Quantum dynamics of 6d (2,0) theories on a circle

June Nahmgoong

Seoul National University

APCTP - **Strings, Branes and Gauge Theories**

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- Talk based on: Seok Kim, JN [1702.04058] + recent progress

Introduction

- 6d SCFTs
  - 6d $\mathcal{N}=(2,0)$: ADE classfication [Witten](95)
  - 6d $\mathcal{N}=(1,0)$: Atomic classification [Heckman, Morrison, Rudelius, Vafa](15)
  - No known Lagrangian description
- N^3 mystery
  - Degrees of freedom $\propto N^3$ [Klebanov, Tseytlin](96)
  - Microscopic computation is challenging
- Strategy
  - Worldvolume theory of N M5 branes: 6d $\mathcal{N}=(2,0)$ A_{N-1} SCFT
  - Put 6d SCFT at tensor branch and compactify on a circle.
  - Use massive objects to explore N^3 behavior of 6d SCFT at massless limit

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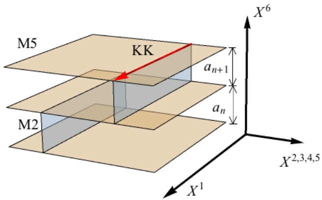
6d $(2,0)$ A_{N-1} SCFT



	0	1	2,3,4,5	6	7,8,9,10
M2	X	X		X	
M5	X	X	X		
		S^1	$SO(4)$	$SO(5)_R$	

- **Tensor branch**
  - Tensor multiplet $(B_{\mu\nu}, \psi, \phi^{6,7,8,9,10})$ with $SO(5)_R$
  - Self-dual string coupled to $B_{\mu\nu}$, $(\star dB = dB)$
  - Non zero VEV to scalar: $\langle \phi^6 \rangle = \text{diag}(a_1, ..., a_{N-1}) \sim$ Coulomb branch
  - Self-dual string has a tension $\propto a_i$

- **Circle compactification**
  - KK momentum along $X^1 \sim X^1 + 2\pi R_1$ (M-theory circle)
  - 5d $U(N)$ MSYM with W-bosons (SD string) and instantons (KK)
  - Instanton has a mass $\propto 1/R_1$

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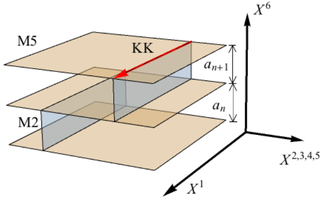

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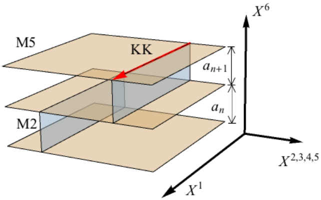

	0	1	2,3,4,5	6	7,8,9,10
M2	X	X		X	
M5	X	X	X		
	$\overbrace{}^{T^2}$		$\overbrace{}^{SO(4)}$	$\overbrace{}^{SO(5)_R}$	

- BPS index
  - Counts BPS bound states of W-bosons and instantons

$$Z(\tau, a, m, \epsilon_{1,2}) = \text{Tr}\Big[(-1)^F e^{2\pi i\tau\frac{H+P}{2}} e^{2\pi i\bar{\tau}\frac{H-P}{2}} e^{\epsilon_1(J_1+J_R)} e^{\epsilon_1(J_2+J_R)} e^{2mJ_L} e^{-\alpha\cdot a}\Big]$$

  - τ: complex structure of T^2 ($\tau = i\frac{R_0}{R_1}$, Im$[\tau] \sim$ (coupling constant)$^{-2}$)
  - a_i: tensor VEVs
  - m: chemical potential for $SU(2)_L \subset SO(5)_R$, $\mathcal{N} = 2^*$ mass deformation in 4d
  - $\epsilon_{1,2}$: chemical potentials for spatial $SO(4)$ locked with R-symmetry (Ω-deformation)

