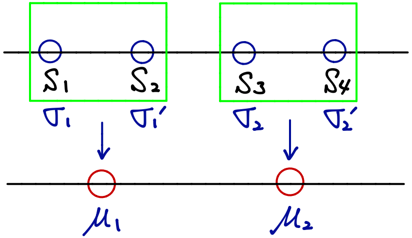
$\lim_{\ell \to \infty} \vec{K}_\ell$: fixed point corresponding to physical system

cf.) we may use $\bar{\ell} \equiv \ln \ell$

$$\frac{d\vec{K}_{\bar{\ell}}}{d\bar{\ell}} = \vec{B}[\vec{K}_{\bar{\ell}}] \qquad \bar{\ell} : 0 \to \infty$$

(example) 1D Ising model — decimation

$$\bar{K} \equiv -\beta \mathcal{H} = K \sum_{i=1}^{N} S_i S_{i+1} + h \sum_{i=1}^{N} S_i \quad (S_{N+1} = S_1)$$



projection operator

$$P(\mu, \sigma) = \prod_{I=1}^{N/2} \delta_{\mu_I, \sigma_I'}$$

partial trace over $\{\sigma_I\}$

i) $H = 0$

$$e^{\bar{K}[\mu]} = \sum_{\{\sigma_I\}} \sum_{\{\sigma_I'\}} e^{\bar{K}[\sigma_I, \sigma_I']} P(\mu, \sigma) = \sum_{\{\sigma_I\}} \sum_{\{\sigma_I'\}} e^{K \sum_{I=1}^{N/2} \sigma_I (\overset{/\!/ \mu_{I-1}}{\sigma_{I-1}'} + \overset{/\!/ \mu_I}{\sigma_I'})} \underline{P(\mu, \sigma)}$$

$$= \prod_{I=1}^{N/2} \sum_{\sigma_I = -1}^{1} e^{K \sigma_I (\mu_{I-1} + \mu_I)} = \prod_{I=1}^{N/2} \{ 2 \cosh [K(\mu_{I-1} + \mu_I)] \}$$

$$2 \cosh [K(\mu_{I-1} + \mu_I)] \equiv A e^{K' \mu_{I-1} \mu_I}$$

$\mu_{I-1} = \mu_I = +1$: $2 \cosh(2K) = A e^{K'}$
$\mu_{I-1} = \mu_I = -1$: $2 \cosh(-2K) = A e^{K'}$ } same

$\mu_{I-1} = -\mu_I = \pm 1$: $2 = A e^{-K'}$

$\rightarrow$

$$A = 2 e^{K'}$$

$$e^{-2K'} = \frac{1}{2 \cosh(2K)}$$

RGT with $\ell = 2$

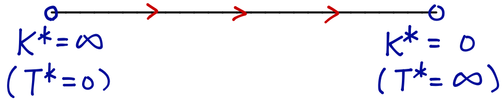
$$e^{-2K'} = \frac{1}{\cosh(2K)} \quad \text{or} \quad K' = \frac{1}{2}\ln\left[\cosh(2K)\right]$$

fixed point $\quad e^{-2K^*} = \dfrac{1}{\cosh(2K^*)} \rightarrow K^* = 0\ ,\ \infty$

$$w \equiv e^{-2K}\ , \quad w' = \frac{2}{w + w^{-1}} = \frac{2w}{w^2+1}$$

For $K^* = \infty$ (ferromagnetic FP)
$(T^* = 0)$ (critical)

$$w' \simeq 2w\ , \quad \Lambda^w = 2 = 2^1 \quad \left(\begin{array}{l} \ell = 2 \\ y_w = 1 \end{array}\right.$$



$$\xi(w) = \overset{\ell}{2}\, \xi(\overset{\Lambda^w}{2} w) \sim w^{-1} \sim e^{2K}$$

agree with exact result at low temperature

ii) $H \neq 0$

$$e^{\bar{K}[\mu]} = \sum_{\{\sigma_I\}} e^{K\sum_{I=1}^{N/2} \sigma_I(\mu_{I-1}+\mu_I) + h\sum_{I=1}^{N/2}(\sigma_I+\mu_I)}$$

$$= \prod_{I=1}^{N/2} \left[2\cosh\{K(\mu_{I-1}+\mu_I)\} + h \right] e^{\frac{1}{2}h(\mu_{I-1}+\mu_I)}$$

$$\equiv \prod_{I=1}^{N/2} \left[A\, e^{K'\mu_{I-1}\mu_I + \frac{1}{2}h(\mu_{I-1}+\mu_I)} \right]$$

