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Asia Pacific Center for Theoretical Physics

MIT bag model detail

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The MIT bag model (A. W. Thomas, Adv. Nucl. Phys. 13, 1 (1984))

$\vec{\alpha} \equiv \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$, $\beta \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^\mu = (\gamma^0, \vec{\gamma}) \equiv (\beta, \beta\vec{\alpha})$

$= \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \right)$

$\vec{\sigma} \equiv (\sigma^1, \sigma^2, \sigma^3) = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)$

$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$

Dirac equation $*\left(\vec{\sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \right)$

$$H\psi = i\frac{\partial \psi}{\partial t} = E\psi, \quad H = \vec{\alpha}\cdot\vec{p} + \beta m \quad (\text{free})$$

$$p^\mu = i\partial^\mu = i\frac{\partial}{\partial x_\mu} = \left(i\frac{\partial}{\partial t}, -i\vec{\nabla}\right) \qquad (x^\mu = (x^0, \vec{x}),\ x_\mu = (x^0, -\vec{x}))$$
$$= (p^0, \vec{p}) = (E, \vec{p})$$

$H\psi = E\psi$ (free with mass m)

$(\vec{\alpha}\cdot\vec{p} + \beta m)\psi = E\psi$

$(E - \vec{\alpha}\cdot\vec{p} - \beta m)\psi = 0 \quad \rightarrow$ multiply β from l.h.s.

$(\beta E - \beta\vec{\alpha}\cdot\vec{p} - m)\psi = 0 \quad \leftarrow \beta^2 = 1$

$\gamma^\mu \downarrow$

$$\left(\begin{array}{l} (\gamma^0 p_0 - \vec{\gamma}\cdot\vec{p} - m)\psi = 0 \qquad p_\mu = i\partial_\mu \\ (\gamma^\mu p_\mu - m)\psi = 0 \rightleftarrows (\gamma^\mu i\partial_\mu - m)\psi = 0 \\ (\not{p} - m)\psi = 0 \rightleftarrows (i\not{\partial} - m)\psi = 0 \end{array} \right)$$

$$\left\{ \begin{array}{l} H = \vec{\alpha}\cdot\vec{p} + \beta m \\ \vec{j} = \vec{l} + \frac{1}{2}\vec{\sigma} \\ K = \beta(\vec{\sigma}\cdot\vec{l} + 1) \end{array} \right. \qquad \text{—— (0.1)}$$

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$$\boxed{g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)}$$

$$i\frac{\partial}{\partial t}\psi = H\psi = (\vec{\alpha}\cdot\vec{p} + m)\psi$$

$$\beta\left(i\frac{\partial}{\partial t} - \vec{\alpha}\cdot\vec{p} - m\right)\psi = 0 \qquad \left(\nabla^a \equiv \frac{\partial}{\partial x^a}\right)$$

$$(\beta i\partial_0 - \beta\vec{\alpha}\cdot(-i\vec{\nabla}) - m)\psi = 0$$

$$(\gamma^0 i\partial_0 - \gamma^j(-i\partial_j) - m)\psi = 0$$

$$\underline{(\gamma^\mu i\partial_\mu - m)\psi = 0}$$

$$(\gamma^\mu p_\mu - m)\psi = (\not{p} - m)\psi = 0$$

$$p^\mu = i\partial^\mu = i\frac{\partial}{\partial x_\mu} = i\left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_j}\right)$$

$$= i(\partial^0, \partial^j) = i(\partial^0, -\partial_j)$$

$$= (i\partial^0, -i\partial_j) = (p^0, \vec{p})$$

$$= (i\partial^0, -i\vec{\nabla}) = (i\partial_0, -i\vec{\nabla})$$

$$\nabla_j = \partial_j = \frac{\partial}{\partial x^j}$$

$$x^j = (x^1, x^2, x^3) = (x, y, z)$$

$$\partial_\mu \to D_\mu \equiv \partial_\mu + i g_v V_\mu$$

$$V^\mu = (V^0, \vec{V}) = (V^0, V^j) = (V^0, -V_j)$$

① $(\gamma^\mu i\partial_\mu - m)\psi = 0$, $\partial_\mu \to D_\mu$

$$\longrightarrow (\gamma^\mu i(\partial_\mu + i g_v V_\mu) - m)\psi = 0$$

$$(\gamma^\mu i\partial_\mu \mp \gamma^\mu g_v V_\mu - m)\psi = 0$$

② $m \to m + W(r)$ $\quad$ <u>W: Lorentz scalar</u>

$$x^\mu = (x^0, x^1, x^2, x^3) = (t, x, y, z)$$

$$r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} = \sqrt{x^2 + y^2 + z^2}$$

$$(\gamma^\mu i\partial_\mu \mp \gamma^\mu g_v V_\mu - (m + W(r)))\psi = 0$$

$$(\gamma^0 i\partial_0 + \gamma^j i\partial_j \mp \gamma^0 g_v V_0 \mp \gamma^j g_v V_j - (m + W(r)))\psi = 0$$

$$\gamma^0 = \beta, \quad \gamma^j = \beta\alpha^j$$

$$\beta\left(i\frac{\partial}{\partial t} + i\alpha^j\partial_j \mp g_v V^0 + \alpha^j g_v V^j - \beta(m + W(r))\right)\psi = 0$$

$E\psi$ $\quad (\beta^2 = 1)$

$$i\frac{\partial}{\partial t}\psi = (-i\alpha^j\partial_j + g_v V^0 \mp \alpha^j g_v V^j + \beta(m + W(r)))\psi$$

$$\boxed{E\psi = (-i\vec{\alpha}\cdot\vec{\nabla} \mp \vec{\alpha}\cdot g_v\vec{V} + \beta(m + W(r)) + g_v V^0)\psi} \quad \text{--- (1.1)}$$

$$\left(\vec{\nabla} = \frac{\partial}{\partial\vec{x}} \Leftrightarrow \nabla^a = \frac{\partial}{\partial x^a}\right)$$

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(1.1) → $\vec{V} = 0,\quad V^0 = V(r)$

$$\boxed{\left(-i\vec{\alpha}\cdot\vec{\nabla} + g_v V^0(r) + \beta(m + W(r))\right)\psi = E\psi} \quad \text{— (2.1)}$$

$$\boxed{H\psi = E\psi}$$

$$\begin{cases} H = -i\vec{\alpha}\cdot\vec{\nabla} - g_v V^0(r) + \beta(m + W(r)) \\ \vec{j} = \vec{l} + \frac{1}{2}\vec{\sigma} \qquad (\vec{s} = \frac{1}{2}\vec{\sigma}) \\ K \equiv \beta(\vec{\sigma}\cdot\vec{l} + 1) \end{cases}$$

$$\boxed{[\vec{j}, K] = [H, \vec{j}] = [H, K] = 0} \longrightarrow$$

$$\vec{j}^2 = (\vec{l} + \tfrac{1}{2}\vec{\sigma})^2 = \vec{l}^2 + \vec{\sigma}\cdot\vec{l} + \tfrac{1}{4}\vec{\sigma}^2 = \vec{l}^2 + \vec{\sigma}\cdot\vec{l} + \tfrac{3}{4}$$

$$\left(= \vec{l}^2 + (\vec{\sigma}\cdot\vec{l} + 1) - \tfrac{1}{4} \right)$$

$$\left(\vec{\sigma}^2 = 3 \qquad (\sigma^1)^2 + (\sigma^2)^2 + (\sigma^3)^2 = \sigma_x^2 + \sigma_y^2 + \sigma_z^2 \right)$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\beta(\vec{\sigma}\cdot\vec{l} + 1) = \begin{pmatrix} \vec{\sigma}\cdot\vec{l} + 1 & 0 \\ 0 & -(\vec{\sigma}\cdot\vec{l} + 1) \end{pmatrix}$$

$$\begin{aligned} K^2 &= \beta(\vec{\sigma}\cdot\vec{l} + 1)\,\beta(\vec{\sigma}\cdot\vec{l} + 1) \\ &= (\vec{\sigma}\cdot\vec{l} + 1)(\vec{\sigma}\cdot\vec{l} + 1) \\ &= (\vec{\sigma}\cdot\vec{l})(\vec{\sigma}\cdot\vec{l}) + 2\vec{\sigma}\cdot\vec{l} + 1 \\ &= \sigma^i l^i \sigma^j l^j + 2\vec{\sigma}\cdot\vec{l} + 1 \\ &= \sigma^i \sigma^j l^i l^j + 2\vec{\sigma}\cdot\vec{l} + 1 \\ &= (\delta^{ij} + i\varepsilon^{ijk}\sigma^k) l^i l^j + 2\vec{\sigma}\cdot\vec{l} + 1 \\ &= \cancel{\vec{l}^2 + 2\vec{\sigma}\cdot\vec{l} + 1} \\ &= l^i l^i + i\vec{\sigma}\cdot(\vec{l}\times\vec{l}) + 2\vec{\sigma}\cdot\vec{l} + 1 \\ &= \vec{l}^2 + i\vec{\sigma}\cdot(i\vec{l}) + 2\vec{\sigma}\cdot\vec{l} + 1 \\ &= \vec{l}^2 + \vec{\sigma}\cdot\vec{l} + 1 \\ &= \vec{j}^2 + \tfrac{1}{4} \quad \left(= j(j+1) + \tfrac{1}{4} = (j + \tfrac{1}{2})^2 \right) \end{aligned}$$

$$\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$$

$$(\vec{l}\times\vec{l})$$

$$\therefore \boxed{K^2 = \vec{j}^2 + \tfrac{1}{4} \left(= (j + \tfrac{1}{2})^2 \right)} \qquad (2.2)$$

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$\boxed{\vec{l}\times\vec{l} = i\vec{l}}$ MITB-2-2

$$p^a x^b = -i\frac{\partial}{\partial x^a} x^b = -i\delta^{ab}$$

$$\begin{aligned}
\vec{l}\times\vec{l} &= \hat{e}_a (\vec{l}\times\vec{l})^a \\
&= \hat{e}_a\, \varepsilon^{abc} l^b l^c \\
&= \hat{e}_a\, \varepsilon^{abc} (\varepsilon^{bjk} x^j p^k)(\varepsilon^{clm} x^l p^m) \\
&= \hat{e}_a\, \varepsilon^{abc} \varepsilon^{bjk} \varepsilon^{clm}\, x^j \underbrace{p^k x^l}_{=\,-i\delta^{kl} + x^l p^k} p^m
\end{aligned}$$

$$= -i\delta^{kl} x^j p^m + x^j x^l p^k p^m$$

$$= \hat{e}_a\, \varepsilon^{abc} \varepsilon^{bjk} \varepsilon^{clm} (-i\delta^{kl} x^j p^m + x^j x^l p^k p^m)$$

$$\left(= \hat{e}_a\, \varepsilon^{abc} \left(-i\, \varepsilon^{bjk} \varepsilon^{ckm} x^j p^m + \varepsilon^{bjk}\varepsilon^{clm} x^j x^l p^k p^m\right) \right)$$

$$= \hat{e}_a\, \varepsilon^{abc} \varepsilon^{bjk} \left(-i\, \varepsilon^{ckm} x^j p^m + \varepsilon^{clm} x^j x^l p^k p^m\right)$$

$$\varepsilon^{abc}\varepsilon^{bjk} = +\varepsilon^{bca}\varepsilon^{bjk} = \delta^{cj}\delta^{ak} - \delta^{ck}\delta^{aj}$$

$$= \hat{e}_a (\delta^{cj}\delta^{ak} - \delta^{ck}\delta^{aj}) \left(-i\, \varepsilon^{ckm} x^j p^m \quad \text{①} \quad + \varepsilon^{clm} x^j x^l p^k p^m \right) \quad \text{②}$$

$$\begin{aligned}
&= \hat{e}_a \big(-i\, \varepsilon^{cam} x^c p^m + \underbrace{i\, \varepsilon^{ccm} x^a p^m}_{\to\, 0} \quad \text{①} \\
&\qquad + \underbrace{\varepsilon^{clm} x^c x^l p^a p^m}_{0} - \underbrace{\varepsilon^{clm} x^a x^l p^c p^m}_{\to\, 0} \big) \\
&= \hat{e}_a (-i\, \varepsilon^{cam} x^c p^m) \\
&= i\, \hat{e}_a\, \varepsilon^{acm} x^c p^m = i\vec{l} \qquad \therefore \boxed{\vec{l}\times\vec{l} = i\vec{l}}
\end{aligned}$$

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$\boxed{[\vec{J}, K] = 0}$

MITB-3

in Cartesian an arbitrary vector $\vec{B}$ can be written

$$\vec{B} = B^i \hat{e}_i = B^1\hat{e}_1 + B^2\hat{e}_2 + B^3\hat{e}_3 = B^x\hat{e}_x + B^y\hat{e}_y + B^z\hat{e}_z$$

component $\vec{B} = (B^1, B^2, B^3) = (B^x, B^y, B^z)$

$\hat{e}_i, \hat{e}_j$ satisfy

$$\boxed{\hat{e}_i \times \hat{e}_j = i\varepsilon_{ijk}\hat{e}_k, \quad \hat{e}_i \cdot \hat{e}_j = \delta_{ij}} \quad \text{——} (3.1)$$

$$\hat{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \hat{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \hat{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\boxed{\vec{x} = (x^1, x^2, x^3) = \vec{r} = (r^1, r^2, r^3) = (x, y, z)}$$

$$[\vec{J}, K] = [\underbrace{\vec{l} + \tfrac{1}{2}\vec{\sigma}}_{(\vec{l}+\frac{1}{2}\vec{\sigma})I_4}, \beta(\vec{\sigma}\cdot\vec{l} + 1)]$$

$$= \beta[\vec{l} + \tfrac{1}{2}\vec{\sigma}, \vec{\sigma}\cdot\vec{l} + 1]$$

$$= \beta(\underbrace{[\vec{l}, \vec{\sigma}\cdot\vec{l}]}_{1} + \tfrac{1}{2}\underbrace{[\vec{\sigma}, \vec{\sigma}\cdot\vec{l}]}_{2})$$

$$1 = [l^a\hat{e}_a, \sigma^b l^b] = \hat{e}_a\sigma^b[l^a, l^b]$$
$$= \hat{e}_a\sigma^b\, i\varepsilon^{abc} l^c$$
$$= i\varepsilon^{abc}\hat{e}_a\sigma^b l^c = i\vec{\sigma}\times\vec{l}$$

$$2 = \tfrac{1}{2}[\sigma^a\hat{e}_a, \sigma^b l^b] = \tfrac{1}{2}\hat{e}_a l^b[\sigma^a, \sigma^b]$$
$$= \tfrac{1}{2}\hat{e}_a l^b\, 2i\varepsilon^{abc}\sigma^c$$
$$= i\varepsilon^{abc}\hat{e}_a l^b\sigma^c = i\vec{l}\times\vec{\sigma} = -i\vec{\sigma}\times\vec{l}$$

$$\therefore\ 1 + 2 = 0 \longrightarrow \boxed{[\vec{J}, K] = 0} \quad \text{——} \quad (3.2)$$

