# Supersymmetric gauge theories and elliptic integrable systems

Antonio Sciarappa

KIAS

Based on arXiv:????.????? (with ??? ??? and ??? ???)

Strings, Branes and Gauge Theories 2018 - APCTP

# Overview

Consider mass-deformed maximal $SU(N)$ Super Yang-Mills in 4d, 5d

- Conjecturally, related to elliptic integrable systems

$$4\text{d }\ \mathcal{N}=2^*\ \ SU(N)\ \text{ on }\ \mathbb{R}^4_{\epsilon_{1,2}} \qquad \Longleftrightarrow \qquad N\text{ - particle Calogero-Moser}$$

$$5\text{d }\ \mathcal{N}=1^*\ \ SU(N)\ \text{ on }\ \mathbb{R}^4_{\epsilon_{1,2}}\times S^1_R \qquad \Longleftrightarrow \qquad N\text{ - particle Ruijsenaars-Schneider}$$

- Precise relation depends on values of $\epsilon_{1,2}$

$\epsilon_1 = 0,\ \epsilon_2 = 0$ : classical, time-independent (Seiberg-Witten)

$\epsilon_1 = \hbar,\ \epsilon_2 = 0$ : quantum, time-independent (NS limit)

$\epsilon_1 \neq 0,\ \epsilon_2 \neq 0$ : quantum, time-dependent (general case)

Goal: test this conjectural relation and study its implications

Focus on NS limit $\epsilon_1 = \hbar$, $\epsilon_2 = 0$: Bethe/Gauge correspondence between

Quantum Integrable System $\Longleftrightarrow$ Supersymmetric Gauge Theory

What can we learn from this?

- If conjecture true, get exact analytic solution to quantum integrable system (usually hard to construct, especially for models with elliptic potential)

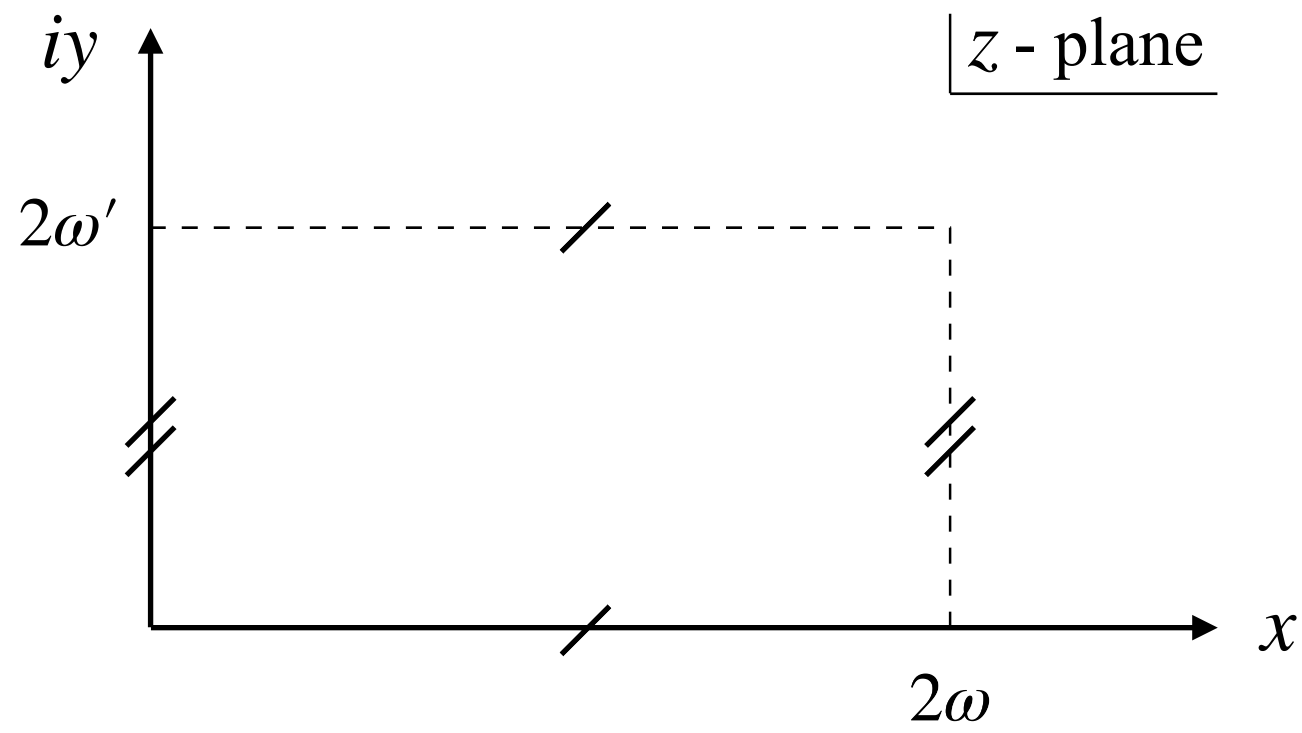
- Allows us to study possible gauge theory effects non-perturbative in $\epsilon_1$, related to quantum mechanical instanton effects non-perturbative in $\hbar$