$\mu_I = \mu_{I-1} = +1$: $\quad 2\cosh(2K+h)e^{h} = A e^{K'+h'} \quad \cdots$ ①

$\mu_I = \mu_{I-1} = -1$: $\quad 2\cosh(-2K+h)e^{-h} = A e^{K'-h'} \quad \cdots$ ②

$\mu_I = -\mu_{I-1} = \pm 1$: $\quad 2\cosh(h) = A e^{-K'} \quad \cdots$ ③

$$x \equiv e^{-4K}, \quad y \equiv e^{-2h}$$

②/① : $\quad y' = e^{-2h} \dfrac{\cosh(-2K+h)}{\cosh(2K+h)} = \dfrac{y(x+y)}{1+xy}$

③²/(①×②) : $\quad x' = \dfrac{\cosh^2(h)}{\cosh(2K+h)\cosh(-2K+h)} = \dfrac{x(1+y)^2}{(x+y)(1+xy)}$

③²×①×② : $\quad A^4 = \dfrac{16(1+y)^2(1+xy)(x+y)}{x y^2}$

$$x' = \frac{x(1+y)^2}{(x+y)(1+xy)} \qquad y' = \frac{y(x+y)}{1+xy} \qquad x \equiv e^{-4K}, \quad y \equiv e^{-2h}$$

fixed points $0 \leq x \leq 1, \quad 0 \leq y \leq 1$
$(0 \leq T < \infty, \ \infty > h \geq 0)$

$x = 0, \ y = 0$: $T = 0, \ H = \infty$

$x = 0, \ y = 1$: $T = 0, \ H = 0$ ferromagnetic FP

$x = 1, \ 0 \leq y \leq 1$: fixed line $(T = \infty)$ paramagnetic

i) $x^* = y^* = 0$

$$x' \simeq \frac{x}{x+y}, \quad y' \simeq y(x+y)$$

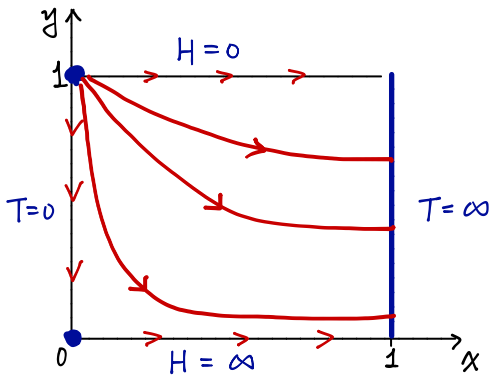
$$\Rightarrow x' \cdot y' \simeq xy$$

ii) $x^* = 0, \ y^* = 1$

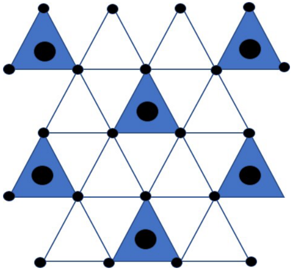
$$x' \simeq 4x \quad (y_x = 2), \quad y' - 1 \simeq 2(y-1) \quad (y_y = 1)$$

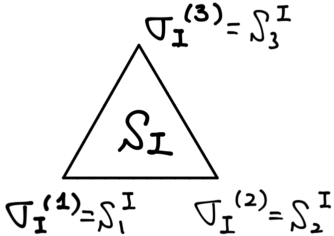
iii) $x^* = 1$

$$x' - 1 \simeq -(x-1)^2 y, \quad y' \simeq y$$



⟨2D Ising model in triangular lattice⟩





$\sigma_I \equiv \{ S_1^I, S_2^I, S_3^I \}$

majority rule

$S_I = +1 \qquad \sigma_I^{(+)} = \{+\ +\ +\},\ \{+\ +\ -\},\ \{+\ -\ +\},\ \{-\ +\ +\}$

$S_I = -1 \qquad \sigma_I^{(-)} = \{-\ -\ -\},\ \{-\ -\ +\},\ \{-\ +\ -\},\ \{+\ -\ -\}$

$$e^{\bar{K}[S_I]} = \underbrace{\sum_{\{\sigma_I\}}'}_{\text{sum over } \sigma_I^{(S_I)}} e^{\bar{K}[\sigma_I]}$$

i) $h=0$

$$\bar{K} = \bar{K}_0 + \bar{V}, \qquad \bar{K}_0 = K \sum_I \sum_{i,j\in I} S_i S_j, \qquad \bar{V} = K \sum_{\langle I,J\rangle} \sum_{\substack{i\in I, j\in J \\ \langle i,j\rangle}} S_i S_j$$

$$e^{\bar{K}'[S_I]} = \sum_{\{\sigma_I\}}' e^{\bar{K}_0 + \bar{V}} = \langle e^{\bar{V}} \rangle_0 \underbrace{\sum_{\{\sigma_I\}}' e^{\bar{K}_0}}_{= Z_0[K]^{N/3}}$$

$$Z_0[K] = \sum_{S_1,S_2,S_3} e^{K[S_1S_2 + S_2S_3 + S_3S_1]} = e^{3K} + 3e^{-K}$$

