Wilson loops, Wilson surfaces and S-duality

Antonio Sciarappa

KIAS

Based on arXiv:1804.09932 (with Joonho Kim, Seok Kim, Prarit Agarwal)

\+ arXiv:1806.09636 (with Benjamin Assel)

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Overview

Goal: study Wilson loops in 5d supersymmetric gauge theories on $\mathbb{R}^4_{\epsilon_{1,2}} \times S^1_R$

Set-up: Lagrangian theories engineered by (p,q) 5-brane webs in type IIB

- How to realize Wilson loops in the brane web picture?
- How to compute VEVs of Wilson loops in generic representations?
- What are their properties (S-duality, enhanced flavor symmetry)?

Focus on $\mathcal{N} = 1^*$ $SU(N)$ theories or $\mathcal{N} = 1$ $SU(N) + N_F$ fundamental

(p,q) 5-brane webs

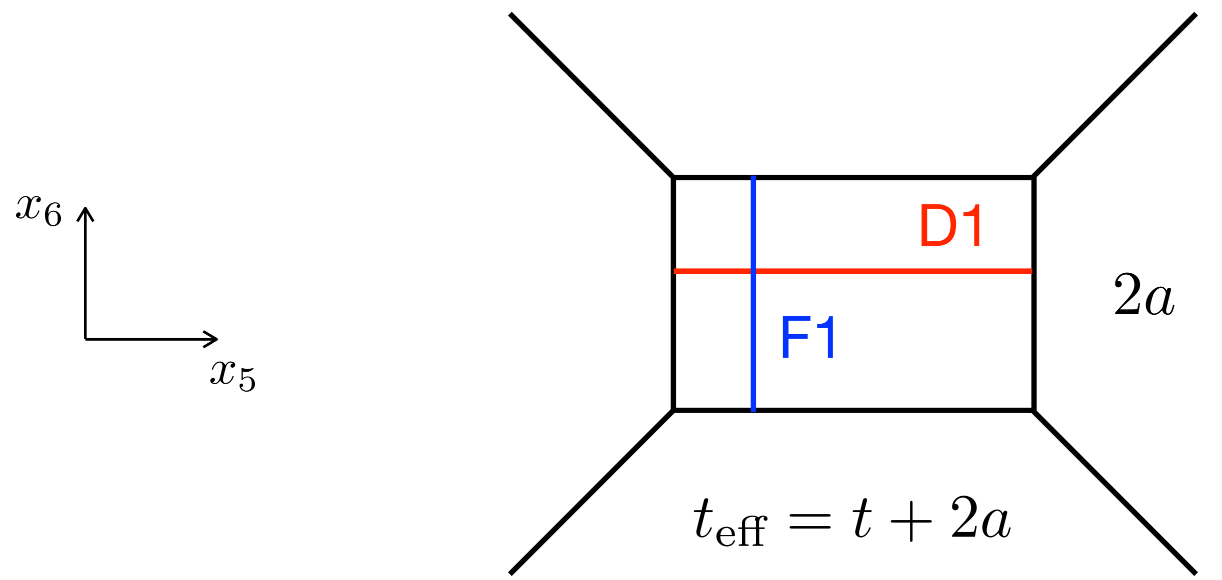
5d $\mathcal{N} = 1$ gauge theories: ill-defined in UV (dimensionful gauge coupling)

- Can think of it to emerge in IR as a deformation of UV 5d SCFT
- UV 5d SCFT engineered via brane systems, such as (p,q) 5-brane webs

	0	1	2	3	4	5	6	7	8	9
D5	X	X	X	X	X	X				
NS5	X	X	X	X	X		X			
$5_{(p,q)}$	X	X	X	X	X	θ	θ			
F1	X						X			
D1	X					X				

After deformation, 5d gauge theory realized on the D5 branes

Example - pure $SU(2)$:

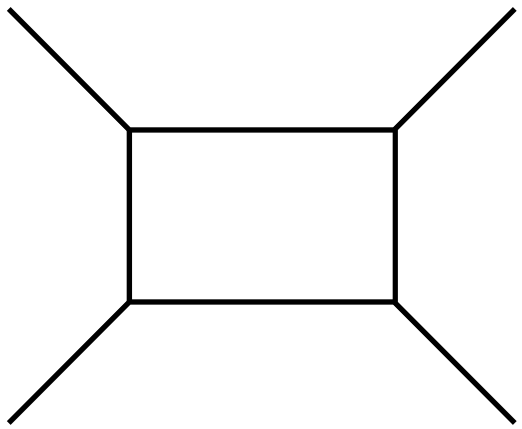


UV: 5d rank 1 $E_1 = SU(2)$ SCFT

IR: 5d $\mathcal{N} = 1$ pure $SU(2)$ theory

	0	1	2	3	4	5	6	7	8	9
D5	X	X	X	X	X	X				
NS5	X	X	X	X	X		X			
$5_{(p,q)}$	X	X	X	X	X	θ	θ			
F1	X						X			
D1	X					X				

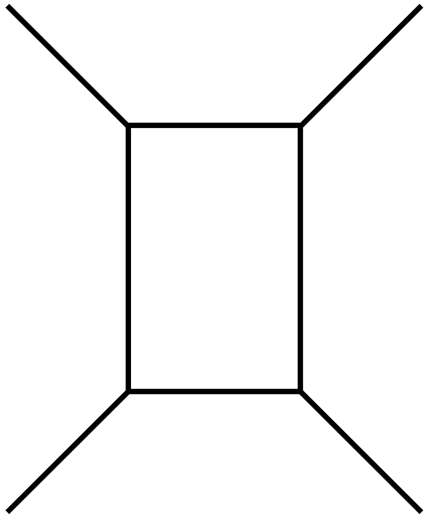
Type IIB S-duality: equivalence between brane webs with (p,q) $\longrightarrow$ (-q,p)



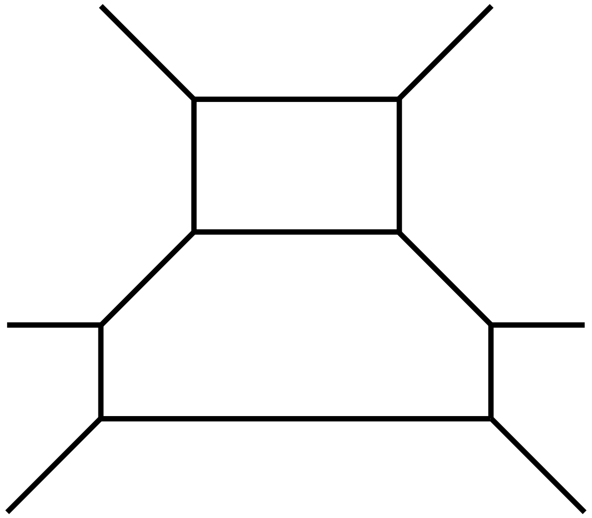
$SU(2)$

$\Longleftrightarrow$

self-dual

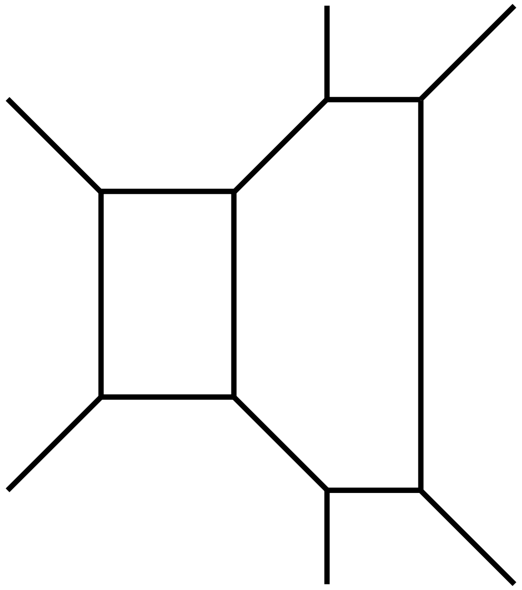


$SU(2)$



$SU(3)\ \ N_F = 2$

$\Longleftrightarrow$



$SU(2) \times SU(2)$

Partition functions

Lagrangian theories: partition function Z_{5d} on $\mathbb{R}^4_{\epsilon_{1,2}} \times S^1_R$ via localization

- Final result factorizes into perturbative + instanton part:

$$Z_{5d} = Z_{5d}^{pert} Z_{5d}^{inst}$$

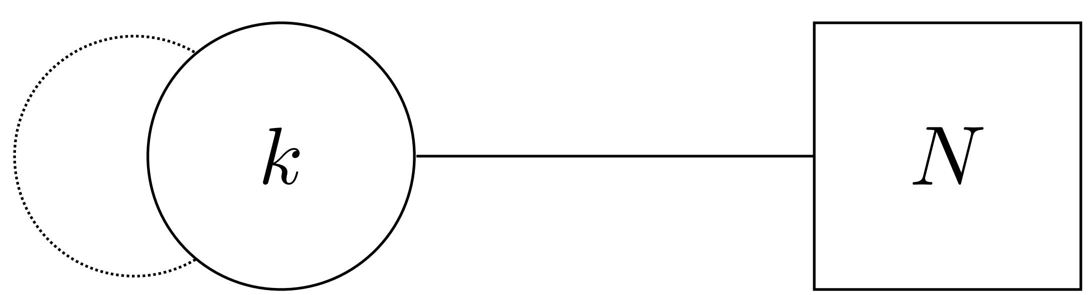
- Instanton part: sum over all instanton number sectors ($Q = e^{-t}$)

$$Z_{5d}^{inst} = \sum_{k \geqslant 0} Q^k Z_{ADHM}^{(k)}$$

with $Z_{ADHM}^{(k)}$ partition function auxiliary ADHM Quantum Mechanics

ADHM QM for

$\mathcal{N} = 1 \ SU(N)$:



$Z^{(k)}_{ADHM}$ evaluated via 1d localization as a Jeffrey-Kirwan residue integral:

$$Z^{(k)}_{ADHM} = \frac{1}{k!}\oint\left[\prod_{s=1}^{k}\frac{d\phi_s}{2\pi i}\right] Z^{(k)}_{vec}\, Z^{(k)}_{fund}\, Z^{(k)}_{CS}$$

$$Z^{(k)}_{vec} = \prod_{s\neq t}^{k}{}' \frac{\sinh(\phi_s-\phi_t)\,\sinh(\phi_s-\phi_t+2\epsilon_+)}{\sinh(\phi_s-\phi_t+\epsilon_1)\,\sinh(\phi_s-\phi_t+\epsilon_2)}\prod_{s=1}^{k}\prod_{r=1}^{N}\frac{1}{4\sinh(\pm\phi_s\mp a_r+\epsilon_+)}\,,$$

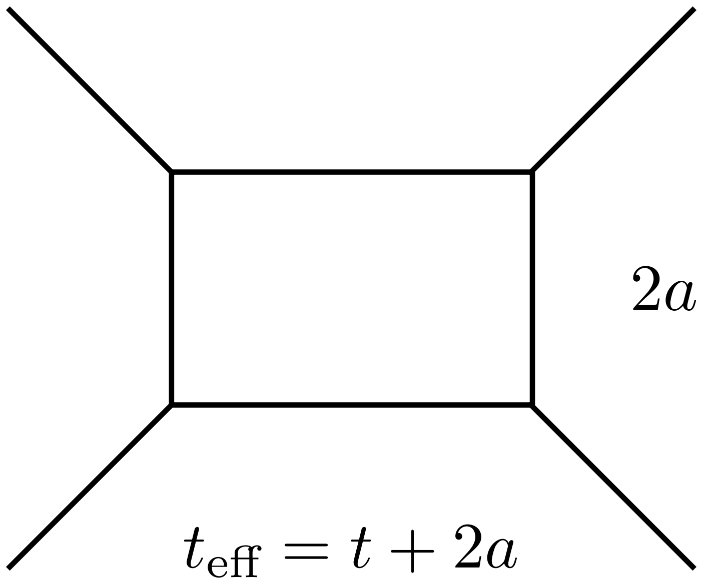
$$Z^{(k)}_{fund} = \prod_{s=1}^{k}\prod_{b=1}^{N_F} 2\sinh(\phi_s-m_b)\,, \qquad Z^{(k)}_{CS} = \prod_{s=1}^{k} e^{-\kappa\phi_s}$$

