On Geometric Classification of 5d SCFTs

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POSTECH

Based on

arXiv : 1705.05836 with Patrick Jefferson, Cumrun Vafa, Gabi Zafrir
arXiv : 1801.04036 with Patrick Jefferson, Sheldon Katz, Cumrun Vafa

Plan of talk

- Review of 5d N=1 SUSY CFTs.
- Geometric construction of 5d SCFTs.
- Rank 2 classification of 5d SCFTs.

Higher dimensional QFTs

Interacting conformal field theories (CFTs) in higher dimensions (d=5,6) have been constructed in string theory. [Witten 95], [Strominger 95], [Seiberg 96], …

They are all SUSY theories preserving 8 or 16 supersymmetries.

- Classification of 6d SCFTs

All N=(2,0) (or 16 SUSYs) CFTs can be constructed in Type IIB string theory on ADE singularities. [Witten 95]

Most N=(1,0) CFTs can be engineered using F-theory on elliptic CY3.
[Heckman, Morrison, Vafa 13], [Heckman, Morrison, Rudelius, Vafa 15], [Bhardwaj, Morrison, Tachikawa, Tomasiello 18],…

We are now interested in classification of 5d SCFTs.

5d Supersymmetric CFTs

Basic properties of 5d SCFTs

- N=1 SUSY (8 supersymmetries)
- Superconformal algebra F_4
  $SO(1,4)$ Lorentz symmetry + $SU(2)_R$ symmetry
- No marginal deformation.
- Relevant (mass) deformations associated to global symmetries.

Some 5d SCFTs admit gauge theory descriptions at low energy upon mass deformations $m \sim 1/g^2$ of global instanton symmetry.

$$\delta \mathcal{L} = \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu}$$

In this case, SCFT arises in UV fixed point when $g^2 \to \infty$ of this low energy effective gauge theory.

[Seiberg 96], [Intriligator, Morrison, Seiberg 97]

- Large class of 5d SCFTs can be engineered by

  (p,q) 5-brane webs in Type IIB — [Aharony, Hanany 97], [Aharony, Hanany, Kol 97], [DeWolfe, Iqbal, Hanany, Katz 99], …

  M-theory on non-compact CY3 — [Morrison, Seiberg 96], [Douglas, Katz, Vafa 96], [Katz, Klemm, Vafa 96], [Intriligator, Morrison, Seiberg 97], …

  - Their low energy (gauge theory) descriptions exhibit interesting strong coupling phenomena such as dualities and symmetry enhancements.

- 5d Kaluza-Klein (KK) theories

  Some special cases lead to 5d theories showing emergent spacetime (or Kaluza-Klein circle) symmetry and they are promoted to 6d SCFTs on a circle. So their UV completions are not 5d SCFTs.

  ex) $\mathcal{N}=2$ gauge theories $\longrightarrow$ 6d (2,0) ADE CFTs on a circle

  [Seiberg 96], [Douglas 10], [Lambert, Papageorgakis 10], …

Effective Prepotential on Coulomb branch

Coulomb branch moduli space of 5d SCFTs is parametrized by vacuum expectation values of the real scalar fields ϕ_i in vector multiplet (or dynamical Kahler parameters of CY3).

Low energy physics on Coulomb branch is characterized by "prepotential" (or triple intersections of CY3).

$$\mathcal{F} = \frac{1}{2g_0^2} h_{ij}\phi^i\phi^j + \frac{\kappa}{6} d_{ijk}\phi^i\phi^j\phi_k + \frac{1}{12}\left(\sum_{e\in\mathrm{root}} |e\cdot\phi|^3 - \sum_f \sum_{w\in\mathbf{r}_f} |w\cdot\phi + m_f|^3\right)$$

g_0 : gauge coupling, m_f : masses, $h_{ij} = \mathrm{Tr}(T_iT_j)$, $d_{ijk} = \mathrm{Tr}T_{(i}T_jT_{k)}$, κ : CS − level for $SU(N>2)$

[Witten 96], [Seiberg 96], [Intriligator, Morrison, Seiberg 97]

- Effective coupling : $\tau_{ij} = \partial_i\partial_j\mathcal{F}$
- Metric on Coulomb branch : $ds^2 = \tau_{ij}d\phi^i d\phi^j$
- Tension of magnetic monopole string : $T_{M_i} \sim \phi_{Di} \equiv \partial_i\mathcal{F}$

Classification of 5d gauge theories

5d gauge theories having UV fixed points were classified using the condition that metric on Coulomb branch is non-negative everywhere.