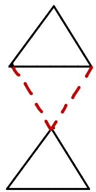
perturbation theory.

cumulant expansion

$\ln(1+x) \simeq x - \frac{1}{2}x^2 + \mathcal{O}(x^3)$

$$\ln\langle e^{\bar{V}}\rangle_0 = \ln\left(1+\langle\bar{V}\rangle_0 + \frac{1}{2!}\langle\bar{V}^2\rangle_0 + \cdots\right) \simeq \langle\bar{V}\rangle_0 + \frac{1}{2}\left(\langle\bar{V}^2\rangle_0 - \langle\bar{V}\rangle_0^2\right) + \mathcal{O}(\bar{V}^3)$$

$$\Rightarrow \bar{K}' = \frac{N}{3}\ln Z_0[K] + \langle\bar{V}\rangle_0 + \frac{1}{2}\left(\langle\bar{V}^2\rangle_0 - \langle\bar{V}\rangle_0^2\right)$$



$$\bar{V} = K\sum_{\langle I,J\rangle}\sum_{\substack{i\in I, j\in J \\ \langle i,j\rangle}} S_i S_j, \quad \langle\bar{V}\rangle_0 = K\sum_{\langle I,J\rangle} 2\langle S_i\rangle_0\langle S_j\rangle_0$$

$$\langle S_i\rangle_0 = \frac{e^{3K}+e^{-K}}{e^{3K}+3e^{-K}}\cdot S_I \equiv \Phi(K) S_I.$$

$$\Rightarrow \langle V\rangle_0 = 2K\Phi(K)^2\sum_{\langle I,J\rangle} S_I S_J$$

$$\therefore \bar{K}'[S_I] = \frac{N}{3}\ln Z_0[K] + K'\sum_{\langle I,J\rangle} S_I S_J + \mathcal{O}(\bar{V}^2)$$

$$K' = 2K\Phi(K)^2$$

Fixed Points

$$K^* = 2K^* \Phi(K^*)^2 \Rightarrow K^* = 0, \infty, \text{ and } \Phi(K_c) = \frac{1}{\sqrt{2}}$$

$$\hookrightarrow K_c = \frac{1}{4}\ln(1+2\sqrt{2}) \simeq 0.34$$

$$\Lambda^t = \left.\frac{dK'}{dK}\right|_{K=K_c} = 2\Phi(K_c)^2 + 4K_c\Phi(K_c)\Phi'(K_c) \simeq 1.62$$

cf.) exact results : $\sinh(2K_c) = \frac{1}{\sqrt{3}} \rightarrow K_c = \frac{1}{4}\ln 3 \simeq 0.27$

$l = \sqrt{3}$ and $\nu = 1 \rightarrow \Lambda^t = \sqrt{3} \simeq 1.73$

ii) $h \neq 0$

fixed point $h^* = 0$

$$e^{\bar{K}'[S_I]} = \sum_{\{\sigma_I\}}' e^{\bar{K}[\sigma_I]}$$

near the fixed point

$$e^{\bar{K}'[S_I] + \delta\bar{K}'[S_I]} = \sum_{\{\sigma_I\}}' e^{\bar{K}[\sigma_I] + \delta\bar{K}[\sigma_I]}$$

$$\delta\bar{K} \equiv \delta h \sum_I (\sigma_I^{(1)} + \sigma_I^{(2)} + \sigma_I^{(3)})$$ due to small field δh

$$\delta\bar{K}' = \delta h' \sum_I S_I$$

$$e^{\bar{K}'[S_I] + \delta \bar{K}'[S_I]} = \sum_{\{\sigma_I\}}' e^{\bar{K}[\sigma_I] + \delta \bar{K}[\sigma_I]}$$

up to linear order in δh

$$e^{\bar{K}'[S_I]}(1 + \delta \bar{K}'[S_I]) = \sum_{\{\sigma_I\}}' e^{\bar{K}[\sigma_I]}(1 + \delta \bar{K}[\sigma_I])$$