# Part I:

# Four-dimensional case

# Integrable systems: conventions

We will consider systems of $N$ particles living on a rectangular torus $T^2_\tau$




$$\tau = \omega'/\omega$$

$$Q_{4d} = e^{2\pi i \tau}$$

Complex coordinate variables ($j = 1, \ldots, N$):

$$z_j = x_j + iy_j, \qquad x_j, y_j \in \mathbb{R}$$

# Elliptic Calogero-Moser system

Hamiltonians $H_j$ of $N$ - particle classical elliptic Calogero-Moser (complex):

$$H_1 = \sum_{j=1}^{N} p_j \qquad \text{(total momentum)}$$

$$H_2 = \sum_{j=1}^{N} p_j^2 + g^2 \sum_{j<k}^{N} \wp(z_j - z_k \,|\, \omega, \omega')$$

$$H_3 = \ldots , \qquad \ldots \qquad , H_N = \ldots$$

with $g$ coupling constant and $\{H_j, H_k\} = 0$

Classical Calogero-Moser related to 4d $\mathcal{N} = 2^*$ $SU(N)$ gauge theory at $\epsilon_{1,2} = 0$:

$$Q_{4d} = e^{2\pi i \tau} , \qquad g = m \text{ adjoint mass} \in \mathbb{C}$$

Why relation to gauge theory at $\epsilon_{1,2} = 0$ ?

- Original observation: the Seiberg-Witten curve of 4d $\mathcal{N} = 2^*$ $SU(N)$ theory coincides with the spectral curve of $N$ - particle classical Calogero-Moser (auxiliary object useful to prove classical integrability of the system)

- Later: all 4d theories of class $\mathcal{S}$ are associated to Hitchin integrable system

From Seiberg-Witten curve + differential $\lambda_{SW}$ + 1-cycle basis $\{A_j, B_j\}$ compute classical periods (action/angle variables) providing IR solution to gauge theory

$$a_j^{(0)} = \oint_{A_j} \lambda_{SW} \ , \qquad a_{D,j}^{(0)} = \frac{\partial F_0}{\partial a_j^{(0)}} = \oint_{B_j} \lambda_{SW} \quad \Longrightarrow \quad \tau_{ij}^{IR} = \frac{\partial a_{D,i}^{(0)}}{\partial a_j^{(0)}}$$

S-duality $\tau_{IR} \longleftrightarrow -1/\tau^{IR}$ of gauge coupling: exchange $a_j^{(0)} \longleftrightarrow a_{D,j}^{(0)}$

Quantum Calogero-Moser ("complex"): canonical quantization $[\hat{z}_j\,,\,\hat{p}_k] \;=\; i\hbar\delta_{j,k}$

$$\hat{H}_1 \;= -\,i\hbar \sum_{j=1}^{N} \partial_{z_j} \quad \text{(total momentum)}$$

$$\hat{H}_2 \;= -\,\hbar^2 \sum_{j=1}^{N} \partial_{z_j}^2 \;+\; \left(g^2 - \hbar^2/4\right) \sum_{j<k}^{N} \wp(z_j - z_k \,|\, \omega, \omega')$$

$$\hat{H}_3 \;=\; \ldots\,, \qquad \ldots \qquad , \hat{H}_N \;=\; \ldots$$

with $\hat{H}_j$ commuting differential operators: $[\hat{H}_j\,,\,\hat{H}_k] = 0$

Complex Quantum Mechanical problem: find $\psi(\vec{z})\,,\,E_j$ such that

$$\boxed{\hat{H}_j\,\psi(\vec{z}) \;=\; E_j\,\psi(\vec{z})}$$

What is the analogue of quantization on the gauge theory side?