Observables

- **Instanton partition function**
  - Instanton expansion: $q = e^{2\pi i\tau}$ [Nekrasov](02) [H.-C.Kim, S.Kim, E.Koh, K.Lee, S.Lee](11)

$$Z(\tau, a, m, \epsilon_{1,2}) = Z_{\text{pert}}(a, m, \epsilon_{1,2}) \cdot \Big(\sum_{k=0}^{\infty} Z_k(a, m, \epsilon_{1,2}) \cdot q^k \Big)$$

$$Z_k(a, m, \epsilon_{1,2}) = \sum_{\sum |Y_i| = k} \prod_{i,j=1}^{N} \sum_{s \in Y_i} \frac{\sinh \frac{E_{ij}(s)+m-\epsilon_+}{2} \cdot \sinh \frac{E_{ij}(s)-m-\epsilon_+}{2}}{\sinh \frac{E_{ij}(s)}{2} \cdot \sinh \frac{E_{ij}(s)-2\epsilon_+}{2}}$$

- **Elliptc genus of self-dual string**
  - W-boson expansion: e^{-a_i} [Haghighat, Iqbal, Kozcaz, Lockhart, Vafa](13)
  - Manifest modular property

$$Z(\tau, a, m, \epsilon_{1,2,}) = [Z_{U(1)}(\tau, m, \epsilon_{1,2})]^N \cdot \sum_{\{n_i\}=\{0\}}^{\{\infty\}} Z_{\{n_i\}}(\tau, m, \epsilon_{1,2}) \cdot e^{\sum n_i a_i}$$

$$Z_{\{n_i\}}(\tau, m, \epsilon_{1,2}) = \sum_{|Y_i|=n_i} \prod_{i=1}^{N} \prod_{s \in Y_i} \frac{\theta_1(\tau | \frac{E_{i,i+1}(s)-m+\epsilon_-}{2\pi i}) \cdot \theta_1(\tau | \frac{E_{i,i-1}(s)+m+\epsilon_-}{2\pi i})}{\theta_1(\tau | \frac{E_{i,i}(s)+\epsilon_1}{2\pi i}) \cdot \theta_1(\tau | \frac{E_{i,i}(s)-\epsilon_2}{2\pi i})}$$

Goal: BPS index at massless limit $(\tau, a_i \to 0)$

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Goal: BPS index at massless limit $(\tau, a_i \to 0)$

S-duality

- **S-duality of 4d theories**
  - Duality between two QFTs with strong coupling and weak coupling
  - Complex coupling $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$: $\tau \leftrightarrow -\frac{1}{\tau}$
- **S-duality of 6d theories on T^2**
  - Temporal circle (R_0) and M-theory circle (R_1) form T^2 of complex structure $\tau = i\frac{R_0}{R_1}$
  - $SL(2,\mathbb{Z})$ of T^2 includes S-duality $\tau \leftrightarrow -\frac{1}{\tau}$ $(R_0 \leftrightarrow R_1)$
  - $R_1 \ll R_0$ $(\tau \to i \cdot \infty$, weak coupling): 5d SYM $\longrightarrow N^2$ d.o.f.
  - $R_1 \gg R_0$ $(\tau \to i \cdot 0^+$, strong coupling): 6d SCFT $\longrightarrow N^3$ d.o.f.

  } **S-dual**
  - S-duality in 6d should be anomalous

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- **Chiral anomaly**

– Anomaly 8-form of 6d (2,0) A_{N-1} theory [Ohmori, Shimizu, Tachikawa, Yonekura](14)

$$I_8 = N I_8(1) + \frac{N^3 - N}{24} p_2(SO(5)_R), \quad p_2 = \text{second Pontryagin class}$$

– Inflow to self-dual string gives anomaly 4-form [H.-C.Kim, S.Kim, J.Park](16)

$$I_4 = -\Omega^{ij} n_i n_j \chi(SO(4)) - 2\Omega^{ij} \rho_i n_j \chi(SO(4)_R \subset SO(5)_R),$$

$$\Omega = U(N) \text{ Cartan matrix}, \quad \rho = U(N) \text{ Weyl vector}, \quad \chi = \text{Euler character}$$

– Modular anomaly of the elliptic genus is determined from I_4

$$Z_{\{n_i\}}(-\frac{1}{\tau}, \frac{m}{\tau}, \frac{\epsilon_{1,2}}{\tau}) = \exp\Big[\frac{\hat{I}_4}{4\pi i \tau}\Big] \cdot Z_{\{n_i\}}(\tau, m, \epsilon_{1,2})$$

$$\hat{I}_4 = I_4(F \to z): \quad \chi(SO(4)) \to \epsilon_-^2 - \epsilon_+^2, \quad \chi(SO(4)_R) \to m^2 - \epsilon_+^2$$