- $\mathrm{eigen}(\tau_{ij}(\phi)) > 0$ with $\phi \in \mathcal{C}$ [Seiberg 96], [Intriligator, Morrison, Seiberg 97]

  where $\mathcal{C}$ is Coulomb branch where all perturbative states have $m^2 \geq 0$

This old condition has been generalized such as

> 5d gauge theory has an interacting CFT fixed point when
>
> $\mathrm{eigen}(\tau_{ij}(\phi)) > 0$ with $\phi \in \mathcal{C}_{\mathrm{phys}}$ [Jefferson, H-C. Kim, Vafa, Zafrir 17]
>
> Here, $\mathcal{C}_{\mathrm{phys}} \subset \mathcal{C}$ is subset of Coulomb branch where all states have $m^2 \geq 0$.

Since 5d gauge theories involve non-perturbative instanton states, it would be very difficult to perform full classification using this generalized criterion.

Classification of 5d gauge theories with single gauge node

5d gauge theory has an interacting CFT fixed point when

$$\mathrm{eigen}(\tau_{ij}(\phi)) > 0 \quad \text{with} \quad \phi \in \mathcal{C}_{\rm phys}$$ [Jefferson, H-C. Kim, Vafa, Zafrir 17]

Here, $\mathcal{C}_{\rm phys} \subset \mathcal{C}$ is subset of Coulomb branch where all states have $m^2 \geq 0$.

Instead, we consider relaxed condition as

$$\mathrm{eigen}(\tau_{ij}(\phi)) > 0 \quad \text{with} \quad \phi \in \tilde{\mathcal{C}}$$

where $\tilde{\mathcal{C}}$ is subset of Coulomb branch with non-negative masses (or tensions) of all perturbative states and monopole strings, i.e. $m^2_{\rm pert} \geq 0,\ T \geq 0$.

Using this condition, "possibly" non-trivial gauge theories with single gauge node are classified in [Jefferson, H-C. Kim, Vafa, Zafrir 17] .

Ex : All rank 2 theories

$N_{\mathbf{Sym}}$	$N_{\mathbf{F}}$	$\lvert k \rvert$
1	0	$\frac{3}{2}$
1	1	0
0	10	0
0	9	$\frac{3}{2}$
0	6	4
0	3	$\frac{13}{2}$
0	0	9

(a) Marginal $SU(3)$ theories with CS level k, $N_{\mathbf{Sym}}$ symmetric and $N_{\mathbf{F}}$ fundamental hypermultiplets.

$N_{\mathbf{AS}}$	$N_{\mathbf{F}}$
3	0
2	4
1	8
0	10

(b) Marginal $Sp(2)$ gauge theories with $N_{\mathbf{AS}}$ anti-symmetric, $N_{\mathbf{F}}$ fundamental hypermultiplets. The theory with $N_{\mathbf{AS}} = 3$ can have $\theta = 0, \pi$.

$N_{\mathbf{F}}$
6

(c) A marginal G_2 gauge theory with $N_{\mathbf{F}}$ fundamental matters.

- These are 5d KK theories whose UV completions (if exist) are 6d SCFTs on a circle.
- Their descendants by integrating out matter hypers give rise to 5d SCFTs.
- New predictions : ex) $SU(3)_{\kappa=6,7,8}, \; G_2 \text{ w/ } N_{\mathbf{F}} = 5, \cdots$

Q) Do they all really have UV CFT fixed points?

Geometric construction of 5d CFTs

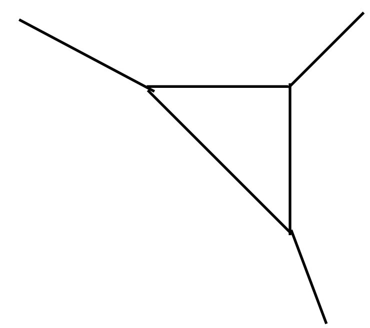
M-theory on Calabi-Yau threefold

11d M-theory compactified on a ‘contractible’ Calabi-Yau threefold X_6 will engineer a 5d SCFT.

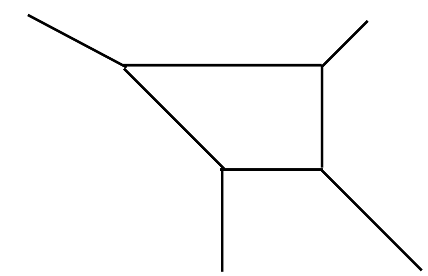
[Morrison, Sieberg 96], [Douglas, Katz, Rafa 96], [Intriligator, Morrison, Seiberg 97]

- Contractible CY3 : Surfaces $S_i \subset X_6$ can contract to a singular point.