$$\Rightarrow \delta \bar{K}'[S_I] = \frac{\sum_{\{\sigma_I\}}' e^{\bar{K}[\sigma_I]} \delta \bar{K}[\sigma_I]}{e^{\bar{K}'[\sigma_I]}} = \frac{\sum_{\{\sigma_I\}}' e^{\bar{K}[\sigma_I]} \delta \bar{K}[\sigma_I]}{\sum_{\{\sigma_I\}}' e^{\bar{K}[\sigma_I]}}$$

up to zeroth order in $\bar{V}$

$$\delta h' \sum_I S_I = \langle \delta \bar{K} \rangle_0 = \delta h \sum_I \langle \sigma_I^{(1)} + \sigma_I^{(2)} + \sigma_I^{(3)} \rangle_0 = \delta h \cdot 3 \Phi(K) \sum_I S_I$$

$$\Rightarrow \delta h' = 3 \Phi(K) \delta h$$

$$\Lambda^h = 3 \Phi(K_c) = \frac{3}{\sqrt{2}} \simeq 2.1 \qquad \left(\text{exact value } \Lambda^h = \ell^{y_h} = \ell^{\beta\delta/\nu} = \sqrt{3}^{\frac{1}{8} \cdot 15/1} \approx 2.8 \right)$$

phase diagram

i) C $(K=K_c)$

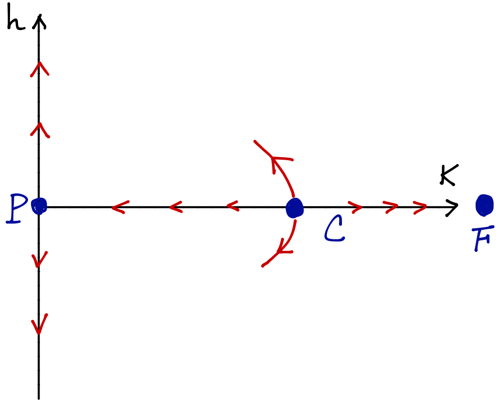
$$\begin{cases} \Lambda^t \simeq 1.62\,, & y_t \simeq 0.88 \\ \Lambda^h \simeq 2.1 & y_h \simeq 1.4 \end{cases}$$

ii) P $(K=0)$

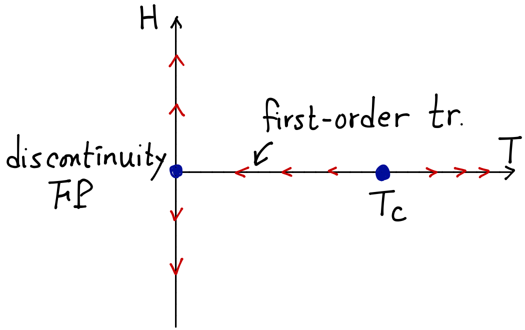
$$\begin{cases} \Lambda^t = 2\Phi(0)^2 + 4\cdot 0\cdot \Phi(0)\,\Phi'(0) = \frac{1}{2}, & y_t \simeq -1.3 \\ \Lambda^h = 3\Phi(0) = \frac{3}{2}\,, & y_h \simeq 0.74 \end{cases}$$

iii) F $(K=\infty)$

$$\begin{cases} \Lambda^t = 2\Phi(\infty)^2 + 4\lim_{K\to\infty} K\Phi(K)\,\Phi'(K) = 2, & y_t \simeq 1.3 \\ \Lambda^h = 3\Phi(\infty) = 3\,, & y_h = 2 \end{cases}$$



<discontinuity fixed point>



magnetization $m[K^{(0)}] = \left.\frac{\partial g}{\partial h}\right|_{h=0^+} = \ell^{-d}\left.\frac{\partial g[K']}{\partial h}\right|_{h=0^+}$

$$= \ell^{-d} \underbrace{\frac{\partial h'}{\partial h}}_{a[K]} \underbrace{\left.\frac{\partial g[K']}{\partial h'}\right|_{h=0^+}}_{m[K']}$$

$$m[K^{(0)}] = \frac{a[K^{(0)}]}{\ell^d} m[K^{(1)}] = \prod_{i=0}^{n} \frac{a[K^{(i)}]}{\ell^d} m[K^{(i+1)}]$$

for $T^{(0)} < T_c$, $T^{(\infty)} = 0$, $m[T^{(\infty)}, h = 0^+] = 1$

$$m[K^{(0)}] = \frac{a[K^{(0)}]}{\ell^d} m[K^{(1)}] = \prod_{i=0}^{n} \frac{a[K^{(i)}]}{\ell^d}$$: nonzero and bounded