– S-duality anomaly is determined from the anomaly polynomial

Modular anomaly equation

- **Quasimodular forms**
  - Elliptic genus is quasimodular: spanned by Eisenstein series $E_{2,4,6}(\tau)$
  - Only E_2 has modular anomaly: $E_2(-\frac{1}{\tau}) = \tau^2\left(E_2(\tau) + \frac{6}{\pi i \tau}\right)$
  - Modular anomaly of the elliptic genus determines E_2 dependence

$$\frac{\partial Z_{\{n_i\}}}{\partial E_2} = \frac{1}{24}\Big[- (\epsilon_-^2 - \epsilon_+^2)\Omega^{ij} n_i n_j - 2(m^2 - \epsilon_+^2)\Omega^{ij}\rho_i n_j \Big] Z_{\{n_i\}}$$

- **Modular anomaly equation**
  - BPS index as a grand canonical ensemble: $Z = [Z_{U(1)}]^N \sum_{\{n_i\}} Z_{\{n_i\}} e^{-n_i a_i}$

$$\frac{\partial Z}{\partial E_2} = \frac{1}{24}\Big[- (\epsilon_-^2 - \epsilon_+^2)\Omega^{ij} \frac{\partial}{\partial a_i}\frac{\partial}{\partial a_j} + \underbrace{2(m^2 - \epsilon_+^2)\Omega^{ij}\rho_i \frac{\partial}{\partial a_j}}_{\text{extra}} \Big] Z$$

  - $E_2 \sim$ time, $a_i \sim$ space, $Z \sim$ temperature $\rightarrow$ **Heat equation** + extra part
  - S-duality: $Z(-\frac{1}{\tau}, \frac{z}{\tau}; E_2(-\frac{1}{\tau})) = Z(\tau, z; E_2(\tau) + \frac{6}{\pi i \tau}) \rightarrow$ **Time evolution**
  - Gaussian heat kernel convolution

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- **S-duality kernel**
  - Decompose Z: $Z = [Z_{U(1)}]^N \cdot \exp\Big[\frac{m^2-\epsilon_+^2}{\epsilon_1\epsilon_2}\rho\cdot a - E_2(\tau)\frac{N^3-N}{288}\frac{(m^2-\epsilon_+^2)^2}{\epsilon_1\epsilon_2}\Big]\cdot Z_{\text{S-dual}}$
  - S-duality kernel: Gaussian heat kernel for tensor VEV space ($\epsilon_1\epsilon_2 < 0$)

$$Z_{\text{S-dual}}(-\frac{1}{\tau}, a^D, \frac{z}{\tau}) = \int_{-\infty}^{\infty}\prod_{i=1}^{N-1} da_i \exp\Big[-\frac{\pi i\tau}{\epsilon_1\epsilon_2}\Omega^{-1}_{ij}(a^i_D - a^i)(a^j_D - a^j)\Big] Z_{\text{S-dual}}(\tau, a, z)$$

  - Generalization of 4d prepotential's S-duality (Legendre transform)
  - Expect nonperturbative correction for $\epsilon_{1,2}$ [Hosomichi, S.Lee, J.Park](10)
- **Large volume limit**
  - Turning off IR regulator $\epsilon_{1,2} \to 0$
  - Quantum prepotential f: $Z \sim \exp\Big[-\frac{f}{\epsilon_1\epsilon_2}\Big]$ [Nekrasov](02)
  - Saddle point approximation ($f = f(\tau, a, m)$, $f^D = f(-\frac{1}{\tau}, a^D, \frac{m}{\tau})$)

$$a_i^D = a_i + \frac{\Omega^{ij}}{2\pi i\tau}\Big(\frac{\partial f}{\partial a_j} + m^2\rho_j\Big), \quad \tau^2 f^D = f + \frac{1}{4\pi i\tau}\Omega^{ij}\frac{\partial f}{\partial a_i}\frac{\partial f}{\partial a_j} + N(\tau^2 f^D_{U(1)} - f_{U(1)})$$

  - Compute prepotential at strong coupling (f^D) from prepotential at weak coupling (f)

- **S-duality kernel**
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- **Weak coupling side**
  - Perturbative prepotential is the leading part ($q = e^{2\pi i\tau} << 1$)

$$f = \sum_{\alpha>0}\Big(2\text{Li}_3(e^{-\alpha(a)}) - \text{Li}_3(e^{-\alpha(a)+m}) - \text{Li}_3(e^{-\alpha(a)-m})\Big) + \sum_{k=1}^{\infty} f_k q^k, \quad f_k \sim e^{\pm kNm}$$