Z^{inst}_{5d} : series expansion in Q , coefficients _exact_ in $\alpha_r = e^{a_r}$, $q_i = e^{\epsilon_i}$

$SU(2)$: $$Z^{inst}_{5d} = 1 + Q\frac{q_1q_2(1+q_1q_2)}{(1-q_1)(1-q_2)(1-\alpha^2q_1q_2)(1-\alpha^{-2}q_1q_2)} + \ldots$$

Z_{5d} also computable via topological vertex (even non-Lagrangian cases);

same as before, but double series expansion in $Q_F = e^{-2a}$, $Q_B = e^{-t_{eff}}$



$$SU(2): \qquad Z_{5d} = 1 - \frac{1 + q_1 q_2}{(1 - q_1)(1 - q_2)}(Q_F + Q_B) + \dots$$

S-duality manifest: symmetry under exchange $Q_F \longleftrightarrow Q_B$ (fiber/base)

Z_{5d} also knows about flavor symmetry UV SCFT (enhanced from IR one)

- Manifest IR flavor symmetry usually smaller than UV SCFT symmetry;

  $SU(2) + N_F$ fundamental example: IR $SO(2N_F) \times U(1)_{inst} \longrightarrow$ UV E_{N_F+1}

- Z_{5d} can be decomposed into characters of UV SCFT symmetry,

  when expanded in an opportune set of variables

- Pure $SU(2)$ example: characters of E_1 (for $Q_F = A^2 y$, $Q_B = A^2 y^{-1}$)

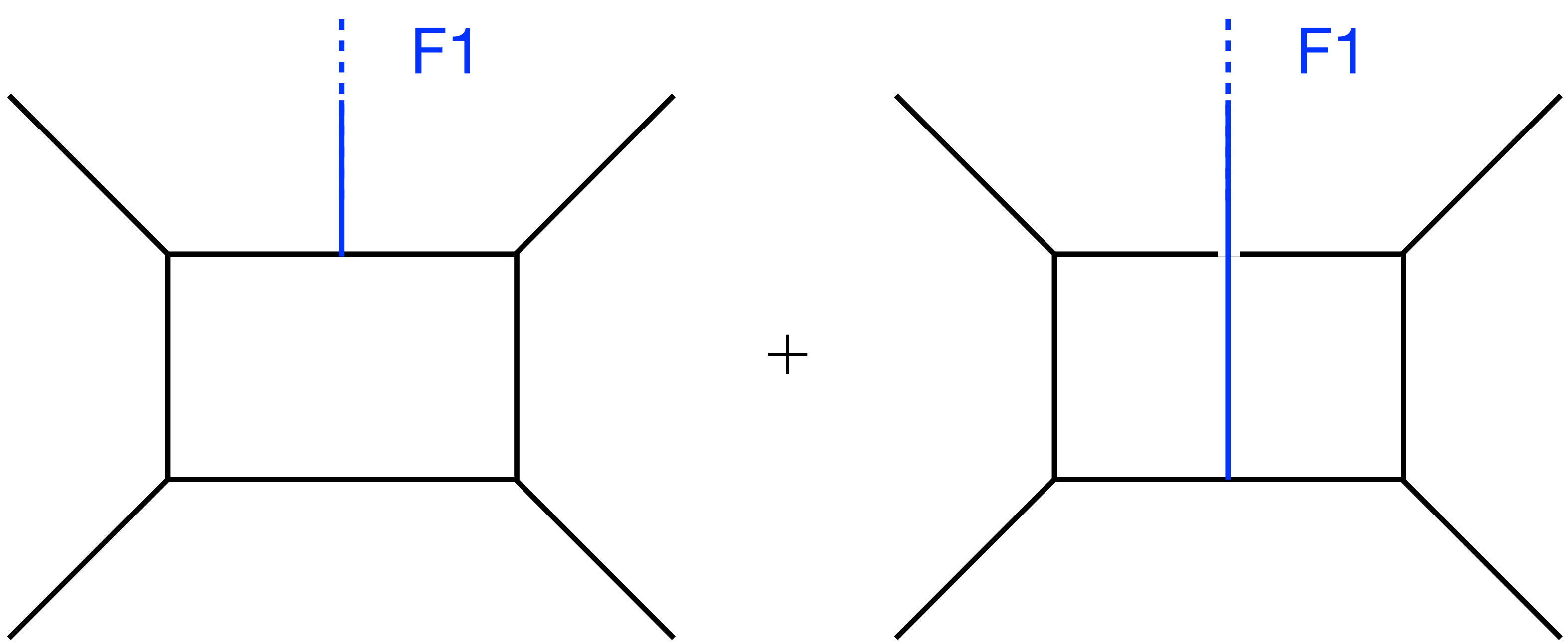
$$SU(2): \qquad Z_{5d} = 1 - \frac{1 + q_1 q_2}{(1-q_1)(1-q_2)} \chi^{E_1}(y) A^2 + \dots$$

Wilson loops

How to realize Wilson loops? Naively, semi-infinite F1 ending on D5

$$W_{\mathbf{2}}^{SU(2)} = \alpha + \alpha^{-1} + O(Q)$$

$W_{\mathbf{2}}^{SU(2)}$:



$+$ (D1, F1)

α $\quad$ α^{-1} $\quad$ $O(Q)$

Naive localization computation of Wilson loop VEV:

$$W_{\mathbf{R}} = Z_{5d}^{pert}\, W_{\mathbf{R}}^{inst}\,, \qquad W_{\mathbf{R}}^{inst} = \sum_{k\geqslant 0} Q^k W_{\mathbf{R},\,ADHM}^{(k)}$$

with $W_{\mathbf{R},\,ADHM}^{(k)}$ observable VEV in auxiliary ADHM Quantum Mechanics

$$W_{\mathbf{R},\,ADHM}^{(k)} = \frac{1}{k!}\oint \left[\prod_{s=1}^{k}\frac{d\phi_s}{2\pi i}\right] Z_{vec}^{(k)}\, Z_{fund}^{(k)}\, Z_{CS}^{(k)}\, Ch_{\mathbf{R}}$$

$Ch_{\mathbf{R}}$ equivariant Chern character of the universal bundle in rep. $\mathbf{R}$

However, technical regularization problems for generic representation $\mathbf{R}$:

poles at infinity, unclear extra corrections, … $\Longrightarrow$ unreliable approach

How can we alternatively approach the problem?

Proposal: add N' D3 at finite distance to the (p,q) 5-brane web

	0	1	2	3	4	5	6	7	8	9
D5	X	X	X	X	X	X				
NS5	X	X	X	X	X		X			
$5_{(p,q)}$	X	X	X	X	X	θ	θ			
F1	X						X			
D1	X					X				
D3	X							X	X	X

D3 intersect the various 5-branes only along S^1_R (0-th direction)

$\Longrightarrow$ insert loop operator $\mathscr{L}^{(N')}_{SQM}$ for the 5d theory on D5

What kind of loop observable is $\mathscr{L}^{(N')}_{SQM}$?

- Not a Wilson loop, but preserves same supercharges

- Proper interpretation: coupling 5d theory to an $\mathcal{N} = (0{,}4)$ SQM, involving D3-D5 fermionic low-energy modes of mass M_j

$$S_\chi = \int dt \sum_{j=1}^{N'} \bar{\chi}_j^a \left(i\partial_t \delta_a^b - A_a^b + \varphi_a^b + M_j \delta_a^b\right)\chi_b^j$$

- Also known as qq-character, fundamental ($N' = 1$) or higher ($N' > 1$)

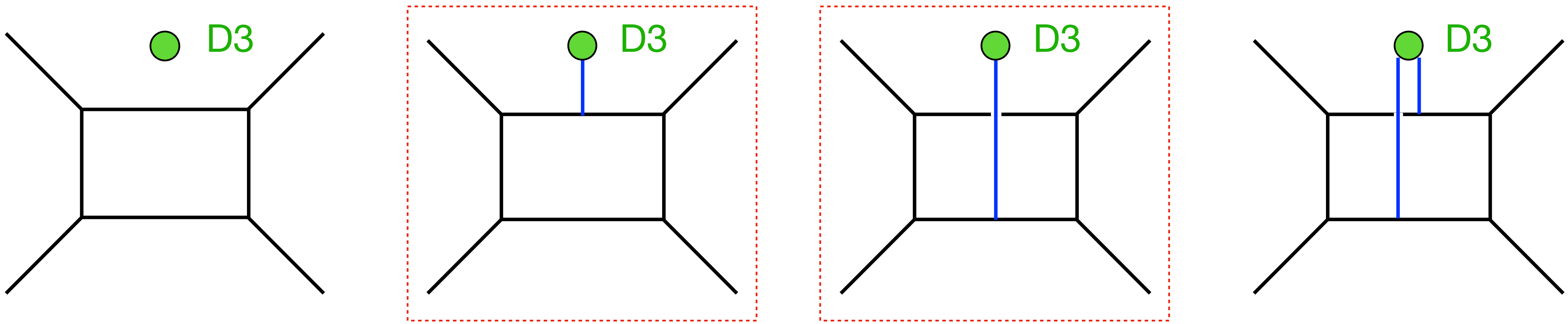
- Remark: loop operator also for 4d $\mathcal{N} = 2^*$ $U(N')$ theory on D3 $(\mathbb{R}^3_{-\epsilon_+} \times S^1_R)$ (contains information on Wilson, 't Hooft loops in 4d $\mathcal{N} = 2^*$ theory?)

Why should we consider $\mathscr{L}^{(N')}_{SQM}$? (More on this later)

Claim: Wilson loops in tensor product of minuscule (antisymmetric) reps.

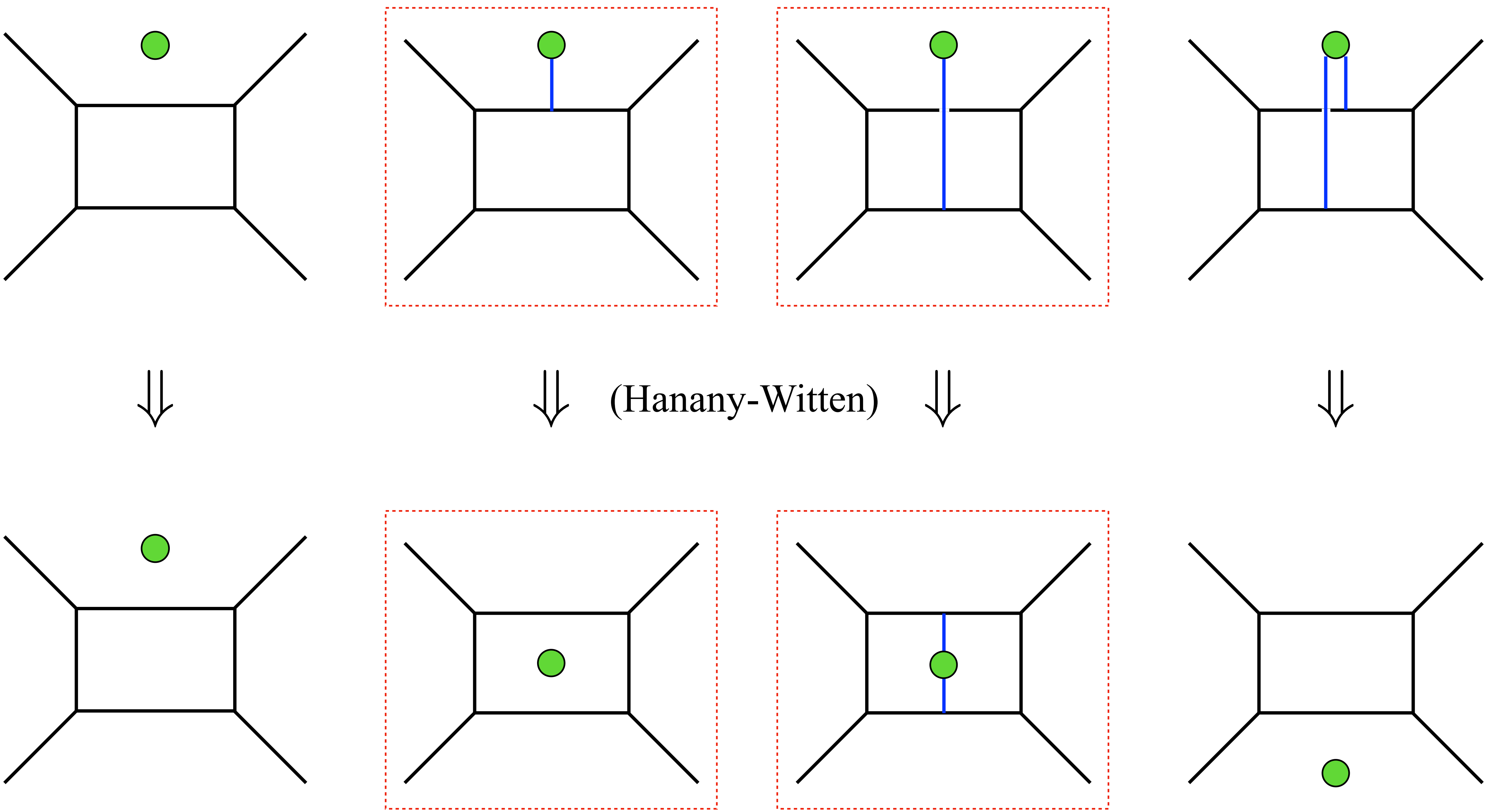
are contained as special sectors of the whole $\mathscr{L}^{(N')}_{SQM}$ loop observable