$\therefore \lim_{i\to\infty} \frac{a[K^{(i)}]}{\ell^d} = 1 \Rightarrow \left.\frac{\partial h'}{\partial h}\right|_{\text{discontinuity FP}} = \ell^d$, $y_h = d$

"Nienhuis - Nauenberg criterion"

< crossover phenomena >

$$f_s = |t|^{2-\alpha}\, \mathcal{F}_f^{(\pm)}(h/|t|^{\Delta})$$

for $h=0$, $f_s = |t|^{2-\alpha}\mathcal{F}_f^{(\pm)}(0) \sim |t|^{2-\alpha}$ as $t \to 0$

$\chi_T \sim |t|^{-\gamma}$ $\quad (-\gamma = 2-\alpha-2\Delta)$

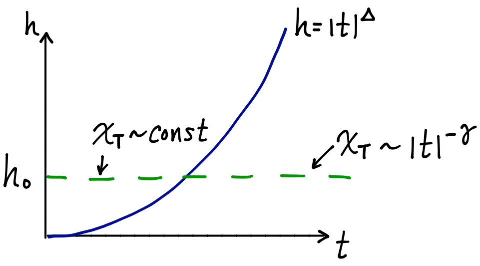
for small field $h\ (\neq 0)$

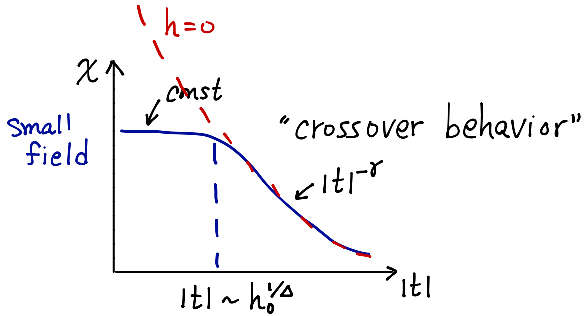
as $t \to 0$, $h/|t|^{\Delta} \to \infty$ $\quad (h^{1/\Delta} \gg |t|)$

$\mathcal{F}_{\pm}(h/|t|^{\Delta}) \sim \left(\frac{h}{|t|^{\Delta}}\right)^{1/\delta+1}$ $\quad \left(\because m \sim \frac{\partial f_s}{\partial h} \sim h^{1/\delta}\right)$

$\therefore f_s \sim |t|^{2-\alpha-\Delta-\Delta/\delta} \cdot h^{1/\delta+1} \sim h^{1/\delta+1} \Rightarrow \chi_T \sim \frac{\partial^2 f_s}{\partial h^2} \sim h^{1/\delta-1} \sim \text{const}$

for $h^{1/\Delta} \ll |t| \ll 1$, $\chi \sim |t|^{-\gamma}$





< Finite-size scaling >

for the system of finite size L

$$f_s([K], L^{-1}) = \ell^{-d} f_s([K'], \ell L^{-1}) \quad : \; y_L = 1 \qquad 「 f_s([K]) = \lim_{L\to\infty} f_s([K], L^{-1}) 」$$

$$\Rightarrow f_s(t, L^{-1}) = |t|^{2-\alpha} \underbrace{F_f^{\pm}(L^{-1}|t|^{-1/y_t})}_{F_f^{\pm}(\xi_\infty/L) = F^{\pm}(|t| L^{1/\nu})} \qquad (1/y_t = \nu)$$

specific heat

$$C(t, L^{-1}) = |t|^{-\alpha} F_f^{\pm}(L^{-1}|t|^{-\nu}) = L^{\alpha/\nu} D^{\pm}(|t| L^{1/\nu})$$

$$= L^{\alpha/\nu} D(t L^{1/\nu}) \qquad D(x) = \begin{cases} D^{+}(x) & x>0 \\ D^{-}(x) & x<0 \end{cases}$$

i) bounded $\sim L^{\alpha/\nu}$

ii) $T_c(L) = T_{c,\infty} + A L^{-1/\nu}$: peak temperature

<Reading Materials>

1. Nigel Goldenfeld, "Lectures on Phase Transitions and the Renormalization Group", (Westview, 1992).

2. 김두철, "상전이와 임계 현상" (민음사, 1983) : 대우 학술 총서

3. John Cardy, "Scaling and Renormalization in Statistical Physics", (Cambridge, 1996)