Witten, Cardy, and Holonomy Saddles

PILJIN YI
Korea Institute for Advanced Study

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K. Hori, H. Kim, P.Y. 2014
S-J. Lee, P.Y. 2016
S-J. Lee, P.Y. 2017
C. Hwang, P.Y. 2017
C. Hwang, S. Lee, P.Y. 2018

an old problem and an old puzzle

1d witten index via localization, or not

how "H-saddles" resolve the puzzle

gluing gauge theories across dimensions

back in 1997

$$\frac{5}{4} = 1 + \frac{1}{4}$$

P.Y. 1997
Sethi, Stern 1997

in an effort to confirm the M-theory hypothesis, of course

M on $\mathbf{S}^1 \times \mathcal{R}^{9+1}$ = IIA on $\mathcal{R}^{9+1}$

IIA theory must remember
this M-theory origin

by forming an infinite tower of
multi D-particle bound states
moving freely on $\mathcal{M}_{9+1}$

$$\mathcal{I}_{\mathcal{N}=16}^{SU(N)} = 1$$

Witten 1995

back in 1997

$$\mathcal{I} = \lim_{\beta\to\infty} \mathrm{Tr}(-1)^{\mathcal{F}} e^{-\beta H}$$

↑

$$\mathcal{I}^{SU(2)}_{\mathcal{N}=16:\mathrm{bulk}} = \mathcal{I}^{SU(2)}_{\mathcal{N}=16} - \delta\mathcal{I}^{SU(2)}_{\mathcal{N}=16}$$

$$\frac{5}{4} = 1 + \frac{1}{4}$$

which is, perhaps, one of the most convoluted ways to obtain '1'

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↑

$$\lim_{\beta\to 0} \mathrm{Tr}(-1)^{\mathcal{F}} e^{-\beta H} \rightarrow \mathcal{Z}^{SU(2)}_{\mathcal{N}=16} = \int_{SU(2)/Z_2} dX\, d\Psi\;\; e^{-[X,X]^2/4 + X_\mu \Psi \Gamma_\mu \Psi/2}$$

P.Y. / Sethi, Stern 1997

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$$\mathcal{I}^{U(1)/Z_2}_{\mathcal{N}=16;\mathrm{bulk}}$$

P.Y. 1997

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P.Y. / Sethi, Stern 1997

the continuum contribution $-\delta\mathcal{I}$ localizes to the boundary
→ $-\delta\mathcal{I}$ is computable via the asymptotic Coulomb dynamics

P.Y. 1997

$$-\delta\mathcal{I}_{\mathcal{N}}^{SU(2)}$$

$$= -\delta\mathcal{I}_{\mathcal{N}}^{U(1)/Z_2}$$

$$= \quad \mathcal{I}_{\mathcal{N};\mathrm{bulk}}^{U(1)/Z_2}$$

arbitrary high rank cases followed, soon

$$\mathcal{I}^{SU(N)}_{\mathcal{N}=16:\mathrm{bulk}} = \mathcal{I}^{SU(N)}_{\mathcal{N}=16} - \delta\mathcal{I}^{SU(N)}_{\mathcal{N}=16}$$

$$\mathcal{Z}^{SU(N)}_{\mathcal{N}=16} = \sum_{p|N;p\geq 1} \frac{1}{p^2} = 1 + \sum_{p|N;p>1} \frac{1}{p^2}$$

Nekrasov, Moore, Shatashvili 1998

Green, Gutperle 1997
Kac, Smilga 1999

$$\mathcal{I}^{SU(N)}_{\mathcal{N}=16} = 1$$

similar problems with smaller supersymmetry address
Seiberg-Witten vs. IIA theory on local Calabi-Yau conifold

$$\mathcal{I}^{SU(N)}_{\mathcal{N}=4,8:\mathrm{bulk}} = \mathcal{I}^{SU(N)}_{\mathcal{N}=4,8} - \delta\mathcal{I}^{SU(N)}_{\mathcal{N}=4,8}$$

$$\boxed{\mathcal{Z}^{SU(N)}_{\mathcal{N}=4,8} = \frac{1}{N^2}} = 0 + \boxed{\frac{1}{N^2} = \mathcal{I}^{U(1)^{N-1}/S_N}_{\mathcal{N}=4,8:\mathrm{bulk}}}$$

$$\Downarrow$$

$$\mathcal{I}^{SU(N)}_{\mathcal{N}=4,8} = 0$$

P.Y. 1997
Sethi, Stern 1997
Gutperle, Green 1997
Moore, Nekrasov, Shatashvili 1998

one would have naturally expected,
for other simple gauge groups…

$$\mathcal{I}^G_{\mathcal{N}=4,8:\text{bulk}} = \mathcal{I}^G_{\mathcal{N}=4,8} - \delta\mathcal{I}^G_{\mathcal{N}=4,8}$$

$$\mathcal{Z}^G_{\mathcal{N}=4,8} = 0 + \mathcal{I}^{U(1)^r/W_G}_{\mathcal{N}=4,8:\text{bulk}}$$

yet, …

⬆

$$\mathcal{I}^G_{\mathcal{N}=4,8} = 0$$

P.Y. 1997
Green, Gutperle 1997
Kac, Smilga 1999

P.Y. / Sethi, Stern 1997
Moore, Nakrasov, Shatashvili 1998
Staudacher 2000 / Pestun 2002

$\mathcal{N}=4$	$\mathcal{I}^G_{\text{bulk}} = -\delta\mathcal{I}^G$	$\mathcal{I}^G_{\text{bulk}} = \mathcal{Z}^G$
$SU(N)$	$\frac{1}{N^2}$	$\frac{1}{N^2}$
$Sp(2)$	$\frac{5}{32}$	$\frac{9}{64}$
$Sp(3)$	$\frac{15}{128}$	$\frac{51}{512}$
$Sp(4)$	$\frac{195}{2048}$	$\frac{1275}{16384}$
$Sp(5)$	$\frac{663}{8192}$	$\frac{8415}{131072}$
$Sp(6)$	$\frac{4641}{65536}$	$\frac{115005}{2097152}$
$Sp(7)$	$\frac{16575}{262144}$	$\frac{805035}{16777216}$
$SO(7)$	$\frac{15}{128}$	$\frac{25}{256}$
$SO(8)$	$\frac{59}{1024}$	$\frac{117}{2048}$
$SO(9)$	$\frac{195}{2048}$	$\frac{613}{8192}$
$SO(10)$	$\frac{27}{512}$	$\frac{53}{1024}$
$SO(11)$	$\frac{663}{8192}$	$\frac{1989}{32768}$
$SO(12)$	$\frac{1589}{32768}$	$\frac{6175}{131072}$
$SO(13)$	$\frac{4641}{65536}$	$\frac{26791}{524288}$
$SO(14)$	$\frac{1471}{32768}$	$\frac{5661}{131072}$
$SO(15)$	$\frac{16575}{262144}$	$\frac{92599}{2097152}$
G_2	$\frac{35}{144}$	$\frac{151}{864}$
F_4	$\frac{30145}{165888}$	$\frac{493013}{3981312}$

$$\mathcal{I}^G_{\mathcal{N}=4,8:\text{bulk}}$$

$$= -\delta\mathcal{I}^G_{\mathcal{N}=4,8}$$

$$= \mathcal{I}^{U(1)^r/W_G}_{\mathcal{N}=4,8;\text{bulk}}$$

?

$$\mathcal{I}^G_{\mathcal{N}=4,8:\text{bulk}}$$

$$= \mathcal{Z}^G_{\mathcal{N}=4,8}$$

$\mathcal{N}=4$	$\mathcal{I}^G_{\text{bulk}} = -\delta\mathcal{I}^G$	$\mathcal{I}^G_{\text{bulk}} = \mathcal{Z}^G$
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$SO(8)$	59	117

IIA with an orienti-point

M on $\mathbf{S}^1 \times \mathcal{R}^{0+1} \times \mathcal{R}^9/Z_2$ = IIA on $\mathcal{R}^{0+1} \times \mathcal{R}^9/Z_2$

anomaly cancelation requires a single chiral fermion supported on $\mathbf{S}^1 \times \mathcal{R}^{0+1}$

➡

Kaluza-Klein reduction generates two towers of fermionic harmonic oscillators, resulting in four Hilbert spaces whose partition functions constitute the two generating functions above

Dasgupt, Mukhi 1995

Kol, Hanany, Rajaraman 1999

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IIA theory must remember
this M-theory origin

by forming an infinite tower of
multi D-particle bound states
along fixed points of the orienti-point

Dasgupta, Mukhi 1995
Kol, Hanany, Rajaraman 1999
Kac, Smilga 1999
⋮
S.J.Lee, P.Y. 2016 & 2017

which requires

$$\sum_N \mathcal{I}^{SU(N)}_{\mathcal{N}=16} t^N = \frac{1}{1-t}$$

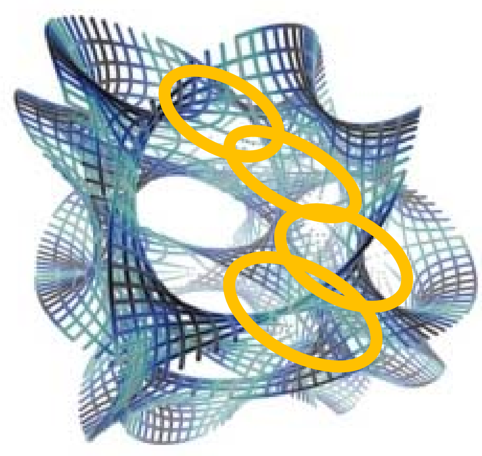
$$\sum_N \mathcal{I}^{O(N)}_{\mathcal{N}=16} t^N = \prod_{n=1}^{\infty} (1 + t^{2n-1})$$

$$\sum_N \mathcal{I}^{Sp(N)}_{\mathcal{N}=16} t^{2N} = \prod_{n=1}^{\infty} (1 + t^{2n})$$

but the program for proving the latter two was stuck, well before we come to this maximal supersymmetry then … jumping forward some 15 years

wall-crossing, rational invariants, quiver invariants, localization, black hole microstates, …