Very rough idea: $\mathscr{L}^{(N')}_{SQM}$ contains all possible ways of stretching F1's



$W^{SU(2)}_{\mathbf{2}}$

Remark: by Hanany-Witten brane creation / annihilation effect



Wilson loops: diagrams with D3 in the interior and zero net number of F1

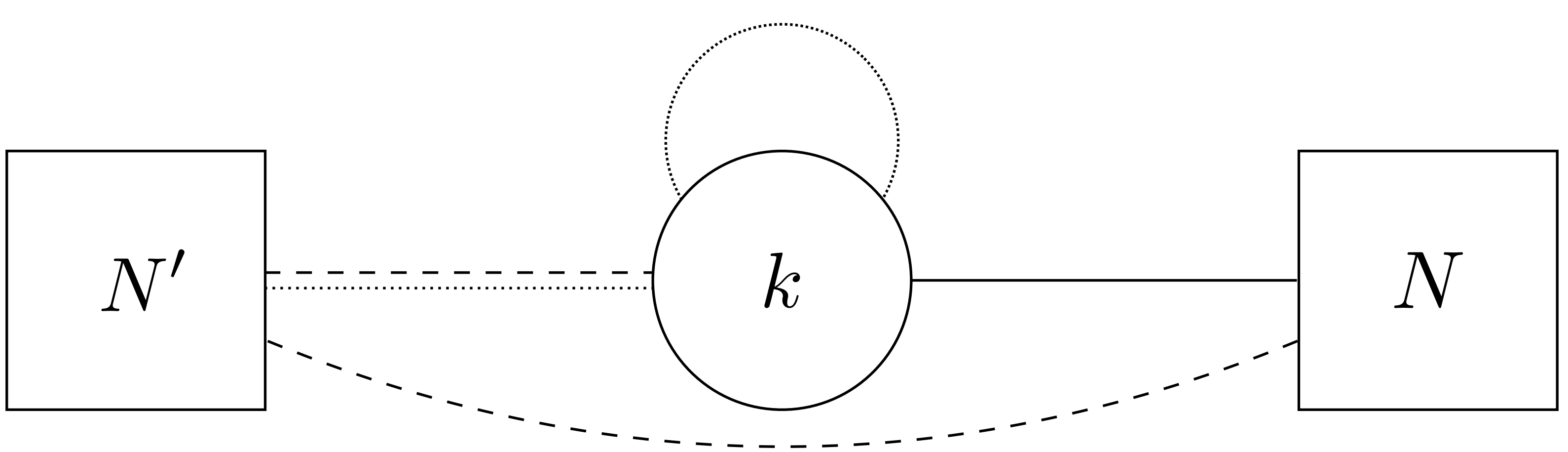
Advantage of $\mathscr{L}^{(N')}_{SQM}$: no issues with localization computation

Localization result once more factorized as

$$\mathscr{L}^{(N')}_{SQM} = Z^{pert}_{5d}\, \mathscr{L}^{(N'),\,inst}_{SQM}\,, \qquad \mathscr{L}^{(N'),\,inst}_{SQM} = \sum_{k\geqslant 0} Q^k Z^{(k)}_{ADHM'}$$

with $Z^{(k)}_{ADHM'}$ partition function of modified ADHM Quantum Mechanics

ADHM' QM for $\mathcal{N}=1$ $SU(N)$:



$Z^{(k)}_{ADHM'}$ evaluated via 1d localization as a Jeffrey-Kirwan residue integral:

$$Z^{(k)}_{ADHM'} = \frac{1}{k!} \oint \left[\prod_{s=1}^{k} \frac{d\phi_s}{2\pi i} \right] Z^{(k)}_{vec} \, Z^{(k)}_{fund} \, Z^{(k)}_{CS} \, Z^{(k)}_{SQM}$$

$$Z^{(k)}_{SQM} = \prod_{r=1}^{N} \prod_{l=1}^{N'} 2 \sinh(M_l - a_r) \prod_{s=1}^{k} \prod_{l=1}^{N'} \frac{\sinh(\pm\phi_s \mp M_l + \epsilon_-)}{\sinh(\pm\phi_s \mp M_l + \epsilon_+)}$$

No poles at infinity, regularization problems, … $\Longrightarrow$ reliable computation;

Jeffrey-Kirwan selects more poles than usual ADHM ones (Young tableaux)

and for convenience we will consider the normalized observable

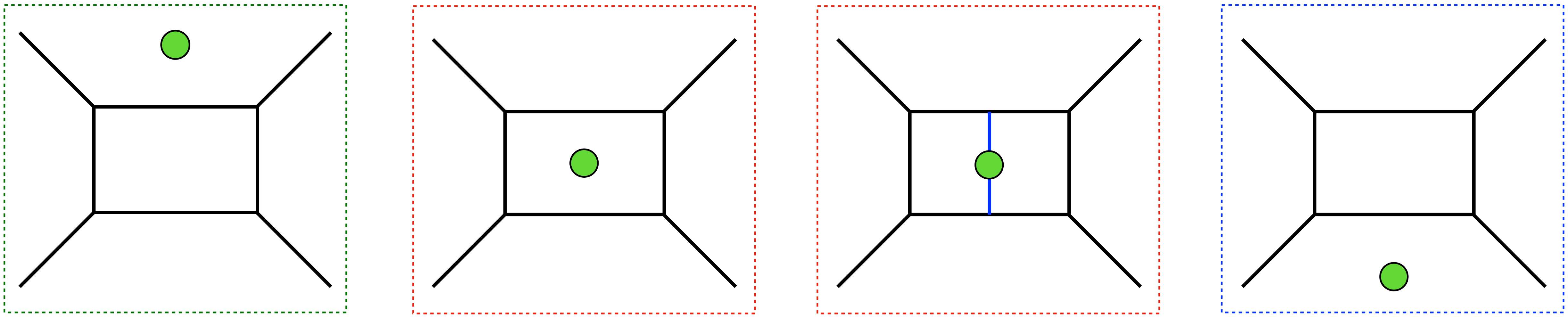
$$\langle \mathscr{L}^{(N')}_{SQM} \rangle = \mathscr{L}^{(N')}_{SQM} / Z_{5d}$$

Disadvantage: often hard to extract Wilson loops

How to recover Wilson loops out of $\langle \mathscr{L}^{(N')}_{SQM} \rangle$?

For $N' = 1$, it was shown that $\langle \mathscr{L}^{(1)}_{SQM} \rangle$ is a Laurent polynomial in $x = e^M$ whose coefficients are Wilson loops in in rank-l antisymmetric reps.

Example: $\mathcal{N} = 1$ pure $SU(2)$ theory ("classical" diagrams: no D1, F1)

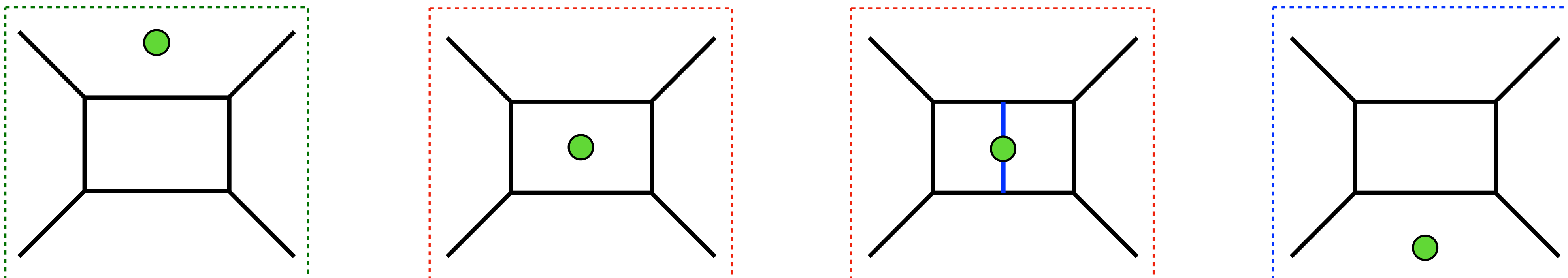


$$\langle \mathscr{L}^{(1)}_{SQM} \rangle = \boxed{x} - \boxed{\langle W_{\mathbf{2}} \rangle} + \boxed{x^{-1}}$$

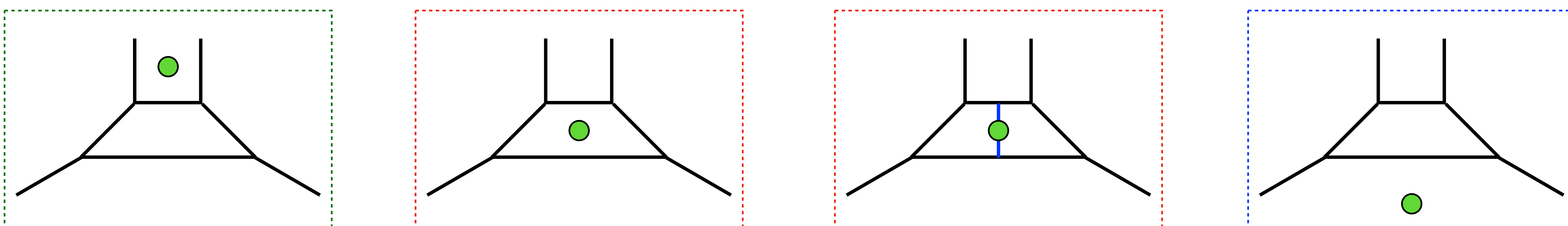
Remark: different (p,q) 5-brane webs realizing the same 5d gauge theory

may have different line operators $\langle \mathcal{L}^{(N')}_{SQM} \rangle$, but still same Wilson loops

$\mathcal{N}=1$ pure $SU(2)$ theory, $\theta=0$: $\langle \mathcal{L}^{(1)}_{SQM} \rangle = x - \langle W_2 \rangle + x^{-1}$