Ex : del Pezzo surfaces dP_n in CY3
$n < 9$



$dP_0 = \mathbb{P}^2$



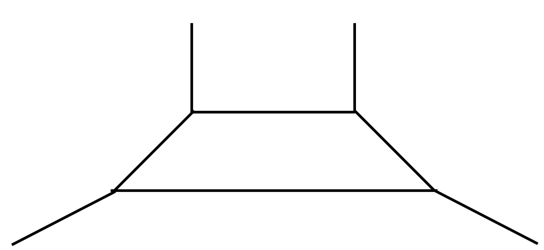
$dP_1 \;\rightarrow\; SU(2)_{\theta=\pi}$

- In fact, ‘contractible CY3’ can be generalized to ‘shrinkable CY3’.

- Shrinkable CY3 : $S_i \subset X_6$ can contract to a point or non-compact 2-cycles.

[Jefferson, Katz, H-C. Kim, Vafa 2018]

Ex :



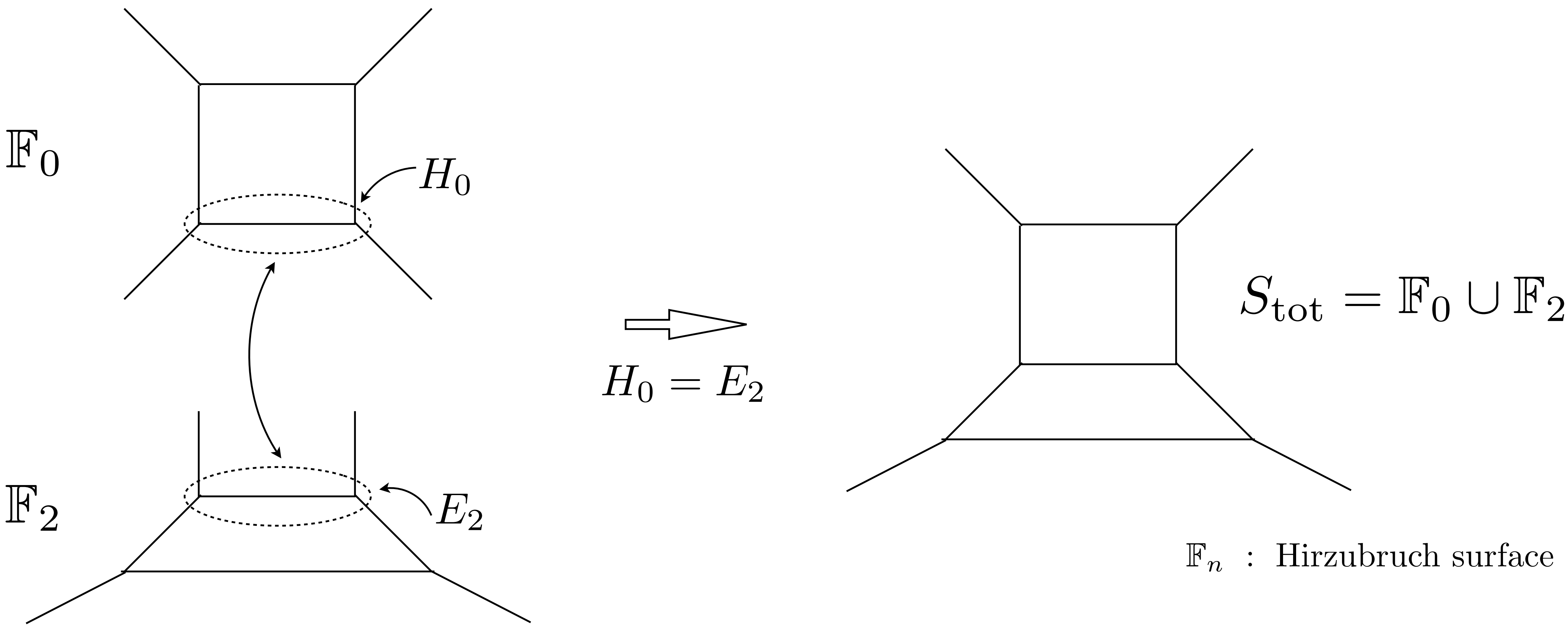
$\mathbb{F}_2 \;\rightarrow\; SU(2)_{\theta=0}$

$\mathbb{F}_n \;:\; \text{Hirzubruch surface}$

Gluing surfaces in Calabi-Yau threefold

Generic shrinkable CY3 can be constructed by gluing rank 1 surfaces.

Ex) Gluing two surfaces $\mathbb{F}_0$ and $\mathbb{F}_2$ yields a rank 2 surface $S_{\text{tot}} = \mathbb{F}_0 \cup \mathbb{F}_2$.



$\mathbb{F}_n$: Hirzubruch surface

- We glue base class H_0 in $\mathbb{F}_0$ and section E_2 in $\mathbb{F}_2$.
- Final CY3 embedding $S_{\text{tot}} = \mathbb{F}_0 \cup \mathbb{F}_2$ is a smooth CY3 corresponding to $SU(3)$ gauge theory with CS-level $\kappa = 1$.