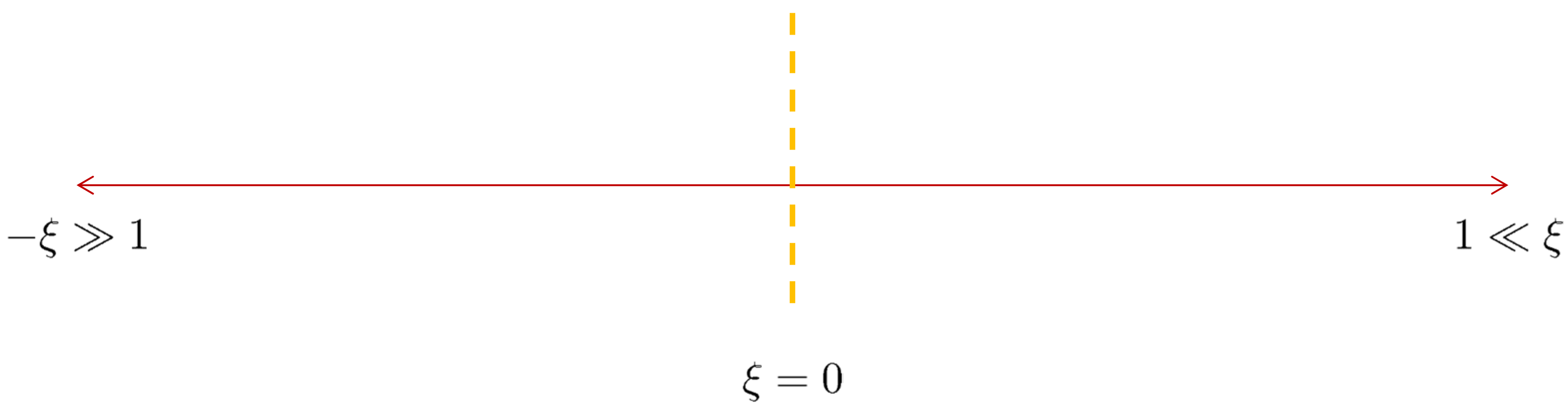
$$\underline{R^1} \times R^3 \times$$



1d/2d Gauged Linear Sigma Models with 4 Supercharges

$SU(2)_R \times U(1)_R$ gauge fields $(A_0, \lambda_\alpha, X_i, D)^a$ FI constants ξ^i for U(1)'s

$J_{1,2,3}$ R chirals $(X, \psi_\alpha, F)^I$



$\mathcal{I}$ as Ω

$$\mathcal{I}(\mathbf{y};x) \equiv \mathrm{Tr}_{\mathrm{kernel}(Q)}\left[(-1)^{2J_3}\mathbf{y}^{2(R+J_3)}x^{G_F}\right]$$

$$\Omega(\mathbf{y};x) \equiv \lim_{e^2\to 0}\mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right]$$

a sweeping generalization of geometric index theorem
via path-integral by Alvarez-Gaume, ~1983, to gauged systems

Hori, Kim, P.Y. 2014

with the naïve invariance of index under continuous deformation, or under the banner of "localization"

$$\mathcal{L}_{\text{vector}} = \frac{1}{e^2}\,\mathrm{Re}\left(\int d\theta^2\ \mathrm{tr}\, W_\alpha W^\alpha\right)$$

$$\mathcal{L}_{\text{chiral}} = \frac{1}{g^2}\int d\theta^2 d\bar{\theta}^2\ \mathrm{tr}\,\bar{\Phi} e^V \Phi$$

$$\mathcal{L}_{\text{usperpotential}} = \int d\theta^2 W(\Phi) + c.c.$$

$$\mathcal{L}_{\text{FI}} = \xi \int d\theta^2 d\bar{\theta}^2 \mathrm{tr}\, V$$

Benini, Eager, Hori, Tachikawa 2013
Hori, Kim, P.Y. 2014

scale up FI to send $e\xi$ to infinite for a reason to be explained,

then, after a long, long, long song and dance, ……

a Jeffrey-Kirwan contour integral

$$\Omega \equiv \lim_{e^2\to 0} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right]$$

$$= \sum \text{JK-Res}_{\eta:\{Q_i\}} g(u,\bar{u};0)$$

$$g(u,\bar{u};D=0) = \left(\frac{1}{\mathbf{y}-\mathbf{y}^{-1}}\right)^{\mathrm{rank}} \prod_\alpha \frac{t^{-\alpha/2}-t^{\alpha/2}}{t^{\alpha/2}\mathbf{y}^{-1}-t^{-\alpha/2}\mathbf{y}}$$

$$\times \prod_i \frac{t^{-Q_i/2}x^{-F_i/2}\mathbf{y}^{-(R_i/2-1)}-t^{Q_i/2}x^{F_i/2}\mathbf{y}^{R_i/2-1}}{t^{Q_i/2}x^{F_i/2}\mathbf{y}^{R_i/2}-t^{-Q_i/2}x^{-F_i/2}\mathbf{y}^{-R_i/2}}$$

Hori, Kim, P.Y. 2014
Hwang, Kim, Kim, Park 2014

Szenes, Vergne 2004
Brion, M.Vergne 1999
Jeffrey, Kirwan 1993

N=4 CP(N -1)

chirals	$U(1)$	$U(N)_F$
X	1	N

null

$\xi < 0$

$0 < \xi$

1
0 0
0 1 0
0 0 0 0
0 0 1 0 0
0 0 0 0 0 0
0 0 1 0 0
0 0 0 0
0 1 0
0 0
1

$N = 6$

quintic CY3 hypersurface in CP4

	P	$X_{1,2,3,4,5}$
$U(1)$	-5	1

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array}$$

$\xi < 0$

$0 < \xi$

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

N=4 rank 2 GLSM Q.M. for CY3 in WCP$_{(11222)}$

	P	$X_{1,2}$	$Y_{1,2,3}$	Z
$U(1)_1$	-4	0	1	1
$U(1)_2$	0	1	0	-2

hybrid

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 1 & & 86 & & 86 & & 1 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array}$$

geometric

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 2 & & 0 & \\ 1 & & 86 & & 86 & & 1 \\ & 0 & & 2 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Landau-Ginsburg

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & \\ 1 & & 83 & & 83 & & 1 \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{array}$$

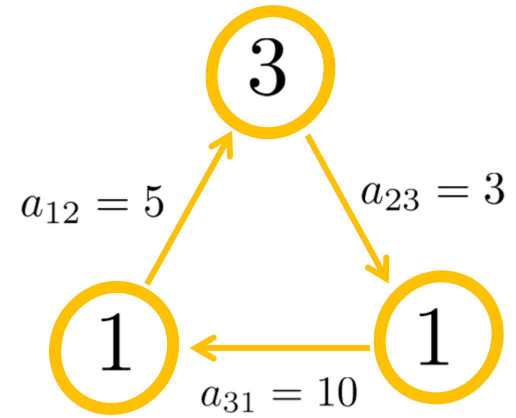
orbifold

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 83 & & 83 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

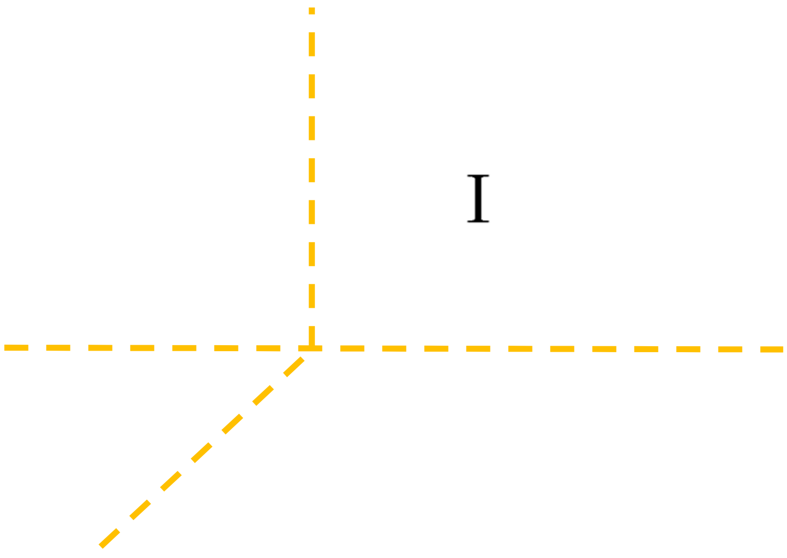
for the class of all N=4 quiver quantum mechanics,
the entire Hodge diamonds can be recursively read off
from such Hirzebruch indices
in each and every wall-crossing chambers!!!

J.Manschot, B.Pioline, A.Sen 2010~2013
S.J. Lee, Z.L. Wang, P.Y. 2012~2014

N = 4 (3,1,1) triangle quiver

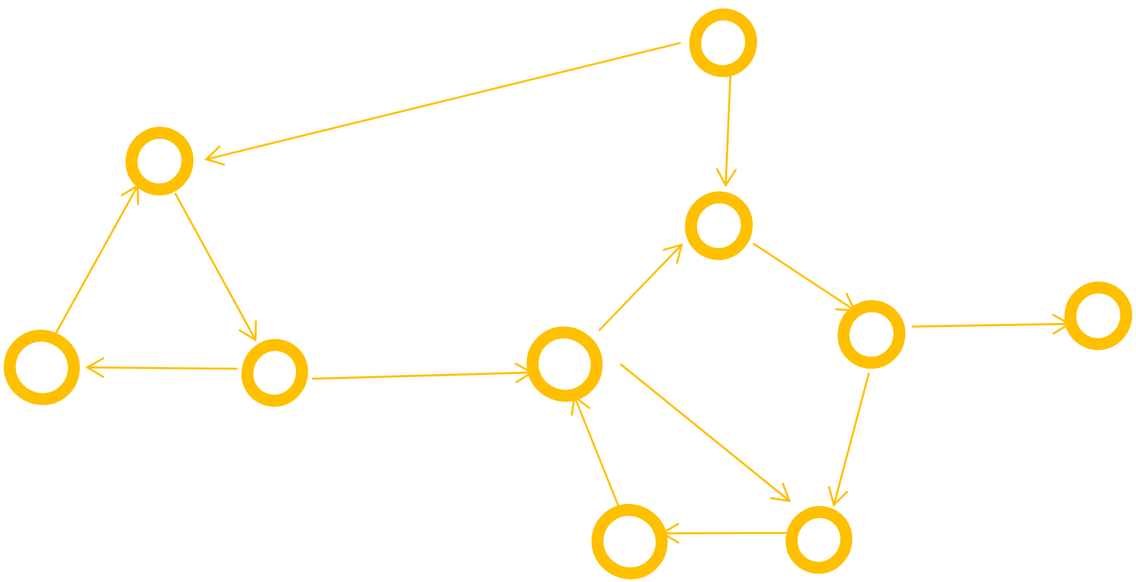


1
0 0
0 2 0
0 0 0 0
0 0 4 0 0
0 0 6 6 0 0
0 0 0 5 0 0 0
0 0 6 6 0 0
0 0 4 0 0
0 0 0 0
0 2 0
0 0
1