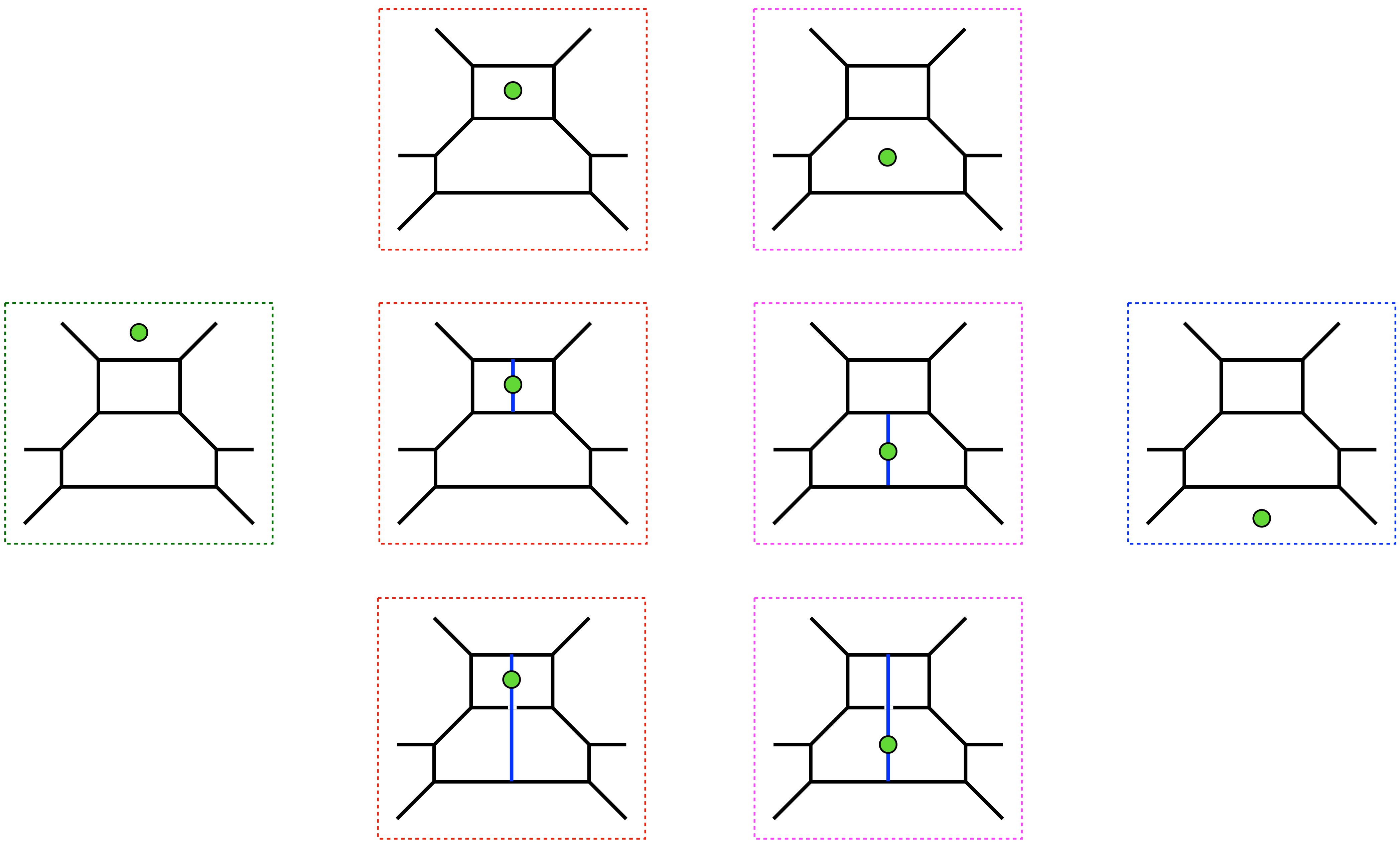


$\mathcal{N}=1$ pure $SU(2)$ theory, $\theta=2\pi$: $\langle \mathcal{L}^{(1)}_{SQM} \rangle = x(1+Q) - \langle W_2 \rangle + x^{-1}$



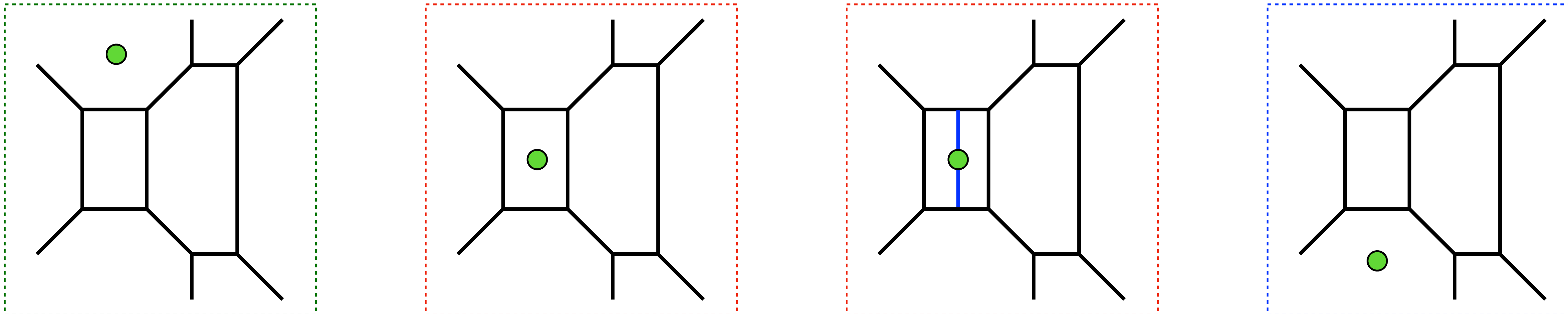
where usual extra factors from parallel NS5 are removed by normalization

Example: $\mathcal{N} = 1\ \ SU(3)\ \ N_F = 2$ theory

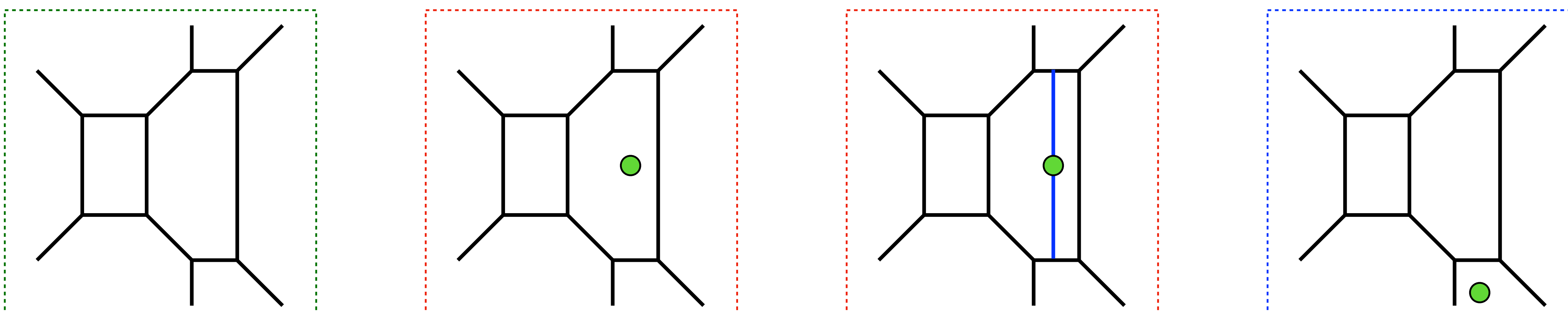


$$\langle \mathscr{L}^{(1)}_{SQM} \rangle = x^{3/2} - \langle W_{\mathbf{3}} \rangle\, x^{1/2} + \langle W_{\bar{\mathbf{3}}} \rangle\, x^{-1/2} - x^{-3/2}$$

Example: $\mathcal{N}=1\ \ SU(2)\times SU(2)$ theory



$$\langle \mathscr{L}^{(1,0)}_{SQM}\rangle = x - \langle W_{(\mathbf{2},\mathbf{1})}\rangle + x^{-1}$$



$$\langle \mathscr{L}^{(0,1)}_{SQM}\rangle = x - \langle W_{(\mathbf{1},\mathbf{2})}\rangle + x^{-1}$$

What do we learn from the $N' = 1$ case?

- By using Hanany-Witten moves, Wilson loops can be associated to diagrams with D3 branes in the interior region and zero F1 net number
- These diagrams have specific charge under 4d $U(1)$ gauge group on D3; $U(1)$ charge: (number D5 below D3) - (number D5 above D3)

This suggests to isolate Wilson loops by selecting sectors of such charge

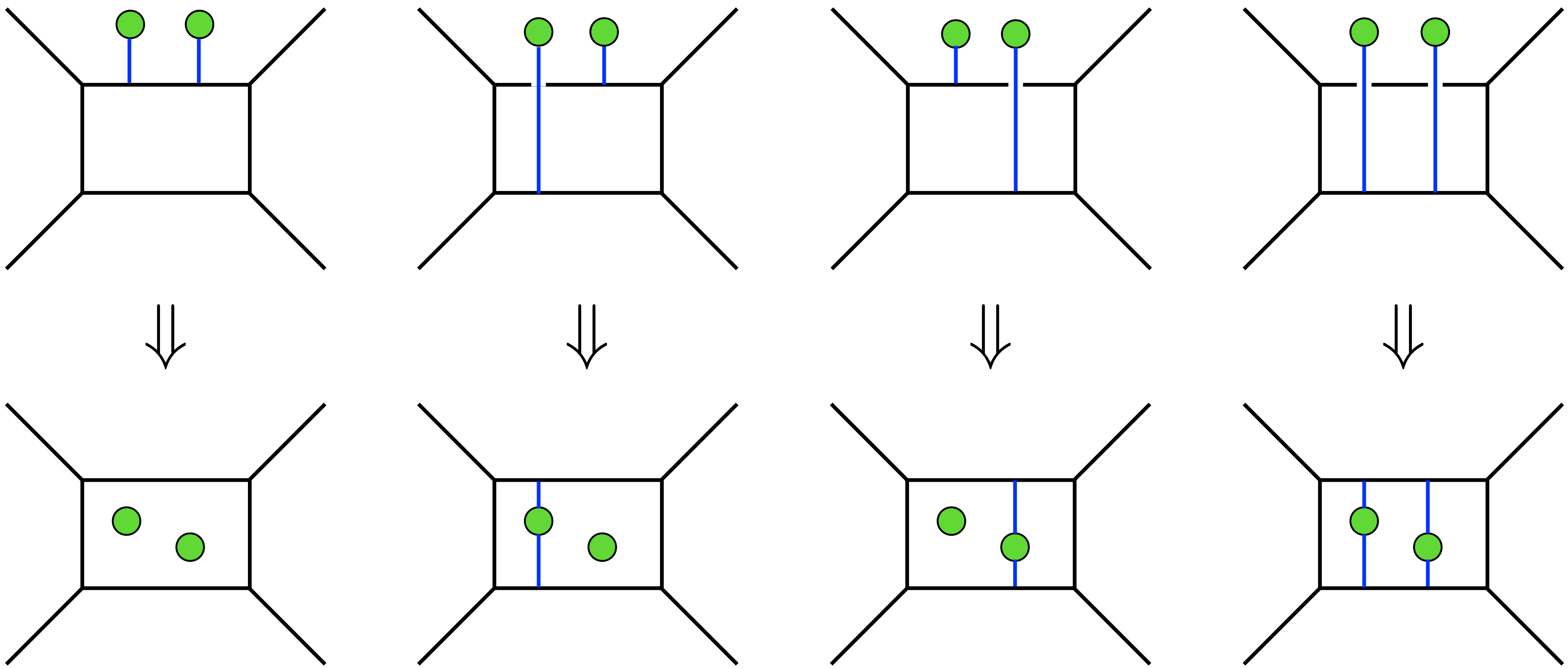
$$SU(2): \qquad \langle \mathscr{L}^{(1)}_{SQM} \rangle = x - \langle W_\mathbf{2} \rangle + x^{-1} \quad \Longrightarrow \quad \langle W_\mathbf{2} \rangle = - \oint \frac{dx}{x} \langle \mathscr{L}^{(1)}_{SQM} \rangle$$

$$SU(3)\ N_F = 2: \qquad \langle \mathscr{L}^{(1)}_{SQM} \rangle = x^{3/2} - \langle W_\mathbf{3} \rangle x^{1/2} + \langle W_{\overline{\mathbf{3}}} \rangle x^{-1/2} - x^{-3/2}$$

$$\Longrightarrow \quad \langle W_\mathbf{3} \rangle = - \oint \frac{dx}{x} \frac{1}{x^{1/2}} \langle \mathscr{L}^{(1)}_{SQM} \rangle , \qquad \langle W_{\overline{\mathbf{3}}} \rangle = \oint \frac{dx}{x} \frac{1}{x^{-1/2}} \langle \mathscr{L}^{(1)}_{SQM} \rangle$$

What about $\langle \mathscr{L}^{(N')}_{SQM} \rangle$ for $N' > 1$? Unclear…

- Expected to contain Wilson loops in tensor product of antisym. reps., at least from 5-brane web picture:



$\langle W_{\mathbf{2}\otimes\mathbf{2}} \rangle$ in $\mathcal{N} = 1$ pure $SU(2)$

- However, from computations $\langle \mathscr{L}^{(N')}_{SQM} \rangle$ is a rational function of $x_1, \ldots, x_{N'}$ rather than a Laurent polynomial; how to extract Wilson loops out of it?