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MITB - 4

$\boxed{[H, \vec{J}]=0}$

$$[H, \vec{J}] = [-i\vec{\alpha}\cdot\vec{\nabla} + g_v V^0(r) + \beta(m+W(r)), \vec{l} + \tfrac{1}{2}\vec{\sigma}]$$

$$\vec{l} = \vec{r}\times\vec{p} = \vec{r}\times(-i\vec{\nabla}) = -i\hat{e}_a \varepsilon^{abc} r^b \frac{\partial}{\partial x^c} = -i\hat{e}_a \varepsilon^{abc} x^b \frac{\partial}{\partial x^c}$$

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

$\vec{r} = (r^1, r^2, r^3) = (x^1, x^2, x^3) = (x, y, z)$

$$= \underbrace{[-i\vec{\alpha}\cdot\vec{\nabla}, \vec{l} + \tfrac{1}{2}\vec{\sigma}]}_{①} \cancel{+\beta[W}$$
$$+ \underbrace{[+g_v V^0(r), \vec{l}]}_{②} + \underbrace{[W(r), \vec{l}]}_{③}$$

$$= \underbrace{[-i\vec{\alpha}\cdot\vec{\nabla}, \vec{l}]}+ \underbrace{\tfrac{1}{2}[-i\vec{\alpha}\cdot\vec{\nabla}, \vec{\sigma}]}$$
$$+ g_v \underbrace{[\vec{l}, V^0(r)]}_{②} - \underbrace{[\vec{l}, W(r)]}_{③}$$

$x^a = r^a$

$$[\vec{l}, f(r)] = -i\hat{e}_a \varepsilon^{abc} x^b \left[\frac{\partial}{\partial x^c}, f(r)\right]$$
$$= -i\hat{e}_a \varepsilon^{abc}\left(\frac{\partial f}{\partial r}\frac{\partial r}{\partial x^c}\right) x^b \qquad f' = \frac{\partial f}{\partial r}$$
$$= -i\hat{e}_a \varepsilon^{abc} f'(r)\, x^b \frac{x^c}{r}$$
$$= -i\hat{e}_a \frac{f'(r)}{r}\varepsilon^{abc} x^b x^c = 0$$

$$\therefore\ ② = ③ = 0$$

$$-i\vec{\alpha}\cdot\vec{\nabla} = \begin{pmatrix} 0 & -i\vec{\sigma}\cdot\vec{\nabla} \\ -i\vec{\sigma}\cdot\vec{\nabla} & 0 \end{pmatrix}, \quad \vec{l} = \vec{l}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[-i\vec{\alpha}\cdot\vec{\nabla}, \vec{l}] = \begin{pmatrix} 0 & [-i\vec{\sigma}\cdot\vec{\nabla}, \vec{l}] \\ [-i\vec{\sigma}\cdot\vec{\nabla}, l] & 0 \end{pmatrix} \quad \text{--- } (4.1)$$

$$\tfrac{1}{2}[-i\vec{\alpha}\cdot\vec{\nabla}, \vec{\sigma}] = \tfrac{1}{2}\begin{pmatrix} 0 & [-i\vec{\sigma}\cdot\vec{\nabla}, \vec{\sigma}] \\ [-i\vec{\sigma}\cdot\vec{\nabla}, \vec{\sigma}] & 0 \end{pmatrix} \quad \text{--- } (4.2)$$

!! !!! "component = a"

$$\boxed{[\vec{l}, f(r)] = 0} = [l^a, f(r)] \quad \text{--- } (4.3)$$

!!! (a = 1, 2, 3 also!)

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(4.1) $[-i\vec{\sigma}\cdot\vec{\nabla}, \vec{l}\,]$

$$[-i\vec{\sigma}\cdot\vec{\nabla}, \vec{l}\,]$$
$$= [-i\sigma^a \frac{\partial}{\partial x^a}, \hat{e}_j \varepsilon^{jkl} x^k (-i\frac{\partial}{\partial x^l})]$$
$$= -\hat{e}_j \varepsilon^{jkl}\, \sigma^a [\frac{\partial}{\partial x^a}, x^k \frac{\partial}{\partial x^l}]$$
$$= -\hat{e}_j \varepsilon^{jkl} \sigma^a (\delta^{ak}\frac{\partial}{\partial x^l})$$
$$= -\hat{e}_j \varepsilon^{jal} \sigma^a \frac{\partial}{\partial x^l} = -\vec{\sigma}\times\vec{\nabla} \quad \text{—— (5.1)}$$

(4.2) $\frac{1}{2}[-i\vec{\sigma}\cdot\vec{\nabla}, \vec{\sigma}\,]$

$$\frac{1}{2}[-i\vec{\sigma}\cdot\vec{\nabla}, \vec{\sigma}\,]$$
$$= \frac{1}{2}[-i\sigma^a\frac{\partial}{\partial x^a}, \sigma^b \hat{e}_b]$$
$$= \frac{1}{2}(-i\frac{\partial}{\partial x^a})\hat{e}_b [\sigma^a, \sigma^b]$$
$$= \frac{1}{2}(-i\frac{\partial}{\partial x^a})\hat{e}_b\, 2i\varepsilon^{abc}\sigma^c$$
$$= +\frac{\partial}{\partial x^a}\hat{e}_b \varepsilon^{abc}\sigma^c$$
$$= \hat{e}_b \varepsilon^{bca}\sigma^c\frac{\partial}{\partial x^a} = \vec{\sigma}\times\vec{\nabla} \quad \text{—— (5.2)}$$

$$\therefore [H, \vec{J}\,] = [-i\vec{\alpha}\cdot\vec{\nabla}, \vec{l}\,] + \frac{1}{2}[-i\vec{\alpha}\cdot\vec{\nabla}, \vec{\sigma}\,] \underbrace{+ ② + ③}_{=0}$$
$$= (4.1) + (4.2)$$
$$= \begin{pmatrix} 0 & [-i\vec{\sigma}\cdot\vec{\nabla}, \vec{l}\,] + \frac{1}{2}[-i\vec{\sigma}\cdot\vec{\nabla}, \vec{\sigma}\,] \\ [-i\vec{\sigma}\cdot\vec{\nabla}, \vec{l}\,] + \frac{1}{2}[-i\vec{\sigma}\cdot\vec{\nabla}, \vec{\sigma}\,] & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -\vec{\sigma}\times\vec{\nabla} + \vec{\sigma}\times\vec{\nabla} \\ -\vec{\sigma}\times\vec{\nabla} + \vec{\sigma}\times\vec{\nabla} & 0 \end{pmatrix} = 0$$

$$\therefore \boxed{[H, \vec{J}\,] = 0}$$

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$$\boxed{[H,K]=0}$$

$$[H,K]=[-i\vec{\alpha}\cdot\vec{\nabla}+g_v V^0(r)+\beta(m+W(r)),\ \beta(\vec{\sigma}\cdot\vec{l}+1)]$$

$$=[-i\vec{\alpha}\cdot\vec{\nabla},\ \beta(\vec{\sigma}\cdot\vec{l}+1)] \qquad \vec{\alpha}=\begin{pmatrix}0&\vec{\sigma}\\ \vec{\sigma}&0\end{pmatrix}$$

$$\left(\because\ (4.3)\to[l^a,f(r)]=0 \quad (a=1,2,3)\right) \qquad \beta=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$\beta^2=1$$

$$=\left[\begin{pmatrix}0&-i\vec{\sigma}\cdot\vec{\nabla}\\-i\vec{\sigma}\cdot\vec{\nabla}&0\end{pmatrix},\ \begin{pmatrix}\vec{\sigma}\cdot\vec{l}+1&0\\0&-(\vec{\sigma}\cdot\vec{l}+1)\end{pmatrix}\right]$$

$$\cancel{e\ (\ldots -i\vec{\sigma}\cdot\vec{\nabla}(\vec{\sigma}\cdot\vec{l}+1))}$$

$$=\begin{pmatrix}0&-i\vec{\sigma}\cdot\vec{\nabla}\\-i\vec{\sigma}\cdot\vec{\nabla}&0\end{pmatrix}\begin{pmatrix}\vec{\sigma}\cdot\vec{l}+1&0\\0&-(\vec{\sigma}\cdot\vec{l}+1)\end{pmatrix}-\begin{pmatrix}\vec{\sigma}\cdot\vec{l}+1&0\\0&-(\vec{\sigma}\cdot\vec{l}+1)\end{pmatrix}\begin{pmatrix}0&-i\vec{\sigma}\cdot\vec{\nabla}\\-i\vec{\sigma}\cdot\vec{\nabla}&0\end{pmatrix}$$

$$=\begin{pmatrix}0&i\vec{\sigma}\cdot\vec{\nabla}(\vec{\sigma}\cdot\vec{l}+1)+i(\vec{\sigma}\cdot\vec{l}+1)\vec{\sigma}\cdot\vec{\nabla}\\-i\vec{\sigma}\cdot\vec{\nabla}(\vec{\sigma}\cdot\vec{l}+1)-i(\vec{\sigma}\cdot\vec{l}+1)\vec{\sigma}\cdot\vec{\nabla}&0\end{pmatrix}$$

$$=\begin{pmatrix}0&i\vec{\sigma}\cdot\vec{\nabla}\,\vec{\sigma}\cdot\vec{l}+i\vec{\sigma}\cdot\vec{l}\,\vec{\sigma}\cdot\vec{\nabla}+2i\vec{\sigma}\cdot\vec{\nabla}\\-i\vec{\sigma}\cdot\vec{\nabla}\,\vec{\sigma}\cdot\vec{l}-i\vec{\sigma}\cdot\vec{l}\,\vec{\sigma}\cdot\vec{\nabla}-2i\vec{\sigma}\cdot\vec{\nabla}&0\end{pmatrix} \qquad \text{—— (6.1)}$$

$$i\vec{\sigma}\cdot\vec{\nabla}\,\vec{\sigma}\cdot\vec{l}$$

$$=i\sigma^a\frac{\partial}{\partial x^a}\sigma^b l^b$$

$$=i\sigma^a\sigma^b\frac{\partial}{\partial x^a}\epsilon^{bjk}x^j\left(-i\frac{\partial}{\partial x^k}\right)$$

$$=\sigma^a\sigma^b\epsilon^{bjk}\frac{\partial}{\partial x^a}x^j\frac{\partial}{\partial x^k} \qquad \delta^{aj}\frac{\partial}{\partial x^k}$$

$$=\sigma^a\sigma^b\epsilon^{bjk}\left(\delta^{aj}\frac{\partial}{\partial x^k}+x^j\frac{\partial}{\partial x^a}\frac{\partial}{\partial x^k}\right)$$

$$=\sigma^a\sigma^b\left(\epsilon^{bak}\frac{\partial}{\partial x^k}+\epsilon^{bjk}x^j\frac{\partial}{\partial x^k}\frac{\partial}{\partial x^a}\right)$$

$$=(\delta^{ab}+i\epsilon^{abc}\sigma^c)\left(\epsilon^{bak}\frac{\partial}{\partial x^k}+\epsilon^{bjk}x^j\frac{\partial}{\partial x^k}\frac{\partial}{\partial x^a}\right)$$

$$\longrightarrow \times$$

other method !! better !!

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(6.1) → consider the term

$$\underbrace{i\vec\sigma\cdot\vec\nabla\,\vec\sigma\cdot\vec l + i\vec\sigma\cdot\vec l\,\vec\sigma\cdot\vec\nabla + 2i\vec\sigma\cdot\vec\nabla} \qquad \boxed{\nabla^a \equiv \frac{\partial}{\partial x^a} \;\; (\equiv \nabla_a)} \quad (7.1)$$

$$\begin{aligned}\vec\sigma\cdot\vec A\,\vec\sigma\cdot\vec B &= \sigma^a\sigma^b A^a B^b \\ &= (\delta^{ab} + i\varepsilon^{abc}\sigma^c) A^a B^b \\ &= \vec A\cdot\vec B + i\vec\sigma\cdot(\vec A\times\vec B)\end{aligned}$$