elliptic genus & witten index

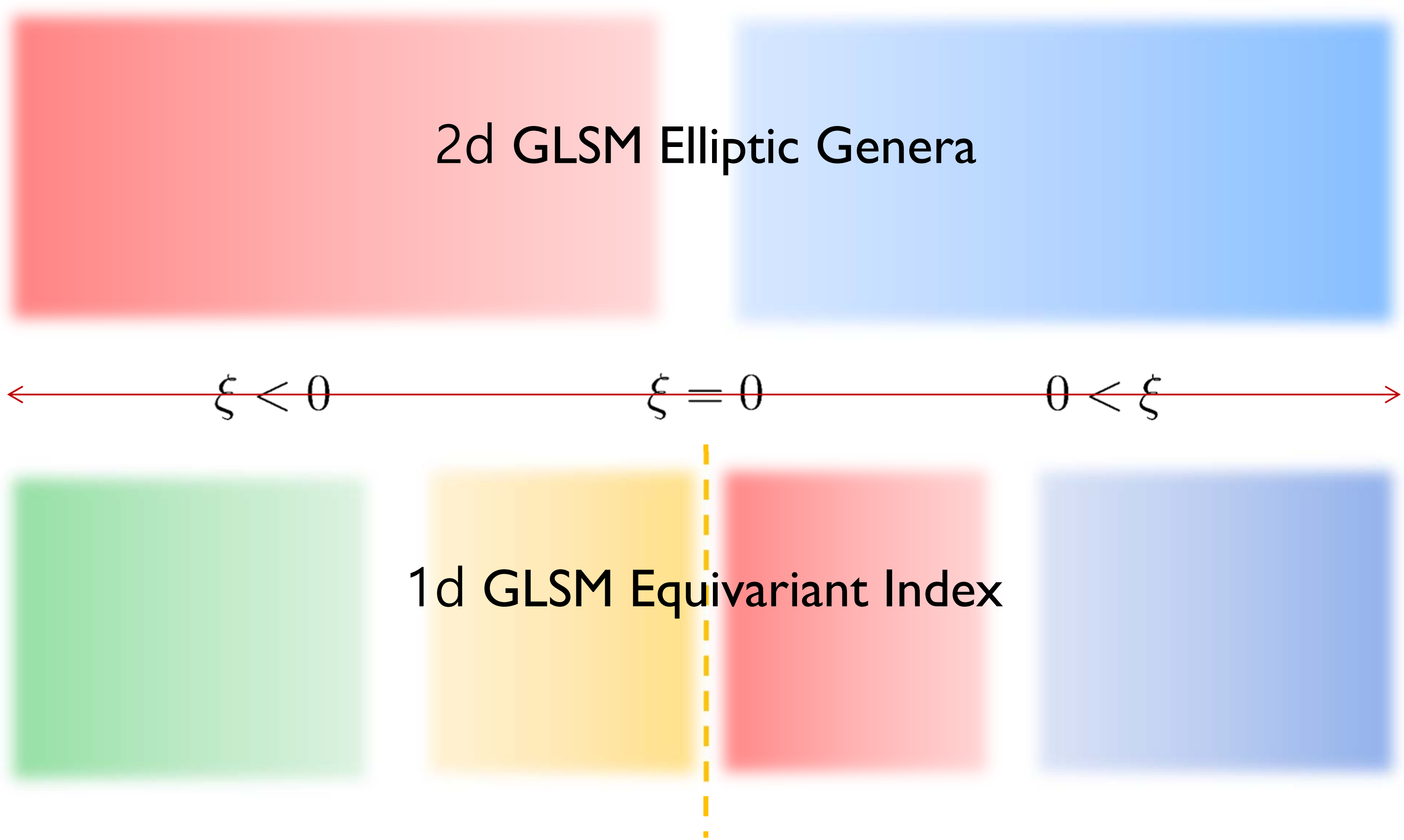
cohomology



wall-crossing

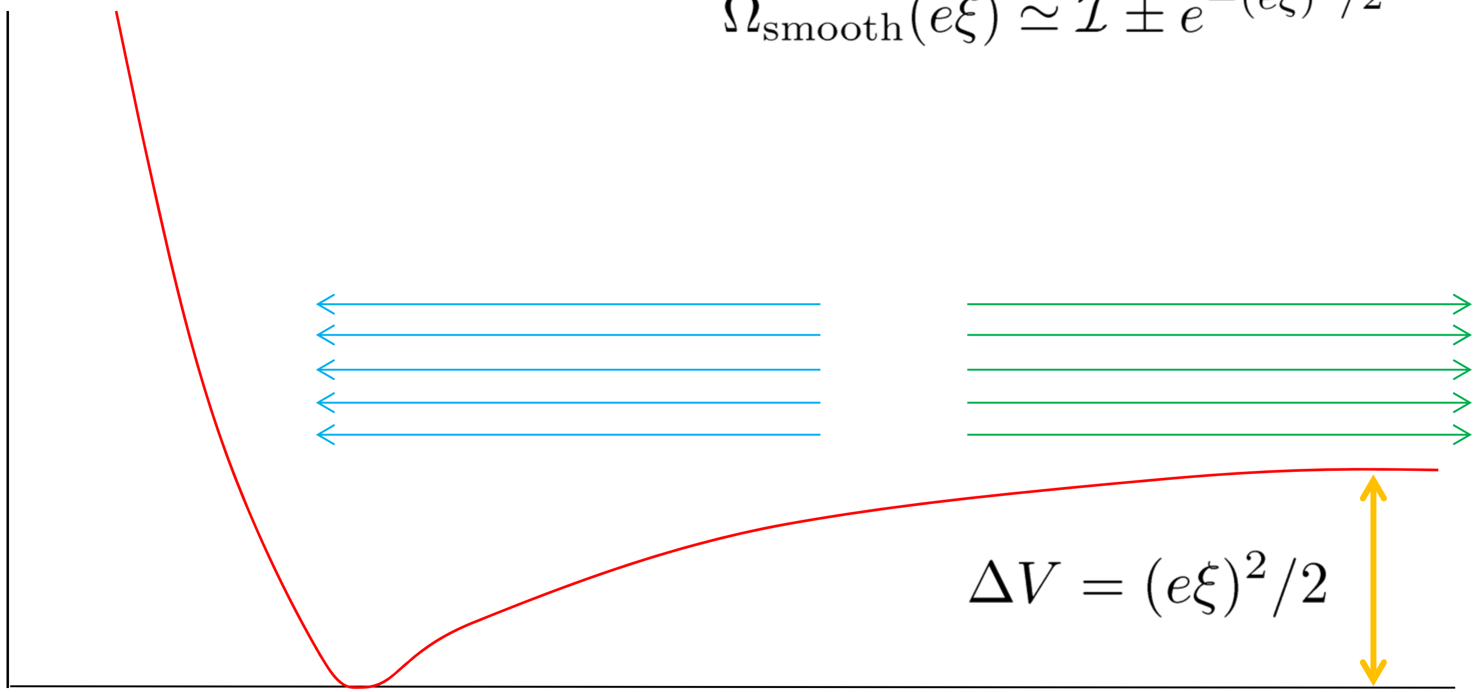
quiver invariants

4d N=2 black hole microstates


2d GLSM Elliptic Genera
$\xi < 0$
$\xi = 0$
$0 < \xi$
1d GLSM Equivariant Index

wall-crossing happens at $\xi = 0$ because
a continuum touches the ground state,
invalidating the naive invariance

$$\Omega_{\text{smooth}}(e\xi) \simeq \mathcal{I} \pm e^{-(e\xi)^2/2}$$



Hori, Kim, P.Y. 2014

what if such asymptotic flat directions
cannot be lifted by a parameter tuning?
can we still count the relevant Witten index reliably via path integral?

the answer has to be "NO" generally and
higher supersymmetry, e.g. ADHM, does not help either

$\mathcal{I}$ from Ω

rational invariants to the rescue

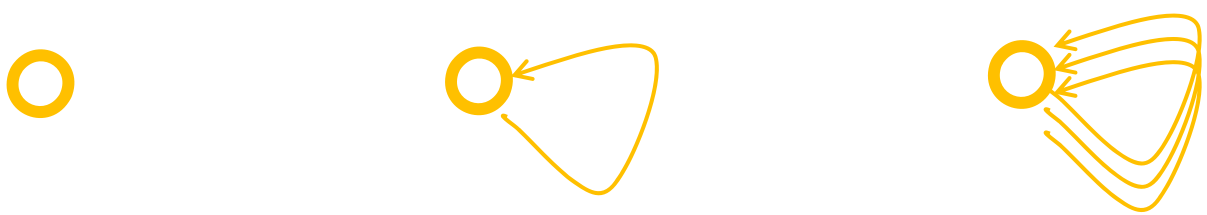
when the asymptotic flatness arises from the Coulomb side

& how this fixed a road to M-theory hypothesis

S.J. Lee + P.Y., 2016

back to supersymmetric pure Yang-Mills quantum mechanics

$$\mathcal{N} = 4, 8, 16$$



S.J. Lee + P.Y., 2016

after rigorous applications of HKY procedure,

$$\Omega^{SU(2)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{2}\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})} \rightarrow \frac{1}{2^2}$$

$$\Omega^{SU(3)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{3}\frac{1}{(\mathbf{y}^{-2}+1+\mathbf{y}^{2})} \rightarrow \frac{1}{3^2}$$

$$\Omega^{SU(4)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{4}\frac{1}{(\mathbf{y}^{-3}+\mathbf{y}^{-1}+\mathbf{y}+\mathbf{y}^{3})} \rightarrow \frac{1}{4^2}$$

$$\Omega^{SU(N)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{N}\frac{\mathbf{y}^{-1}-\mathbf{y}}{\mathbf{y}^{-N}-\mathbf{y}^{N}} \rightarrow \frac{1}{N^2}$$

S.J. Lee + P.Y., 2016

other rank 2 examples

$$\Omega^{SO(4)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{4}\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^2}$$

$$\Omega^{SO(5)/Sp(2)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{8}\left[\frac{2}{\mathbf{y}^{-2}+\mathbf{y}^2}+\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^2}\right]$$

$$\Omega^{G_2}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{12}\left[\frac{2}{\mathbf{y}^{-2}-1+\mathbf{y}^2}+\frac{2}{\mathbf{y}^{-2}+1+\mathbf{y}^2}+\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^2}\right]$$

S.J. Lee + P.Y., 2016

higher rank examples

$$\Omega^{SU(4)/SO(6)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{4}\frac{1}{(\mathbf{y}^{-3}+\mathbf{y}^{-1}+\mathbf{y}+\mathbf{y}^{3})}$$

$$\Omega^{SO(7)/Sp(3)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{48}\left[\frac{8}{\mathbf{y}^{-3}+\mathbf{y}^{3}}+\frac{6}{(\mathbf{y}^{-2}+\mathbf{y}^{2})(\mathbf{y}^{-1}+\mathbf{y})}+\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^{3}}\right]$$

$$\Omega^{SO(8)}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{192}\left[\frac{32}{(\mathbf{y}^{-3}+\mathbf{y}^{3})(\mathbf{y}^{-1}+\mathbf{y})}+\frac{12}{(\mathbf{y}^{-2}+\mathbf{y}^{2})^{2}}+\frac{1}{(\mathbf{y}^{-1}+\mathbf{y})^{4}}\right]$$

S.J. Lee, P.Y. 2016

which can be organized into

elliptic Weyl elements only

$$0 \neq \mathrm{Det}\,(1 - w)$$

$$\Omega^{G}_{\mathcal{N}=4}(\mathbf{y}) = \frac{1}{|W_G|} \sum_{w \in W_G}{}' \frac{1}{\mathrm{Det}\,(\mathbf{y}^{-1} - \mathbf{y} \cdot w)}$$