Proposal: Wilson loops still isolated by selecting sectors of specific charge under the Cartan $U(1)^{N'} \subset U(N')$ of the 4d gauge group on D3 (separated)

Example: $\mathcal{N} = 1$ pure $SU(2)$, case $N' = 2$

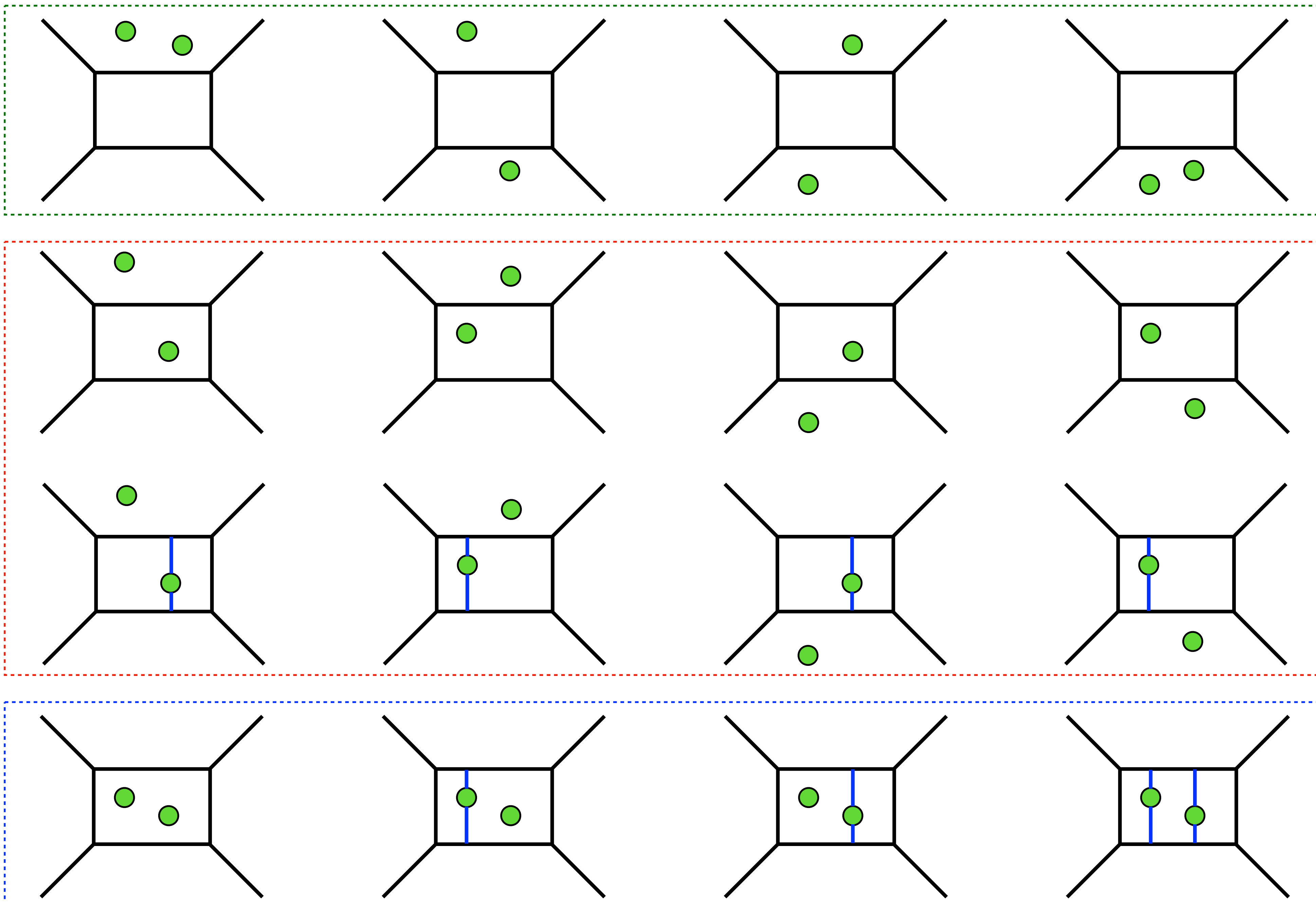
$$\langle \mathscr{L}^{(2)}_{SQM} \rangle \;=\; x_1 x_2 + x_1 x_2^{-1} + x_1^{-1} x_2 + x_1^{-1} x_2^{-1} \;-\; (x_1 + x_2 + x_1^{-1} + x_2^{-1}) \langle W_{\mathbf{2}} \rangle$$

$$+ \langle W_{\mathbf{2} \otimes \mathbf{2}} \rangle - Q \frac{(1-q_1)(1-q_2)(1+q_1 q_2) x_1 x_2}{(x_1 - q_1 q_2 x_2)(x_2 - q_1 q_2 x_1)}$$

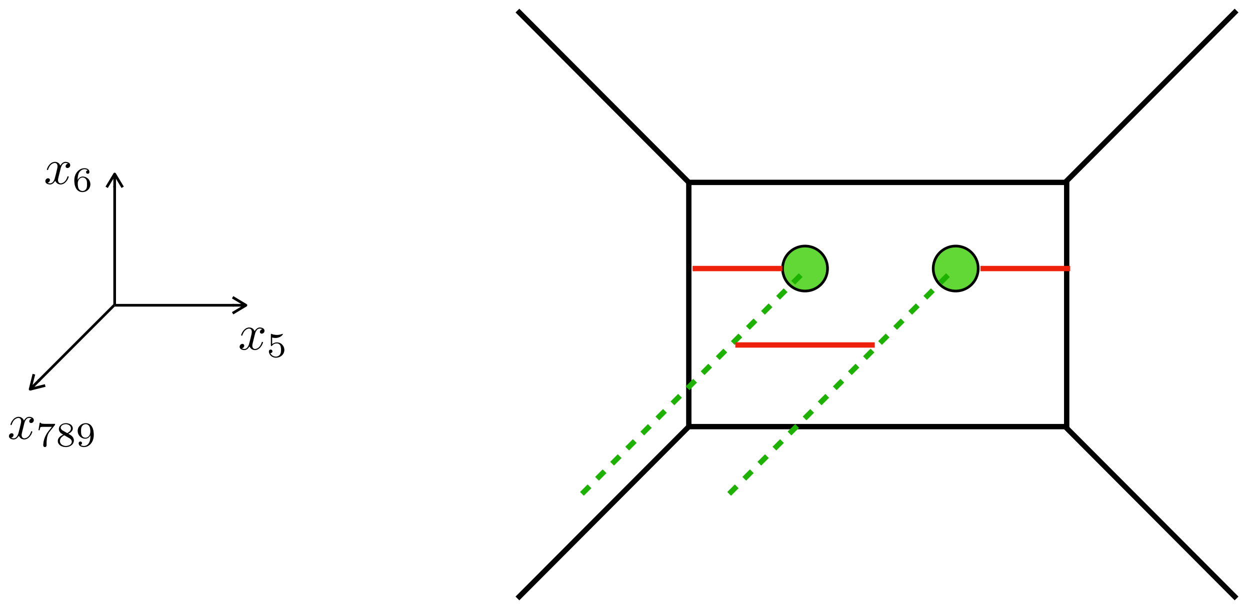
with (as natural generalization of $N' = 1$) $\quad \langle W_{\mathbf{2} \otimes \mathbf{2}} \rangle = \oint \frac{dx_1}{x_1} \frac{dx_2}{x_2} \langle \mathscr{L}^{(2)}_{SQM} \rangle$

$$\langle \mathscr{L}^{(2)}_{SQM} \rangle = x_1x_2 + x_1x_2^{-1} + x_1^{-1}x_2 + x_1^{-1}x_2^{-1} - (x_1 + x_2 + x_1^{-1} + x_2^{-1})\langle W_{\mathbf{2}} \rangle$$

$$+ \langle W_{\mathbf{2}\otimes\mathbf{2}} \rangle - Q\frac{(1-q_1)(1-q_2)(1+q_1q_2)x_1x_2}{(x_1 - q_1q_2x_2)(x_2 - q_1q_2x_1)}$$



Extra rational terms: possible breaking of D1 when D3 collinear



In some cases, related to monopole bubbling 4d $\mathcal{N} = 2^*$ theory on D3:

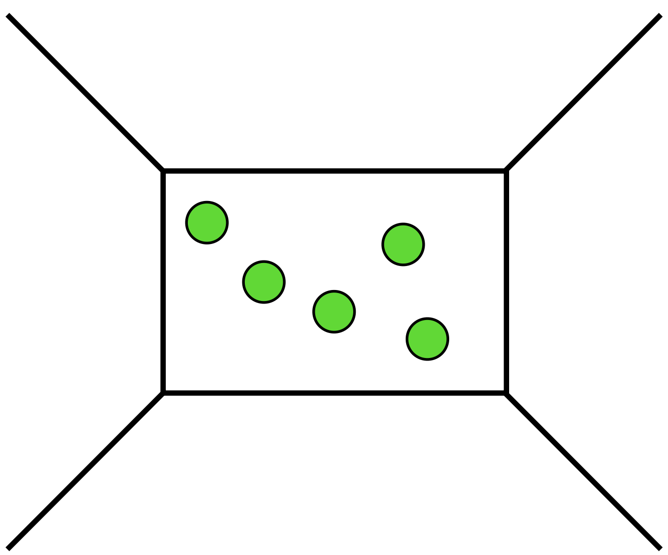
$$\oint \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\langle \mathscr{L}^{(2)}_{SQM}\rangle \quad \Longrightarrow \quad SU(2) \text{ monopole bubbling}$$

more general interpretation however still unclear

Remark: with some care, we can also isolate other representations

$$\langle W_{\mathbf{2}\otimes\mathbf{2}}\rangle = \oint \frac{dx_1}{x_1}\frac{dx_2}{x_2}\langle \mathscr{L}^{(2)}_{SQM}\rangle \qquad \text{vs.} \qquad \langle W_{\mathbf{3}}\rangle = \frac{1}{2}\oint \frac{dx_1}{x_1}\frac{dx_2}{x_2}\left(1-\frac{x_1}{x_2}\right)\left(1-\frac{x_2}{x_1}\right)\langle \mathscr{L}^{(2)}_{SQM}\rangle$$

More generally, for $SU(2)$ theories with N_F fundamental:



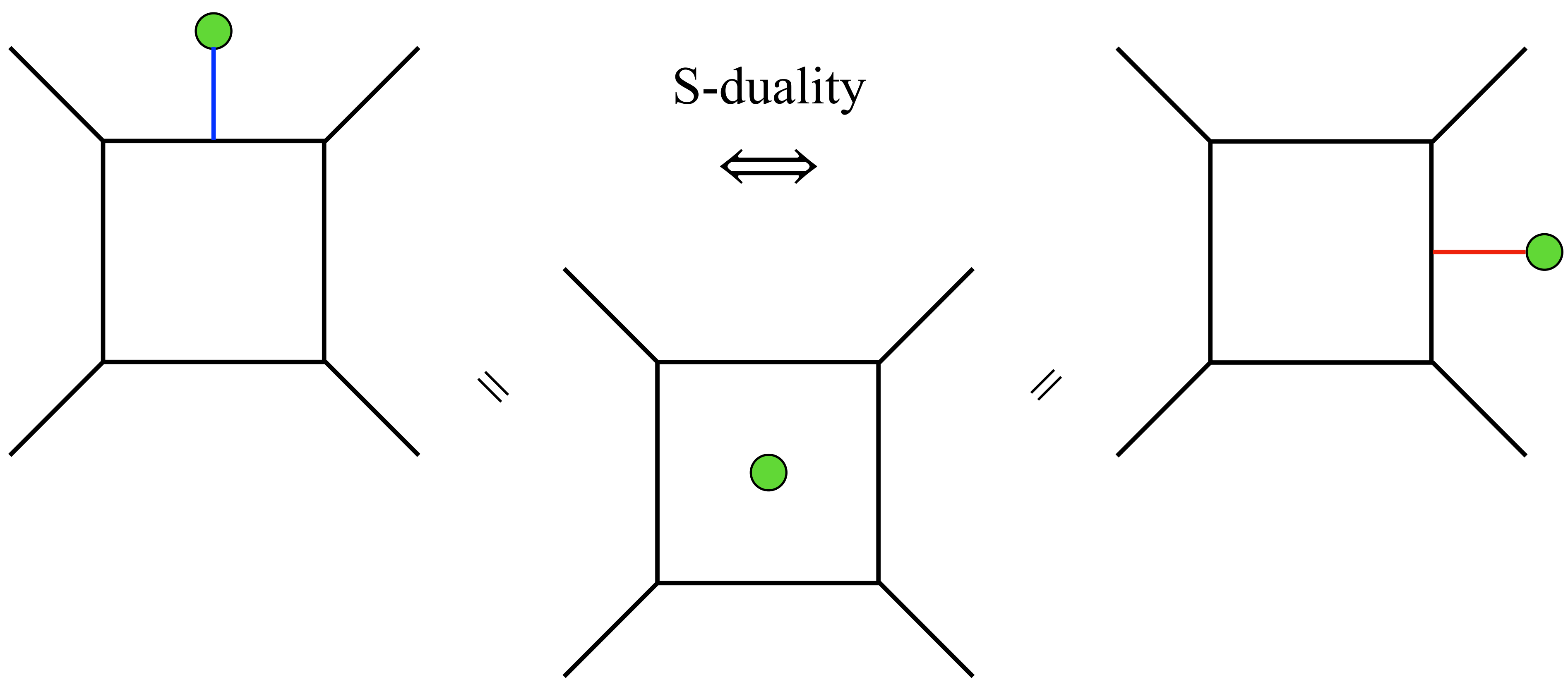
$$\langle W_{\mathbf{2}\otimes N'}\rangle = (-1)^{N'}\oint \prod_{i=1}^{N'} \frac{dx_i}{x_i}\langle \mathscr{L}^{(N')}_{SQM}\rangle$$

A similar story is valid for higher rank theories and quiver theories

S-duality and enhanced flavor symmetry

How to **test** our prescription for extracting Wilson loops? S-duality

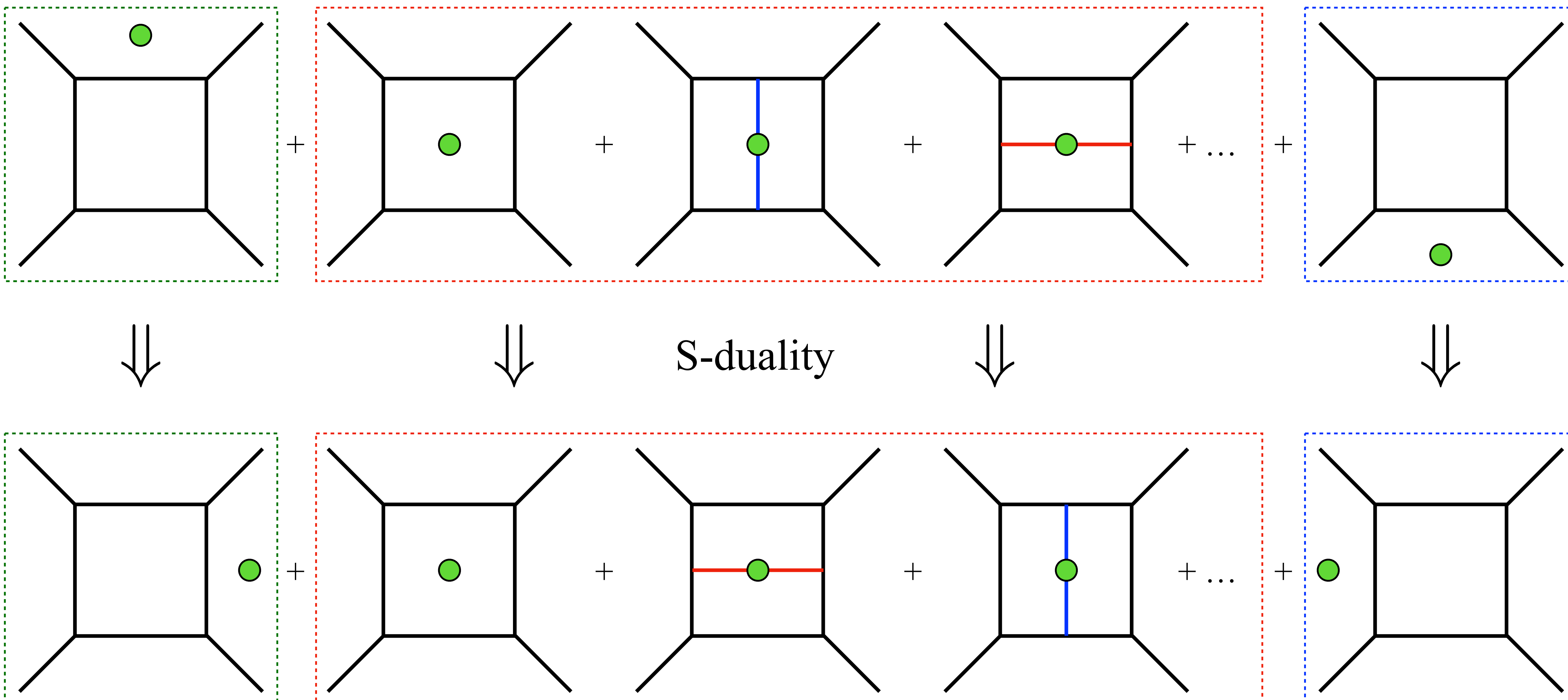
Main point: Wilson loops mapped to Wilson loops under S-duality



(better: mapped to some “instanton” loop, equivalent to a Wilson loop)

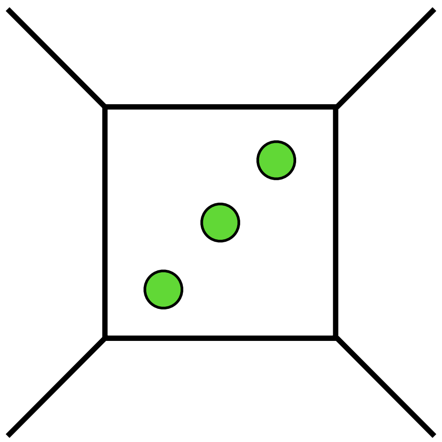
Remark: only tensor product Wilson loops have nice S-duality properties, while other Wilson loops and the whole loop observable $\langle \mathscr{L}^{(N')}_{SQM} \rangle$ do not:

$$\langle \mathscr{L}^{(1)}_{SQM} \rangle = \boxed{x} - \boxed{\langle W_{\mathbf{2}} \rangle} + \boxed{x^{-1}}$$

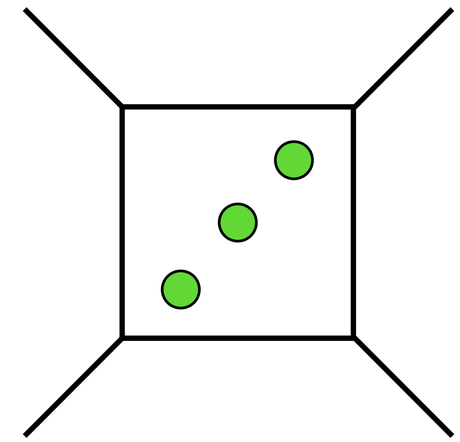


Our Wilson loop prescription nicely reproduces brane picture expectation

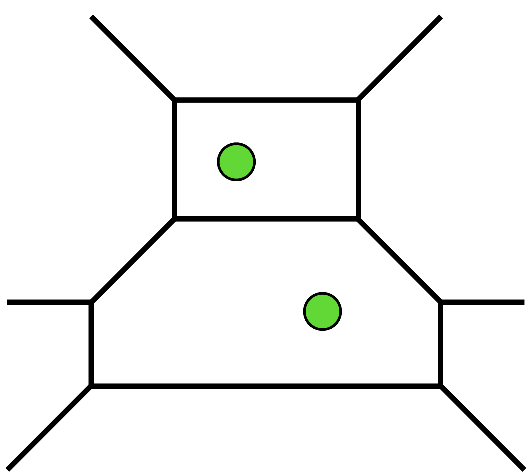
- $SU(2) + N_F$ fundamental (expanded in "top. strings" variables Q_F, Q_B):



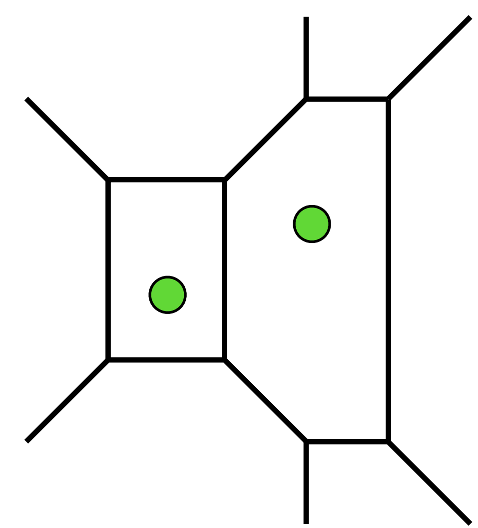
$$\langle W_{\mathbf{2}\otimes N'} \rangle \quad \Longleftrightarrow \quad \langle W_{\mathbf{2}\otimes N'} \rangle$$



- $SU(3)$ $N_F = 2$ versus $SU(2) \times SU(2)$:



$$\langle W_{\mathbf{3}^{\otimes n_1} \otimes \bar{\mathbf{3}}^{\otimes n_2}} \rangle \quad \Longleftrightarrow \quad \langle W_{(\mathbf{2}^{\otimes n_1}, \mathbf{2}^{\otimes n_2})} \rangle$$



(actually covariant map: mapped up to a phase, unless at SCFT point)

Further **test**: tensor product Wilson loops exhibit enhanced flavor symmetry

- Pure $SU(2)$ case - $E_1 = SU(2)$ symmetry ($Q_F = A^2 y,\ Q_B = A^2 y^{-1}$):

$$A\, y^{1/2} \langle W_{\mathbf{2}} \rangle \;=\; 1 + \boxed{\chi_{\mathbf{2}}^{E_1}(y)} A^2 + \chi_{\mathbf{3}}^{A_1}(q_+)\, A^4 + \chi_{\mathbf{5}}^{A_1}(q_+) \boxed{\chi_{\mathbf{2}}^{E_1}(y)} A^6 + \ldots$$

$$A^2 y \langle W_{\mathbf{2}\otimes\mathbf{2}} \rangle \;=\; 1 + \boxed{2\chi_{\mathbf{2}}^{E_1}(y)} A^2 + \left(\boxed{\chi_{\mathbf{3}}^{E_1}(y)} + \chi_{\mathbf{3}}^{A_1}(q_+) + \chi_{\mathbf{2}}^{A_1}(q_+)\chi_{\mathbf{2}}^{A_1}(q_-) \right) A^4 + \ldots$$