$\therefore$

$$\begin{cases} i\vec\sigma\cdot\vec\nabla\,\vec\sigma\cdot\vec l = i(\vec\nabla\cdot\vec l + i\vec\sigma\cdot(\vec\nabla\times\vec l)) \\ i\vec\sigma\cdot\vec l\,\vec\sigma\cdot\vec\nabla = i(\vec l\cdot\vec\nabla + i\vec\sigma\cdot(\vec l\times\vec\nabla)) \end{cases}$$

$$\begin{aligned}\vec\nabla\cdot\vec l &= \nabla^a l^a = \nabla^a \varepsilon^{abc} x^b(-i\nabla^c) \\ &= \varepsilon^{abc}\big(\delta^{ab}(-i\nabla^c) + x^b\nabla^a(-i\nabla^c)\big) \\ &= 0 + 0 \qquad \uparrow a,b \text{ symmetric} \quad \uparrow a,c \text{ symmetric} \\ &= \underline{0} \qquad \varepsilon^{abc}:\ a,b,c \text{ antisymmetric}\end{aligned}$$

$$\begin{aligned}\vec l\cdot\vec\nabla &= l^a\nabla^a \\ &= \varepsilon^{abc} x^b(-i\nabla^c)\nabla^a \\ &= \underline{0} \qquad (\because a, c \text{ symmetric})\end{aligned}$$

$$\left.\begin{cases} i\vec\sigma\cdot\vec\nabla\,\vec\sigma\cdot\vec l = -\vec\sigma\cdot(\vec\nabla\times\vec l) \\ i\vec\sigma\cdot\vec l\,\vec\sigma\cdot\vec\nabla = -\vec\sigma\cdot(\vec l\times\vec\nabla) \end{cases}\right\} \quad (7.2)$$

$$\begin{aligned}\vec\sigma\cdot(\vec\nabla\times\vec l) &= \sigma^a(\vec\nabla\times\vec l)^a \\ &= \sigma^a\varepsilon^{abc}\nabla^b l^c \\ &= \sigma^a\varepsilon^{abc}\nabla^b(\varepsilon^{cjk}x^j(-i\nabla^k)) \\ &= \sigma^a\varepsilon^{abc}\varepsilon^{cjk}(\nabla^b x^j(-i\nabla^k)) \\ &= \sigma^a\varepsilon^{abc}\varepsilon^{cjk}(\delta^{bj}(-i\nabla^k) + x^j\nabla^b(-i\nabla^k)) \\ &= \sigma^a\big(\varepsilon^{abc}\varepsilon^{cjk}\delta^{bj}(-i\nabla^k) \\ &\qquad + \varepsilon^{abc}\varepsilon^{cjk}x^j(-i\nabla^k)\nabla^b\big) \\ &= \sigma^a\big(\varepsilon^{abc}\varepsilon^{cbk}(-i\nabla^k) \\ &\qquad - \varepsilon^{acb}\varepsilon^{cjk}x^j(-i\nabla^k)\nabla^b\big) \\ &= \sigma^a\big((\delta^{ac}\delta^{ck} - \delta^{ak}\delta^{cc})(-i\nabla^k) \\ &\qquad - \varepsilon^{acb}l^c\nabla^b\big) \\ &= \sigma^a\big((\underset{\uparrow\,\delta^{ak}}{\delta^{ak}} - 3\delta^{ak})(-i\nabla^k) - (\vec l\times\vec\nabla)^a\big) \rightsquigarrow \to \text{※}\end{aligned}$$

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∇^a

$$\cancel{A} = \sigma^a \left(2i\nabla^a - (\vec{l}\times\vec{\nabla})^a \right)$$
$$= 2i\,\vec{\sigma}\cdot\vec{\nabla} - \vec{\sigma}\cdot(\vec{l}\times\vec{\nabla}) \qquad \text{—— } (8.1)$$

(8.1), (7.2) $\longrightarrow$ (7.1)

$$\therefore\ i\vec{\sigma}\cdot\vec{\nabla}\,\vec{\sigma}\cdot\vec{l} + i\vec{\sigma}\cdot\vec{l}\,\vec{\sigma}\cdot\vec{\nabla} + 2i\,\vec{\sigma}\cdot\vec{\nabla}$$
$$= -\vec{\sigma}\cdot(\vec{\nabla}\times\vec{l}) - \vec{\sigma}\cdot(\vec{l}\times\vec{\nabla}) + 2i\,\vec{\sigma}\cdot\vec{\nabla}$$
$$= -\left(2i\vec{\sigma}\cdot\vec{\nabla} - \vec{\sigma}\cdot(\vec{l}\times\vec{\nabla})\right) - \vec{\sigma}\cdot(\vec{l}\times\vec{\nabla}) + 2i\,\vec{\sigma}\cdot\vec{\nabla}$$
$$= 0$$

$$\therefore\ \underline{i\vec{\sigma}\cdot\vec{\nabla}\,\vec{\sigma}\cdot\vec{l} + i\vec{\sigma}\cdot\vec{l}\,\vec{\sigma}\cdot\vec{\nabla} + 2i\vec{\sigma}\cdot\vec{\nabla} = 0} \quad //$$
$$\downarrow$$
$$\therefore\ \underline{[H, K] = 0}$$

$$\cancel{A}\ \boxed{[\vec{J}, K] = [H, \vec{J}] = [H, K] = 0} \quad // \qquad \text{—— } (8.2)$$

$$(2.2) \longrightarrow \quad K^2 = \vec{J}^2 + \frac{1}{4}$$

eigenvalues of $K \to \kappa$

$$\kappa^2 = j(j+1) + \frac{1}{4}$$
$$= j^2 + j + \frac{1}{4}$$
$$= \left(j + \frac{1}{2}\right)^2$$

// $\cancel{A}$

$$\therefore\ \boxed{\kappa = \pm\left(j + \frac{1}{2}\right)} \qquad \text{—— } (8.3)$$

$$K = \beta(\vec{\sigma}\cdot\vec{l} + 1) = \begin{pmatrix} \vec{\sigma}\cdot\vec{l}+1 & 0 \\ 0 & -(\vec{\sigma}\cdot\vec{l}+1) \end{pmatrix} \equiv \begin{pmatrix} \hat{K} & 0 \\ 0 & -\hat{K} \end{pmatrix}$$
$$= \begin{pmatrix} \hat{K} & 0 \\ 0 & -\hat{K} \end{pmatrix} \quad (\text{operator } \hat{K}$$

$$\longrightarrow \left(\hat{K}\psi^{\mu}_{\kappa} = -\kappa\,\psi^{\mu}_{\kappa} \right)$$

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$$H = -i\vec{\alpha}\cdot\vec{\nabla} - g_v V(\vec{r}) + \beta\,(m + W(r))$$

$$\psi(t,\vec{r}) = \psi(\vec{r})\, e^{-iEt}$$

$$H\,\psi(\vec{r}) = \left[-i\vec{\alpha}\cdot\vec{\nabla} + g_v V^0(r) + \beta\,(m+W(r))\right]\psi(\vec{r}) = E\,\psi(\vec{r})$$

$$(H, K, \vec{J}^2, J_z) \leftrightarrow (E, -\kappa, j(j+1), j_z = \mu)$$

$$0 = [\vec{J}, K] = [H, \vec{J}] = [H, K]$$

$$\boxed{\begin{cases} \vec{J}^2\,\psi^\mu_\kappa = j(j+1)\,\psi^\mu_\kappa \\ J_z\,\psi^\mu_\kappa = \mu\,\psi^\mu_\kappa \\ K\,\psi^\mu_\kappa = -\kappa\,\psi^\mu_\kappa \end{cases}} \qquad \text{———} \quad (9.0)$$

$$\cancel{H\psi(\vec{r})}$$

$$\boxed{\left[-i\vec{\alpha}\cdot\vec{\nabla} + \beta\,(m+W)\right]\psi(\vec{r}) = (E - g_v V^0)\,\psi(\vec{r})}$$

$$\text{———} \quad (9.1)$$

$$\begin{cases} K = \beta\,(\vec{\sigma}\cdot\vec{l} + 1) = \begin{pmatrix} \vec{\sigma}\cdot\vec{l}+1 & 0 \\ 0 & -(\vec{\sigma}\cdot\vec{l}+1) \end{pmatrix} \\ \qquad \equiv \begin{pmatrix} \hat{k} & 0 \\ 0 & -\hat{k} \end{pmatrix} \qquad \hat{k} \ (\text{or } \hat{\kappa}) \\ K\,\psi^\mu_\kappa = -\kappa\,\psi^\mu_\kappa \end{cases}$$

$$\hat{r} \equiv \frac{\vec{r}}{|\vec{r}|}$$

$$\psi^\mu_\kappa = \begin{pmatrix} g(r)\,\chi^\mu_\kappa(\hat{r}) \\ i f(r)\,\chi^\mu_{-\kappa}(\hat{r}) \end{pmatrix} \qquad \text{———} \quad (9.2)$$

$\chi^\mu_\kappa(\hat{r})$, $\chi^\mu_{-\kappa}(\hat{r})$ → later !!

!!

$\boxed{1}$

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$\vec{a} = \hat{e}_i a_i \quad (i=1,2,3)$ MITB-10

(9.1) $\to -i\vec{\alpha}\cdot\vec{\nabla} \to$? $\qquad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$

① $\hat{r}\times(\hat{r}\times\vec{\nabla}) = (\hat{r}_a\hat{e}_a)\times(\hat{r}_b\hat{e}_b\times\hat{e}_c\nabla_c)$

$= \hat{r}_a\hat{r}_b\nabla_c\,\hat{e}_a\times(\hat{e}_b\times\hat{e}_c)$

$= \hat{r}_a\hat{r}_b\nabla_c\,\hat{e}_a\times(\varepsilon_{bcd}\hat{e}_d) \quad \leftarrow \hat{e}_a\times\hat{e}_d = \varepsilon_{adl}\hat{e}_l$

$= \hat{r}_a\hat{r}_b\nabla_c\,\varepsilon_{bcd}\,\varepsilon_{adl}\,\hat{e}_l \qquad \varepsilon_{bcd}\varepsilon_{ald}$

$= -\hat{r}_a\hat{r}_b\nabla_c\,\varepsilon_{bcd}\,\varepsilon_{ald}\,\hat{e}_l$

$= -\hat{r}_a\hat{r}_b\nabla_c\,(\delta_{ba}\delta_{cl} - \delta_{bl}\delta_{ca})\,\hat{e}_l$

$= -\hat{r}_a\hat{r}_a\nabla_c\hat{e}_c + \hat{r}_a\hat{r}_b\nabla_a\hat{e}_b \qquad \nabla_a\hat{e}_b = \hat{e}_b\nabla_a$

$= -1\,\vec{\nabla} + \hat{r}_b\hat{e}_b\,\hat{r}_a\nabla_a$

$= -\vec{\nabla} + \hat{r}\,(\hat{r}\cdot\vec{\nabla}) \qquad \hat{r}\cdot\vec{\nabla} = \frac{\partial}{\partial r}$

$\therefore\ \vec{\nabla} = -\hat{r}\times(\hat{r}\times\vec{\nabla}) + \hat{r}\,(\hat{r}\cdot\vec{\nabla}) \quad \text{— (10.1)}$

$\therefore$

$-i\vec{\alpha}\cdot\vec{\nabla} = -i\vec{\alpha}\cdot\left(-\hat{r}\times(\hat{r}\times\vec{\nabla}) + \hat{r}\,(\hat{r}\cdot\vec{\nabla})\right)$

$= i\vec{\alpha}\cdot\left(\hat{r}\times i\,(\hat{r}\times(-i\vec{\nabla}))\right) - i\vec{\alpha}\cdot\hat{r}\frac{\partial}{\partial r}$

$= i\vec{\alpha}\cdot\frac{1}{r}(\hat{r}\times i\vec{l}) - i\vec{\alpha}\cdot\hat{r}\frac{\partial}{\partial r}$

$\Big(= -\vec{\alpha}\cdot\frac{1}{r}(\frac{\vec{r}}{r}\times\vec{l}) - i\vec{\alpha}\cdot\hat{r}\frac{\partial}{\partial r}$

$= -\frac{1}{r^2}\vec{\alpha}\cdot(\vec{r}\times\vec{l}) - i\vec{\alpha}\cdot\hat{r}\frac{\partial}{\partial r} \Big)$

$= -\vec{\alpha}\cdot\frac{1}{r}(\hat{r}\times\vec{l}) - i\vec{\alpha}\cdot\hat{r}\frac{\partial}{\partial r}$

$\vec{l} = \vec{r}\times\vec{p}$
$= \frac{1}{r}(\hat{r}\times\vec{p})$
$\hookrightarrow \hat{r}\cdot\vec{l} = 0$

$\vec{\alpha}\cdot\hat{r}\ \vec{\sigma}\cdot\vec{l} = \begin{pmatrix} 0 & \vec{\sigma}\cdot\hat{r} \\ \vec{\sigma}\cdot\hat{r} & 0 \end{pmatrix}\vec{\sigma}\cdot\vec{l} \qquad \sigma_i\sigma_j$

$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\vec{\sigma}\cdot\hat{r}\ \vec{\sigma}\cdot\vec{l}$

$-\frac{1}{r}\vec{\alpha}\cdot(\hat{r}\times\vec{l})$

$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\hat{r}_i l_j\,(\delta_{ij} + i\varepsilon_{ijk}\sigma_k)$

$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}(\underbrace{\hat{r}\cdot\vec{l}}_{=0} + i\varepsilon_{kij}\sigma_k\hat{r}_i l_j)$

$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} i\vec{\sigma}\cdot(\hat{r}\times\vec{l}) = i\vec{\alpha}\cdot(\hat{r}\times\vec{l})$

$\therefore\ \vec{\alpha}\cdot(\hat{r}\times\vec{l}) = -i\vec{\alpha}\cdot\hat{r}\ \vec{\sigma}\cdot\vec{l}$

$= -\frac{1}{r}(-i\vec{\alpha}\cdot\hat{r}\ \vec{\sigma}\cdot\vec{l}) - i\vec{\alpha}\cdot\hat{r}\frac{\partial}{\partial r}$

$= \frac{i}{r}\vec{\alpha}\cdot\hat{r}\ \vec{\sigma}\cdot\vec{l} - i\vec{\alpha}\cdot\hat{r}\frac{\partial}{\partial r} \longrightarrow$ (10.2) ✱

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$(0.1) \rightarrow \quad K = \beta(\vec{\sigma}\cdot\vec{l} + 1)$

$\therefore \beta K = \beta^2(\vec{\sigma}\cdot\vec{l}+1) = \vec{\sigma}\cdot\vec{l}+1 \qquad (\beta^2 = 1) \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\vec{\sigma}\cdot\vec{l} = \beta K - 1$

(10.2)

$\hookrightarrow \quad -i\vec{\alpha}\cdot\vec{\nabla} = -i\vec{\alpha}\cdot\hat{r}\frac{\partial}{\partial r} + \frac{i}{r}\vec{\alpha}\cdot\hat{r}\,\vec{\sigma}\cdot\vec{l}$

$= -i\vec{\alpha}\cdot\hat{r}\frac{\partial}{\partial r} + \frac{i}{r}\vec{\alpha}\cdot\hat{r}\,(\beta K - 1)$

$$\therefore \boxed{-i\vec{\alpha}\cdot\vec{\nabla} = -i\vec{\alpha}\cdot\hat{r}\frac{\partial}{\partial r} + \frac{i}{r}\vec{\alpha}\cdot\hat{r}\,(\beta K - 1)}$$

(2.1)