Construction algorithm of 'Shrinkable CY3's

All shrinkable CY3 are constructed by a gluing rank 1 surfaces $S_{\text{tot}} = \cup_i S_i$.

1. Building blocks S_i

   a. Hirzebruch surfaces and their blowups $Bl_p(\mathbb{F}_n)$. p : # of blowups

   b. del Pezzo surfaces dP_n .

2. Two surfaces S_i and S_j are glued along curve $C_g = S_1 \cap S_2$

   a. C_g is a smooth irreducible rational curve.

   b. $(C_g|_1)^2 + (C_g|_2)^2 = -2$.

3. All 2-cycles have non-negative volumes (when all masses are turned off).

$$Vol(C) = -C \cdot J \geq 0 \ , \quad C \subset S_{\text{tot}} \qquad J = \sum_i \phi_i S_i, \quad \phi_i > 0$$

4. At least one 4-cycle has positive volume.

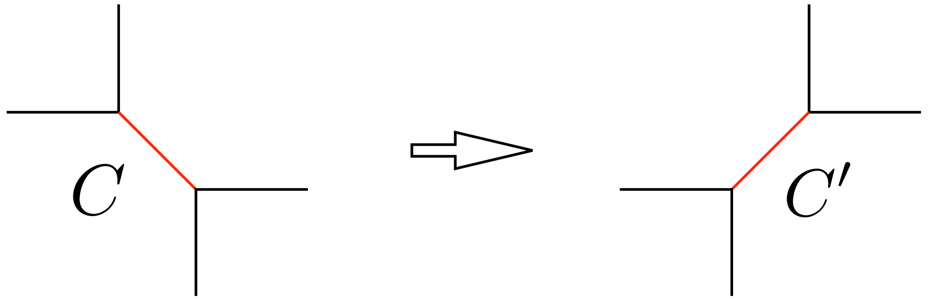
- Dimension (or rank) of Coulomb branch = number of compact surfaces

Deformation Equivalence of CY3's

Different geometries can give the same SCFT (up to decoupled free sector) when all Kahler parameters are turned off.

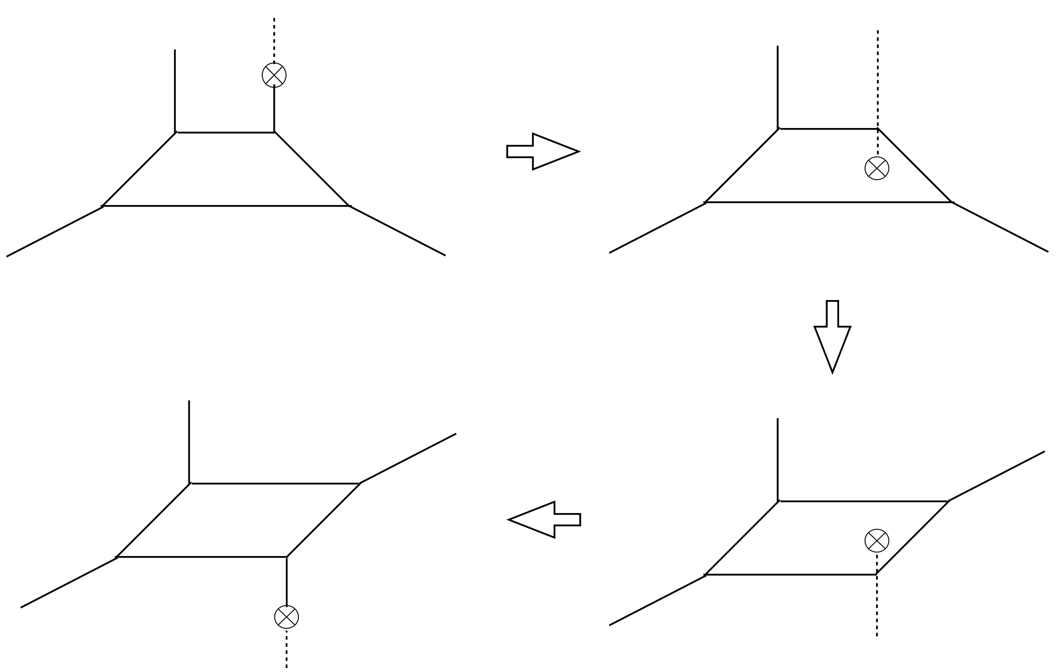
- We claim that geometries are 'Deformation Equivalent' having the same CFT fixed point if they are related by