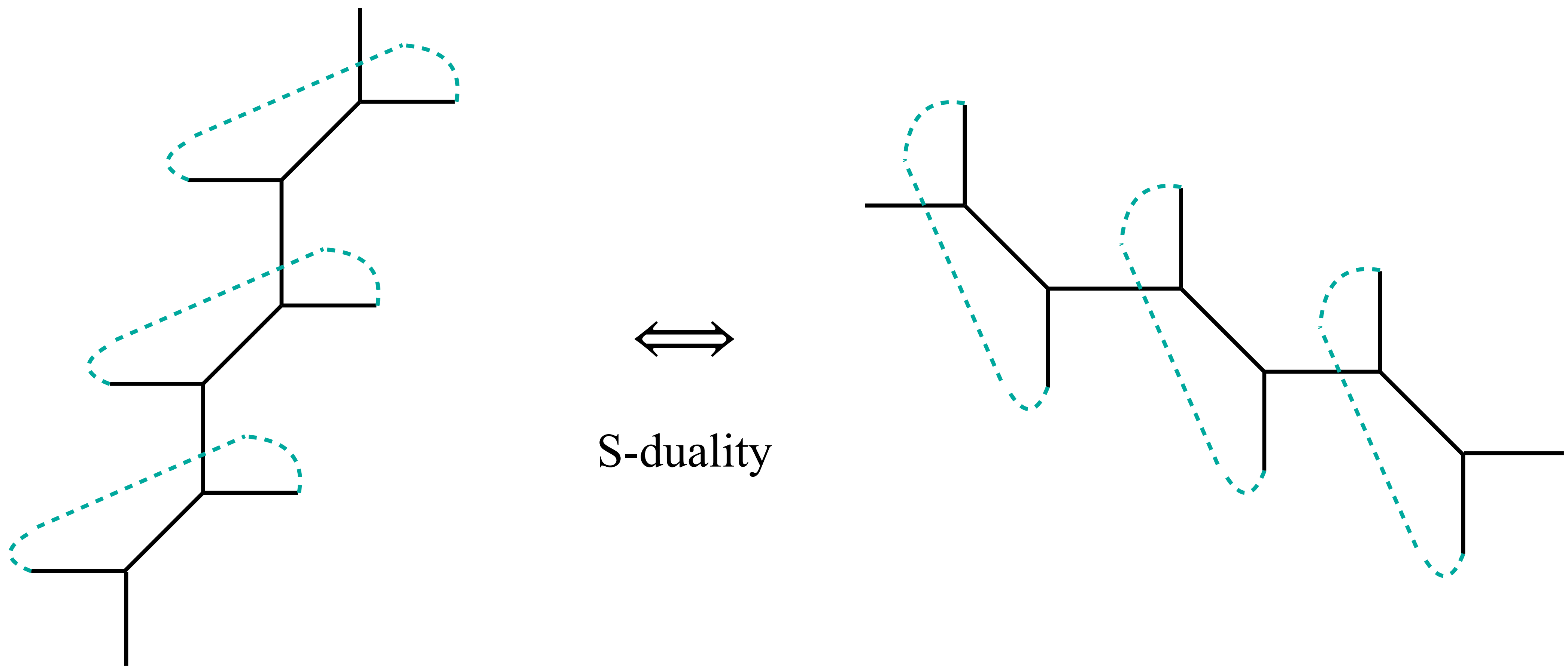
- $SU(2)\ N_F = 4$ case - $E_5 = SO(10)$ symmetry:

$$A\, y_1^{1/2} \langle W_{\mathbf{2}} \rangle \;=\; 1 + \boxed{\chi_{\mathbf{10}}^{E_5}(\vec{y})} A^2 - \chi_{\mathbf{2}}^{A_1}(q_+) \boxed{\chi_{\overline{\mathbf{16}}}^{E_5}(\vec{y})} A^3 + \ldots$$

$$A^2 y_1 \langle W_{\mathbf{2}\otimes\mathbf{2}} \rangle \;=\; 1 + \boxed{2\chi_{\mathbf{10}}^{E_5}(\vec{y})} A^2 - \left(\chi_{\mathbf{2}}^{A_1}(q_+) + \chi_{\mathbf{2}}^{A_1}(q_-) \right) \boxed{\chi_{\overline{\mathbf{16}}}^{E_5}(\vec{y})} A^3 + \ldots$$

Wilson loops and Wilson surfaces

Consider $\mathcal{N} = 1^*$ $SU(N)$; S-dual: 6d Abelian theory (M-strings set-up)



5d $\mathcal{N} = 1^*$ $SU(N)$

on $\mathbb{R}^4_{\epsilon_{1,2}} \times S^1_R$

$\Longleftrightarrow$

6d A_{N-1} $\mathcal{N} = (2,0)$

(tensor branch)

on $\mathbb{R}^4_{\epsilon_{1,2}} \times T^2$

The theories have same partition function; what happens to Wilson loops?

- Wilson loops in 5d: codimension 4 defects (line operator)

- Natural analogue codimension 4 defect in 6d: Wilson surface

$$\langle S_{\mathbf{R}} \rangle \;\sim\; Tr_{\mathbf{R}}\left[\exp \int_{T^2} \left(iB + \Phi\, ds \wedge d\tau\right)\right]$$

(formally; to be better defined, for example via brane construction)

- S-duality maps 5d Wilson loops to 6d Wilson surfaces

- How to compute 6d Wilson surfaces? Construct 2d analogue of $\langle \mathscr{L}^{(N')}_{SQM} \rangle$

Revisit first the Wilson loops / $\langle \mathscr{L}^{(N')}_{SQM} \rangle$ computation for 5d $\mathcal{N} = 1^* \ SU(N)$

More natural brane set-up: D4-D4' intersecting along S^1_R

background $S^1_R \times \mathbb{R}^2_{\epsilon_+ + \epsilon_-} \times \mathbb{R}^2_{\epsilon_+ - \epsilon_-} \times \mathbb{R}^2_{-\epsilon_+ + m} \times \mathbb{R}^2_{-\epsilon_+ - m} \times \mathbb{R}$

	0	1	2	3	4	5	6	7	8	9
N D4	X	X	X	X	X					
N' D4'	X					X	X	X	X	
F1	X									X
D0	X									

System completely symmetric under exchange $\epsilon_+ \longleftrightarrow -\epsilon_+ \,,\ \epsilon_- \longleftrightarrow m$

$\Longrightarrow$ $\langle\langle \mathscr{L}^{(N')}_{SQM} \rangle\rangle$ contains Wilson loops for both D4 and D4' 5d theories

$$\langle\langle \mathscr{L}^{(N')}_{SQM} \rangle\rangle = \frac{\mathscr{L}^{(N')}_{SQM}}{Z_{5d}\, Z_{5d'}}$$

- Example: $N = 2\,,\ N' = 1$ (for $\alpha_1 = \alpha_2^{-1} = \alpha$)

$$\langle\langle \mathscr{L}^{(1)}_{SQM} \rangle\rangle = x - \langle W^{SU(2)}_{\mathbf{2}} \rangle + x^{-1}$$

- Example: $N = 1\,,\ N' = 2$ (for $x_1 = x_2^{-1} = x$)

$$\langle\langle \mathscr{L}^{(2)}_{SQM} \rangle\rangle = \alpha - \langle \widetilde{W}^{SU(2)}_{\mathbf{2}} \rangle + \alpha^{-1}$$

- Example: $N = 2\,,\ N' = 2$, with $\langle W^{SU(2)}_{\mathbf{2}\otimes\mathbf{2}} \rangle = \oint \frac{dx_1}{x_1}\frac{dx_2}{x_2} \langle\langle \mathscr{L}^{(2)}_{SQM} \rangle\rangle$

$$\langle\langle \mathscr{L}^{(2)}_{SQM} \rangle\rangle = \frac{x_1 x_2}{\alpha_1\alpha_2} + \frac{\alpha_1\alpha_2}{x_1 x_2} - \left(\sqrt{\frac{x_1 x_2}{\alpha_1\alpha_2}} + \sqrt{\frac{\alpha_1\alpha_2}{x_1 x_2}}\right) \langle W^{SU(2)}_{\mathbf{2}} \rangle \langle \widetilde{W}^{SU(2)}_{\mathbf{2}} \rangle + {''}\langle W^{SU(2)}_{\mathbf{2}\otimes\mathbf{2}} \rangle + \langle \widetilde{W}^{SU(2)}_{\mathbf{2}\otimes\mathbf{2}} \rangle{''}$$

where tensor product Wilson loops have a common, *shared* part

$${''}\langle W^{SU(2)}_{\mathbf{2}\otimes\mathbf{2}} \rangle + \langle \widetilde{W}^{SU(2)}_{\mathbf{2}\otimes\mathbf{2}} \rangle{''} = \langle W^{SU(2)}_{\mathbf{2}\otimes\mathbf{2}} \rangle + \left(\langle \widetilde{W}^{SU(2)}_{\mathbf{2}\otimes\mathbf{2}} \rangle - f(Q, \epsilon_\pm, m)\right) = \left(\langle W^{SU(2)}_{\mathbf{2}\otimes\mathbf{2}} \rangle - f(Q, \epsilon_\pm, m)\right) + \langle \widetilde{W}^{SU(2)}_{\mathbf{2}\otimes\mathbf{2}} \rangle$$

What is the analogue of $\langle \mathscr{L}^{(N')}_{SQM} \rangle$ for the 6d A_{N-1} $\mathcal{N} = (2{,}0)$ theory?

Start from brane realization 6d A_{N-1} $\mathcal{N} = (2{,}0)$, add codim. 4 defect

(from previous set-up with $\mathbb{R}^4_{5678} \longrightarrow TN$, applying TST on 5 circle)

	0	1	2	3	4	5	6	7	8	9
N NS5	X	X	X	X	X	X				
1 D6	X	X	X	X	X	X				X
D2	X	X								X
N' D4'	X	X					X	X	X	

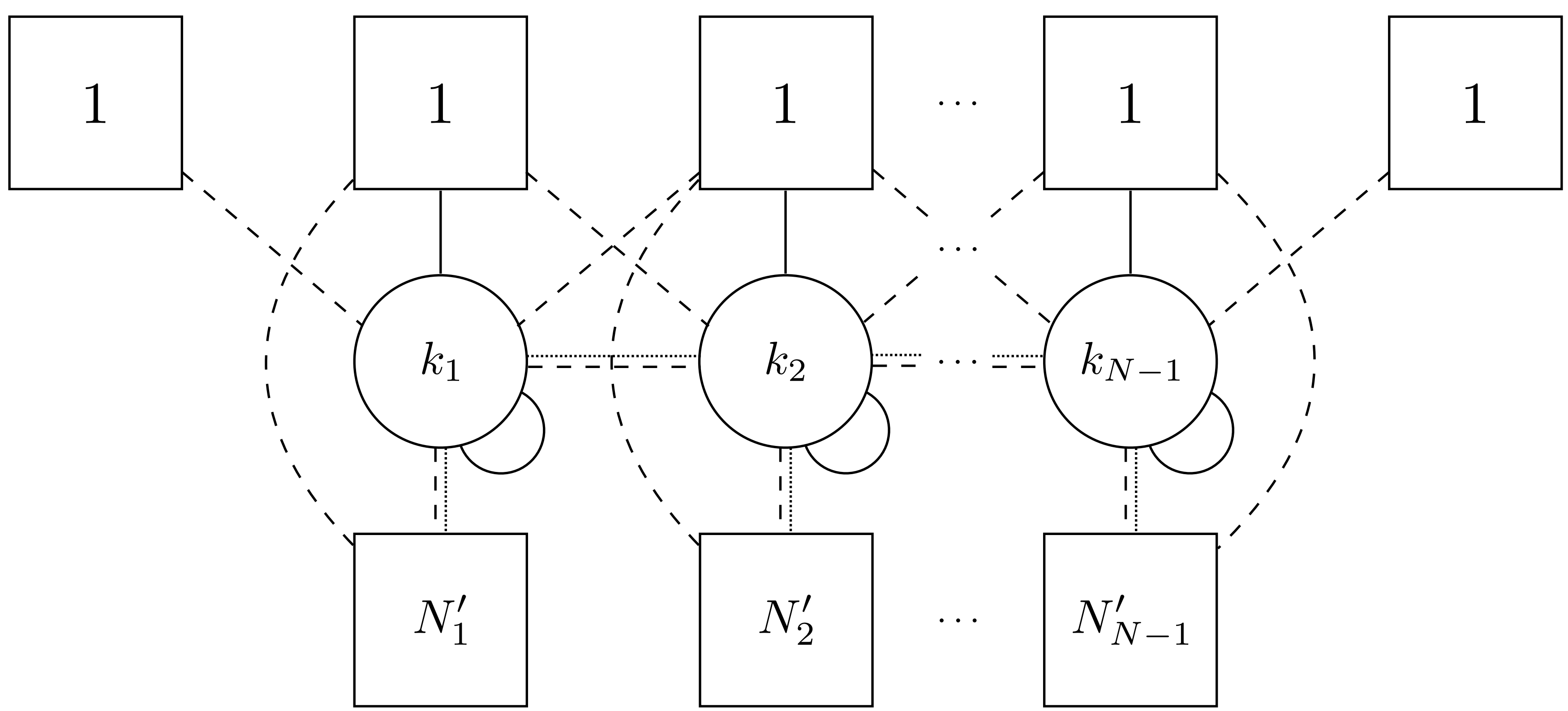
Proposal: brane set-up for $\langle \mathcal{S}^{(N')}_{2d} \rangle$, containing 6d Wilson surfaces

$$\langle \mathcal{S}^{(N')}_{2d} \rangle = \mathcal{S}^{(N')}_{2d} / Z_{6d}$$

Localization computation result factorized as

$$\mathcal{S}_{2d}^{(N')} = Z_{6d}^{pert}\, \mathcal{S}_{2d}^{(N'),\, str}\,, \qquad \mathcal{S}_{2d}^{(N'),\, str} = \sum_{k_1,\ldots,k_{N-1}\geqslant 0} \prod_{i=1}^{N-1} \alpha_{i,i+1}^{-k_i}\, \mathbb{E}_{M_{str}{}'}^{(k_1,\ldots,k_{N-1})}$$