Weyl group

analog of Kontsevich-Soibelman's rational invariants,
or analog of Gopakumar-Vafa's multicover formulae,
meaning that these captures multi-particle plane-wave sectors

elliptic Weyl elements for some classical groups

G	W	Elliptic Weyl Elements
$SU(N)$	S_N	$(123\cdots N)$
$SO(4)$	$Z_2 \times S_2$	$(\dot{1})(\dot{2})$
$SO(5)/Sp(2)$	$(Z_2)^2 \times S_2$	$(1\dot{2}), \quad (\dot{1})(\dot{2})$
$SO(6)$	$(Z_2)^2 \times S_3$	$(1\dot{2})(\dot{3})$
$SO(7)/Sp(3)$	$(Z_2)^3 \times S_3$	$(\dot{1}\dot{2}\dot{3}), \quad (12\dot{3}), \quad (1\dot{2})(\dot{3}), \quad (\dot{1})(\dot{2})(\dot{3})$
$SO(8)$	$(Z_2)^3 \times S_4$	$(\dot{1}\dot{2}\dot{3})(\dot{4}), \quad (12\dot{3})(\dot{4}), \quad (1\dot{2})(3\dot{4}), \quad (\dot{1})(\dot{2})(\dot{3})(\dot{4})$

why? because the localization implicitly computes the bulk part!

$$\mathcal{I} \quad = \quad \mathcal{I}_{\rm bulk} \quad + \quad \delta\mathcal{I}$$

⬇

$$\lim_{\beta\to 0} {\rm Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right]$$

$e^{2/3}\beta \to 0$ ⬆

$$\Omega \equiv \lim_{e^2\to 0} {\rm Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right]$$

no ground state → an asymptotic orbifold problem

$$0 = \mathcal{I}^G_{\mathcal{N}=4,8} = \mathcal{I}^G_{\mathcal{N}=4,8;\text{bulk}} + \delta\mathcal{I}^G_{\mathcal{N}=4,8}$$

⬇

$$\Omega^G_{\mathcal{N}=4,8} = \mathcal{I}^G_{\mathcal{N}=4,8;\text{bulk}} = -\delta\mathcal{I}^G_{\mathcal{N}=4,8}$$

$$= -\delta\mathcal{I}^{U(1)^r/W}_{\mathcal{N}=4,8}$$

$$= \mathcal{I}^{U(1)^r/W}_{\mathcal{N}=4,8;\text{bulk}}$$

P.Y. 1997

Green+Gutperle 1997
Kac+Smilga 1999

S.J. Lee, P.Y. 2016

$\mathcal{N} = 16$ with general simple Lie groups

$$\Omega^{G}_{\mathcal{N}=16}(\mathbf{y}, x) \quad = \quad \mathcal{I}^{G}_{\mathcal{N}=16} + \sum_{G' \subset G} \# \cdot \Delta^{G'}_{\mathcal{N}=16}$$

$$\Delta^{G}_{\mathcal{N}=16}(\mathbf{y}, x) = \frac{1}{|W|} {\sum_{w}}' \frac{1}{\mathrm{Det}\,(\mathbf{y}^{-1} - \mathbf{y} \cdot w)} \cdot \prod_{a=1,2,3} \frac{\mathrm{Det}\,\left(\mathbf{y}^{R_a - 1} x^{F_a/2} - \mathbf{y}^{1-R_a} x^{-F_a/2} \cdot w\right)}{\mathrm{Det}\,\left(\mathbf{y}^{R_a} x^{F_a/2} - \mathbf{y}^{-R_a} x^{-F_a/2} \cdot w\right)}$$

elliptic Weyl elements only

$0 \neq \mathrm{Det}\,(1 - w)$

S.J. Lee, P.Y. 2016

$$\Omega^{SU(N)}_{\mathcal{N}=16} = 1 + \sum_{p|N;p>1} 1 \cdot \Delta^{SU(p)}_{\mathcal{N}=16}$$

$$\Omega^{SO(5)/Sp(2)}_{\mathcal{N}=16} = 1 + 2\Delta^{SO(3)/Sp(1)}_{\mathcal{N}=16} + \Delta^{SO(5)/Sp(2)}_{\mathcal{N}=16}$$

$$\Omega^{G_2}_{\mathcal{N}=16} = 2 + 2\Delta^{SU(2)}_{\mathcal{N}=16} + \Delta^{G_2}_{\mathcal{N}=16}$$

$$\Omega^{SO(7)}_{\mathcal{N}=16} = 1 + 3\Delta^{SO(3)}_{\mathcal{N}=16} + \left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^2 + \Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(7)}_{\mathcal{N}=16}$$

$$\Omega^{Sp(3)}_{\mathcal{N}=16} = 2 + 3\Delta^{Sp(1)}_{\mathcal{N}=16} + \left(\Delta^{Sp(1)}_{\mathcal{N}=16}\right)^2 + \Delta^{Sp(2)}_{\mathcal{N}=16} + \Delta^{Sp(3)}_{\mathcal{N}=16}$$

$$\Omega^{SO(8)}_{\mathcal{N}=16} = 2 + 4\Delta^{SO(3)}_{\mathcal{N}=16} + 2\left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^2 + \left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^3 + 3\Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(8)}_{\mathcal{N}=16}$$

$$\Omega^{SO(9)}_{\mathcal{N}=16} = 2 + 4\Delta^{SO(3)}_{\mathcal{N}=16} + 2\left(\Delta^{SO(3)}_{\mathcal{N}=16}\right)^2 + 2\Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(3)}_{\mathcal{N}=16} \cdot \Delta^{SO(5)}_{\mathcal{N}=16} + \Delta^{SO(7)}_{\mathcal{N}=16} + \Delta^{SO(9)}_{\mathcal{N}=16}$$

$$\Omega^{Sp(4)}_{\mathcal{N}=16} = 2 + 5\Delta^{Sp(1)}_{\mathcal{N}=16} + 2\left(\Delta^{Sp(1)}_{\mathcal{N}=16}\right)^2 + 2\Delta^{Sp(2)}_{\mathcal{N}=16} + \Delta^{Sp(1)}_{\mathcal{N}=16} \cdot \Delta^{Sp(2)}_{\mathcal{N}=16} + \Delta^{Sp(3)}_{\mathcal{N}=16} + \Delta^{Sp(4)}_{\mathcal{N}=16}$$

$$\Delta^{G}_{\mathcal{N}=16}(\mathbf{y}, x) = \frac{1}{|W|} \sum_{w}{}' \frac{1}{\mathrm{Det}\left(\mathbf{y}^{-1} - \mathbf{y} \cdot w\right)} \cdot \prod_{a=1,2,3} \frac{\mathrm{Det}\left(\mathbf{y}^{R_a - 1} x^{F_a/2} - \mathbf{y}^{1-R_a} x^{-F_a/2} \cdot w\right)}{\mathrm{Det}\left(\mathbf{y}^{R_a} x^{F_a/2} - \mathbf{y}^{-R_a} x^{-F_a/2} \cdot w\right)}$$

the results suffice for reading off the Witten index $\mathcal{I}^{G}_{\mathcal{N}=16}$ from the unique integral part

$$\begin{aligned}
\mathcal{I}^{SU(N)}_{\mathcal{N}=16} &= 1 \\
\mathcal{I}^{SO(5)/Sp(2)}_{\mathcal{N}=16} &= 1 \\
\mathcal{I}^{G_2}_{\mathcal{N}=16} &= 2 \\
\mathcal{I}^{SO(7)}_{\mathcal{N}=16} &= 1 \\
\mathcal{I}^{Sp(3)}_{\mathcal{N}=16} &= 2 \\
\mathcal{I}^{SO(8)}_{\mathcal{N}=16} &= 2 \\
\mathcal{I}^{SO(9)}_{\mathcal{N}=16} &= 2 \\
\mathcal{I}^{Sp(4)}_{\mathcal{N}=16} &= 2 \\
&\vdots
\end{aligned}$$

S.J. Lee, P.Y. 2016

and similarly

$$\mathcal{I}^{SU(N)}_{\mathcal{N}=16} = 1$$

$$\mathcal{I}^{O(5)}_{\mathcal{N}=16} = \mathcal{I}^{Sp(2)}_{\mathcal{N}=16} = 1$$

$$\mathcal{I}^{G_2}_{\mathcal{N}=16} = 2$$

$$\mathcal{I}^{O(7)}_{\mathcal{N}=16} = 1$$

$$\mathcal{I}^{Sp(3)}_{\mathcal{N}=16} = 2$$

$$\mathcal{I}^{O(8)}_{\mathcal{N}=16} = 2$$

$$\mathcal{I}^{O(9)}_{\mathcal{N}=16} = 2$$

$$\mathcal{I}^{Sp(4)}_{\mathcal{N}=16} = 2$$

$$\vdots$$

S.J. Lee, P.Y. 2017

which can be organized into the generating functions

$$\sum_N \mathcal{I}^{SU(N)}_{\mathcal{N}=16} t^N = \frac{1}{1-t}$$

$$\sum_N \mathcal{I}^{O(N)}_{\mathcal{N}=16} t^N = \prod_{n=1}^{\infty} (1+t^{2n-1})$$

$$\sum_N \mathcal{I}^{Sp(N)}_{\mathcal{N}=16} t^{2N} = \prod_{n=1}^{\infty} (1+t^{2n})$$

S.J. Lee + P.Y., 2017

which fit precisely, yet again, the M-theory hypothesis then, what went wrong 15 years ago?

P.Y. 1997
Green, Gutperle 1997
Kac, Smilga 1999

P.Y. / Sethi, Stern 1997
Moore, Nakrasov, Shatashvili 1998
Staudacher 2000 / Pestun 2002

$\mathcal{N}=4$	$\mathcal{I}^G_{\text{bulk}} = -\delta\mathcal{I}^G$	$\mathcal{I}^G_{\text{bulk}} = \mathcal{Z}^G$
$SU(N)$	$\frac{1}{N^2}$	$\frac{1}{N^2}$
$Sp(2)$	$\frac{5}{32}$	$\frac{9}{64}$
$Sp(3)$	$\frac{15}{128}$	$\frac{51}{512}$
$Sp(4)$	$\frac{195}{2048}$	$\frac{1275}{16384}$
$Sp(5)$	$\frac{663}{8192}$	$\frac{8415}{131072}$
$Sp(6)$	$\frac{4641}{65536}$	$\frac{115005}{2097152}$
$Sp(7)$	$\frac{16575}{262144}$	$\frac{805035}{16777216}$
$SO(7)$	$\frac{15}{128}$	$\frac{25}{256}$
$SO(8)$	$\frac{59}{1024}$	$\frac{117}{2048}$
$SO(9)$	$\frac{195}{2048}$	$\frac{613}{8192}$
$SO(10)$	$\frac{27}{512}$	$\frac{53}{1024}$
$SO(11)$	$\frac{663}{8192}$	$\frac{1989}{32768}$
$SO(12)$	$\frac{1589}{32768}$	$\frac{6175}{131072}$
$SO(13)$	$\frac{4641}{65536}$	$\frac{26791}{524288}$
$SO(14)$	$\frac{1471}{32768}$	$\frac{5661}{131072}$
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G_2	$\frac{35}{144}$	$\frac{151}{864}$
F_4	$\frac{30145}{165888}$	$\frac{493013}{3981312}$

$$\mathcal{I}^G_{\mathcal{N}=4,8:\text{bulk}}$$

$$= -\delta\mathcal{I}^G_{\mathcal{N}=4,8}$$

$$= \mathcal{I}^{U(1)^r/W_G}_{\mathcal{N}=4,8;\text{bulk}}$$

$$\mathcal{I}^G_{\mathcal{N}=4,8:\text{bulk}}$$

$$= \mathcal{Z}^G_{\mathcal{N}=4,8:\text{matrix}}$$

recall that the localization implicitly computes the bulk part

$$\mathcal{I} \quad = \quad \mathcal{I}_{\text{bulk}} \quad + \quad \delta\mathcal{I}$$

$$\Downarrow$$

$$\lim_{\beta\to 0} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right]$$

$$e^{2/3}\beta \to 0 \qquad \Uparrow$$

$$\Omega \equiv \lim_{e^2\to 0} \mathrm{Tr}\left[(-1)^{2J_3}\mathbf{y}^{2J_3+2R}x^{G_F}e^{-\beta Q^2}\right]$$