$H\psi = E\psi$

$$\boxed{\begin{aligned} H &= -i\vec{\alpha}\cdot\vec{\nabla} + g_v V^0(r) + \beta(m + W(r)) \\ &= -i\vec{\alpha}\cdot\hat{r}\frac{\partial}{\partial r} + \frac{i}{r}\vec{\alpha}\cdot\hat{r}(\beta K - 1) + g_v V^0(r) + \beta(m + W(r)) \end{aligned}}$$

$— (11.1)$

$K\psi = -\kappa\psi \quad \longrightarrow \quad K\psi^{\mu}_{\kappa} = -\kappa\,\psi^{\mu}_{\kappa}$

$$\psi^{\mu}_{\kappa} = \begin{pmatrix} g(r)\,\chi^{\mu}_{\kappa}(\hat{r}) \\ if(r)\,\chi^{\mu}_{-\kappa}(\hat{r}) \end{pmatrix} \qquad — (11.2)$$

$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$$= -i\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}\cdot\hat{r}\frac{\partial}{\partial r} + \frac{i}{r}\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}\cdot\hat{r}\,\Big(\beta\,\overbrace{\beta(\vec{\sigma}\cdot\vec{l} - 1)}^{K}\Big)$$

$$+ g_v V^0(r) + \beta(m + W(r))$$

$$= \begin{pmatrix} 0 & -i\vec{\sigma}\cdot\hat{r}\frac{\partial}{\partial r} \\ -i\vec{\sigma}\cdot\hat{r}\frac{\partial}{\partial r} & \end{pmatrix} + \frac{i}{r}\begin{pmatrix} 0 & \vec{\sigma}\cdot\hat{r} \\ \vec{\sigma}\cdot\hat{r} & 0 \end{pmatrix}(\beta K - 1)$$

$$+ \begin{pmatrix} g_v V^0(r) & 0 \\ 0 & g_v V^0(r) \end{pmatrix} + \begin{pmatrix} m + W(r) & 0 \\ 0 & -(m + W(r)) \end{pmatrix}$$

$— (11.3)$

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(11.2)

$$K\psi^{\mu}_{\kappa} = -\kappa\,\psi^{\mu}_{\kappa} \rightsquigarrow (\beta K - 1)\psi^{\mu}_{\kappa} = (\beta(-\kappa)-1)\psi^{\mu}_{\kappa}$$

$$= (-\beta\kappa - 1)\psi^{\mu}_{\kappa}$$

$$= \left\{-\begin{pmatrix}\kappa & 0\\ 0 & -\kappa\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}\psi^{\mu}_{\kappa}$$

$$= \underline{\begin{pmatrix}-\kappa-1 & 0\\ 0 & \kappa-1\end{pmatrix}\psi^{\mu}_{\kappa}}$$

(11.3),

$$H\psi^{\mu}_{\kappa} = E\psi^{\mu}_{\kappa}, \quad (\beta K - 1)\psi^{\mu}_{\kappa} = \underbrace{\begin{pmatrix}-\kappa-1 & 0\\ 0 & \kappa-1\end{pmatrix}}\psi^{\mu}_{\kappa}$$

$$H\psi^{\mu}_{\kappa} = \left\{\begin{pmatrix}0 & -i\vec{\sigma}\cdot\hat{r}\frac{\partial}{\partial r}\\ -i\vec{\sigma}\cdot\hat{r}\frac{\partial}{\partial r} & 0\end{pmatrix} + \frac{i}{r}\begin{pmatrix}0 & \vec{\sigma}\cdot\hat{r}\\ \vec{\sigma}\cdot\hat{r} & 0\end{pmatrix}\begin{pmatrix}-\kappa-1 & 0\\ 0 & \kappa-1\end{pmatrix}\right.$$
$$\left. + \begin{pmatrix}m+W(r)+g_v V^0(r) & 0\\ 0 & -(m+W(r))+g_v V^0(r)\end{pmatrix}\right\}\psi^{\mu}_{\kappa}$$

$$= \begin{pmatrix}m+W+g_v V^0 & -i\vec{\sigma}\cdot\hat{r}\frac{\partial}{\partial r} + \frac{i(\kappa-1)\,\vec{\sigma}\cdot\hat{r}}{r}\\ -i\vec{\sigma}\cdot\hat{r}\frac{\partial}{\partial r} - \frac{i(\kappa+1)\,\vec{\sigma}\cdot\hat{r}}{r} & -m-W+g_v V^0\end{pmatrix}\psi^{\mu}_{\kappa}$$

$$\psi^{\mu}_{\kappa} = \begin{pmatrix}g(r)\,\chi^{\mu}_{\kappa}\\ i f(r)\,\chi^{\mu}_{-\kappa}\end{pmatrix} \qquad \chi^{\mu}_{\kappa}(\hat{r})$$

$$\begin{cases} E(g\chi^{\mu}_{\kappa}) = (m+W+g_v V^0)(g\chi^{\mu}_{\kappa}) + \left(-i\vec{\sigma}\cdot\hat{r}\frac{\partial}{\partial r} + \frac{i(\kappa-1)\vec{\sigma}\cdot\hat{r}}{r}\right)(if\chi^{\mu}_{-\kappa})\\ E(if\chi^{\mu}_{-\kappa}) = \left(-i\vec{\sigma}\cdot\hat{r}\frac{\partial}{\partial r} - \frac{i(\kappa+1)\vec{\sigma}\cdot\hat{r}}{r}\right)(g\chi^{\mu}_{\kappa}) - (m+W-g_v V^0)(if\chi^{\mu}_{-\kappa})\end{cases}$$

$$\boxed{\vec{\sigma}\cdot\hat{r}\,\chi^{\mu}_{\kappa} = -\chi^{\mu}_{-\kappa}} \quad \boxed{2} \rightarrow \boxed{\text{later proof}}$$

$$\begin{cases} E(g\chi^{\mu}_{\kappa}) = (m+W+g_v V^0)(g\chi^{\mu}_{\kappa}) + \left(-i\frac{\partial}{\partial r} + \frac{i(\kappa-1)}{r}\right)(-if\chi^{\mu}_{\kappa})\\ E(if\chi^{\mu}_{-\kappa}) = \left(-i\frac{\partial}{\partial r} - \frac{i(\kappa+1)}{r}\right)(-g\chi^{\mu}_{-\kappa}) - (m+W-g_v V^0)(if\chi^{\mu}_{-\kappa})\end{cases} \quad \text{--- (12.1)}$$

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(12.1) $\rightarrow$ $\quad f' = \frac{d}{dr} f(r), \quad g' = \frac{d}{dr} g(r)$

$$\begin{cases} E g = (m + W + g_v V^0)\, g + \left(-f' + \frac{K-1}{r} f\right) \\ E(if) = \left(i g' + i \frac{K+1}{r} g\right) - (m + W - g_v V^0)(if) \end{cases}$$

$$\begin{cases} (E - m - W - g_v V^0)\, g + f' - \frac{K-1}{r} f = 0 \\ g' + \frac{K+1}{r} g - (E + m + W - g_v V^0) f = 0 \end{cases}$$

$$\begin{cases} V_s \equiv m + W(r) \\ V_v \equiv g_v V^0(r) \end{cases} \qquad \begin{cases} V_s(r) \equiv m + W(r) \\ V_v(r) \equiv g_v V^0(r) \end{cases}$$

$\downarrow$

$$\boxed{\begin{aligned} &(E - V_s - V_v)\, g + f' - \frac{K-1}{r} f = 0 \\ &g' + \frac{K+1}{r} g - (E + V_s - V_v) f = 0 \end{aligned}}$$

— (13.1)
— (13.2)

$$\begin{cases} \psi^{\mu}_{K}(r) = \begin{pmatrix} g(r)\, \chi^{\mu}_{K}(\hat{r}) \\ i f(r)\, \chi^{\mu}_{-K}(\hat{r}) \end{pmatrix} \\ V_s(r) = V_s \equiv m + W(r) \\ V_v(r) = V_v \equiv g_v V^0(r) \end{cases}$$

(8.3) $\rightarrow$ $K = \pm (j + \frac{1}{2})$

— (13.3)

$$\boxed{\begin{aligned} &\left(\frac{d}{dr} + \frac{1-K}{r}\right) f(r) = -\left(E - V_s(r) - V_v(r)\right) g(r) \\ &\left(\frac{d}{dr} + \frac{1+K}{r}\right) g(r) = \left(E + V_s(r) - V_v(r)\right) f(r) \end{aligned}}$$

— (13.4)
— (13.5)

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(13.5) → $f = \frac{1}{E+V_s-V_v}\left(\frac{d}{dr}+\frac{1+\kappa}{r}\right)g = \frac{1}{E+V_s-V_v}\left(g'+\frac{1+\kappa}{r}g\right)$

$= f$

(13.4)

$$\left(\frac{d}{dr}+\frac{1-\kappa}{r}\right)\left\{\frac{1}{E+V_s-V_v}\left(\frac{d}{dr}+\frac{1+\kappa}{r}\right)g\right\} = -(E-V_s-V_v)\,g$$

$$\frac{d}{dr}\{\ \} = \frac{1}{(E+V_s-V_v)^2}\left\{\left[\frac{d}{dr}\left(\frac{d}{dr}+\frac{1+\kappa}{r}\right)g\right](E+V_s-V_v) - \left(\frac{d}{dr}+\frac{1+\kappa}{r}\right)g\,(V_s'-V_v')\right\}$$

$$= \frac{1}{E+V_s-V_v}\left(\frac{d^2}{dr^2}-\frac{1+\kappa}{r^2}+\frac{1+\kappa}{r}\frac{d}{dr}\right)g - \frac{(V_s'-V_v')}{(E+V_s-V_v)^2}\left(\frac{d}{dr}+\frac{1+\kappa}{r}\right)g$$

2.

$$\frac{1}{E+V_s-V_v}\left(\frac{d^2}{dr^2}-\frac{1+\kappa}{r^2}+\frac{1+\kappa}{r}\frac{d}{dr}\right)g - \frac{V_s'-V_v'}{(E+V_s-V_v)^2}\left(\frac{d}{dr}+\frac{1+\kappa}{r}\right)g + \frac{1-\kappa}{r}\cdot\frac{1}{E+V_s-V_v}\left(\frac{d}{dr}+\frac{1+\kappa}{r}\right)g = -(E-V_s-V_v)\,g$$

$$\left(\frac{d^2}{dr^2}-\frac{1+\kappa}{r^2}+\frac{1+\kappa}{r}\frac{d}{dr}+\frac{1-\kappa}{r}\frac{d}{dr}+\frac{1-\kappa^2}{r^2}\right)g + (E+V_s-V_v)(E-V_s-V_v)\,g - \frac{V_s'-V_v'}{E+V_s-V_v}\left(\frac{d}{dr}+\frac{1+\kappa}{r}\right)g = 0$$

$$\boxed{\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}-\frac{\kappa(\kappa+1)}{r^2}+\left[(E-V_v)^2-V_s^2\right]\right)g - \frac{V_s'-V_v'}{E+V_s-V_v}\left(\frac{d}{dr}+\frac{1+\kappa}{r}\right)g = 0}$$

— (14.1)

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(14.1) in the case

$V_s' = V_v' = 0$ (constant)

$$\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{K(K+1)}{r^2}\right)g = -\left[(E-V_v)^2 - V_s^2\right]g$$

$$\boxed{\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{K(K+1)}{r^2}\right)g = -\left[(E-V_v)^2 - V_s^2\right]g}$$

$$\left(\frac{d}{dr}V_s = \frac{d}{dr}V_v = 0\right) \qquad (15.1)$$

$\tilde{p} \equiv \left[(E-V_v)^2 - V_s^2\right]^{1/2}$, $\tilde{p}^2 = (E-V_v)^2 - V_s^2$

$z = \tilde{p}r$ $\qquad \frac{d}{dr} = \frac{\partial z}{\partial r}\frac{d}{dz} = \tilde{p}\frac{d}{dz}$

$r = \frac{z}{\tilde{p}}$ $\qquad \frac{d^2}{dr^2} = \left(\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right)\right)\frac{d}{dz} + \left(\frac{\partial z}{\partial r}\right)^2\frac{d^2}{dz^2}$

$$= \tilde{p}^2\frac{d^2}{dz^2}$$

$$\left(\tilde{p}^2\frac{d^2}{dz^2} + \frac{2}{\left(\frac{z}{\tilde{p}}\right)}\tilde{p}\frac{d}{dz} - \frac{K(K+1)}{\left(\frac{z}{\tilde{p}}\right)^2}\right)g = -\tilde{p}^2 g$$

$$\left(\frac{d^2}{dz^2} + \frac{2}{z}\frac{d}{dz} - \frac{K(K+1)}{z^2} + 1\right)g(z) = 0$$

$$\begin{cases} \tilde{p} = \left[(E-V_v)^2 - V_s^2\right]^{1/2} \\ z = \tilde{p}r, \end{cases}$$

$$\cancel{\left(\frac{d^2}{dz^2} + \frac{2}{z}\frac{d}{dz}\right.}$$

$(V_s' = V_v' = 0)$

$$\boxed{\left(\frac{d^2}{dz^2} + \frac{2}{z}\frac{d}{dz} + 1 - \frac{K(K+1)}{z^2}\right)g(z) = 0}$$

!! Spherical Bessel diff. equation !!! $\qquad (15.2)$

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MITB-16

• no current flow through the surface of the confining region.

in the MIT bag model, $\psi_\kappa^\mu(\vec r) \equiv q(\vec r)$ in the static spherical case $\quad n_\mu \bar q \gamma^\mu q = 0 \quad$ — (16.1)

n_μ; a unit four vector normal to the surface of the confining region

(16.1) is imposed through a linear boundary condition

$$\boxed{i\gamma\cdot n\, q = q} \quad \text{— (16.2)}$$

$$q^\dagger = -i q^\dagger \gamma^\dagger\cdot n \rightarrow \bar q = -i \bar q \gamma\cdot n \qquad (\gamma^0\gamma^0 = 1)$$

$$(q^\dagger\gamma^0 = \bar q = -i q^\dagger \gamma^0\gamma^0\gamma^\dagger\cdot n \gamma^0)$$

$$i n_\mu j^\mu = i n_\mu \bar q \gamma^\mu q$$

$$= (\bar q\, i\gamma\cdot n) q = (-\bar q) q$$

$$= \bar q (i\gamma\cdot n q) = \bar q (q) \qquad \therefore -\bar q q = \bar q q = 0$$

$$\therefore \boxed{i n_\mu j^\mu = 0}$$

$n^\mu = (0, \hat r)$ at $r = R$ (bag surface)