- Natural proposal: gauge theory in NS limit

$$\epsilon_{1,2} = 0 \quad \text{(classical)} \quad \Longrightarrow \quad \epsilon_1 = \hbar,\, \epsilon_2 = 0 \quad \text{(quantum)}$$

- Bethe/Gauge correspondence: dictionary between

Quantum Integrable Systems $\Longleftrightarrow$ 4d Supersymmetric Gauge Theory

- Key point: eigenfunctions, eigenvalues and other Integrable System quantities (often hard to compute within standard Quantum Mechanical techniques) can be computed via localization as vev of various Gauge Theory observables

- Bethe/gauge conjecture tested and partially proved in some special cases, but general story largely unexplored (especially for elliptic models)

The conjectural Bethe/Gauge dictionary:

| **Quantum Integrable System** | $\Longleftrightarrow$ | **Supersymmetric Gauge Theory** |
|---|---|---|
| $N$ particles system | | $SU(N)$ gauge theory |
| Planck constant $\hbar$ | | "NS limit" $\epsilon_1 \sim \hbar$, $\epsilon_2 = 0$ |
| parameter $Q_{4d} = e^{2\pi i \tau}$ | | instanton fugacity $Q_{4d}$ |
| coupling $g$ | | adjoint mass $m$ |
| quantum period $a_j$ | | vector multiplet scalar vev $a_j$ |
| quantum period $a_{D,j} = \partial_{a_j} F$ | | superpotential $F_{NS} = -\ln \langle Z_{4d} \rangle_{NS}$ |
| quantization conditions | | SUSY vacua equations |
| energies $E_I(\overrightarrow{a})$ | | codim. 4 observable $\langle Tr(\phi^{I+1}) \rangle_{NS}$ |
| eigenfunction $\psi(\vec{z})$ | | codim. 2 observable $\langle Z_{2d/4d}(\vec{z}) \rangle_{NS}$ |

To test it, focus for simplicity on 2-particle elliptic Calogero-Moser

$$\Big[ -\partial_z^2 + \big(g^2/\hbar^2 - 1/4\big)\,\wp(z\,|\,\omega,\omega')\Big]\,\psi(z) \;=\; E/\hbar^2\,\psi(z)$$

Remark: *not yet* a true Quantum Mechanical problem since $z\,,\hbar\,,g\,,E\;\in\;\mathbb{C}$ ;

we still need to specify the Hilbert space / reality conditions for all parameters

such that the Hamiltonian is self-adjoint on some domain, for example:

- B-model: Hilbert space $\mathscr{H} = L^2_x([0\,,2\omega])$

$$(z\,,\hbar\,,g)\;\in\;\mathbb{C}\quad\longrightarrow\quad(x\,,\hbar_x\,,g_x)\;\in\;\mathbb{R}_+\,,\quad E = E^{(B)}\;\in\;\mathbb{R}_+$$

- A-model: Hilbert space $\mathscr{H} = L^2_y([0\,,2\omega'])$

$$(z\,,\hbar\,,g)\;\in\;\mathbb{C}\quad\longrightarrow\quad(iy\,,i\hbar_y\,,ig_y)\;\in\;i\mathbb{R}_+\,,\quad E = E^{(A)}\;\in\;\mathbb{R}_+$$

We can therefore consider two “different” Quantum Mechanical problems

- B-model:

$$\left[ -\partial_x^2 + \left( g_x^2/\hbar_x^2 - 1/4 \right) \wp(x\,|\,\omega,\omega') \right] \psi^{(B)}(x) = E^{(B)}/\hbar_x^2\, \psi^{(B)}(x)$$

- A-model:

$$\left[ -\partial_y^2 - \left( g_y^2/\hbar_y^2 - 1/4 \right) \wp(iy\,|\,\omega,\omega') \right] \psi^{(A)}(y) = E^{(A)}/\hbar_y^2\, \psi^{(A)}(y)$$

which are actually related by S-duality, i.e. exchange real / imaginary period

$$-\partial_y^2 - \left( g_y^2/\hbar_y^2 - 1/4 \right) \wp(iy\,|\,\omega,\omega') \quad = \quad -\partial_y^2 + \left( g_y^2/\hbar_y^2 - 1/4 \right) \wp(y\,|\,-i\omega', i\omega)$$

$$\Longrightarrow \quad \boxed{E^{(A)}(\omega,\omega') \; = \; E^{(B)}(-i\omega', i\omega)}$$

Can we solve them via Bethe/Gauge? How to test the gauge theory solution?

