USES of String Field Theory

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I will introduce the covariant string field theories and the Polyakov string path integrals in the proper-time gauge. In the proper-time gauge, the string path integral can be written as integrals over the proper-times in a way similar to the Schwinger's proper time representation of Feynman integrals of quantum field theory. For this reason, it becomes feasible in the proper-time gauge to identify the field theoretical expressions of the string path integrals which depict multiple string scatterings. I will discuss the four-massless-particle scattering amplitude in open string theory and the four-graviton scattering amplitude in the closed string theory. The covariant string field theories are useful to study the ultraviolet behavior and the entanglement entropy of strings which are quite different from their counterparts of point particles. I will also discuss applications of the string field theory in the super-gravity, AdS/CFT and related topics.

Contents

Entanglement entropy

Von Neumann entanglement entropy

For a pure state $\rho_{AB} = |\Psi\rangle\langle\Psi|_{AB}$

$$\begin{aligned} S(\rho_A) &= -\mathrm{tr}\,(\rho_A \log \rho_A) = -\mathrm{tr}\,(\rho_B \log \rho_B) = S(\rho_B) \\ \rho_A &= \mathrm{tr}_B\,(\rho_{AB})\,, \quad \rho_B = \mathrm{tr}_A\,(\rho_{AB})\,. \end{aligned}$$

Renyi entanglement entropies and replica trick

$$\begin{aligned} S_\alpha(\rho_A) &= \frac{1}{1-\alpha} \log \mathrm{tr}(\rho^\alpha) = S_a(\rho_B), \\ S(\rho_A) &= \lim_{\alpha\to 1} S_\alpha(\rho_A). \end{aligned}$$

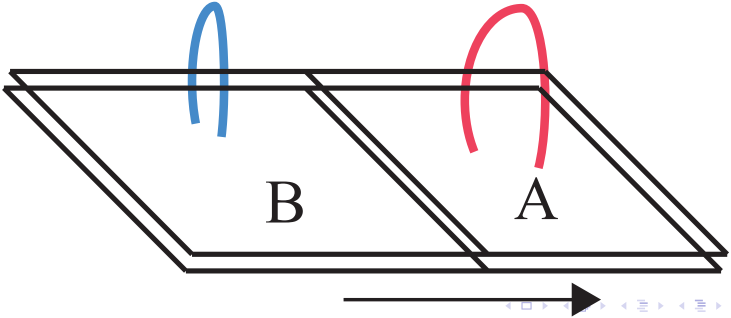
Entanglement entropy is proportional to the area of boundary between the two partitions.

L. Bombelli, R. Koul, J. Lee, and R. Sorkin, Phys. Rev. D **34**, 373 (1986).

M. Srednicki, Phys. Rev. Lett. **71**, 666 (1993).

Locality and Fock space representation of open string field

$$|\Psi\rangle = \sum_{\{N^B_n, N^{gh}_n, n=1,2,3,\dots\}} \sum_a \Psi^a_{\{N^B_n, N^{gh}_n\}}(x^\mu) T^a |\{N^B_n, N^{gh}_n, n=1,2,3,\dots\}\rangle$$
$$= \sum_a \left(\phi^a(x) + A^a_\mu(x) a_1^{\mu\dagger} + \varphi^a_i(x) a_1^{i\dagger} + \dots\right) T^a|0\rangle.$$



Density matrix for string field

String vacuum functional and density matrix

$$\Phi[\Psi] = \langle 0|\Psi\rangle = \frac{1}{N}\int_{\phi_{\{N_n\}}(x,-\infty)=0}^{\phi_{\{N_n\}}(x,0)=\psi_{\{N_n\}}(x)} D[\Phi]e^{-S_E(\Phi)}$$
$$\rho[\Psi,\Psi'] = \langle\Psi|0\rangle\langle 0|\Psi'\rangle = \Phi[\Psi]^*\Phi[\Psi'].$$

We consider the functionals $\Psi = \Psi_A \oplus \Psi_B$ and $\Psi' = \Psi'_A \oplus \Psi_B$ which coincide on the half line $x^1 < 0$. We shall sum over all possible functional Ψ_B. Taking two copies of the half planes, we write the reduced density matrix as

$$\rho_A\left(\Psi,\Psi'\right) = \int D[\Phi_B]\Phi[\Psi_A \oplus \Psi_B]^*\Phi[\Psi'_A \oplus \Psi_B]$$
$$= \frac{1}{N}\int_{\phi_{\{N_n\}}(x,0^-)=\psi'_{\{N_n\}}(x),\ x\in A}^{\phi_{\{N_n\}}(x,0^+)=\psi_{\{N_n\}}(x),\ x\in A} D[\Phi]e^{-S_E(\Phi)}.$$