(16.2)

$$i\gamma\cdot n\, \psi_\kappa^\mu(\vec r) = \psi_\kappa^\mu(\vec r) \qquad \text{at } |\vec r| = R$$

$$-i\hat r\cdot\vec\gamma\, \psi_\kappa^\mu(\vec r) = \psi_\kappa^\mu(\vec r) \qquad \vec\gamma = \begin{pmatrix} 0 & \vec\sigma \\ -\vec\sigma & 0 \end{pmatrix}$$

$$-i\begin{pmatrix} 0 & \vec\sigma\cdot\hat r \\ -\vec\sigma\cdot\hat r & 0 \end{pmatrix}\begin{pmatrix} g(r)\chi_\kappa^\mu(\hat r) \\ i f(r)\chi_{-\kappa}^\mu(\hat r) \end{pmatrix} = \begin{pmatrix} g(r)\chi_\kappa^\mu(\hat r) \\ i f(r)\chi_{-\kappa}^\mu(\hat r) \end{pmatrix} \quad \text{— (16.3)}$$

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$$(16.3) \longrightarrow \begin{cases} \vec{\sigma}\cdot\hat{r}\, f\, \chi_{-\kappa}^{\mu} = g\, \chi_{\kappa}^{\mu} \\ i\vec{\sigma}\cdot\hat{r}\, g\, \chi_{\kappa}^{\mu} = i\, f\, \chi_{-\kappa}^{\mu} \end{cases} \qquad \vec{\sigma}\cdot\hat{r}\, \chi_{\kappa}^{\mu} = -\chi_{-\kappa}^{\mu}$$

$$\begin{cases} -f\, \chi_{\kappa}^{\mu} = g\, \chi_{\kappa}^{\mu} \\ -i\, g\, \chi_{-\kappa}^{\mu} = i\, f\, \chi_{-\kappa}^{\mu} \end{cases} \qquad \begin{matrix} -f(r) = g(r) & \text{at } r = R \\ -g(r) = f(r) & \text{at } r = R \end{matrix}$$

$$\therefore \boxed{f(r) = -g(r) \quad \text{at } r = R} \quad -(17.1)$$

$$(15.2) \longrightarrow \boxed{\left(\frac{d^2}{dz^2} + \frac{2}{z}\frac{d}{dz} + 1 - \frac{\kappa(\kappa+1)}{z^2}\right) g(z) = 0} \quad -(17.2)$$

$$\begin{cases} \kappa > 0 \quad \text{case} \quad l \equiv l_\kappa = \kappa \\ \kappa < 0 \quad \text{case} \quad l \equiv l_\kappa = -\kappa - 1, \quad \kappa = -(l+1) \end{cases}$$

$$\begin{cases} \kappa > 0, \ \kappa(\kappa+1) = l(l+1) = l_\kappa(l_\kappa+1) \\ \kappa < 0, \ \kappa(\kappa+1) = -(l+1)(-(l+1)+1) = l(l+1) = l_\kappa(l_\kappa+1) \end{cases}$$

MITB − 15 $\Rightarrow$

$$\tilde{p} = [(E - V_v)^2 - V_s^2]^{1/2}, \qquad z = \tilde{p}\, r$$

so we may consider the same way as in vacuum

as $p = [E^2 - m^2]^{1/2}, \quad z = pr$ $\quad \begin{pmatrix} V_s = m + W(r) \\ V_v = g_v V^0(r) \end{pmatrix}$

$$\boxed{E + V_s - V_v \Rightarrow E + m}$$

and $W(r) = W$, $V^0(r) = V^0$ constant !!

(17.2) $\longrightarrow$ Spherical Bessel function, regular at $z = 0$

$$\kappa(\kappa+1) = l_\kappa(l_\kappa+1)$$

$$\boxed{g(z) = N_{l_\kappa}\, j_{l_\kappa}(z) = N_{l_\kappa}\, j_{l_\kappa}(\tilde{p}\, r)} \quad -(17.3)$$

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(13.5) $\rightarrow$ $\left(\frac{d}{dr} + \frac{1+\kappa}{r}\right) g(r) = (E+m) f(r)$

$\kappa > 0$, $\kappa = l_\kappa$

$$f(r) = \frac{1}{E+m}\left(\frac{d}{dr} + \frac{1+l_\kappa}{r}\right) g(r)$$

$$f(z) = \frac{p}{E+m}\left(\frac{d}{dz} + \frac{l_\kappa+1}{z}\right)\left(N_{l_\kappa} j_{l_\kappa}(z)\right)$$

sign $\kappa > 0$

$$= +N_{l_\kappa}\frac{p}{E+m} j_{l_\kappa - 1}(z). \qquad (18.1)$$

$z = pr$

$\kappa < 0$, $\kappa = -(l_\kappa + 1)$

$$f(z) = \frac{p}{E+m}\left(\frac{d}{dz} - \frac{l_\kappa}{z}\right)\left(N_{l_\kappa} j_{l_\kappa}(z)\right)$$

sign $\kappa < 0$

$$= -N_{l_\kappa}\frac{p}{E+m} j_{l_\kappa + 1}(z) \qquad (18.2)$$

$$\left(\frac{p}{E+m} = \sqrt{\frac{p^2}{(E+m)^2}} = \sqrt{\frac{E^2-m^2}{(E+m)^2}} = \sqrt{\frac{(E+m)(E-m)}{(E+m)^2}} = \sqrt{\frac{E-m}{E+m}}\right)$$

(17.1) $\rightarrow$ boundary condition at $r = R$, $g(R) = -f(R)$
$z = pr$

$$\begin{cases} \kappa > 0 & j_{l_\kappa}(pR) = -\sqrt{\dfrac{E-m}{E+m}}\, \underbrace{j_{l_\kappa - 1}}_{= l_{-\kappa}}(pR) \\ \kappa < 0 & j_{l_\kappa}(pR) = +\sqrt{\dfrac{E-m}{E+m}}\, \underbrace{j_{l_\kappa + 1}}_{= l_{-\kappa}}(pR) \end{cases}$$

$$\beta_q \equiv \sqrt{\frac{E-m}{E+m}},$$

$\chi_{-\kappa}^{\mu}(\hat{r}) = -\vec{\sigma}\cdot\hat{r}\,\chi_{\kappa}^{\mu}(\hat{r})$

$$\therefore \quad j_{l_\kappa}(pR) = -(\text{sign}\,\kappa)\beta_q\, j_{l_{-\kappa}}(pR)$$

$$\psi_{n\kappa}^{\mu}(\vec{r}) = N_{n\kappa\mu}\begin{pmatrix} g(r)\,\chi_\kappa^\mu(\hat{r}) \\ i f(r)\,\chi_{-\kappa}^\mu(\hat{r}) \end{pmatrix} = N_{n\kappa\mu}\begin{pmatrix} j_{l_\kappa}(pr)\,\chi_\kappa^\mu(\hat{r}) \\ +i(\text{sign}\,\kappa)\,\beta_q\, j_{l_{-\kappa}}(pr)\,\chi_{-\kappa}^\mu(\hat{r}) \end{pmatrix}$$

$$= N_{n\kappa\mu}\begin{pmatrix} j_{l_\kappa}(pr)\,\chi_\kappa^\mu(\hat{r}) \\ +i(\text{sign}\,\kappa)\,\beta_q\, j_{l_{-\kappa}}(pr)\left(-\vec{\sigma}\cdot\hat{r}\,\chi_\kappa^\mu(\hat{r})\right) \end{pmatrix}$$

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$\therefore$

$$\psi^{\mu}_{n\kappa}(\vec{r}) = N_{n\kappa\mu}\begin{pmatrix} j_{l_\kappa}(pr) \\ -i(\mathrm{sign}\,\kappa)\,\beta_q\,\vec{\sigma}\cdot\hat{r}\,j_{l_{-\kappa}}(pr)\end{pmatrix}\chi^{\mu}_{\kappa}(\hat{r})$$

For the ground state quark $\quad \kappa = -1, \quad l_\kappa = -\kappa + 1 = 0$ (S-state)

$$\hat{\kappa} \equiv \mathrm{sign}\,\kappa \;(= -1) \qquad l_{-\kappa} = l_\kappa + 1 = 1$$

$$\psi^{\mu}_{n\kappa}(\vec{r}) = N_{n\kappa\mu}\begin{pmatrix} j_{l_\kappa}(pr) \\ -i\hat{\kappa}\,\beta_q\,\vec{\sigma}\cdot\hat{r}\,j_{l_{-\kappa}}(pr)\end{pmatrix}\chi^{\mu}_{\kappa}(\hat{r})$$

$\kappa = -1$

$$\chi^{\mu}_{-1}(\hat{r}) = \sum_{m,m_s}\langle 0\,\tfrac{1}{2}\,0\,m_s|\tfrac{1}{2}\,\mu\rangle\, Y_{00}(\hat{r})\,\chi_{m_s} = \frac{1}{\sqrt{4\pi}}\chi_\mu \qquad (\mu = \pm\tfrac{1}{2})$$

$$\psi_{n\,-1}(\vec{r}) = N_{n,-1\mu}\begin{pmatrix} j_0(pr) \\ i\beta_q\,\vec{\sigma}\cdot\hat{r}\;j_1(pr)\end{pmatrix}\frac{\chi_\mu}{\sqrt{4\pi}}$$

$n = 0$

$|||$

$\psi_q(\vec{r})$ ground state quark wave function

$$\psi_q(\vec{r}) = N_q\begin{pmatrix} j_0(pr) \\ i\beta_q\,\vec{\sigma}\cdot\hat{r}\,j_1(pr)\end{pmatrix}\frac{\chi_\mu}{\sqrt{4\pi}} \qquad (\mu = \pm\tfrac{1}{2})$$

boundary condition

$$j_0(pR) = \beta_q\, j_1(pR)$$

In medium $\quad p \to \tilde{p}, \quad \beta_q \to \beta_q^*, \quad E \to E^* \qquad$ — (19.1)

$$\tilde{p} = (E^{*2} - M_q^{*2})^{1/2} = \left((E - V_v)^2 - (m_q - V_s)^2\right)^{1/2}$$

$$E^* = \sqrt{M_q^{*2} + \vec{p}^{\,2}}$$

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$$\boxed{\begin{array}{l} \hat{K} \equiv \operatorname{sign} K \\ \psi_{nK}^{\mu}(\vec{r}) = N_{nK\mu} \begin{pmatrix} j_{l_K}(pr) \\ -i\hat{K}\beta_q \vec{\sigma}\cdot\hat{r}\, j_{l_{-K}}(pr) \end{pmatrix} \mathcal{X}_K^{\mu}(\hat{r}) \\ \beta_q = \left(\frac{E-m}{E+m}\right)^{1/2}, \qquad K>0 \begin{cases} l_K = K = j+1/2 \\ l_{-K} = K-1 = j-1/2 \end{cases} \\ \qquad\qquad\qquad\qquad\quad K<0 \begin{cases} l_K = -K-1 = j-1/2 \\ l_{-K} = -K = j+1/2 \end{cases} \\ \mathcal{X}_K^{\mu}(\hat{r}) = \sum_{m,m_s} \langle l_K \tfrac{1}{2} m\, m_s | j \mu \rangle Y_{l_K m}(\hat{r}) \mathcal{X}_{m_s} \end{array}} \quad \text{—— (20.1)}$$

Normalization R; bag radius

$$1 = \int_0^R d^3r\, \psi_{nK}^{\mu\dagger}(\vec{r})\, \psi_{nK}^{\mu}(\vec{r}) \qquad \vec{\sigma}\cdot\hat{r}\,\vec{\sigma}\cdot\hat{r} = 1$$

$$= N_{nK\mu}^2 \int_0^R d^3r\, \mathcal{X}_K^{\mu\dagger}(\hat{r}) \left[j_{l_K}^2(pr) + \hat{K}^2 \beta_q^2 j_{l_{-K}}^2(pr) \right] \mathcal{X}_K^{\mu}(\hat{r})$$

$$= N_{nK\mu}^2 \int d\Omega \sum_{\substack{m\, m_s \\ m'\, m_s'}} \mathcal{X}_{m_s'}^{\dagger} \mathcal{X}_{m_s} Y_{l_K m'}^{*}(\hat{r}) Y_{l_K m} \quad \leftarrow \delta_{m_s' m_s} \delta_{m' m} \; \sum_{\substack{m, m_s \\ m', m_s'}}$$

$$\cdot \int_0^R dr\, r^2 \left[j_{l_K}^2(pr) + \beta_q^2 j_{l_{-K}}^2(pr) \right]$$

$$= N_{nK\mu}^2 \int_0^R dr\, r^2 \left[j_{l_K}^2(pr) + \beta_q^2 j_{l_K}^2(pr) \right] \quad \text{—— (20.2)}$$

* ground state $K = -1$ $l_K = 0$, $l_{-K} = 1$
S state P state

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(19.1) → ground state wave function

(20.2) →

$$1 = N_q^2 \int_0^R dr\, r^2 \left[j_0^2(pr) + \beta_q^2 j_1^2(pr) \right] \qquad \text{—— (21.1)}$$

$$|\vec{p}| = p = \frac{x}{R}, \quad z = pr = \frac{xr}{R}$$

$$E = (p^2+m^2)^{1/2} = \left(\frac{x^2}{R^2} + m^2\right)^{1/2} = \frac{1}{R}\left(x^2 + (mR)^2\right)^{1/2}$$

$$\equiv \frac{1}{R}\Omega \qquad \Omega \equiv \left(x^2 + (mR)^2\right)^{1/2}$$

$$z = \frac{x}{R} r, \quad dz = \frac{x}{R} dr$$

$$r = \frac{R}{x} z, \quad dr = \frac{R}{x} dz$$

r	$0 \longrightarrow R$
z	$0 \longrightarrow x$

$$\beta_q^2 = \frac{E-m}{E+m} = \frac{\frac{1}{R}\Omega - m}{\frac{1}{R}\Omega + m} = \frac{\Omega - mR}{\Omega + mR}$$

$$= N_q^2 \frac{R^3}{x^3} \int_0^x dz\, z^2 \left[j_0^2(z) + \beta_q^2 j_1^2(z) \right]$$