- Given the form of the elliptic potential (confining for both B- and A-model), we expect a discretized set of energy levels $E_n^{(B)}, E_n^{(A)}$ (and eigenfunctions)

- According to Bethe/Gauge, energy quantized according to vacua equations

  - B-model: $a(E) \sim 2\pi n \hbar_x, \quad n \in \mathbb{Z}$

  - A-model: $a_D(E) = \partial_a F_{NS}\big(a(E)\big) \sim 2\pi n \hbar_y, \quad n \in \mathbb{Z}$

  with inverse function $E(a) \sim \langle Tr\big(\phi^2(a)\big)\rangle_{NS}$ computable via localization as a convergent series in $Q_{4d}$ with coefficients exact in $\hbar$ (exact WKB)

- Results can be tested agains numerical diagonalization of the Hamiltonian

A little more in detail:

- From localization we find, putatively (for $\epsilon_1\,, m \in \mathbb{C}$ )

$$E(a) \,=\, \frac{\pi^2}{\omega^2}\left[a^2 \,-\, \frac{1}{12}(m^2{-}\frac{\epsilon_1^2}{4}) \,+\, \frac{1}{6}(m^2{-}\frac{\epsilon_1^2}{4})\big(1 - E_2(\omega,\omega')\big) \,-\, Q_{4d}(m^2{-}\frac{\epsilon_1^2}{4})\frac{16a^2 - 4m^2 - 3\epsilon_1^2}{2(4a^2 - \epsilon_1^2)} \,+\, O(Q_{4d}^2)\right]$$

- Get $E_n^{(B)}, E_n^{(A)}$ imposing reality conditions on $\epsilon_1\,, m$ + quantization conditions

- B-model: $(\epsilon_1\,, m) \in \mathbb{C} \longrightarrow (\hbar_x\,, g_x) \in \mathbb{R}_+\,, \qquad a \sim 2\pi n \hbar_x$

- A-model: $(\epsilon_1\,, m) \in \mathbb{C} \longrightarrow (i\hbar_y\,, ig_y) \in i\mathbb{R}_+\,, \qquad a_D \,=\, \partial_a F_{NS}(a) \,\sim\, 2\pi n \hbar_y$

- Non-trivial consistency check: S-duality relation must be valid

$$E_n^{(A)}(\omega, \omega') \,=\, E_n^{(B)}(-i\omega', i\omega)$$

## Gauge theory results indeed seem to match numerical ones and satisfy S-duality

- B-model:

| $\omega' = -\frac{2\pi^2 i}{\ln(0.0025)}$, $\omega = \pi$, $\frac{g_x}{\hbar_x} = \frac{3}{2}$ | $E_0^{(\mathrm{B})}/\hbar_x^2$ | $E_1^{(\mathrm{B})}/\hbar_x^2$ |
|---|---|---|
| Gauge theory - $O(Q_{\mathrm{4d}}^3)$ | 0.8425256585406518... | 2.0902404697872537... |
| Gauge theory - $O(Q_{\mathrm{4d}}^6)$ | 0.8425256308465472... | 2.0902404238633428... |
| Gauge theory - $O(Q_{\mathrm{4d}}^9)$ | 0.8425256308465468... | 2.0902404238633425... |
| Numerical | 0.8425256308465468... | 2.0902404238633425... |

- A-model:

| $\omega' = i\pi$, $\omega = -\frac{2\pi^2}{\ln(0.0025)}$, $\frac{g_y}{\hbar_y} = \frac{3}{2}$ | $E_0^{(\mathrm{A})}/\hbar_y^2$ | $E_1^{(\mathrm{A})}/\hbar_y^2$ |
|---|---|---|
| Gauge theory - $O(Q_{\mathrm{4d}}^3)$ | 0.8425245641197144... | 2.0902399533002937... |
| Gauge theory - $O(Q_{\mathrm{4d}}^6)$ | 0.8425256308464403... | 2.0902404238631993... |
| Gauge theory - $O(Q_{\mathrm{4d}}^9)$ | 0.8425256308465469... | 2.0902404238633419... |
| Numerical | 0.8425256308465468... | 2.0902404238633425... |

Part I - summary:

- Bethe/Gauge solution of elliptic Calogero-Moser, based on 4d $\mathcal{N} = 2^*$ $SU(N)$,

  appears to be consistent with numerical results (many tests for various $N$)

  $\Longrightarrow$ can use gauge theory to analytically solve quantum mechanical problems

- Gauge theory formulae provide exact WKB solution as convergent series in $Q_{4d}$

- Same gauge theory associated to different quantum problems (A-, B-model);

  S-duality elliptic Calogero-Moser $\Longleftrightarrow$ S-duality 4d $\mathcal{N} = 2^*$ $SU(N)$