Entanglement entropy for massive scalar field: Review

The partition function $Z(n)$ for a massive scalar field in $(p+1)$ dimensions on the n-sheeted Riemann surface

$$\begin{aligned} \ln Z(n) &= -\frac{1}{2} \ln \mathrm{Det}\left[-\Delta + m^2\right], \\ \frac{\partial}{\partial m^2} \ln Z(n) &= -\frac{1}{2} \int_{\mathcal{R}^n} d^{p+1}\mathbf{x}\, G_n(\mathbf{x}, \mathbf{x}'). \end{aligned}$$

where $G_n(\mathbf{x}, \mathbf{x}')$ is the Green's for a massive field in the conical space and $\mathbf{x} = (x^0, x^1, \ldots, x^p) = (x^0, x^1, \mathbf{x}_\perp)$. Making use of the expicit expression of the Green's function in the coincident limit, $\mathbf{x}' \to \mathbf{x}$, we get

$$\begin{aligned} \frac{\partial}{\partial m^2} \ln \mathrm{tr}\, \rho^n &= \frac{\partial}{\partial m^2} \ln \frac{Z(n)}{Z(1)^n} \\ &= -\frac{1-n^2}{24n} A_\perp \int \frac{d^{p-1}\mathbf{p}_\perp}{(2\pi)^{p-1}} \frac{1}{m^2 + p_\perp^2} \end{aligned}$$

where $A_\perp$ is the area of boundary hypersurface, perpendicular to (x^0, x^1)-plane, $A_\perp = \int d^{p-1}\mathbf{x}_\perp$.

Entanglement entropy for open string

Entanglement entropy of a massive field

$$S_A = \frac{A_\perp}{12} \int \frac{ds}{s} \int \frac{d^{p-1}\mathbf{p}_\perp}{(2\pi)^{p-1}} \exp\left\{-s\left(p_\perp^2 + m^2\right)\right\}.$$

A direct extension to string field yields the entanglement entropy of open string in the boson sector ($N_n^{\text{gh}} = 0,\ n = 1, 2, \dots$):

$$\begin{aligned} S_A^{\text{Open}} &= \frac{A_\perp}{12} \int \frac{ds}{s} \int \frac{d^{p-1}\mathbf{p}_\perp}{(2\pi)^{p-1}} \mathrm{Tr} \exp\left\{-s\left(p_\perp^2 + N_B - 1\right)\right\} \\ N_B &= \sum_{n=1} n a_n^{I\dagger} a_n^I, \quad I = 0, 1, \dots, d \end{aligned}$$

where 'Tr' denotes the trace over the Fock space as well as over $U(N)$ group space.

From the bosonic harmonic oscillator algebra

$$\begin{aligned} \mathrm{Tr}\, e^{-sN_B} &= N^2 \sum_{\{N^B_n\}} \exp\left\{-s\sum_{n=1} nN^B_n\right\} \\ &= N^2 e^{-\frac{d+1}{24}s} \frac{1}{\eta\left(\frac{is}{2\pi}\right)^{d+1}} \end{aligned}$$

where $\eta(\tau)$ is the Dedekind eta-function, we find the entanglement entropy for open string in boson sector

$$S^{\text{Open}}_A = \frac{A_\perp}{12} \frac{N^2}{(8\pi^2)^{\frac{p-1}{2}}} \int_0^\infty \frac{dt}{t} \frac{1}{t^{\frac{p-1}{2}}} \exp\left\{\left(1-\frac{d+1}{24}\right)2\pi t\right\} \frac{1}{\eta(it)^{d+1}}$$

where $t = s/(2\pi)$.

Taking into account contributions of ghost sector,

$$
\begin{aligned}
S_A^{\rm Open} &= \frac{A_\perp}{12}\int \frac{ds}{s}\int \frac{d^{p-1}\mathbf{p}_\perp}{(2\pi)^{p-1}}\,\mathrm{Tr}\exp\left\{-s\left(p_\perp^2+N_B+N_{gh}-1\right)\right\}(-1)^{F_{gh}} \\
N_{gh} &= \sum_{n=1} n\left(a^{\dagger\rm gh}_{1n}a^{\rm gh}_{1n}+a^{\dagger\rm gh}_{2n}a^{\rm gh}_{2n}\right), \quad F_{gh}=\sum_{n=1}\left(a^{\dagger\rm gh}_{1n}a^{\rm gh}_{1n}+a^{\dagger\rm gh}_{2n}a^{\rm gh}_{2n}\right).
\end{aligned}
$$