P.Y. 1997
Green, Gutperle 1997
Kac, Smilga 1999

$\mathcal{N}=4$	$\mathcal{I}^G_{\text{bulk}} = \Omega^G$	$\mathcal{I}^G_{\text{bulk}} = -\delta\mathcal{I}^G$	$\mathcal{I}^G_{\text{bulk}} = \mathcal{Z}^G$
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P.Y. /
Sethi, Stern 1997
Moore, Nakrasov, Shatashvili 1998
Staudacher 2000
Pestun 2002

$$\mathcal{I}^G_{\text{bulk}} = \Omega^G$$

Lee, P.Y. 2016

$$\mathcal{I}^G_{\mathcal{N}=4,8:\text{bulk}} = \mathcal{Z}^G_{\mathcal{N}=4,8}$$

P.Y. 1997
Green, Gutperle 1997
Kac, Smilga 1999

$$\mathcal{I}^G_{\text{bulk}} = \Omega^G$$

Lee, P.Y. 2016

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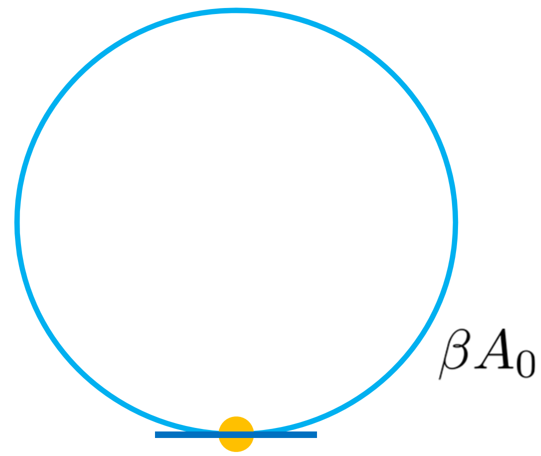
P.Y. /
Sethi, Stern 1997
Moore, Nakrasov, Shatashvili 1998
Staudacher 2000
Pestun 2002

?

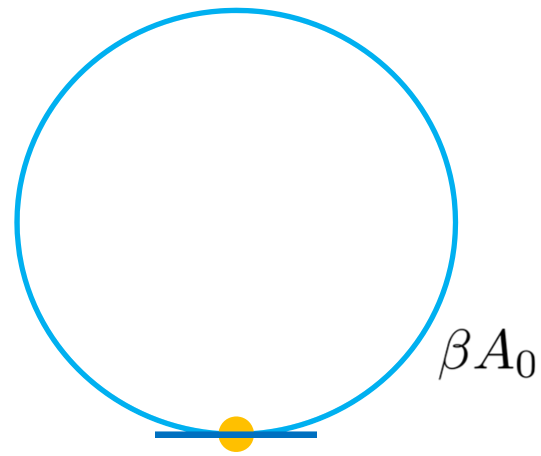
$$\mathcal{I}^G_{\mathcal{N}=4,8:\text{bulk}} = \mathcal{Z}^G_{\mathcal{N}=4,8}$$



$$\mathcal{I}^{G}_{\text{bulk}}\Big|_{\beta\to 0} = \mathcal{Z}^{G}_{\text{matrix integral}}$$

this particular matrix integral is from
the 1d path integral reduced to 0d
in the region near trivial Wilson line

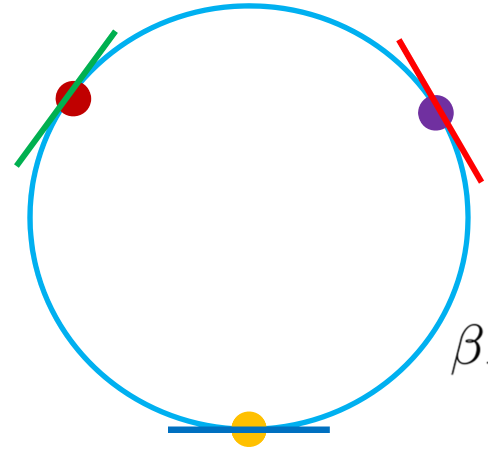
Hwang, P.Y. 2017



$$\mathcal{I}^{G}_{\text{bulk}}\Big|_{\beta\to 0} \neq \mathcal{Z}^{G}_{\text{matrix integral}}$$

this particular matrix integral is from
the 1d path integral reduced to 0d
in the region near trivial Wilson line

Hwang, P.Y. 2017

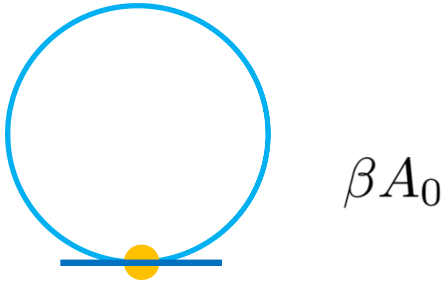


$$\mathcal{I}^G_{\text{bulk}}\Big|_{\beta\to 0} = \sum_{H\subset G}\int dZ\, d\Phi\, \frac{O(\beta^0)}{Z^{2(g-h)}}\, e^{-[Z,Z]^2/4+Z_\mu K_\mu(\Phi)/2}$$

other special Wilson line values,
separated away at distance of order β^{-1},
do contribute generally;
at such *H* saddles the effective 0d theory
must have no decoupled free fermions

Hwang, P.Y. 2017

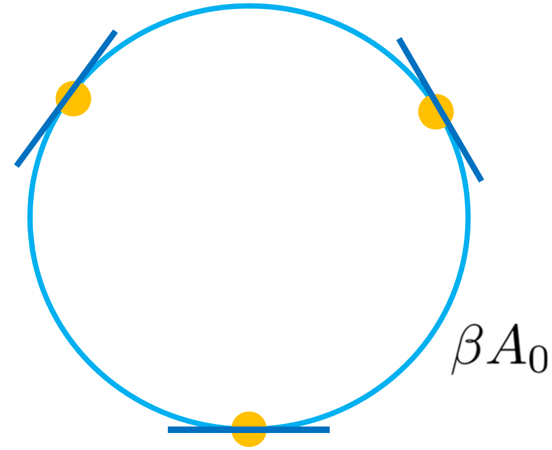
a trivial example
SU(N)



$$\mathcal{I}^G_{\rm bulk}\Big|_{\beta\to 0} = \mathcal{Z}^{SU(N)/Z_N}$$

Hwang, P.Y. 2017

a trivial example
SU(N)

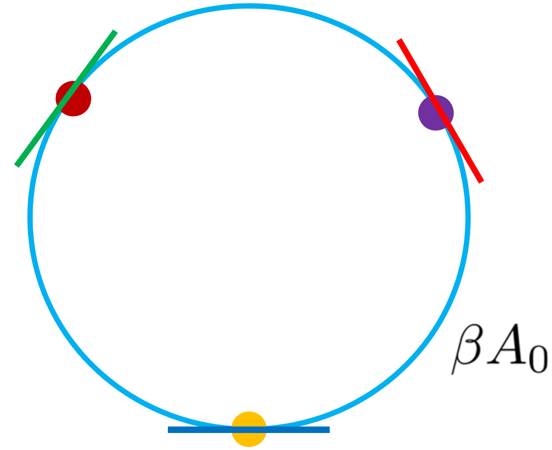


$$\mathcal{I}^G_{\text{bulk}}\Big|_{\beta\to 0} = \sum_{u_{SU(N)}} \int dZ\, d\Phi\; e^{-[Z,Z]^2/4 + Z_\mu K_\mu(\Phi)/2}$$

$$= N \times \int dZ\, d\Phi\; e^{-[Z,Z]^2/4 + Z_\mu K_\mu(\Phi)/2}$$

$$= \mathcal{Z}^{SU(N)/Z_N}$$

Hwang, P.Y. 2017



$$\mathcal{I}^G_{\text{bulk}}(\mathbf{y})\Big|_{\mathbf{y}=e^{\beta z'};\beta\to 0} = \mathcal{Z}^G(z') + \sum_{H<G} d_{G:H}\,\frac{|\det(Q^G)|/|W_G|}{|\det(Q^H)|/|W_H|}\,\mathcal{Z}^H(z')$$

H saddles: maximal non-Abelian subgroups left unbroken by a Wilson line

Hwang, P.Y. 2017

$$\mathcal{I}^{SU(N)}_{\text{bulk}}(\mathbf{y})\Big|_{\mathbf{y}=e^{\beta z'};\beta\to 0} = \mathcal{Z}^{SU(N)}(z')$$

$$\mathcal{I}^{Sp(K)}_{\text{bulk}}(\mathbf{y})\Big|_{\mathbf{y}=e^{\beta z'};\beta\to 0} = \mathcal{Z}^{Sp(K)}(z') + \sum_{m=1}^{K-1}\frac{1}{4}\,\mathcal{Z}^{Sp(m)\times Sp(K-m)}(z')$$

$$\mathcal{I}^{SO(2N)}_{\text{bulk}}(\mathbf{y})\Big|_{\mathbf{y}=e^{\beta z'};\beta\to 0} = \mathcal{Z}^{SO(2N)}(z') + \sum_{m=2}^{N-2}\frac{1}{8}\,\mathcal{Z}^{SO(2m)\times SO(2N-2m)}(z')$$

$$\mathcal{I}^{SO(2N+1)}_{\text{bulk}}(\mathbf{y})\Big|_{\mathbf{y}=e^{\beta z'};\beta\to 0} = \mathcal{Z}^{SO(2N+1)}(z') + \sum_{m=2}^{N}\frac{1}{4}\,\mathcal{Z}^{SO(2m)\times SO(2N+1-2m)}(z')$$

Hwang, P.Y. 2017

P.Y. 1997
Green, Gutperle 1997
Kac, Smilga 1999

$\mathcal{I}^G_{\rm bulk} = \Omega^G$
Lee, P.Y. 2016

$\mathcal{N}=4$	$\mathcal{I}^G_{\rm bulk} = \Omega^G$	$\mathcal{I}^G_{\rm bulk} = -\delta\mathcal{I}^G$	$\mathcal{I}^G_{\rm bulk} = \mathcal{Z}^G$
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P.Y. /
Sethi, Stern 1997
Moore, Nakrasov,
Shatashvili 1998
Staudacher 2000
Pestun 2002