- Eigenfunctions can be worked out along the same lines

# Part II:

# Five-dimensional case

# Elliptic Ruijsenaars-Schneider system

System of $N$ particles on a rectangular torus $T^2_{\tilde{\tau}}$ , with $\tilde{\tau} = \frac{\tilde{\omega}'}{\tilde{\omega}}$ and $Q_{5d} = e^{2\pi i \tilde{\tau}}$

Hamiltonians $H_j$ of $N$ - particle classical Ruijsenaars-Schneider (complex):

$$H_1 = \sum_{j=1}^{N} W_j^* W_j e^{p_j} \quad \text{with} \quad W_j = \sqrt{\frac{\sigma(z_j - z_k + ig \,|\, \tilde{\omega}, \tilde{\omega}')}{\sigma(z_j - z_k \,|\, \tilde{\omega}, \tilde{\omega}')}}$$

$$H_2 = \dots , \qquad \dots \qquad , H_{N-1} = \dots$$

$$H_N = e^{p_1 + \dots + p_N} \qquad \text{(total momentum)}$$

Remark: “relativistic” integrable system (exponential dependence on momenta)

Relation to gauge theory: map to 5d $\mathcal{N} = 1^*$ $SU(N)$ on $\mathbb{R}^4 \times S^1$ at $\epsilon_{1,2} = 0$

$$Q_{5d} = e^{2\pi i\tilde{\tau}}, \qquad g = m \text{ adjoint mass } \in \mathbb{C}$$

Similarly to the 4d case, the Seiberg-Witten curve of 5d $\mathcal{N} = 1^*$ $SU(N)$ theory coincides with the spectral curve of $N$ - particle classical Ruijsenaars-Schneider

From Seiberg-Witten curve + differential $\lambda_{SW}$ + 1-cycle basis $\{A_j, B_j\}$ compute classical periods (action/angle variables) providing IR solution to gauge theory

$$a_j^{(0)} = \oint_{A_j} \lambda_{SW}, \qquad a_{D,j}^{(0)} = \frac{\partial F_0}{\partial a_j^{(0)}} = \oint_{B_j} \lambda_{SW} \qquad \Longrightarrow \qquad \tau_{ij}^{IR} = \frac{\partial a_{D,i}^{(0)}}{\partial a_j^{(0)}}$$

S-duality $\tau_{IR} \longleftrightarrow -1/\tau^{IR}$ of gauge coupling: exchange $a_j^{(0)} \longleftrightarrow a_{D,j}^{(0)}$

Quantum Ruijsenaars-Schneider ("complex"): canonical quantization $[\hat{z}_j\,,\,\hat{p}_k] \;=\; i\hbar\delta_{j,k}$

$$\hat{H}_1 \;=\; \sum_{j=1}^{N} W_j^* \, e^{-i\hbar\partial_{z_j}} \, W_j \quad \text{with} \quad W_j = \sqrt{\frac{\sigma(z_j - z_k + ig\,|\,\tilde{\omega}, \tilde{\omega}')}{\sigma(z_j - z_k\,|\,\tilde{\omega}, \tilde{\omega}')}}$$

$$\hat{H}_2 \;=\; \ldots\,, \qquad \ldots \qquad ,\; \hat{H}_{N-1} \;=\; \ldots$$

$$\hat{H}_N \;=\; e^{-i\hbar\partial_{z_1} + \ldots - i\hbar\partial_{z_N}} \quad \text{(total momentum)}$$

with commuting finite-difference operators: $[\hat{H}_j\,,\,\hat{H}_k] = 0$

Complex Quantum Mechanical problem: find $\psi(\vec{z})\,,\,E_j$ such that

$$\hat{H}_j\,\psi(\vec{z}) \;=\; E_j\,\psi(\vec{z})$$

Everything as before? Not really…

- The quantum Hamiltonians are not differential but finite-difference operators

  $\Longrightarrow$ eigenfunction $\psi(\vec{z})$ ambiguous, defined only up to $i\hbar$-periodic function

How to fix this ambiguity? Faddeev's observation:

- Consider a second set of $N$ commuting finite-difference operators $\hat{\tilde{H}}_j$,

  obtained from the first set by the exchange $2\tilde{\omega} \leftrightarrow \hbar$; by construction

$$[\hat{H}_j\,,\,\hat{H}_k] = 0\,, \qquad [\hat{\tilde{H}}_j\,,\,\hat{\tilde{H}}_k] = 0\,, \qquad [\hat{H}_j\,,\,\hat{\tilde{H}}_k] = 0$$