1. Flop : $Vol(C) = -Vol(C')$
   $C^2 = C'^2 = -1$



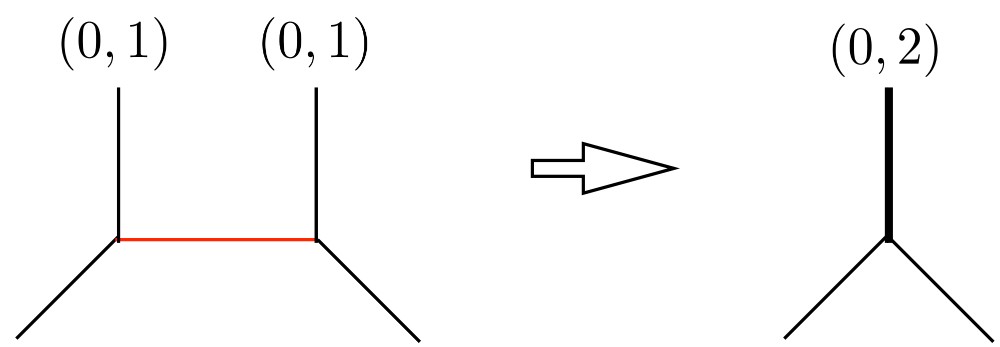
2. Hanany-Witten (HW) transition : a complex structure deformation

ex) $\mathbb{F}_2 \ \rightarrow \ \mathbb{F}_0$

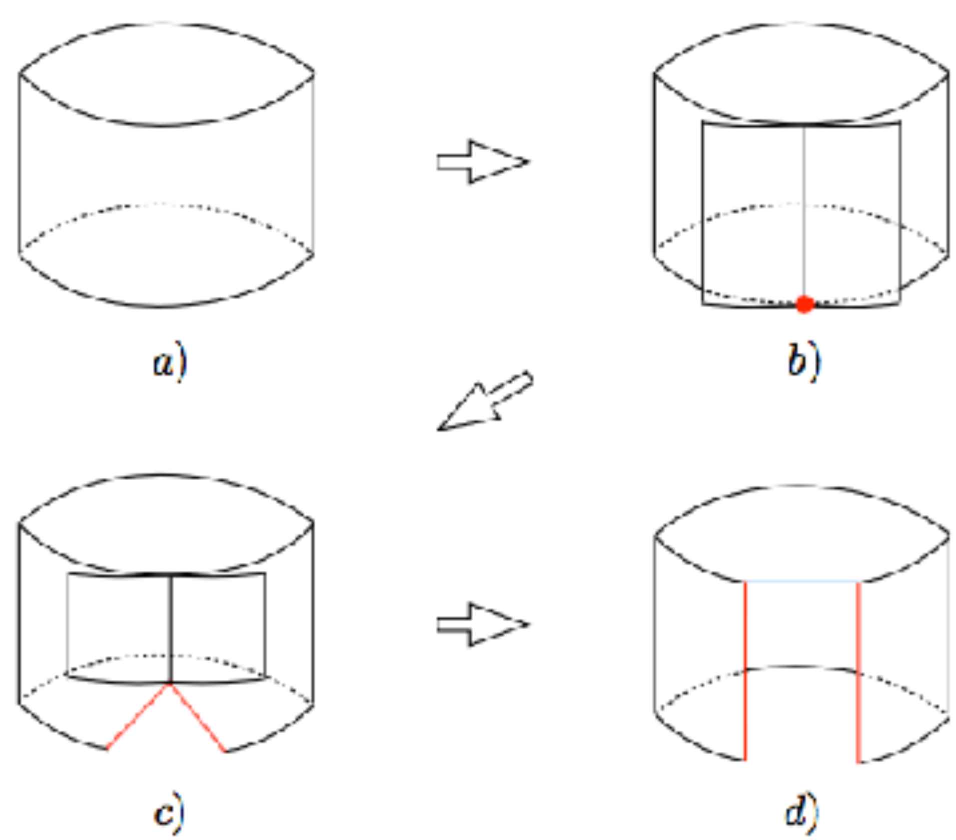


Deformation Equivalence of CY3's

3. Complex structure deformation by tuning mass parameters.



4. Genus reduction : $\mathbb{F}_n^g \rightarrow Bl_{2g}\mathbb{F}_n$ with g self-gluings



Rank 1 classification

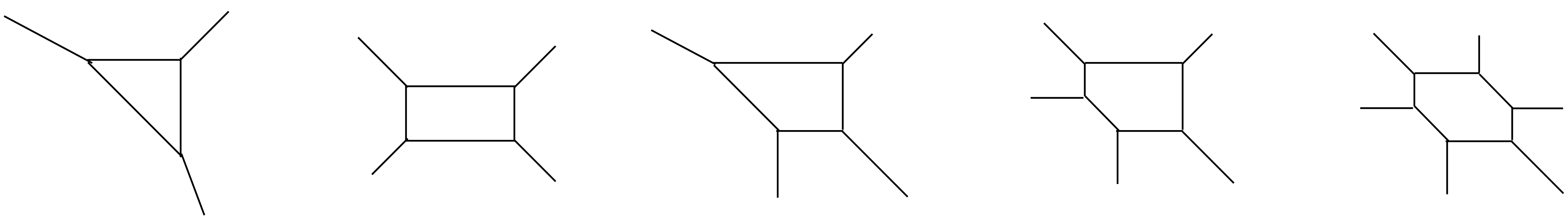
All rank 1 SCFTs are engineered by CY3s of del Pezzo surfaces $dP_{n\leq 7}$ and a Hirzebruch surface $\mathbb{F}_0$.