$\mathcal{I}^G_{\rm bulk} \neq \mathcal{Z}^G_{\rm matrix\ model}$

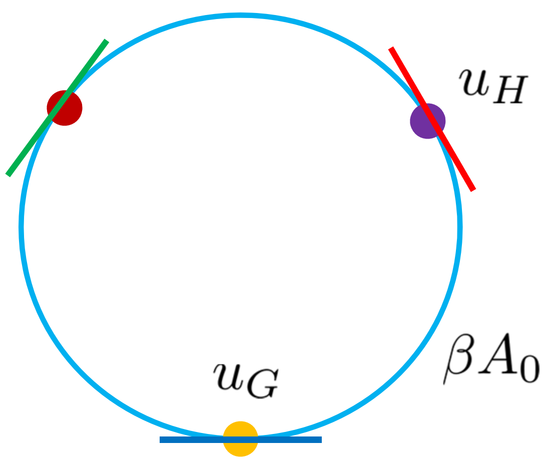
such *H*-saddles appears due to the integration
over the gauge holonomy, and thus are potentially relevant
for all susy partition functions on a vanishing circle,
regardless of space-time dimensions;

they explain many of subtleties out there,
in relating partition functions of susy gauge theories
in two adjacent dimensions

dimensional reduction is multi-branched for susy gauge theories

$S^1 \times \mathcal{M}_{d-1}$ $\mathcal{M}_{d-1}$

$$\Omega_d^G(\beta \tilde{z})\Big|_{\beta \to 0} \quad \to \quad \sum_{u_H} \; c_{G:H}(\beta)\; \mathcal{Z}_{d-1}^H(\tilde{z})$$

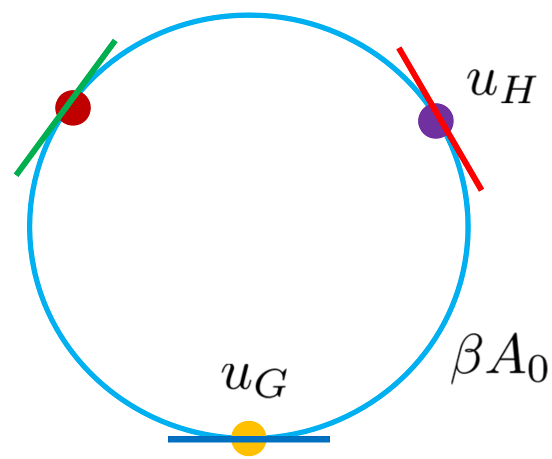


→ (1) equivariant Witten indices of different gauge theories can now be related across dimensions systematically

$S^1 \times T^{d-1}$ $\qquad\qquad\qquad\qquad$ T^{d-1}

$$\mathcal{I}_d^G(\beta\tilde{z})\Big|_{\beta\to 0} \to \sum_{u_H} d_{G:H}\, \mathcal{I}_{d-1}^H(\tilde{z})$$

purely algebraic factors



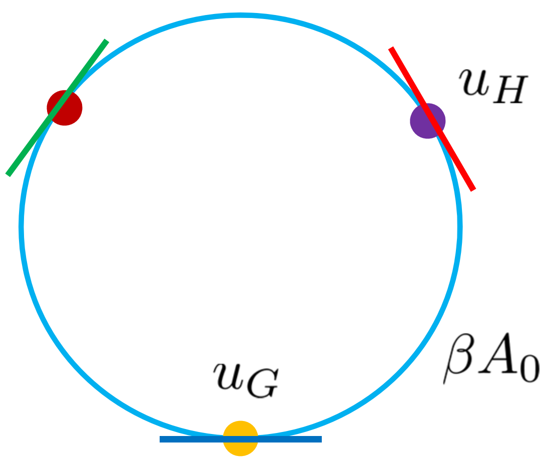
→ (1′) or, more precisely, twisted partition functions can now be related across dimensions systematically

$S^1 \times T^{d-1}$ $\qquad\qquad$ T^{d-1}

$$\mathcal{I}_d^G(\beta\tilde{z})\Big|_{\text{bulk};\beta\to 0} \to \sum_{u_H} d_{G:H}\, \mathcal{I}_{d-1}^H(\tilde{z})\Big|_{\text{bulk}}$$

↓

purely algebraic factors

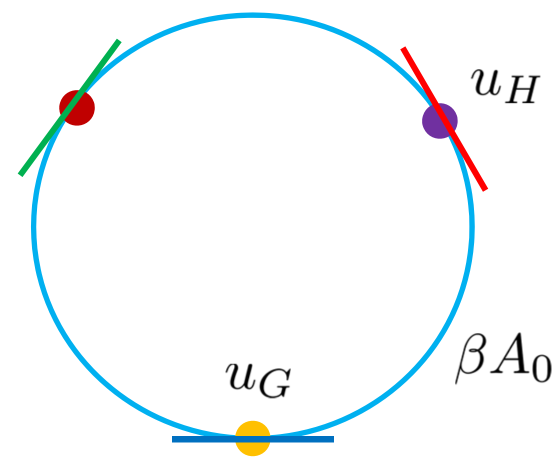


which really characterizes H-saddles and identifies their discrete locations in the space of holonomies

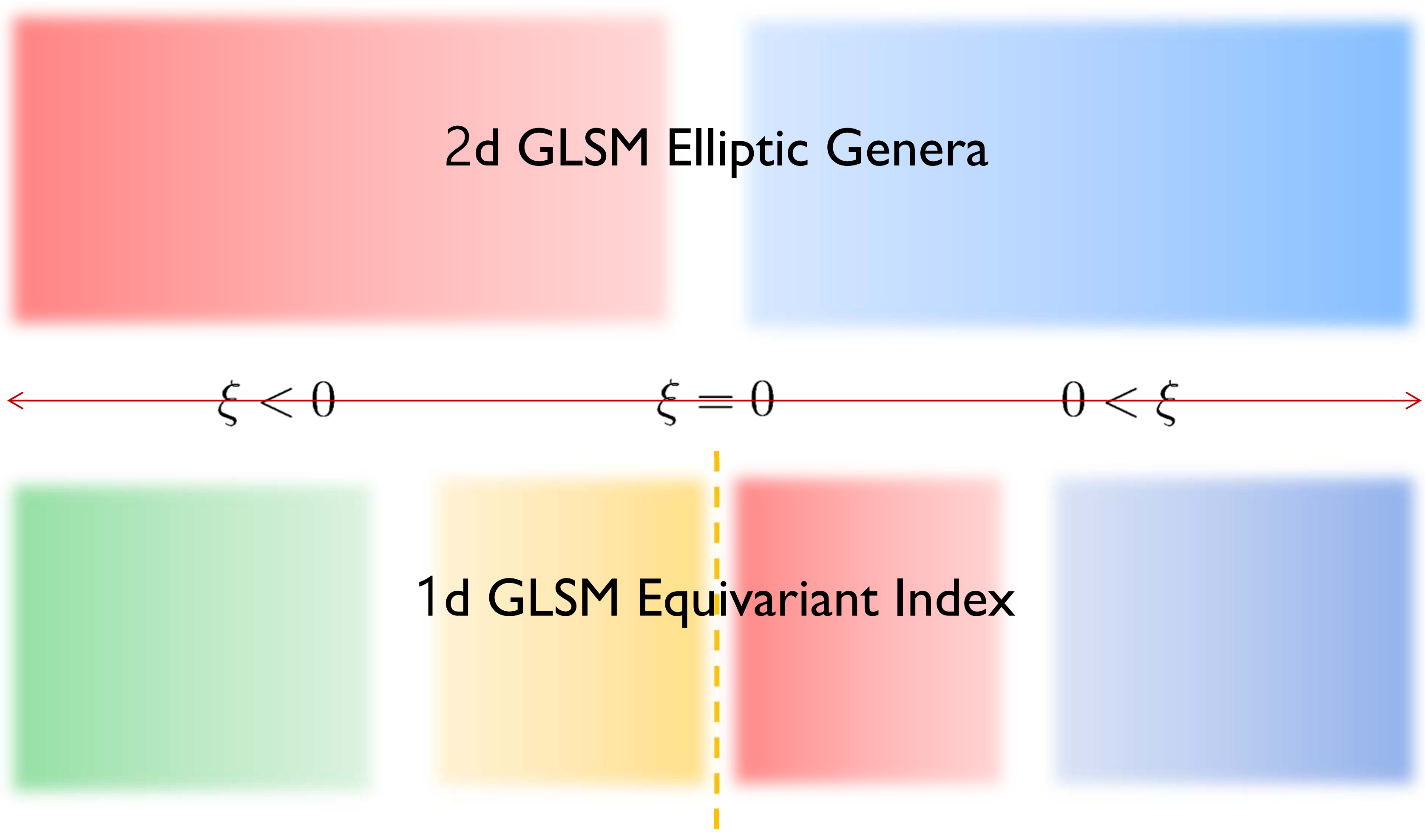
$S^1 \times T^{d-1}$ $\qquad\qquad$ T^{d-1}

$$\mathcal{I}_d^G(\beta\tilde{z})\Big|_{\text{bulk};\beta\to 0} \to \sum_{u_H} d_{G:H}\, \mathcal{I}_{d-1}^H(\tilde{z})\Big|_{\text{bulk}}$$

purely algebraic factors



this phenomenon also underlies why the 2d elliptic genera fail to capture the 1d wall-crossing phenomena



→ (2) dimensional reduction of a dual pair, on a circle, produces many such dual pairs in 1d less, at best

$$S^1 \times \mathcal{M}_{d-1} \qquad\qquad \mathcal{M}_{d-1}$$

$$\Omega_d^G(\beta\tilde{z})\Big|_{\beta\to 0} \to \sum_{u_H} c_{G:H}(\beta)\ \mathcal{Z}_{d-1}^{H}(\tilde{z})$$

⇕ ⇕

$$= \Omega_d^{G'}(\beta\tilde{z})\Big|_{\beta\to 0} \to \sum_{u_{H'}} c_{G':H'}(\beta)\ \mathcal{Z}_{d-1}^{H'}(\tilde{z})$$

Aharony, Razamat, Seiberg, Willett 2013
Aharony, Razamat, Willett 2017

→ (2′) such saddle-by-saddle dualities could fail for d<3 where the holonomy cannot have a vev even in the non-compact limit

$S^1 \times \mathcal{M}_{d-1}$ $\qquad\qquad$ $\mathcal{M}_{d-1}$

$$\Omega_d^G(\beta\tilde{z})\Big|_{\beta\to 0} \to \sum_{u_H} c_{G:H}(\beta)\ \mathcal{Z}_{d-1}^H(\tilde{z})$$

⬍

$$= \ \Omega_d^{G'}(\beta\tilde{z})\Big|_{\beta\to 0} \to \sum_{u_{H'}} c_{G':H'}(\beta)\ \mathcal{Z}_{d-1}^{H'}(\tilde{z})$$

Aharony, Razamat, Willett 2017

→ (3) there may be multiple Cardy exponents and the Dominant one does not generically equal the naïve one