$\vdots$

$$\boxed{N_q^{-2} = 2R^3 j_0^2(x) \left[\Omega(\Omega - 1) + \frac{1}{2} mR \right] \Big/ x^2}$$

(with using, $j_0(x) = \beta_q j_1(x) = \sqrt{\frac{\Omega - mR}{\Omega + mR}}\, j_1(x)$)

(generally) ("Quantum Chromodynamics", Greiner, Schramm, Stein) Springer —— (21.2)

$$1 = N_{n\kappa\mu}^2 \frac{R^3}{x^3} \int_0^x dz\, z^2 \left[j_{l_\kappa}^2(z) + \beta_q^2 j_{l_{-\kappa}}^2(z) \right]$$

with $j_{l_\kappa}(x) = -\hat{\kappa}\, \beta_q\, j_{l_{-\kappa}}(x)$

$$N_{n\kappa\mu}^{-2} = 2R^3 j_{l_\kappa}^2(x) \left[\Omega(\Omega + \kappa) + \frac{1}{2} mR \right] \Big/ x^2$$

$\kappa < 0,\ l_\kappa = -(\kappa+1), \qquad \kappa > 0,\ l_\kappa = \kappa$

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Appendix

MIT bag model

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$\rightarrow$ $\boxed{1}$ (MITB-9) ~~***~~ $\boxed{\text{Appendix}}$

$\boxed{\chi^{\mu}_{\kappa}(\hat{r}),\ \chi^{\mu}_{-\kappa}(\hat{r})}$

$\boxed{1}$

MITB-~~3~~ ~~A~~ A1

$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\beta^2 = 1$

(2.2), (2.3) $\Rightarrow$

$$K = \beta(\vec{\sigma}\cdot\vec{l} + 1),$$
$$= \begin{pmatrix} \vec{\sigma}\cdot\vec{l}+1 & 0 \\ 0 & -(\vec{\sigma}\cdot\vec{l}+1) \end{pmatrix} \equiv \begin{pmatrix} \hat{k} & 0 \\ 0 & -\hat{k} \end{pmatrix}$$

$$\hat{k} = \vec{\sigma}\cdot\vec{l} + 1$$

$$\psi^{\mu}_{\kappa}(\vec{r}) = \begin{pmatrix} g(r)\,\chi^{\mu}_{\kappa}(\hat{r}) \\ if(r)\,\chi^{\mu}_{-\kappa}(\hat{r}) \end{pmatrix}$$

$$\psi \equiv \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} \equiv \psi^{\mu}_{\kappa}(\vec{r})$$

$$\hat{k} = \vec{\sigma}\cdot\vec{l} + 1$$

$$K\psi = -\kappa\psi \rightarrow \begin{pmatrix} \hat{k} & 0 \\ 0 & -\hat{k} \end{pmatrix}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = -\kappa\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

$$\begin{cases} \hat{k}\,\psi_A = -\kappa\,\psi_A \\ -\hat{k}\,\psi_B = -\kappa\,\psi_B \end{cases}; \quad \begin{cases} \hat{k}\psi_A = -\kappa\psi_A \\ \hat{k}\psi_B = \kappa\psi_B \end{cases}$$

$$\hat{k} = \vec{\sigma}\cdot\vec{l} + 1, \qquad \vec{j} = (\vec{l} + \tfrac{1}{2}\vec{\sigma})$$
$$\vec{j}^{\,2} = (\vec{l} + \tfrac{1}{2}\vec{\sigma})^2 = (\vec{l}^{\,2} + \tfrac{1}{4}\vec{\sigma}^{\,2} + \vec{\sigma}\cdot\vec{l} + 1 - 1)$$
$$= (\vec{l}^{\,2} + \tfrac{3}{4} + \hat{k} - 1)$$
$$= \vec{l}^{\,2} - \tfrac{1}{4} + \hat{k}$$

$$\therefore \boxed{\begin{aligned} \hat{k} &= \vec{j}^{\,2} - \vec{l}^{\,2} + \tfrac{1}{4} \\ &= j(j+1) - l(l+1) + \tfrac{1}{4} \end{aligned}} \qquad \text{(A1.1)}$$

~~(2.6.7)~~

$$K^2 = \beta(\vec{\sigma}\cdot\vec{l}+1)\,\beta(\vec{\sigma}\cdot\vec{l}+1) \qquad \text{(2.2) !!}$$
$$= \vec{l}^{\,2} + \vec{\sigma}\cdot\vec{l} + 1 = \vec{j}^{\,2} + \tfrac{1}{4} = j(j+1) + \tfrac{1}{4} = (j + \tfrac{1}{2})^2$$

$$\boxed{K = \pm(j + \tfrac{1}{2})}$$

$$K\psi = -\kappa\psi \iff \begin{cases} \pm(j+\tfrac{1}{2})\,\psi = -\kappa\psi \\ \begin{pmatrix} \hat{k} & 0 \\ 0 & -\hat{k} \end{pmatrix}\psi = -\kappa\psi \end{cases} \qquad \psi \equiv \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

$$\therefore \boxed{\begin{pmatrix} \hat{k} & 0 \\ 0 & -\hat{k} \end{pmatrix}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \pm(j+\tfrac{1}{2})\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}, \quad \hat{k} = j(j+1) - l(l+1) + \tfrac{1}{4}}$$

~~(2.8)~~
(A1.2)

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$$\psi \equiv \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix},$$

$$K\psi = \begin{pmatrix} \hat{k} & 0 \\ 0 & -\hat{k} \end{pmatrix}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = -\kappa\psi = -\kappa\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

$$\overset{(16.1)}{\hat{k}} = j(j+1) - l(l+1) + \frac{1}{4} \qquad = -\left\{\pm\left(j+\frac{1}{2}\right)\right\}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

$$\begin{pmatrix} j(j+1) - l_A(l_A+1) + \frac{1}{4} & 0 \\ 0 & -\left\{j(j+1) - l_B(l_B+1) + \frac{1}{4}\right\} \end{pmatrix}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

$$= -\left\{\pm\left(j+\frac{1}{2}\right)\right\}\begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}$$

$$\left[\begin{aligned} &\left\{j(j+1) - l_A(l_A+1) + \frac{1}{4}\right\}\psi_A = -\left\{\pm\left(j+\frac{1}{2}\right)\right\}\psi_A = -\kappa\psi_A \\ &-\left\{j(j+1) - l_B(l_B+1) + \frac{1}{4}\right\}\psi_B = -\left\{\pm\left(j+\frac{1}{2}\right)\right\}\psi_B = -\kappa\psi_B \end{aligned}\right.$$

—— (A2.1)

① $\kappa = -\left(j+\frac{1}{2}\right)$ case

$$\left\{\begin{aligned} &j(j+1) - l_A(l_A+1) + \frac{1}{4} = j + \frac{1}{2} \\ &-\left\{j(j+1) - l_B(l_B+1) + \frac{1}{4}\right\} = j + \frac{1}{2} \end{aligned}\right.$$

$\uparrow\downarrow$

$$\left\{\begin{aligned} &l_A(l_A+1) = j(j+1) + \frac{1}{4} - \left(j+\frac{1}{2}\right) = j^2 - \frac{1}{4} = \left(j-\frac{1}{2}\right)\left(j+\frac{1}{2}\right) \\ &l_B(l_B+1) = j(j+1) + \frac{1}{4} + \left(j+\frac{1}{2}\right) = j^2 + 2j + \frac{3}{4} = \left(j+\frac{1}{2}\right)\left(j+\frac{3}{2}\right) \end{aligned}\right.$$

$$\therefore l_A = j - \frac{1}{2}, \quad l_B = j + \frac{1}{2}$$

$$\kappa = -\left(j+\frac{1}{2}\right), \longrightarrow \underline{l_A = -\kappa - 1, \quad l_B = -\kappa}$$

$$\left\{\begin{aligned} &l_A \equiv l_\kappa = -\kappa - 1 = j - \frac{1}{2} \\ &l_B \equiv l_{-\kappa} = -\kappa = j + \frac{1}{2} \end{aligned}\right.$$

—— (A2.2)

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② $\kappa = +(j+\frac{1}{2})$ case

$$\begin{cases} l_A(l_A+1) = j(j+1) + \frac{1}{4} + (j+\frac{1}{2}) = (j+\frac{1}{2})(j+\frac{3}{2}) \\ l_B(l_B+1) = j(j+1) + \frac{1}{4} - (j+\frac{1}{2}) = (j-\frac{1}{2})(j+\frac{1}{2}) \end{cases}$$

$$\begin{cases} l_A \equiv l_\kappa = j+\frac{1}{2} = \kappa \\ l_B \equiv l_{-\kappa} = j-\frac{1}{2} = \kappa - 1 \end{cases} \quad \text{—— } (A3.1)$$

(17.1), (18.1) $\leftrightarrow$ $\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} = \begin{pmatrix} g(r)\, \mathcal{X}^\mu_\kappa(\hat{r}) \\ i f(r)\, \mathcal{X}^\mu_{-\kappa}(\hat{r}) \end{pmatrix}$

$$\boxed{\begin{aligned} &\kappa = \pm(j+\tfrac{1}{2}) \\ &\kappa > 0, \quad \begin{cases} l_\kappa = \kappa = j+\frac{1}{2} \\ l_{-\kappa} = \kappa - 1 = j - \frac{1}{2} \end{cases} \\ &\kappa < 0, \quad \begin{cases} l_\kappa = -\kappa - 1 = j - \frac{1}{2} \\ l_{-\kappa} = -\kappa = j+\frac{1}{2} \end{cases} \end{aligned}}$$

l_κ, $l_{-\kappa}$ $\rightleftarrows$ $\begin{pmatrix} \sim l_\kappa \\ \sim l_{-\kappa} \end{pmatrix}$

$$\begin{pmatrix} \mathcal{X}^\mu_\kappa(\hat{r}) \rightleftarrows l_\kappa \\ \mathcal{X}^\mu_{-\kappa}(\hat{r}) \rightleftarrows l_{-\kappa} \end{pmatrix}$$

$$\boxed{l_\kappa - l_{-\kappa} = \text{sign}\, \kappa} \quad !! \text{ —— } (A3.2)$$

* l_κ, $l_{-\kappa}$

$\rightleftarrows$ $\langle l_\kappa \tfrac{1}{2}\, m\, m_s | j \mu \rangle$

$$\begin{cases} \mathcal{X}^\mu_\kappa(\hat{r}) \equiv \sum_{m, m_s} (l_\kappa \tfrac{1}{2}\, m\, m_s | j \mu)\, Y^m_{l_\kappa}(\hat{r})\, \mathcal{X}_{m_s} \\ \mathcal{X}^\mu_{-\kappa}(\hat{r}) = \sum_{m', m_s'} (l_{-\kappa} \tfrac{1}{2}\, m'\, m_s' | j \mu)\, Y^{m'}_{l_{-\kappa}}(\hat{r})\, \mathcal{X}_{m_s'} \end{cases} \quad \left\{ \mathcal{X}_{1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \mathcal{X}_{-1/2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad \text{—— } (A3.2)$$

or $\boxed{\begin{matrix} Y_{l_\kappa m}(\hat{r}) \\ Y_{l_{-\kappa} m'}(\hat{r}) \end{matrix}}$ Spherical harmonics

$\boxed{1}$

2 → 2 (MITB-22) $\chi_\kappa^\mu(\hat r)$

$\bigstar$ $\vec\sigma\cdot\hat r\,\chi_\kappa^\mu(\hat r) = -\chi_{-\kappa}^\mu(\hat r)$

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method No. 1

MITB - A4

$\boxed{\text{Wigner-Eckart theorem}}$

$$\langle \tau J M | T^{(k)}_q | \tau' J' M' \rangle = \frac{(-1)^{2k}}{\sqrt{2J+1}} \langle \tau J \| T^{(k)} \| \tau' J' \rangle \underbrace{\langle J' k M' q | J M \rangle}$$

$$\langle J' k M' q | J M \rangle = \sqrt{2J+1}\,\left((-1)^{J'-k+M}\right)^{-1} \begin{pmatrix} J' & k & J \\ M' & q & -M \end{pmatrix}$$

$$= \sqrt{2J+1}\,(-1)^{-J'+k-M} \times (-1)^{J'+k+J} \begin{pmatrix} J & k & J' \\ -M & q & M' \end{pmatrix}$$

$$= \sqrt{2J+1}\,(-1)^{2k+J-M} \begin{pmatrix} J & k & J' \\ -M & q & M' \end{pmatrix}$$

$$= \frac{(-1)^{2k}}{\sqrt{2J+1}} \cdot \langle \tau J \| T^{(k)} \| \tau' J' \rangle \cdot \sqrt{2J+1}\,(-1)^{2k+J-M} \begin{pmatrix} J & k & J' \\ -M & q & M' \end{pmatrix}$$

$$= (-1)^{4k+J-M} \begin{pmatrix} J & k & J' \\ -M & q & M' \end{pmatrix} \langle \tau J \| T^{(k)} \| \tau' J' \rangle$$

$(-1)^{4k} = 1$!!

$$= (-1)^{J-M} \underbrace{\begin{pmatrix} J & k & J' \\ -M & q & M' \end{pmatrix}}_{\text{3-j symbol}} \langle \tau J \| T^{(k)} \| \tau' J' \rangle$$

$$\therefore \boxed{\langle \tau J M | T^{(k)}_q | \tau' J' M' \rangle = (-1)^{J-M} \begin{pmatrix} J & k & J' \\ -M & q & M' \end{pmatrix} \langle \tau J \| T^{(k)} \| \tau' J' \rangle} \quad \text{(A4.1)}$$

$\bigstar$ $A_{\pm 1} = \mp \frac{1}{\sqrt 2}(A_x \pm i A_y)$, $A_0 = A_z$ $\Longrightarrow$ A_μ $(\mu = \pm 1, 0)$ or A_m $(m = \pm 1, 0)$

$$\vec A \cdot \vec B = A_x B_x + A_y B_y + A_z B_z$$

$$= (-1)^m A_m B_{-m} = (-1)^1 A_1 B_{-1} + (-1)^0 A_0 B_{-0} + (-1)^{-1} A_{-1} B_1$$

$$= (-1)\frac{-1}{\sqrt 2}(A_x + iA_y)\frac{1}{\sqrt 2}(B_x - iB_y) + A_z B_z + (-1)\frac{1}{\sqrt 2}(A_x - iA_y)\frac{-1}{\sqrt 2}(B_x + iB_y)$$

Below $\bigstar$ Admit !!! to use !!!