[Morrison, Seiberg 96], [Douglas, Katz, Vafa 96], [Intriligator, Morrison, Seiberg 97]

- Classification :

S	G	M
$\mathbb{P}^2$	$\cdot$	0
$\mathbb{F}_0$	$SU(2)_{\theta=0}$	1
$dP_1=\mathbb{F}_1$	$SU(2)_{\theta=\pi}$	1
$dP_{n>1}$	$SU(2), N_f=n-1$	n

- Brane constructions



$\mathbb{P}^2$ $\mathbb{F}_0$ dP_1 dP_2 dP_3

Note that all rank 1 SCFTs can be obtained from dP_9 corresponding to 6d E-string theory on a circle by mass deformations.

Rank 2 classification

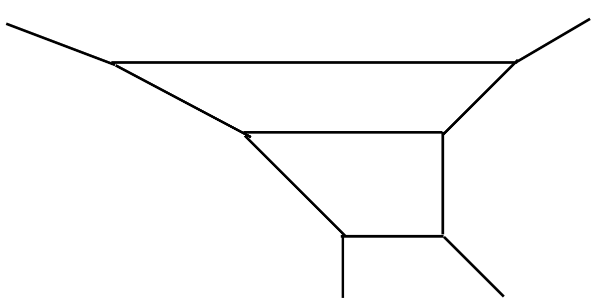
We claim that

> All rank 2 shrinkable CY3 can be realized as $S = S_1 \cup S_2$ for which $S_1 = Bl_p\mathbb{F}_m$ and $S_2 = dP_n$ or $\mathbb{F}_0$.

1. $Bl_p\mathbb{F}_m$ is a blowup of $\mathbb{F}_m$ at p ***generic*** points.
2. Two surfaces are glued along rational curves $C_1 \subset S_1, \; C_2 \subset S_2$.
3. Gluing curves satisfy $C_1^2 + C_2^2 = -2$.
4. $C_1 = E, \; C_1^2 = -m$.

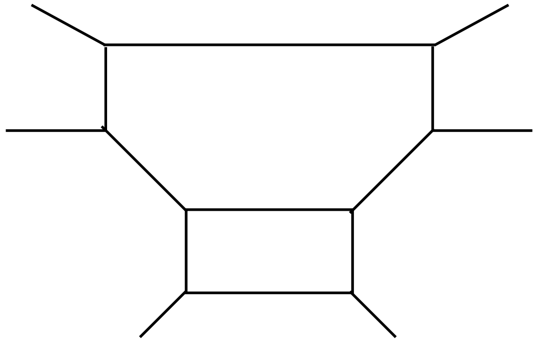
Ex)