- We can then consider the modular double problem involving $2N$ operators:

$$\hat{H}_I\,\psi(\vec{z}) = E_I\,\psi(\vec{z}) \qquad\qquad \hat{\tilde{H}}_I\,\psi(\vec{z}) = \tilde{E}_I\,\psi(\vec{z})$$

Eigenfunctions of the modular double problem will then be unambiguous

(solutions to finite-difference operators in both $i\hbar$ and $2i\tilde{\omega}$ )

Remark 1: Solution to modular double problem expected to have symmetry

$$2\tilde{\omega} \longleftrightarrow \hbar$$

Remark 2: By construction, dual Hamiltonians $\hat{\tilde{H}}_j$ contain information about

non-perturbative corrections in $\hbar$ (i.e. $\epsilon_1$ ) to the quantum mechanical problem:

$$\sigma(z\,|\,\tilde{\omega},\tilde{\omega}') \longrightarrow e^{\frac{\pi z}{2\tilde{\omega}}}, \qquad \sigma(z\,|\,\frac{\hbar}{2},\tilde{\omega}') \longrightarrow e^{\frac{\pi z}{\hbar}}$$

Remark 1+2 $\Rightarrow$ non-perturbative corrections fixed in terms of perturbative ones

Still, quantum mechanical problem hard to solve; can we use Bethe/Gauge again?

The conjectural Bethe/Gauge dictionary (naive 5d uplift):

| **Quantum Integrable System** | | **Supersymmetric Gauge Theory** |
|---|---|---|
| $N$ particles system | | $SU(N)$ gauge theory |
| Planck constant $\hbar$ | | "NS limit" $\epsilon_1 \sim \hbar$, $\epsilon_2 = 0$ |
| parameter $Q_{5d} = e^{2\pi i \tilde{\tau}}$ | | instanton fugacity $Q_{5d}$ |
| coupling $g$ | | adjoint mass $m$ |
| quantum period $a_j$ | $\Longleftrightarrow$ | vector multiplet scalar vev $a_j$ |
| quantum period $a_{D,j} = \partial_{a_j} F$ | | superpotential $F_{NS} = -\ln\langle Z_{5d}\rangle_{NS}$ |
| quantization conditions | | SUSY vacua equations |
| energies $E_I(\overrightarrow{a})$ | | antisym. Wilson loops $\langle W_{\wedge^I}\rangle_{NS}$ |
| eigenfunction $\psi(\vec{z})$ | | codim. 2 observable $\langle Z_{3d/5d}(\vec{z})\rangle_{NS}$ |

To test it, focus again on the “complex” 2-particle case (modular double)

$$\left[\sqrt{\frac{\sigma(z+ig\,|\,\tilde{\omega},\tilde{\omega}')}{\sigma(z\,|\,\tilde{\omega},\tilde{\omega}')}}e^{i\hbar\partial_z}\sqrt{\frac{\sigma(z-ig\,|\,\tilde{\omega},\tilde{\omega}')}{\sigma(z\,|\,\tilde{\omega},\tilde{\omega}')}}+\sqrt{\frac{\sigma(z-ig\,|\,\tilde{\omega},\tilde{\omega}')}{\sigma(z\,|\,\tilde{\omega},\tilde{\omega}')}}e^{-i\hbar\partial_z}\sqrt{\frac{\sigma(z+ig\,|\,\tilde{\omega},\tilde{\omega}')}{\sigma(z\,|\,\tilde{\omega},\tilde{\omega}')}}\right]\psi(z) \;=\; E\,\psi(z)$$

$$\left[\sqrt{\frac{\sigma(z+ig\,|\,\frac{\hbar}{2},\tilde{\omega}')}{\sigma(z\,|\,\frac{\hbar}{2},\tilde{\omega}')}}e^{2i\tilde{\omega}\partial_z}\sqrt{\frac{\sigma(z-ig\,|\,\frac{\hbar}{2},\tilde{\omega}')}{\sigma(z\,|\,\frac{\hbar}{2},\tilde{\omega}')}}+\sqrt{\frac{\sigma(z-ig\,|\,\frac{\hbar}{2},\tilde{\omega}')}{\sigma(z\,|\,\frac{\hbar}{2},\tilde{\omega}')}}e^{-2i\tilde{\omega}\partial_z}\sqrt{\frac{\sigma(z+ig\,|\,\frac{\hbar}{2},\tilde{\omega}')}{\sigma(z\,|\,\frac{\hbar}{2},\tilde{\omega}')}}\right]\psi(z) \;=\; \tilde{E}\,\psi(z)$$