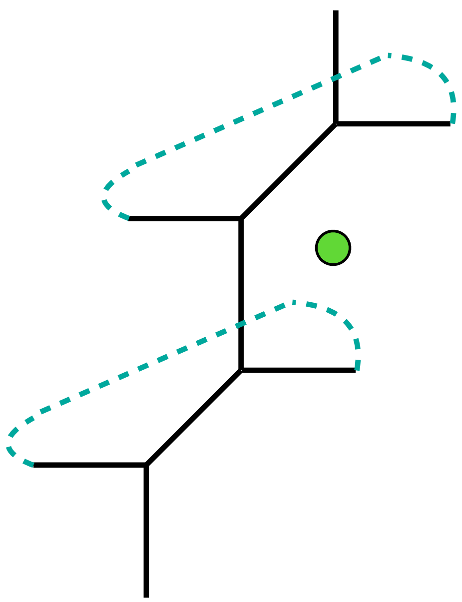
with $\mathbb{E}_{M_{str}{}'}^{(k_1,\ldots,k_{N-1})}$ elliptic genus of modified M-strings ($N' = N'_1 + \ldots + N'_{N-1}$)

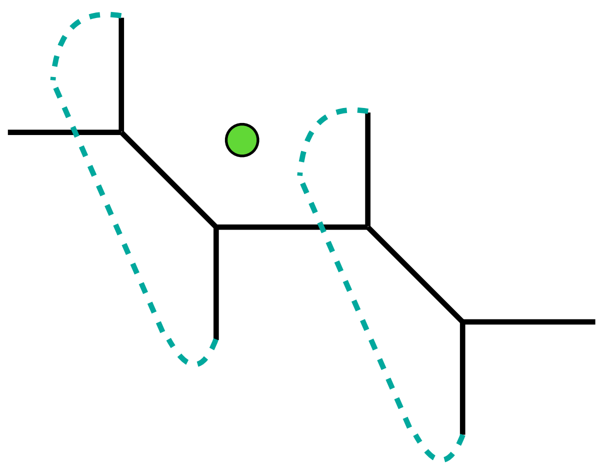


$\langle \mathcal{S}_{2d}^{(N')} \rangle$ expected to contain 6d Wilson surfaces (S-dual to 5d Wilson loops);

explicit computations confirm this expectation, for $N' = 1$ D4':

- Example: $N = 2\,,\ N' = 1$

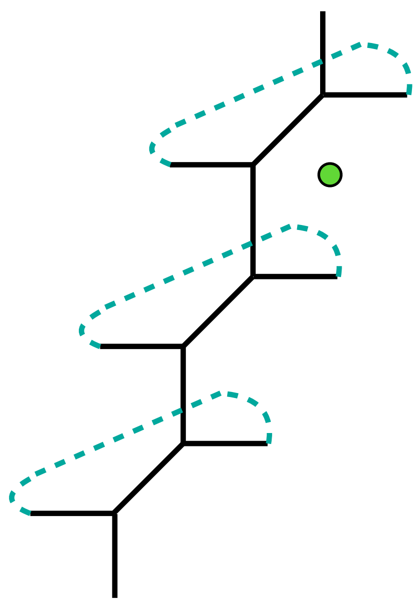




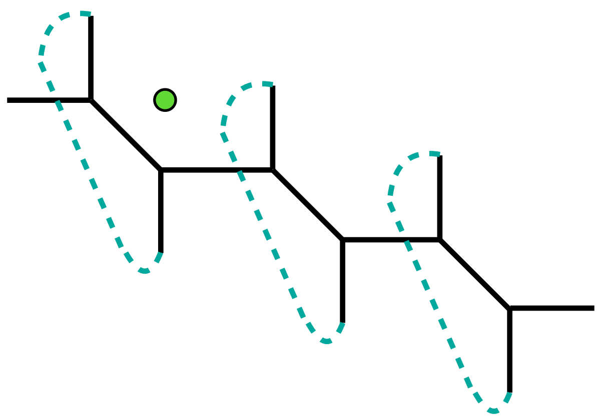
$$\langle W_{\mathbf{2}}^{SU(2)} \rangle = -\oint \frac{dx_1}{x_1} \langle\langle \mathcal{L}_{SQM}^{(1)} \rangle\rangle \qquad\qquad \langle \mathcal{S}_{2d}^{(1)} \rangle \ = \ \theta_1(\ln x_1)\, \langle W_{\mathbf{2}}^{SU(2)} \rangle$$

$\langle W_{\mathbf{2}}^{SU(2)} \rangle$ obtained by removing $\theta_1(\ln x_1)$ or taking $\langle W_{\mathbf{2}}^{SU(2)} \rangle = \oint \frac{dx_1}{x_1} \frac{1}{x_1^{1/2}} \langle \mathcal{S}_{2d}^{(1)} \rangle$

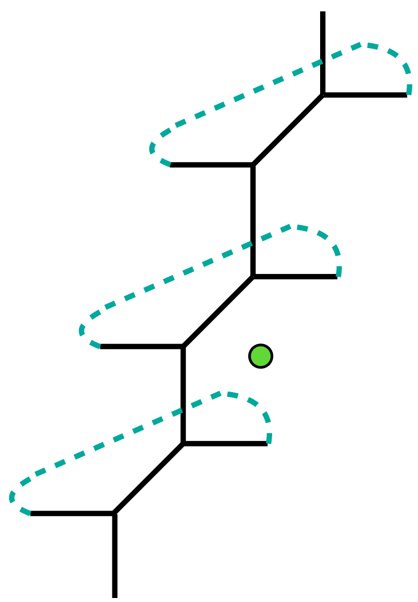
- Example: $N = 3\,,\ N' = 1$



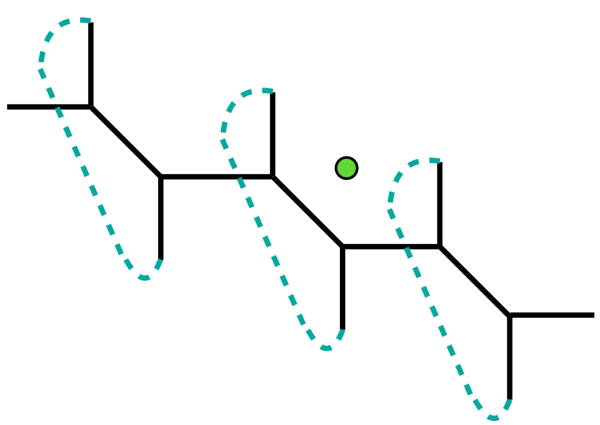
$$\langle W_{\mathbf{3}}^{SU(3)} \rangle = -\oint \frac{dx_1}{x_1} \frac{1}{x_1^{1/2}} \langle\langle \mathscr{L}_{SQM}^{(1)} \rangle\rangle$$



$$\langle \mathcal{S}_{2d}^{(1,0)} \rangle = \theta_1(\ln x_1) \langle W_{\mathbf{3}}^{SU(3)} \rangle$$



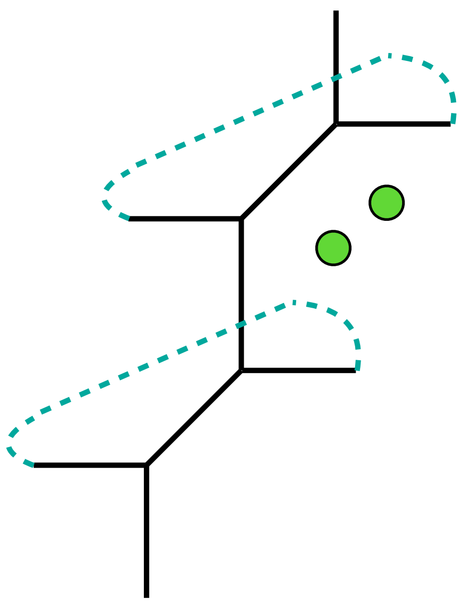
$$\langle W_{\bar{\mathbf{3}}}^{SU(3)} \rangle = \oint \frac{dx_1}{x_1} \frac{1}{x_1^{-1/2}} \langle\langle \mathscr{L}_{SQM}^{(1)} \rangle\rangle$$

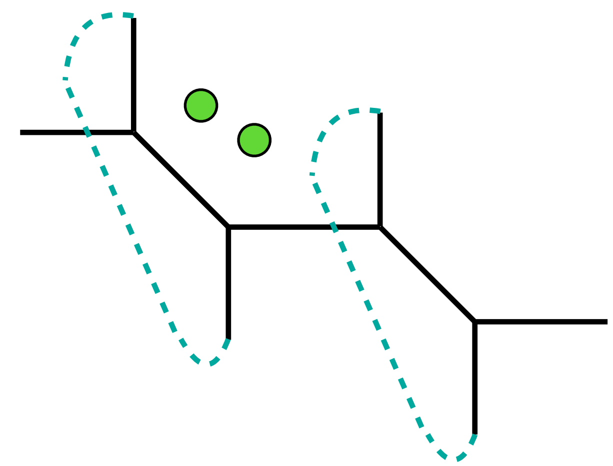


$$\langle \mathcal{S}_{2d}^{(0,1)} \rangle = \theta_1(\ln x_1/\mu) \langle W_{\bar{\mathbf{3}}}^{SU(3)} \rangle$$

The situation is still partially unclear for $N' > 1$ D4':

- Example: $N = 2\,,\ N' = 2$





$$\langle \mathcal{S}^{(2)}_{2d} \rangle \;=\; \text{complicated}$$

$$\langle W^{SU(2)}_{\mathbf{2}\otimes\mathbf{2}} \rangle \;=\; \oint \frac{dx_1}{x_1}\frac{dx_2}{x_2}\, \langle\langle \mathscr{L}^{(2)}_{SQM} \rangle\rangle$$

$$\langle W^{SU(2)}_{\mathbf{2}\otimes\mathbf{2}} \rangle \;\overset{???}{=}\; \oint \frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{1}{x_1^{1/2}}\frac{1}{x_2^{1/2}}\, \langle \mathcal{S}^{(2)}_{2d} \rangle$$

The two objects match, apart from the 1-string sector (wrong measure?)

Summary

Wilson loops for 5d $\mathcal{N}=1\,,\,1^*$ $SU(N)$ theories on $\mathbb{R}^4_{\epsilon_{1,2}}\times S^1_R$:

- In brane picture, properly defined via F1 ending on D3 at finite distance

- D3 branes create 1d defect along S^1_R in 5d theory (coupling to a SQM)

- Partition function $\langle \mathscr{L}^{(N')}_{SQM}\rangle$ of coupled system contains information on

  5d Wilson loops in tensor product of minuscule (antisym) representation

- Nice S-duality transformation properties, enhanced flavor symmetry

- In $\mathcal{N}=1^*$ case, map to Wilson surfaces in M-string set-up (“almost”)

Thanks!