1. $SU(3)_2$



$\mathbb{F}_3 \cup dP_1 , \; C_2 = H, \; H^2 = 1$

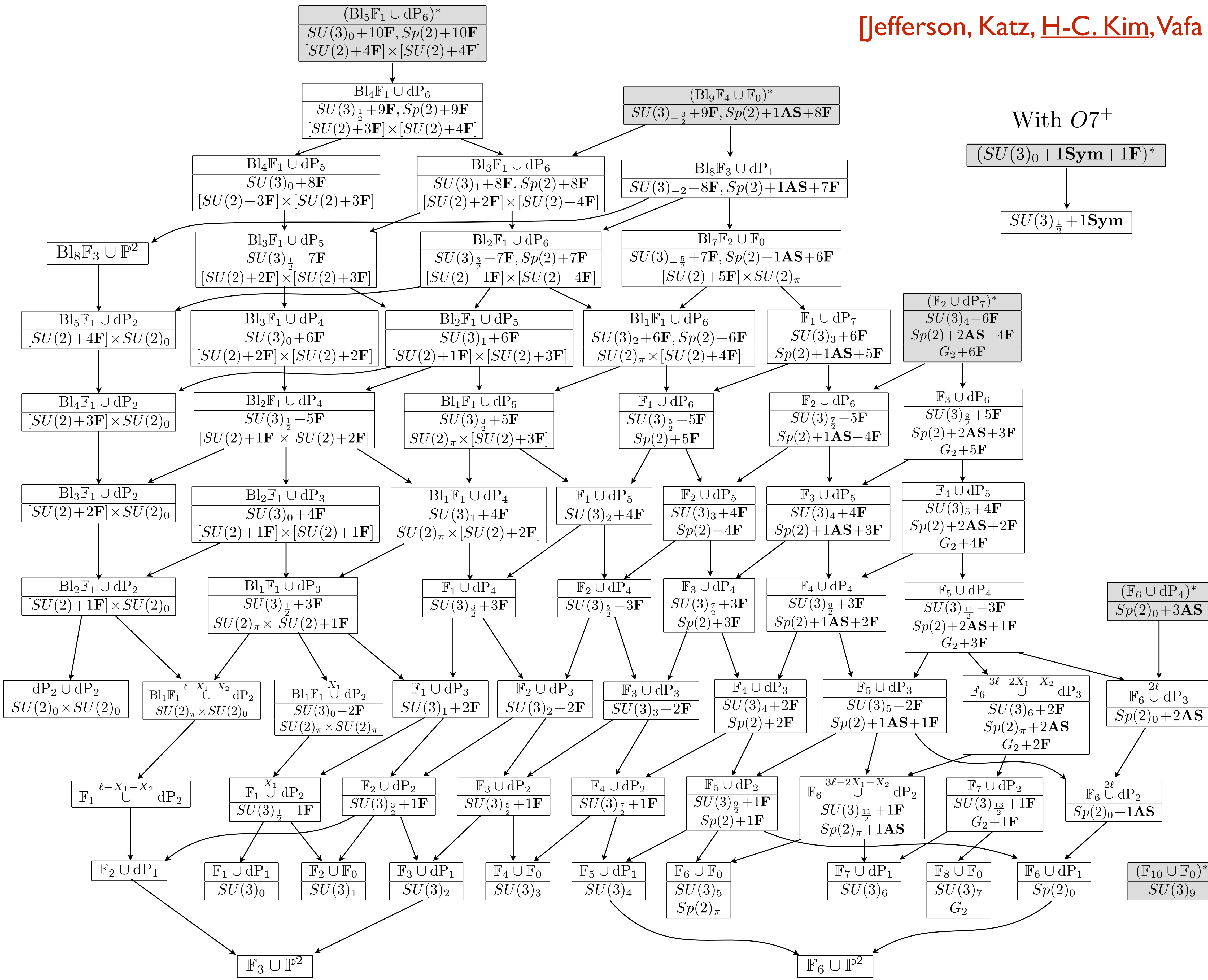
2. $SU(3)_0 + 2F$



$Bl_2\mathbb{F}_2 \cup \mathbb{F}_0, \; C_2 = H, \; H^2 = 0$

Full Classification of rank 2 shrinkable CY3

[Jefferson, Katz, H-C. Kim, Vafa 2018]



- All rank 1 & 2 5d SCFTs arise from mass deformations of 6d SCFTs on a circle!

- Our geometric construction involves all known rank 2 5d SCFTs except one theory, $SU(3)_{\frac{1}{2}} + 1\mathbf{Sym}$ which is realized with frozen singularity $O7^+$.

- All rank 2 SCFTs admit brane construction in Type IIB.

[Bergman, Zafrir 15], [Hayashi, S.S. Kim, K. Lee, Yagi 15, 18]

Geometry and gauge theory : rank 2 with M=1

Gauge theory analysis predicts

$SU(3)_\kappa, 0 \le \lvert\kappa\rvert \le 9$
$Sp(2), \theta = 0, \pi$
G_2

[Jefferson, H-C. Kim, Vafa, Zafrir 2015]

Geometric classification :

$S_1 \cup S_2$	C_1	G	$S_1 \cup S_2$	C_1	G
$\mathbb{F}_0 \cup \mathbb{F}_2$	F_1	$SU(3)_1$	$\mathbb{F}_0 \cup \mathbb{F}_8$	$F_1 + 3H_1$	$SU(3)_7, G_2$
$\mathbb{F}_0 \cup \mathbb{F}_4$	$F_1 + H_1$	$SU(3)_3$	$\mathbb{F}_1 \cup \mathbb{F}_1$	E_1	$SU(3)_0$
$\mathbb{F}_0 \cup \mathbb{F}_6$	$F_1 + 2H_1$	$SU(3)_5, Sp(2)_{\theta=0}$	$\mathbb{F}_1 \cup \mathbb{F}_7$	$2F_1 + H_1$	$SU(3)_6$

(a) Endpoint geometries with $M = 1$. Here $C_2 = E_2$.