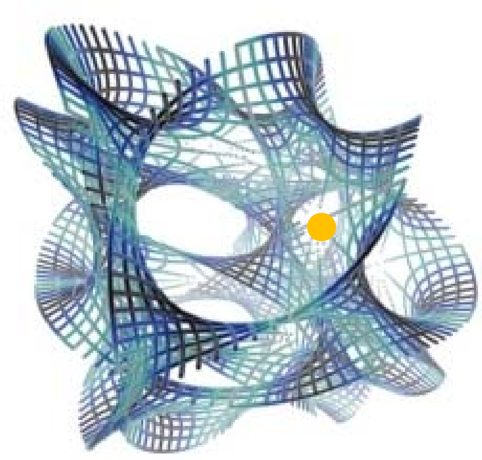
$S^1 \times \mathcal{M}_{d-1}$ $\qquad\qquad$ $\mathcal{M}_{d-1}$

$$\Omega_d^G(\beta\tilde{z})\Big|_{\beta\to 0} \quad\to\quad \sum_{u_H} \; c_{G:H}(\beta)\;\mathcal{Z}_{d-1}^H(\tilde{z})$$

$$\sim e^{S_{\text{Cardy}}^{G:H}/\beta}$$

H-saddles for N=1 supersymmetric gauge theories on compact 4d manifolds with a circle

$$S^1 \times \mathcal{M}_3 \times$$



Hwang, Lee, P.Y. 2017

how *H*-saddles manifest in the Bethe-Ansatz-driven partition functions of massive 4d N=1 theories

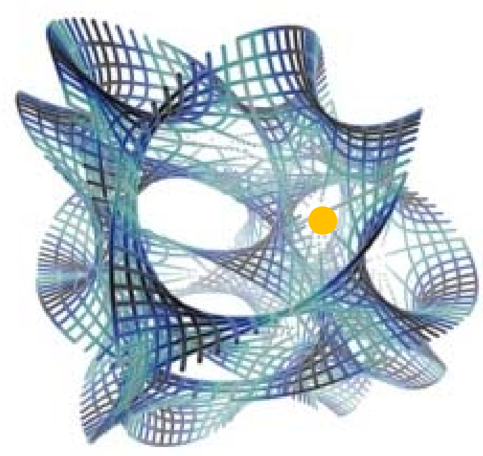
Hwang, Lee, P.Y. 2018

more generally, an entire class of susy partition functions was proposed for Riemannian surfaces with circle bundles

$$T^2 \longrightarrow \Sigma_g$$

A-twisted

$\times$



Closset, Kim, Willett 2017

partition functions as a sum over BAE vacua on A-twisted geometry

$$\Omega_4^G = \sum_{u_* \in \mathcal{S}_{\mathrm{BE}}} \mathcal{F}_1(u_*, \nu; \tau)^{p_1}\, \mathcal{F}_2(u_*, \nu; \tau)^{p_2}\, \mathcal{H}(u_*, \nu; \tau)^{g-1}$$

$$\mathcal{S}_{\mathrm{BE}} = \{u_* \mid \Phi_a(u_*, \nu; \tau) = 1, \forall a, \quad w \cdot u_* \neq u_*, \forall w \in W_G\} / W_G$$

complex structure $\tau \sim i\dfrac{\beta_2}{\beta_1}$
of the fibre torus

complexified
flavor holonomies

complexified
gauge holonomies

Closset, Kim, Willett 2017

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$$\Phi_a(u, \nu; \tau) \equiv \exp(2\pi i \partial_a \mathcal{W})$$

⬇

twisted superpotential in the Coulomb phase
on Σ_g due to the infinite towers
of 2d chiral fields from T^2 compactification

Closset, Kim, Willett 2017

partition functions as a sum over BAE vacua on A-twisted geometry

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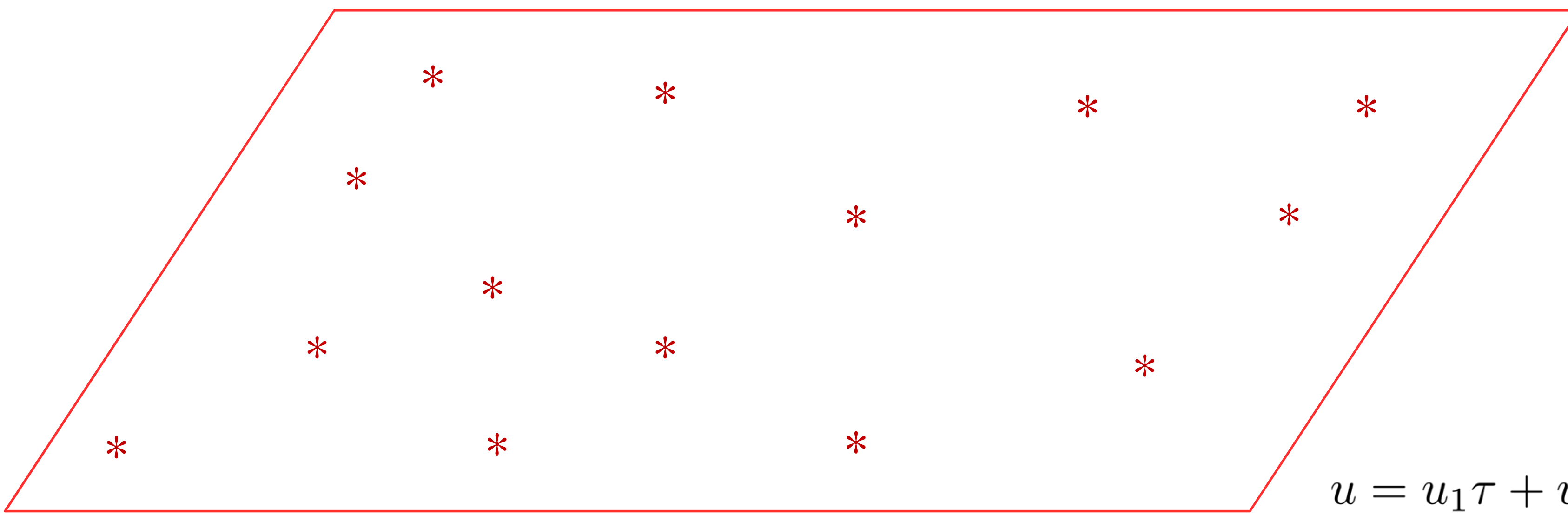


$u = u_1 \tau + u_2$

there are several secret and not-so-secret restrictions

1) integral U(1) r-charges

2) maximal flavor symmetries

3) non-zero flavor holonomies ~ real masses in 3d sense

4) absence of “triples” → SU(N) and Sp(N) only

there are several secret and not-so-secret restrictions

1) integral U(1) r-charges can be bypassed when one chooses "physical" background

2) maximal flavor symmetries

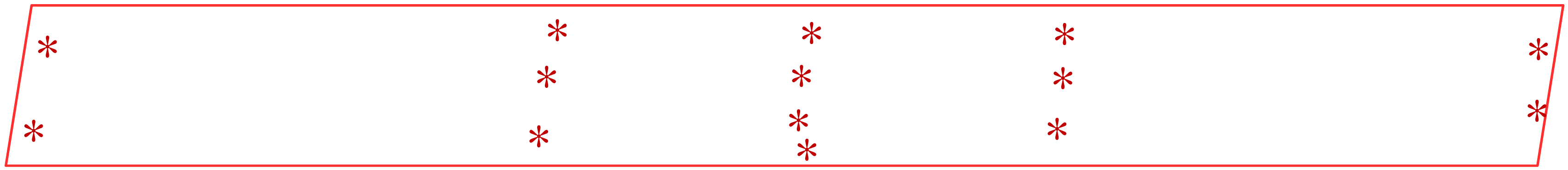
3) non-zero flavor holonomies ~ real masses in 3d sense

4) absence of "triples" → SU(N) and Sp(N) only

small β_2 limit

$$\Omega_4^G = \sum_{u_* \in \mathcal{S}_{\mathrm{BE}}} \mathcal{F}_1(u_*, \nu; \tau)^{p_1} \, \mathcal{F}_2(u_*, \nu; \tau)^{p_2} \, \mathcal{H}(u_*, \nu; \tau)^{g-1}$$

$$\mathcal{S}_{\mathrm{BE}} = \{u_* \mid \Phi_a(u_*, \nu; \tau) = 1, \forall a, \quad w \cdot u_* \neq u_*, \forall w \in W_G\} \, / W_G$$



$$u = u_1 \tau + u_2$$

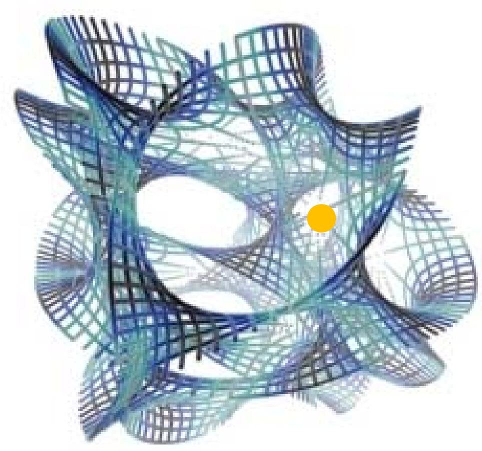
a small radius limit of one geometry may be regarded
as a large radius limit of another geometry

$SL(2, Z)$

T^2

$\downarrow$

A-twisted Σ_g $\times$

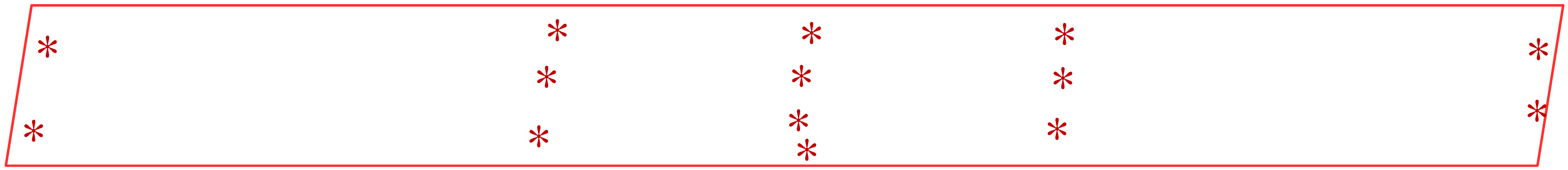


Closset, Kim, Willett 2017

small β_1 limit

$$\Omega_4^G = \sum_{u_* \in \mathcal{S}_{\rm BE}} \mathcal{F}_1(u_*, \nu; \tau)^{p_1} \mathcal{F}_2(u_*, \nu; \tau)^{p_2} \mathcal{H}(u_*, \nu; \tau)^{g-1}$$

$$\mathcal{S}_{\rm BE} = \{u_* \mid \Phi_a(u_*, \nu; \tau) = 1, \forall a, \quad w \cdot u_* \neq u_*, \forall w \in W_G\} / W_G$$