$\bigstar$ $(\vec T^{(k)} \cdot \vec U^{(k)}) = (-1)^k \sqrt{2k+1}\,[\vec T^{(k)} \otimes \vec U^{(k)}]^{(0)}_0$

$k=1$ $\bigstar$ $\vec A \cdot \vec B = (-1)^1 \sqrt 3\,[A^{(1)} \otimes B^{(1)}]^{(0)}_0 = -\sqrt 3\,[A^{(1)} \otimes B^{(1)}]^{(0)}_0$

$\bigstar\bigstar\bigstar$

$\bigstar$ angular momentum tensor algebra !! ($Y_{lm}(\hat r)$ rank l tensor)

→ (nuclear structure calculations ⟺ popular)

$Y_{lm}(\hat r)$ is a rank l tensor !!

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$\chi_{m_s} \equiv \chi_{\frac12 m_s}$ MITB - A5

$$\chi_\kappa^\mu(\hat r) = \sum_{m,m_s} \langle l\ \tfrac12\ m\ m_s | j\ \mu\rangle Y_{lm}(\hat r)\chi_{m_s}$$

$\boxed{\vec\sigma\cdot\hat r\ \chi_\kappa^\mu = -\chi_{-\kappa}^\mu}$ → proof below (method 1)

$$\vec\sigma\cdot\hat r\ \chi_\kappa^\mu = \sum_{\kappa',\mu'} \chi_{\kappa'}^{\mu'} \underbrace{\chi_{\kappa'}^{\mu'\dagger}\vec\sigma\cdot\hat r\ \chi_\kappa^\mu}_{=\langle l'\frac12 j'\mu'|\vec\sigma\cdot\hat r|l\frac12 j\mu\rangle}$$

expand by complete set

$$= \sum_{\kappa'\mu'} \chi_{\kappa'}^{\mu'} \langle l'\tfrac12 j'\mu'|\vec\sigma\cdot\hat r|l\tfrac12 j\mu\rangle \qquad \text{——— (A5.1)}$$

from nuclear structure, angular momentum matrix element

please admit to use below

(⟹ method 2 may be more understandable!)

$\langle l'\frac12 j'\mu'|\vec\sigma\cdot\hat r|l\frac12 j\mu\rangle$ $\boxed{l'\equiv l_{\kappa'},\ l\equiv l_\kappa}$!!

$$= (-1)^{\frac12+j'+l}\begin{Bmatrix} l' & \frac12 & j' \\ \frac12 & l & 1\end{Bmatrix}\langle l'\|\hat r\|l\rangle\langle\tfrac12\|\vec\sigma\|\tfrac12\rangle\delta_{jj'}\delta_{\mu\mu'}$$

$Y_{1m} = \sqrt{\frac{3}{4\pi}}\,\hat r_m^{(1)}$

$\hat r^{(1)}$; rank 1 tensor

$\vec\sigma$; rank 1 tensor

$\sigma_0 = \sigma_z$

(A4.1) ⟼ $\langle\frac12\frac12|\sigma_0|\frac12\frac12\rangle = (-1)^{\frac12-\frac12}\begin{pmatrix}\frac12 & 1 & \frac12\\ -\frac12 & 0 & \frac12\end{pmatrix}\langle\frac12\|\vec\sigma\|\frac12\rangle$

$$1 = (-1)^{\frac12-\frac12}\cdot\frac{1}{\sqrt6}\langle\tfrac12\|\vec\sigma\|\tfrac12\rangle$$

$Y_1 = \sqrt{\frac{3}{4\pi}}\,\hat r^{(1)}$ $\quad\therefore\ \boxed{\langle\frac12\|\vec\sigma\|\frac12\rangle = \sqrt6}$

$$\langle l'\|Y_1\|l\rangle = (-1)^{l'}\sqrt{\frac{(2l'+1)(2\cdot1+1)(2l+1)}{4\pi}}\begin{pmatrix}l' & 1 & l\\ 0 & 0 & 0\end{pmatrix}$$

$$= \sqrt{\frac{3}{4\pi}}(-1)^{l'}\sqrt{(2l'+1)(2l+1)}\begin{pmatrix}l' & 1 & l\\ 0 & 0 & 0\end{pmatrix}$$

$$\boxed{\langle l'\|\hat r\|l\rangle = (-1)^{l'}\sqrt{(2l'+1)(2l+1)}\begin{pmatrix}l' & 1 & l\\ 0 & 0 & 0\end{pmatrix}}$$

$$= \boxed{(-1)^{\frac12+j'+l}\begin{Bmatrix} l' & \frac12 & j' \\ \frac12 & l & 1\end{Bmatrix}\cdot(-1)^{l'}\sqrt{(2l'+1)(2l+1)}\begin{pmatrix}l' & 1 & l\\ 0 & 0 & 0\end{pmatrix}\cdot\sqrt6\,\delta_{jj'}\delta_{\mu\mu'}}$$

——— (A5.2)

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MITB–A6

(A5.2) → $(l' = l'_{\kappa'},\ l = l_\kappa)$

① $\kappa > 0$ case → $l = l_\kappa = j + 1/2$ ← (A3.2)

$\begin{pmatrix} l' & 1 & l \\ 0 & 0 & 0 \end{pmatrix} \neq 0 \longleftrightarrow l' + 1 + l$ even

$l'_{\kappa'} = j' \pm \frac{1}{2} = j \pm \frac{1}{2} \quad (j' = j)$

$= (j + 1/2) \pm \begin{smallmatrix} 0 \\ 1 \end{smallmatrix}$

$= \begin{matrix} l & = (l_\kappa) \\ l-1 & = (l_\kappa - 1) \end{matrix}$

$\therefore l'_{\kappa'} = l$ or $l-1$ (l_κ or $l_\kappa - 1$)

(A3.2)

$l'_{\kappa'} + 1 + l_\kappa$ even $\longleftrightarrow l'_{\kappa'} = l_\kappa - 1$

$= j - 1/2 = l_{-\kappa}$

$l' = l - 1 = j - 1/2 = j' - 1/2 \quad \therefore \boxed{\kappa' = -\kappa}$ —— (A6.1)

② $\kappa < 0$ case → $l_\kappa = l = -\kappa - 1 = j - 1/2$ ← (A3.2)

$\begin{pmatrix} l' & 1 & l \\ 0 & 0 & 0 \end{pmatrix} \neq 0 \longleftrightarrow l' + 1 + l$ even

$l' = j' \pm 1/2 = j \pm 1/2 \quad (j = j')$

$l = l_\kappa = j - 1/2, \quad l' = l'_{\kappa'} = \begin{cases} j + 1/2 = l + 1 = l_\kappa + 1 \\ j - 1/2 = l = l_\kappa \end{cases}$

$\therefore \underset{l'_{\kappa'}}{l'} = l + 1 = (j - 1/2) + 1 = j + 1/2 = l_{-\kappa}$

$\therefore \boxed{\kappa' = -\kappa}$

$l' = l + 1 = j + 1/2 = j' + 1/2$ —— (A6.2)

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$$\begin{cases} ① \ (A6.1), \quad l' = l-1, \ j' = l - 1/2, \quad \kappa' = -\kappa, \quad \delta_{\kappa' -\kappa} \\ ② \ (A6.2), \quad l' = l+1, \ j' = l' - 1/2 = l + 1/2, \quad \kappa' = -\kappa, \quad \delta_{\kappa' -\kappa} \end{cases}$$

①, ② cases $\to$ calculate (A5.2)

$$(A5.2) = (-1)^{1/2 + j' + l + l'} \begin{Bmatrix} l' & 1/2 & j' \\ 1/2 & l & 1 \end{Bmatrix} \sqrt{(2l'+1)(2l+1)} \begin{pmatrix} l' & 1 & l \\ 0 & 0 & 0 \end{pmatrix} \sqrt{6}\, \delta_{jj'} \delta_{\mu\mu'}$$

① phase $(-1)^{1/2 + \overbrace{(l-1/2)}^{j'} + l + \overbrace{(l-1)}^{l'}} = \boxed{(-1)^{3l-1}}$ — (A7.1)

$$\begin{Bmatrix} l' & 1/2 & j' \\ 1/2 & l & 1 \end{Bmatrix} = \begin{Bmatrix} l-1 & 1/2 & l-1/2 \\ 1/2 & l & 1 \end{Bmatrix} \longleftrightarrow \cancel{\begin{Bmatrix} j_1 & j_2 & j_3 \\ 1/2 & j_3 - 1/2 & j_2 - 1/2 \end{Bmatrix} = (-1)^{j_1+j_2+j_3} \left[\frac{(j_1+j_2-j_3)(j_1+j_3-j_2+1)}{2j_2(2j_2+1)(2j_3+1)(2j_3+2)} \right]^{1/2}}$$

use this formula

$$\boxed{\begin{Bmatrix} j_1 & j_2 & j_3 \\ 1/2 & j_3+1/2 & j_2+1/2 \end{Bmatrix} = (-1)^{j_1+j_2+j_3+1} \left[\frac{(j_1+j_2+j_3+2)(j_2+j_3-j_1+1)}{(2j_2+1)(2j_2+2)(2j_3+1)(2j_3+2)} \right]^{1/2}}$$

$j_1 = l-1, \quad j_2 = 1/2, \quad j_3 = l - 1/2,$

$j_1 + j_2 + j_3 + 2 = (l-1) + 1/2 + (l - 1/2) + 2 \ \cancel{= 2(l+1)}$

$= 2l + 1,$

$j_1 + j_2 + j_3 + 1 = 2l, \qquad j_2 + j_3 - j_1 + 1 = 1/2 + (l - 1/2) - (l-1) + 1$

$= 2,$

$2j_2 + 1 = 2, \quad 2j_2 + 2 = 3,$

$2j_3 + 1 = 2(l - 1/2) + 1 = 2l, \quad 2j_3 + 2 = 2(l - 1/2) + 2 = 2l + 1$

$$\begin{Bmatrix} l-1 & 1/2 & l-1/2 \\ 1/2 & l & 1 \end{Bmatrix} = (-1)^{2l} \left[\frac{(2l+1)\, 2}{2 \cdot 3 \cdot 2l\, (2l+1)} \right]^{1/2} = \boxed{\frac{1}{\sqrt{6l}}} \quad \text{— (A7.2)}$$

$$\sqrt{(2l'+1)(2l+1)} = \sqrt{(2(l-1)+1)(2l+1)} = \boxed{\sqrt{(2l-1)(2l+1)}} \quad \text{— (A7.3)}$$

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MITB-A8.

$$\begin{pmatrix} \overset{l-1}{j_1} & \overset{1}{j_2} & \overset{l}{j_3} \\ 0 & 0 & 0 \end{pmatrix} \qquad J = j_1 + j_2 + j_3 \quad \text{even case}$$

$$= (-1)^{\frac{1}{2}J} \left[\frac{(j_1+j_2-j_3)!(j_1+j_3-j_2)!(j_2+j_3-j_1)!}{(j_1+j_2+j_3+1)!} \right]^{1/2}$$

$$\times \frac{(\frac{1}{2}J)!}{(\frac{1}{2}J-j_1)(\frac{1}{2}J-j_2)(\frac{1}{2}J-j_3)}$$

$$= (-1)^{l} \left[\frac{0!\,(2l-2)!\,2!}{(2l+1)!} \right]^{1/2} \cdot \frac{l!}{1!\,(l-1)!\,0!}$$

$$= (-1)^{l} \left[\frac{2}{(2l+1)\,2l\,(2l-1)} \right]^{1/2} l$$

$$= (-1)^{l} \left[\frac{l}{(2l+1)(2l-1)} \right]^{1/2} \qquad \text{——} \quad (A8.1)$$

(A7.1), (A7.2), (A7.3), (A8.1)

$$\rightarrow \quad (A5.2) = (-1)^{3l+1} \cdot \frac{1}{\sqrt{6l}} \cdot \sqrt{\frac{(2l-1)\cancel{(2l+1)}}{(2l+1)}} \cdot (-1)^{l} \left[\frac{l}{(2l+1)(2l-1)} \right]^{1/2}$$

$$\cdot \sqrt{6}\, \delta_{jj'} \delta_{\mu\mu'}$$

$$= (-1)^{4l+1} \delta_{jj'} \delta_{\mu\mu'} = (-1)\, \delta_{jj'} \delta_{\mu\mu'}$$

$\therefore$ ①

$$\boxed{\begin{aligned} \vec{\sigma}\cdot\hat{r}\, \chi_{\kappa}^{\mu} &= \cancel{(\ldots)} \sum_{\kappa',\mu'} \chi_{\kappa'}^{\mu'} (-1)\, \delta_{\kappa',-\kappa}\, \delta_{jj'} \delta_{\mu\mu'} \\ &= -\chi_{-\kappa}^{\mu} \end{aligned}}$$

—— (A8.2)

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MITB-A9

② phase $(-1)^{1/2+j'+l+l'} = (-1)^{1/2+(l+1/2)+l+(l+1)} = (-1)^{3l+2} = (-1)^{3l}$

$= (-1)^{3l'-3}$

$l = l'-1$ $\quad = (-1)^{3l'-1}$ — (A9.1)

$$\begin{Bmatrix} l' & 1/2 & j' \\ 1/2 & l & 1 \end{Bmatrix} = \begin{Bmatrix} l' & 1/2 & l'-1/2 \\ 1/2 & l'-1 & 1 \end{Bmatrix}$$

(A7.2) $l' \leftrightarrow l$

6j symbol symmetry

$$= \begin{Bmatrix} l'-1 & 1/2 & l'-1/2 \\ 1/2 & l' & 1 \end{Bmatrix}$$

$$= \frac{1}{\sqrt{6l'}} \qquad \delta_{\kappa',-\kappa} \qquad \text{— (A9.2)}$$

$$\sqrt{(2l'+1)(2l+1)} = \sqrt{(2l'+1)(2(l'-1)+1)} = \sqrt{(2l'+1)(2l'-1)} \qquad \text{— (A9.3)}$$