$S_1 \cup S_2$	C_1	G	Endpoint
$\mathbb{F}_1 \cup \mathbb{F}_2$	F_1	$SU(2)\hat{\times}SU(2)$	$\mathbb{P}^2 \cup \mathbb{F}_3$
$\mathbb{F}_1 \cup \mathbb{F}_3$	H_1	$SU(3)_2$	$\mathbb{P}^2 \cup \mathbb{F}_3$
$\mathbb{F}_1 \cup \mathbb{F}_5$	$F_1 + H_1$	$SU(3)_4$	$\mathbb{P}^2 \cup \mathbb{F}_6$
$\mathbb{F}_1 \cup \mathbb{F}_6$	$2H_1$	$Sp(2)_{\theta=\pi}$	$\mathbb{P}^2 \cup \mathbb{F}_6$
$\mathbb{F}_1 \cup \mathbb{F}_{10}$	$F_1 + 4H_1$	$SU(3)_9$	6d?

(b) Other geometries of $\mathbb{F}_{n_1} \cup \mathbb{F}_{n_2}$ with M=1.

Note that no shrinkable geometry for $SU(3)_8$ gauge theory!

Dualities from geometry

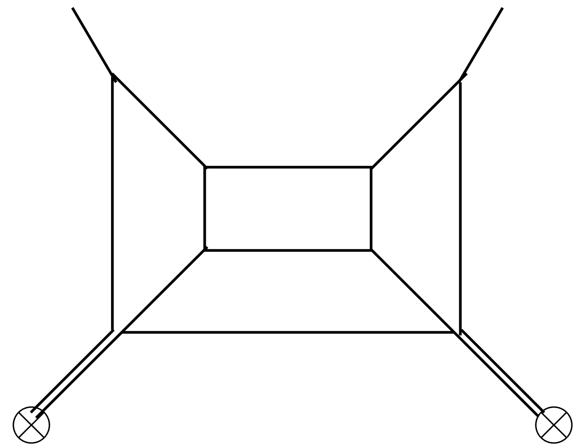
Geometric duality can lead to dualities between gauge theories.

- Fiber class W_i in each surface $S_i \subset S = \cup_i S_i$ can form a Cartan matrix $A_{ij}(G)$ of Lie group G. Namely,

$$-W_i S_j = A_{ij}(G)$$

[Intriligator, Morrison, Seiberg 97]

- Choice of fiber classes is not unique.
- Different choices correspond to different gauge theory descriptions.
- Ex :



$\mathbb{F}_0 \cup \mathbb{F}_6\,,\ C_1 = H_1 + 2F_1$
$,\ C_2 = E_2 \subset \mathbb{F}_6$

Since $F_1^2 = H_1^2 = 0$, we have two choices :

1. $W_1 = F_1,\ W_2 = F_2 \ \rightarrow\ SU(3)_5$
2. $W_1 = H_1,\ W_2 = F_2 \ \rightarrow\ Sp(2)_{\theta=\pi}$

Thus, $F_1 \leftrightarrow H_1$ leads to $SU(3)_5 \leftrightarrow Sp(2)$ duality.

[Gaiotto, H-C. Kim 2015]

New dualities from geometry

- $SU(3)_7 \leftrightarrow G_2$ duality from $\mathbb{F}_0 \cup \mathbb{F}_8$
  - Gluing curves are $C_1 = H_1 + 3F_1 \subset \mathbb{F}_0,\ C_2 = E_2 \subset \mathbb{F}_8$.
  - Two fiber class choices : $1.\ W_1 = F_1,\ W_2 = F_2 \ \rightarrow\ SU(3)_7$
    $2.\ W_1 = H_1,\ W_2 = F_2 \ \rightarrow\ G_2$

- $SU(3)_6 + 2\mathbf{F} \ \leftrightarrow\ G_2 + 2\mathbf{F} \ \leftrightarrow\ Sp(2)_{\theta=\pi} + 2\mathbf{AS}$ duality from $dP_3 \cup \mathbb{F}_6$
  - dP_3 has three exceptional curves X_1, X_2, X_3 with self-intersection '-1'.
  - Gluing curves are $C_1 = 3l - X_1 - 2X_2,\ C_2 = E_2 \subset \mathbb{F}_6$.
  - Three fiber class choices : $1.\ W_1 = l - X_1,\ W_2 = F_2 \ \rightarrow Sp(2), N_A = 2$
    $2.\ W_1 = l - X_2,\ W_2 = F_2 \ \rightarrow SU(3)_6, N_f = 2$
    $3.\ W_1 = l - X_3,\ W_2 = F_2 \ \rightarrow G_2, N_f = 2$

Summary and future directions

- We proposed a systematic way to construct shrinkable Calabi-Yau threefolds which give 5d SCFTs.
- We gave full classification of rank 2 shrinkable CY3.
- Geometric constructions confirm gauge theory predictions and also provide new dualities.

Future directions

- Gauge theory classification including non-perturbative analysis.
- Full classification of 5d SCFTs and shrinkable Calabi-Yau threefolds.

Thank you very much !