  - Suppression condition for the instanton prepotential

$$|qe^{\pm Nm}| < 1 \quad \rightarrow \quad |\text{Im}[m^D]| < \frac{2\pi}{N} \quad (m^D = \frac{m}{\tau})$$

- **Strong coupling side**
  - Massless limit: $\tau^D,\ a_i^D \rightarrow 0$
  - Prepotential at strong coupling ($\beta = -2\pi i\tau^D$)

$$f^D = \frac{N^3}{24\beta}m_D^4 - \text{sgn}(\text{Im}[m_D])\frac{\pi i N^2}{6\beta}m_D^3 - \frac{\pi^2 N}{6\beta}m_D^2 + \mathcal{O}(\beta^0), \quad |\text{Im}[m^D]| < \frac{2\pi}{N}$$

  - Explicit $N^3 m_D^4$ term but validity region is highly limited

Saddle point approximation

$$a_i^D = a_i + \frac{\Omega^{ij}}{2\pi i\tau}\Big(\frac{\partial f}{\partial a_j} + m^2\rho_j\Big), \quad \tau^2 f^D = f + \frac{1}{4\pi i\tau}\Omega^{ij}\frac{\partial f}{\partial a_i}\frac{\partial f}{\partial a_j} + N(\tau^2 f^D_{U(1)} - f_{U(1)})$$

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Beyond the range: $\text{Im}[m^D] > 2\pi/N$

- **Nonperturbative bound states**

– Instanton prepotential is not suppressed even in the weak coupling side

$$U(3):\ f = f_{\text{pert}} - \text{Li}_3(qe^{-a_1-a_2+3m}) + \ldots$$
$$U(4):\ f = f_{\text{pert}} - \text{Li}_3(qe^{-a_1-a_2+3m}) - \text{Li}_3(qe^{-a_2-a_3+3m}) - \text{Li}_3(q^2e^{-a_1-a_2-a_3+5m}) + \ldots$$

– Combined bound states of W-bosons and instantons

– Massless bound states change the saddle point equation

1) Classify nonperturbative bound states
2) Track their massless points

} to explore the full range of m^D

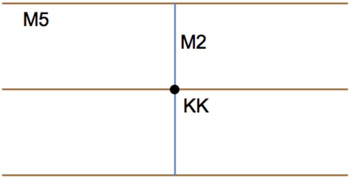
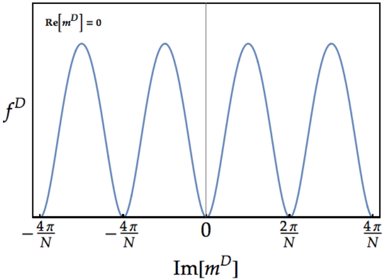


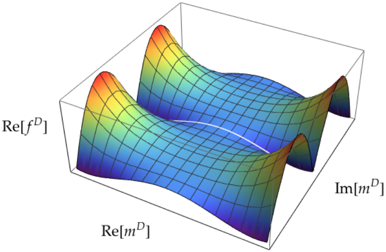
Figure 1: Nonperturbative bound state of U(3)

- Asymptotic prepotential of 6d (2,0) A_{N-1} SCFT
  - Valid expression for all $m^D \in \mathbb{C}$

$$f^D = \frac{1}{N\beta}\Big(\mathrm{Li}_4(1) - \mathrm{Li}_4(e^{Nm^D}) - \mathrm{Li}_4(e^{-Nm^D})\Big) = \frac{N^3 m_D^4}{24\beta} + \mathcal{O}(N^2)$$

  - Periodicity enhancement $m^D \sim m^D + \frac{2\pi i}{N}$
  - Universal $N^3 m_D^4$ behavior → evidence for N^3 d.o.f.





Concluding remarks

- S-duality of 6d theories on a circle
  - 6d chiral anomaly makes S-duality anomalous
  - S-duality kernel is derived from modular anomaly equation

- N^3 behavior
  - Saddle point approximation at large volume limit
  - Universal $N^3 m_D^4$ behavior is observed at 6d SCFT point

- Extension to the other 6d SCFTs
  - 6d (1,0) theories: E-string theories, etc

Thank you!