3j symbol symmetry

$$\begin{pmatrix} l' & 1 & l \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} l' & 1 & l'-1 \\ 0 & 0 & 0 \end{pmatrix} = (-1)^{l'+1+(l'-1)} \begin{pmatrix} l'-1 & 1 & l' \\ 0 & 0 & 0 \end{pmatrix}$$

$l \leftrightarrow l'$ (A8.1)

$$= (-1)^{2l'} \cdot (-1)^{l'} \left[\frac{l'}{(2l'+1)(2l'-1)} \right]^{1/2} \qquad \text{— (A9.4)}$$

$(-1)^{2l'} = 1$

(A9.1) ~ (A9.4)

$\longrightarrow$

$$\vec{\sigma}\cdot\hat{r}\, \chi_\kappa^\mu = \sum_{\kappa'\mu'} \chi_{\kappa'}^{\mu'} \delta_{\kappa',-\kappa}\, \delta_{jj'}\, \delta_{\mu\mu'} \cdot \frac{1}{\sqrt{6l'}}$$

$$\cdot (-1)^{3l'-1} \sqrt{(2l'+1)(2l'-1)} \cdot (-1)^{l'} \left[\frac{l'}{(2l'+1)(2l'-1)} \right] \sqrt{6}$$

$$= (-1)^{4l'-1} \chi_{-\kappa}^\mu = \boxed{-\chi_{-\kappa}^\mu} \qquad \text{— (A9.5)}$$

(A8.2), (A9.5)

$\Longrightarrow$

$$\therefore \boxed{\vec{\sigma}\cdot\hat{r}\, \chi_\kappa^\mu(\hat{r}) = -\chi_{-\kappa}^\mu(\hat{r})} \qquad \text{— (A9.6)}$$

$\Rightarrow$ $\boxed{\text{method 2, No.2 is better !!}}$

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2

$$\boxed{\vec{\sigma}\cdot\hat{r}\,\mathcal{X}_{\kappa}^{\mu} = -\mathcal{X}_{-\kappa}^{\mu}}$$

method No. 2

MITB - A10

(Rose, "Relativistic electron theory")

$$\boxed{\mathcal{X}_{\kappa}^{\mu}(\hat{r}) = \sum_{m,m_s} \langle l\ \tfrac{1}{2}\ m\ m_s | j\ \mu \rangle Y_{lm}(\hat{r})\, \mathcal{X}_{m_s}} \quad \text{— (A10.1)}$$

$\vec{\sigma}\cdot\hat{r}$ is (pseudo) scalar operator in angular momentum space

↓

does not change $\vec{j}$ $(\vec{j}^{\,2})$ $\therefore [\vec{\sigma}\cdot\hat{r}, \vec{j}^{\,2}] = 0$

$$\hat{r} = \frac{\vec{r}}{r}$$

$$[j_z, \vec{\sigma}\cdot\hat{r}] = [l_z + \tfrac{1}{2}\sigma_z,\ \vec{\sigma}\cdot\vec{r}\,\tfrac{1}{r}]$$

$$l_z = (\vec{r}\times\vec{p})_z = r_x p_y - r_y p_x = x\left(-i\frac{\partial}{\partial y}\right) - y\left(-i\frac{\partial}{\partial x}\right)$$
$$= -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$l_z \frac{1}{r} = -i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{1}{\sqrt{x^2+y^2+z^2}} + \frac{1}{r} l_z$$
$$= -i\left(x\frac{y}{r} - y\frac{x}{r}\right) + \frac{1}{r} l_z$$

$$[l_z, \tfrac{1}{r}] = 0, \quad [\sigma_z, \tfrac{1}{r}] = 0$$

$$= \frac{1}{r}[l_z + \tfrac{1}{2}\sigma_z, \vec{\sigma}\cdot\vec{r}]$$
$$= \frac{1}{r}\left\{[l_z, \vec{\sigma}\cdot\vec{r}] + \tfrac{1}{2}[\sigma_z, \vec{\sigma}\cdot\vec{r}]\right\}$$

$$[l_z, \vec{\sigma}\cdot\vec{r}] = \left[-i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right), \sigma_x x + \sigma_y y\right]$$
$$= -i\left\{\left[x\frac{\partial}{\partial y}, \sigma_y y\right] - \left[y\frac{\partial}{\partial x}, \sigma_x x\right]\right\}$$
$$= -i(x\sigma_y - y\sigma_x)$$

$$[\sigma_z, \vec{\sigma}\cdot\vec{r}] = [\sigma_z, \sigma_x x + \sigma_y y]$$
$$= i\sigma_y x - i\sigma_x y = i(x\sigma_y - y\sigma_x)$$

$$= \frac{1}{r}\left(-i(x\sigma_y - y\sigma_x) + i(x\sigma_y - y\sigma_x)\right) = 0$$

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M1TB-A11

$\therefore\ [\hat{J}_z, \vec{\sigma}\cdot\hat{r}] = 0, \quad [\vec{J}^2, \vec{\sigma}\cdot\hat{r}] = 0$ —— (A11.1)

$\vec{\sigma}\cdot\hat{r}$ has odd parity, $\begin{pmatrix} l_\kappa \\ l_{-\kappa}\end{pmatrix} \to l_\kappa - l_{-\kappa} = \pm 1$
parity difference !!

$\therefore\ \vec{\sigma}\cdot\hat{r}\, \chi_\kappa^\mu = a\, \chi_{-\kappa}^\mu$
(a ; constant to be determined)

$\vec{\sigma}\cdot\hat{r}\ \vec{\sigma}\cdot\hat{r}\ \chi_\kappa^\mu = \vec{\sigma}\cdot\hat{r}\ a\chi_{-\kappa}^\mu$
$= a^2 \chi_\kappa^\mu$ $\quad \therefore a^2 = 1$
$a = \pm 1$

(A10.1) $Y_{lm}(\theta,\varphi) = Y_{lm}(\hat{r})$

$$Y_{lm}(\theta,\varphi) = (-1)^m \sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}\cdot\frac{(-e^{i\varphi}\sin\theta)^m}{2^l l!}\left(\frac{d}{d\cos\theta}\right)^{l+m}(\cos^2\theta-1)^l$$ —— (A11.2)

to determine a, we take $\hat{r} = \hat{e}_z$, $\theta = 0$
to have non-zero value, $m = 0$!! $\quad \because (\sin\theta = 0)$

$t = \cos\theta$

$$\left(\frac{d}{dt}\right)^l (t^2-1)^l = \left(\frac{d}{dt}\right)^{l-1} l(2t)(t^2-1)^{l-1}$$
$$= l\left(\frac{d}{dt}\right)^{l-2}\left(2(t^2-1)^{l-1} + (l-1)\,2t\,(t^2-1)^{l-2}\right)$$
$$= l\left(\frac{d}{dt}\right)^{l-3}\left(2\,2t\,(t^2-1)^{l-2}\,2(l-1) + (l-1)\,2(t^2-1)^{l-2} + (l-1)(2t)^2(l-2)(t^2-1)^{l-3}\right)$$

$t \to 1$
survives $2t$

$$= \cdots$$
$$= l(l-1)\cdots 1\cdot (2t)^l\cdot(t^2-1)^0$$

$t = 1$
$$= l!\,2^l$$ —— (A11.2)

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MITB - A12

$$\therefore Y_{lm}(\hat{e}_z) = \delta_{m0}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{l!}{l!}}\cdot\frac{1}{2^l l!}\, l!\, 2^l$$
$$= \sqrt{\frac{2l+1}{4\pi}}\,\delta_{m0}$$

$$\therefore \mathcal{X}_\kappa^\mu(\hat{e}_z) = \sum_{m m_s}\langle l\,\tfrac12\, m\, m_s | j\,\mu\rangle\, Y_{lm}(\hat{e}_z)\,\mathcal{X}_{m_s}$$
$$= \sum_{m m_s}\langle l\,\tfrac12\, m\, m_s | j\,\mu\rangle\cdot\sqrt{\frac{2l+1}{4\pi}}\,\delta_{m0}\,\mathcal{X}_{m_s}$$
$$= \sum_{m_s}\langle l\,\tfrac12\, 0\, m_s | j\,\mu\rangle\cdot\sqrt{\frac{2l+1}{4\pi}}\,\mathcal{X}_{m_s}$$
$$0 + m_s = \mu \qquad \therefore m_s = \mu$$
$$= \sqrt{\frac{2l+1}{4\pi}}\,\langle l\,\tfrac12\, 0\, \mu | j\,\mu\rangle\,\mathcal{X}_\mu$$

$l = l_\kappa$, (A3.2) $\rightarrow$ $l_\kappa - l_{-\kappa} = \mathrm{sign}\,\kappa$

$l_{-\kappa} = l_\kappa - \mathrm{sign}\,\kappa$

$$\mathcal{X}_\kappa^\mu(\hat{e}_z) = \sqrt{\frac{2l+1}{4\pi}}\,\langle l_\kappa\,\tfrac12\, 0\,\mu | j\,\mu\rangle\,\mathcal{X}_\mu \qquad \text{—— (A12.1)}$$

$\sqrt{\frac{2l_\kappa+1}{4\pi}}$

$$\vec{\sigma}\cdot\hat{r}\,\mathcal{X}_\kappa^\mu(\hat{e}_z) = a\,\mathcal{X}_{-\kappa}^\mu(\hat{e}_z)$$
$$= a\sqrt{\frac{2l_{-\kappa}+1}{4\pi}}\,\langle l_{-\kappa}\,\tfrac12\, 0\,\mu | j\,\mu\rangle\,\mathcal{X}_\mu$$

— (A12.2)

$\hat{r} = \hat{e}_z \rightarrow \vec{\sigma}\cdot\hat{r} = \sigma_z$

$\sigma_z\,\mathcal{X}_\mu = 2\mu\,\mathcal{X}_\mu \qquad \mu = \pm\tfrac12$

~~(...)~~ (A12.1), (A12.2) $\rightarrow$

$$2\mu\sqrt{\frac{2l_\kappa+1}{4\pi}}\,\langle l_\kappa\,\tfrac12\, 0\,\mu | j\,\mu\rangle = a\sqrt{\frac{2l_{-\kappa}+1}{4\pi}}\,\langle l_{-\kappa}\,\tfrac12\, 0\,\mu | j\,\mu\rangle$$

— (A12.3)

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MITB - A13

(A12.3) →

$$2\mu\,(2l_\kappa+1)^{1/2}\langle l_\kappa\ \tfrac{1}{2}\ 0\ \mu\,|\,j\ \mu\rangle = a\,(2l_{-\kappa}+1)^{1/2}\langle l_{-\kappa}\ \tfrac{1}{2}\ 0\ \mu\,|\,j\ \mu\rangle \quad \text{— (A13.1)}$$

$\boxed{① \ \kappa<0}$ $\quad j = l_\kappa + \tfrac{1}{2} = l_{-\kappa} - \tfrac{1}{2}$, $\quad l_{-\kappa} = l_\kappa + 1$

$\mu = 1/2$

$$2\cdot\tfrac{1}{2}\,(2l_\kappa+1)^{1/2}\left(\frac{l_\kappa+\frac{1}{2}+\frac{1}{2}}{2l_\kappa+1}\right)^{1/2} = a\,(2l_{-\kappa}+1)^{1/2}\,(-1)\left(\frac{l_{-\kappa}-\frac{1}{2}+\frac{1}{2}}{2l_{-\kappa}+1}\right)^{1/2}$$

$$(l_\kappa+1)^{1/2} = -a\,(l_{-\kappa})^{1/2} = -a\,(l_\kappa+1)^{1/2}$$

$$\therefore\ a = -1 \quad \text{——— (A13.2)}$$

$\mu = -1/2$

$$2(-\tfrac{1}{2})\,(2l_\kappa+1)^{1/2}\left(\frac{l_\kappa+\frac{1}{2}+\frac{1}{2}}{2l_\kappa+1}\right)^{1/2} = a\,(2l_{-\kappa}+1)^{1/2}\left(\frac{l_{-\kappa}-\frac{1}{2}+\frac{1}{2}}{2l_{-\kappa}+1}\right)^{1/2}$$

$$-(l_\kappa+1)^{1/2} = a\,(l_{-\kappa})^{1/2} = a\,(l_\kappa+1)^{1/2}$$

$$\therefore\ a = -1 \quad \text{——— (A13.3)}$$

$\boxed{② \ \kappa>0}$ $\quad j = l_\kappa - \tfrac{1}{2} = l_{-\kappa} + \tfrac{1}{2}$ $\quad l_\kappa = l_{-\kappa}+1$, $\quad l_{-\kappa} = l_\kappa - 1$

$\mu = 1/2$

$$2\cdot\tfrac{1}{2}\,(2l_\kappa+1)^{1/2}\,(-1)\left(\frac{l_\kappa-\frac{1}{2}+\frac{1}{2}}{2l_\kappa+1}\right)^{1/2} = a\,(2l_{-\kappa}+1)^{1/2}\left(\frac{l_{-\kappa}+\frac{1}{2}+\frac{1}{2}}{2l_{-\kappa}+1}\right)^{1/2}$$

$$-(l_\kappa)^{1/2} = a\,(l_{-\kappa}+1)^{1/2} = a\,(l_\kappa-1+1)^{1/2} = a\,(l_\kappa)^{1/2}$$

$$\therefore\ a = -1 \quad \text{— (A13.4)}$$

$\mu = -1/2$

$$2(-\tfrac{1}{2})\,(2l_\kappa+1)^{1/2}\left(\frac{l_\kappa-\frac{1}{2}+\frac{1}{2}}{2l_\kappa+1}\right)^{1/2} = a\,(2l_{-\kappa}+1)^{1/2}\left(\frac{l_{-\kappa}+\frac{1}{2}+\frac{1}{2}}{2l_{-\kappa}+1}\right)^{1/2}$$

$$-(l_\kappa)^{1/2} = a\,(l_{-\kappa}+1)^{1/2} = a\,(l_\kappa-1+1)^{1/2} = a\,(l_\kappa)^{1/2}$$

$$\therefore\ a = -1 \quad \text{— (A13.5)}$$

(A13.2) ~ (A13.5)

$\therefore\ a = -1$ for all the cases

$$\therefore\ \boxed{\vec{\sigma}\cdot\hat{r}\,\chi_\kappa^\mu(\hat{r}) = -\chi_{-\kappa}^\mu(\hat{r})}$$

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