As before, we need to specify the Hilbert space; for example:

- B-model: Hilbert space $\mathscr{H} = L^2_x([0\,,\,2\tilde{\omega}])$

$$(z\,,\,\hbar\,,\,g) \in \mathbb{C} \;\longrightarrow\; (x\,,\,\hbar_x\,,\,g_x) \in \mathbb{R}_+\,, \quad E = E^{(B)} \in \mathbb{R}_+$$

- A-model: Hilbert space $\mathscr{H} = L^2_y([0\,,\,2\tilde{\omega}'])$

$$(z\,,\,\hbar\,,\,g) \in \mathbb{C} \;\longrightarrow\; (iy\,,\,i\hbar_y\,,\,ig_y) \in i\mathbb{R}_+\,, \quad E = E^{(A)} \in \mathbb{R}_+$$

Again A, B problems have discrete energy $E_n^{(B)}$, $E_n^{(A)}$ and are related by S-duality

$$\Longrightarrow \qquad E_n^{(A)}(\omega,\omega') \;=\; E_n^{(B)}(-i\omega', i\omega)$$

Gauge theory putative solution constructed following the same steps as before:

- According to Bethe/Gauge, energy quantized according to vacua equations

  - B-model: $a(E) \;\sim\; 2\pi n \hbar_x\,, \quad n \in \mathbb{Z}$

  - A-model: $a_D(E) \;=\; \partial_a F_{NS}\big(a(E)\big) \;\sim\; 2\pi n \hbar_y\,, \quad n \in \mathbb{Z}$

  with inverse function $E(a) \;\sim\; \langle W_{\square}(a) \rangle_{NS}$ computable via localization

  as a convergent series in $Q_{5d}$ with coefficients exact in $\hbar$ (exact WKB)

- Results can be tested agains numerical diagonalization of the Hamiltonian

However, gauge theory prescription not correct for A-model, S-duality broken:

- B-model:

| $\tilde{\omega}' = i\pi,\ \tilde{\omega} = \frac{2\pi^2}{5},\ g_x = \sqrt{7},\ \hbar_x = \sqrt{2}$ | $E_0^{(\mathrm{B})}$ | $E_1^{(\mathrm{B})}$ |
|---|---|---|
| Gauge theory - $O(Q_{5\mathrm{d}}^2)$ | 3.0186771307022... | 4.8763947499066... |
| Gauge theory - $O(Q_{5\mathrm{d}}^3)$ | 3.0187733076808... | 4.8768251822466... |
| Gauge theory - $O(Q_{5\mathrm{d}}^4)$ | 3.0187773975933... | 4.8768228328517... |
| Numerical | 3.0187773680795... | 4.8768228695812... |

- A-model:

| $\tilde{\omega}' = \frac{2\pi^2 i}{5},\ \tilde{\omega} = \pi,\ g_y = \sqrt{7},\ \hbar_y = \sqrt{2}$ | $E_0^{(\mathrm{A})}$ | $E_1^{(\mathrm{A})}$ |
|---|---|---|
| Gauge theory - $O(Q_{5\mathrm{d}}^2)$ | 3.0190902985544... | 4.8768247463049... |
| Gauge theory - $O(Q_{5\mathrm{d}}^3)$ | 3.0190922181646... | 4.8768255934207... |
| Gauge theory - $O(Q_{5\mathrm{d}}^4)$ | 3.0190922202921... | 4.8768255943602... |
| Numerical (exact) | 3.0187773680795... | 4.8768228695812... |

Why mismatch for the A-model?

- B-model energy ok $\Longrightarrow$ problem must be in A-type quantization conditions

- Conjectural quantization conditions written in terms of $F_{NS} = -\ln\langle Z_{5d}\rangle_{NS}$ :

$$F^{inst}_{NS} = -Q_{5d}\frac{(1-\mu)(1-q\mu^{-1})}{(1-q)(1-q\alpha)(1-q\alpha^{-1})}\left[(1+\mu)(1+q^2\mu^{-1})+q(\mu+\mu^{-1})-2q(\alpha+\alpha^{-1})-2q\right]+O(Q^2_{5d})$$

with $q = e^{\frac{2\pi\epsilon_1}{2\tilde{\omega}}}$, $\mu = e^{\frac{2\pi m}{2\tilde{\omega}}}$, $\alpha = e^{\frac{2\pi a}{2\tilde{\omega}}}$

- For A-model $q \in$ unit circle $\Longrightarrow$ $F_{NS}$ not well-defined function, poles at

$$\hbar_y/2\tilde{\omega} \in \mathbb{Q}$$

which however should be perfectly admissible values in Quantum Mechanics

What are we missing?