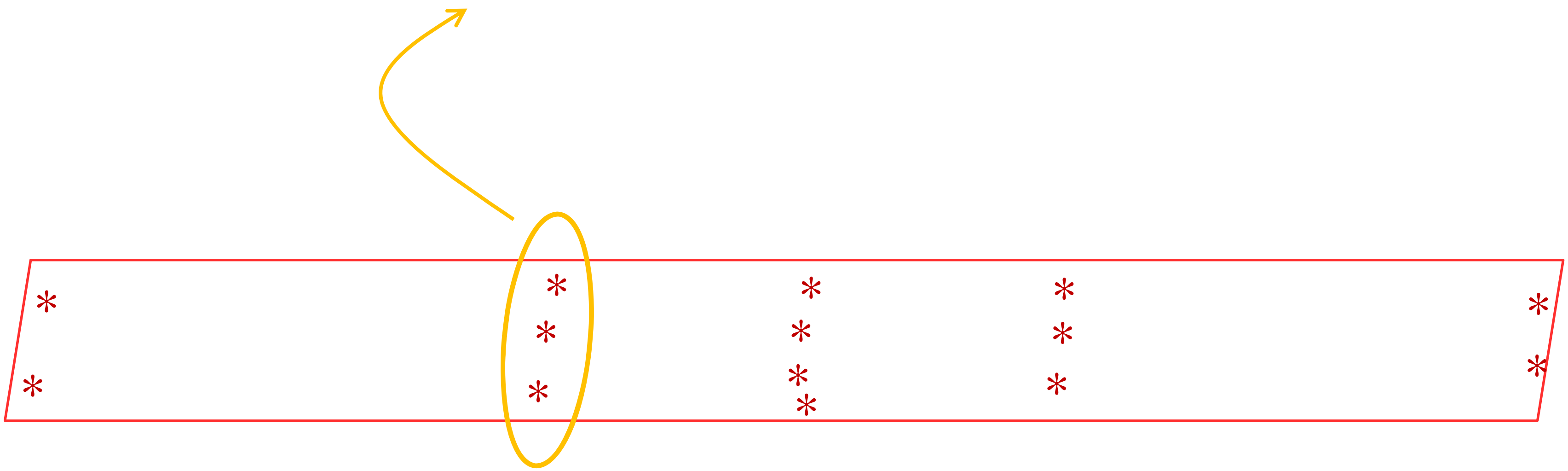


$$\tilde{u} = u_1 + u_2/\tau$$

in either limit

$$\Omega_4^G = \sum_{u_H} \sum_{\sigma_* \in \mathcal{S}_{\mathrm{BE}}^H} \mathcal{F}_1^H(\sigma_*, \nu; \tau)^{p_1} \, \mathcal{F}_2^H(\sigma_*, \nu; \tau)^{p_2} \, \mathcal{H}^H(\sigma_*, \nu; \tau)^{g-1}$$

$$\mathcal{S}_{\mathrm{BE}}^H = \left\{ \sigma_* \mid \Phi_a^H(\sigma_*, \nu; \tau) = 1, \forall a, \quad w \cdot \sigma_* \neq \sigma_*, \forall w \in W_H \right\} / W_H$$

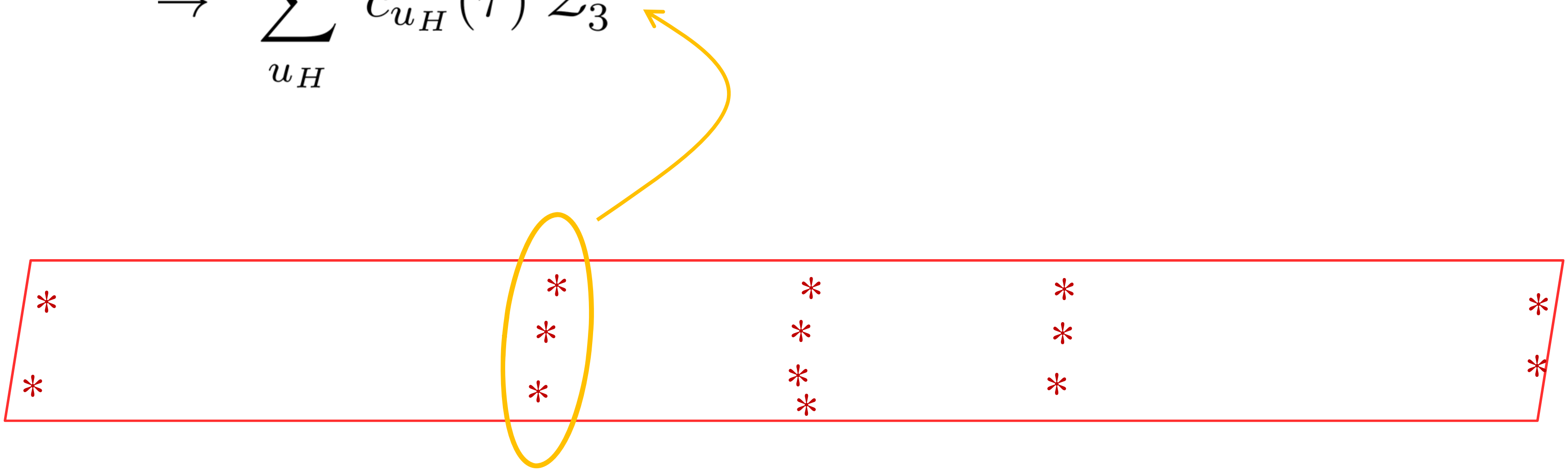


$$u = u_1 \tau + u_2$$
$$\tilde{u} = u_1 + u_2/\tau$$

in either limit

$$\Omega_4^G = \sum_{u_H} \sum_{\sigma_* \in \mathcal{S}_{\mathrm{BE}}^H} \mathcal{F}_1^H(\sigma_*, \nu; \tau)^{p_1} \, \mathcal{F}_2^H(\sigma_*, \nu; \tau)^{p_2} \, \mathcal{H}^H(\sigma_*, \nu; \tau)^{g-1}$$

$$\to \sum_{u_H} c_{u_H}(\tau) \; \mathcal{Z}_3^H$$

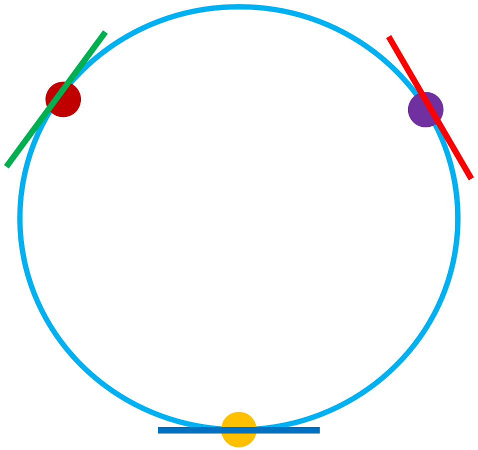


$$u = u_1 \tau + u_2$$
$$\tilde{u} = u_1 + u_2/\tau$$

→ (4) there may be multiple Cardy exponents/Casimir energies and, generically, the dominant ones need not equal the naïve ones

$$\frac{1}{2\pi i\tau}\log\left(c_{u_H}(\tau)\right) \qquad\qquad \frac{1}{2\pi i\tilde\tau}\log\left(c_{\tilde u_H}(\tilde\tau)\right)$$

$$= \quad (g-1)\times\left[-\frac{1}{12}(\mathrm{tr}_f R)+\frac{1}{2}\sum_\alpha \epsilon_\alpha(1-\epsilon_\alpha)+\frac{1}{2}\sum_i (r_i-1)\sum_{\rho_i}\epsilon_{\rho_i}(1-\epsilon_{\rho_i})\right]+\cdots$$



$$\epsilon_Q = \{Q\cdot u_H/\tau\} = \{Q\cdot \tilde u_H/\tilde\tau\}$$

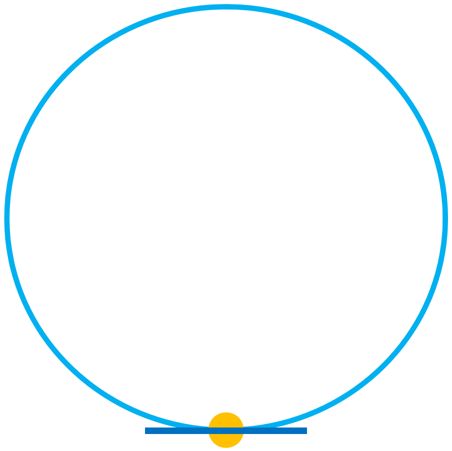
Hwang, Lee, P.Y. 2018

as opposed to more familiar but naïve exponents at $\begin{array}{c} u_H = 0 \\ \tilde{u}_H = 0 \end{array}$:

$$\frac{1}{2\pi i \tau} \log\left(c_{u_H=0}(\tau)\right) \qquad\qquad \frac{1}{2\pi i \tilde{\tau}} \log\left(c_{\tilde{u}_H=0}(\tilde{\tau})\right)$$

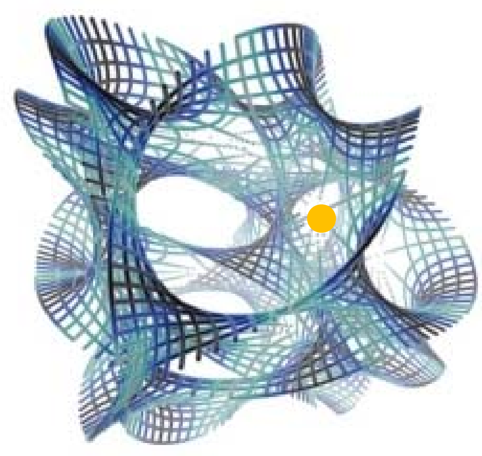
$$= \quad (g-1) \times \left[-\frac{1}{12} (\mathrm{tr}_f R) \right] \; + \; \cdots$$

Di Pietro + Komargodski 2014



something similar can be done for superconformal indices:
the naïve Cardy exponents are generically modified
due to the presence of *H*-saddles

$$S^1 \times S^3 \times$$

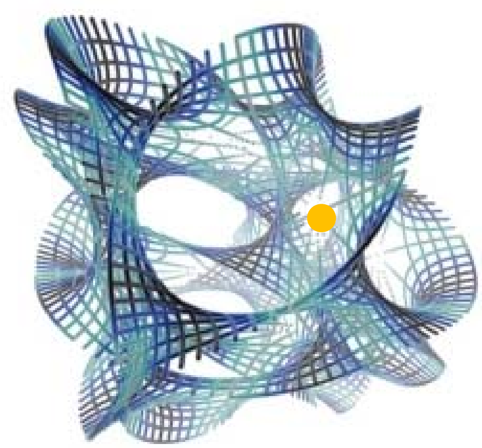


Hwang, Lee, P.Y. 2018

cf) Ardehali 2015
Di Pietro, Honda 2016

something similar can be done for superconformal indices:
the naïve Casimir exponents hold, however,
despite the presence of *H*-saddles

~~$SL(2,Z)$~~

$S^1 \times S^3 \times$ 

Hwang, Lee, P.Y. 2018
Assel, Cassani, Di Pietro, Komrgodski, Lorenzen, Martelli 2015
Bobev, Bullimore, Kim 2015

a chapter closed, in the M-theory hypothesis,
via localization

ubiquitous "H-saddles"

gluing supersymmetric gauge theories
across dimensions

"H-saddles" for (2,0) theories on T^2