- Gauge theory formulae provide exact WKB solution to the A-model problem, that is the solution containing all perturbative corrections in $\hbar$
- However, full solution may involve non-perturbative contributions in $\hbar$ which gauge theory is not able to take into account
- Expectation: exact quantization conditions for A-model will be obtained once we add non-perturbative corrections to the gauge theory (WKB) ones, which should make the poles at $\hbar_y / 2\tilde{\omega} \in \mathbb{Q}$ disappear
- Need a “non-perturbatively” corrected version of Bethe/Gauge for 5d theories

How to determine these non-perturbative corrections? Modular duality

From modular duality structure, expect full solution to be $2\tilde{\omega} \leftrightarrow \hbar$ symmetric;

however, Bethe/Gauge formulae (WKB solution) do not have this property

$\Longrightarrow$ proposal: correct Bethe/Gauge by restoring this symmetry

$$\text{full solution} \;=\; \text{“symmetrized” Bethe/Gauge}$$

The full A-type quantization conditions would then be

$$a_D \;=\; \partial_a F_{full}(a) \;\sim\; 2\pi n \hbar_y \,, \qquad \boxed{F_{full}(a) = F_{NS}(a;\hbar_y, 2\tilde{\omega}) + F_{NS}(a; 2\tilde{\omega}, \hbar_y)}$$

S-duality involves non-perturbatively corrected $F_{full}$ (NS on squashed $S^5$ ?),

which can be shown to be free of poles at $\hbar_y / 2\tilde{\omega} \in \mathbb{Q}$

Correct energies and S-duality restored after adding non-perturbative effects:

- B-model:

| $\tilde{\omega}' = i\pi,\ \tilde{\omega} = \frac{2\pi^2}{5},\ g_x = \sqrt{7},\ \hbar_x = \sqrt{2}$ | $E_0^{(\mathrm{B})}$ | $E_1^{(\mathrm{B})}$ |
|---|---|---|
| Gauge theory - $O(Q_{5\mathrm{d}}^2)$ | $\underline{3.018}6771307022\ldots$ | $\underline{4.876}3947499066\ldots$ |
| Gauge theory - $O(Q_{5\mathrm{d}}^3)$ | $\underline{3.01877}33076808\ldots$ | $\underline{4.87682}51822466\ldots$ |
| Gauge theory - $O(Q_{5\mathrm{d}}^4)$ | $\underline{3.0187773}975933\ldots$ | $\underline{4.8768228}328517\ldots$ |
| Numerical | $\underline{3.0187773680795}\ldots$ | $\underline{4.8768228695812}\ldots$ |

- A-model:

| $\tilde{\omega}' = \frac{2\pi^2 i}{5},\ \tilde{\omega} = \pi,\ g_y = \sqrt{7},\ \hbar_y = \sqrt{2}$ | $E_0^{(\mathrm{A})}$ | $E_1^{(\mathrm{A})}$ |
|---|---|---|
| Gauge theory - $O(Q_{5\mathrm{d}}^2)$ | $\underline{3.01877}54461509\ldots$ | $\underline{4.876822}0215249\ldots$ |
| Gauge theory - $O(Q_{5\mathrm{d}}^3)$ | $\underline{3.01877736}59499\ldots$ | $\underline{4.87682286}86407\ldots$ |
| Gauge theory - $O(Q_{5\mathrm{d}}^4)$ | $\underline{3.0187773680}777\ldots$ | $\underline{4.8768228695}802\ldots$ |
| Numerical | $\underline{3.0187773680795}\ldots$ | $\underline{4.8768228695812}\ldots$ |

Part II - summary:

- Contrary to 4d case, Bethe/Gauge solution of elliptic Ruijsenaars-Schneider based on 5d $\mathcal{N} = 1^*$ $SU(N)$ is inconsistent with numerical results, since gauge theory formulae only provide exact WKB solution to the problem

- Non-perturbative corrections in $\hbar$ (i.e. $\epsilon_1$) completely fixed by modular duality $\Longrightarrow$ provide a "non-perturbatively" corrected version of 5d Bethe/Gauge; analytic solution thus obtained is consistent with numerical results (various $N$)

- 5d vacua equations, S-duality relation receive $\epsilon_1$ non-perturbative corrections: 5d gauge theory in flat space incomplete? Meaningful only in string theory?

Thanks!