Using

$$
\sum_{\{N_{gh}\}} e^{-sN_{gh}}(-1)^{F_{gh}} = \prod_{n=1}\left(1-e^{-sn}\right)^2 = e^{\frac{s}{12}}\eta\left(\frac{is}{2\pi}\right)^2,
$$

we find that the entanglement entropy may be written as

$$
S_A^{\rm Open} = \frac{A_\perp}{12}\frac{N^2}{(8\pi^2)^{\frac{p-1}{2}}}\int_0^\infty \frac{dt}{t}\frac{1}{t^{\frac{p-1}{2}}}\exp\left\{\left(\frac{25-d}{24}\right)2\pi t\right\}\frac{1}{\eta(it)^{d-1}}.
$$

IR behavor of the entanglement entropy for open string

1. IR Behavor: Apart from the leading divergence, the entanglement entropy for open string is at most logarithmically divergent. Its IR behavor is finite for $p \geq 2$ and logarithmically divergent for $p = 1$. In the asypmtotic region $t \to \infty$, the integrand becomes

$$\frac{1}{t^{\frac{p+1}{2}}} \frac{1}{\eta(it)^{24}} = \frac{1}{t^{\frac{p+1}{2}}} \left\{ e^{2\pi t} + 24 + \mathcal{O}(e^{-2\pi t}) \right\}.$$

UV behavors of the entanglement entropy for open string

1. UV Behavor: Except for the tachyon contribution, the entanglement entropy is UV finite for $p \leq 24$ and logarithmically divergent for $p = 25$. Using the modular transformation formula

$$\eta(-1/\tau) = (-i\tau)^{1/2}\eta(\tau),$$

and rewriting the integral for S_A^{Open} in terms of $s = 1/t$, we get

$$S_A^{\rm Open} = \frac{A_\perp}{12}\frac{N^2}{(8\pi^2)^{\frac{p-1}{2}}}\int_0^\infty ds\, s^{\frac{1}{2}(p-27)}\frac{1}{\eta(is)^{24}}.$$

The integrand in UV region where $s \to \infty$, may be expanded asymptotically as

$$s^{\frac{1}{2}(p-27)}\frac{1}{\eta(is)^{24}} = s^{\frac{1}{2}(p-27)}\left\{e^{2\pi s} + 24 + \mathcal{O}(e^{-2\pi s})\right\}.$$

BH entropy as Entanglement entropy

Entanglement entropy in QFT

$$S_E = \frac{A(\Sigma)}{\epsilon^{d-2}}, \quad \langle \phi(x)\phi(y) \rangle \sim \frac{1}{\epsilon^{d-2}}$$

Entanglement entropy measures short-distance correlations between the two partitions (interior and exteria regions of black hole)

G. 'tHooft, Nucl. Phys. B **256**, 727 (1985).
D. Kabat, Nucl. Phys. B **453** 281, (1995).
S. N. Solodukhin, Phys. Rev. Lett. **97**, 201601 (2006).

Puzzle of non-minimal coupling

Renormalization of Newton constant

$$\frac{1}{4G_{\text{ren}}} = \frac{1}{4G} + \sum_s \frac{N_s}{(4\pi)^{\frac{(d-2)}{2}}(d-2)\epsilon^{d-2}} \frac{D_s(d)}{6}.$$

In the presence of the non-minimal coupling ξR for scalar field theory,

$$\frac{1}{4G_{\text{ren}}} = \frac{1}{4G} + \frac{1}{(4\pi)^{\frac{(d-2)}{2}}(d-2)\epsilon^{d-2}} \left(\frac{1}{6} - \xi\right).$$

However, the entanglement entropy in a Ricci flat background ($R = 0$) is not affected by the non-minimal coupling.

S. Solodukhin, Living Rev. Relativity **14**, 8 (2011).

BH entropy for extremal and near extremal black holes in string theory

Microstates of Black Hole

In string theory, extremal black hole is described by BPS states:

$$N_{BPS} = \exp\left(S_{BH}\right).$$

BH entropy is obtained by counting the microstates of the BPS solitons.

A. Strominger and C. Vafa, Phys. Lett. B **379**, 99 (1996).
C. G. Callan and J. M. Maldacena, Nucl. Phys. B **472**, 591 (1996).
G. Horowitz and A. Strominger, Phys. Rev. Lett. **77**, 2368 (1996).

Entanglement entropy and AdS/CFT

Ryu-Takayanagi Formula

If we choose a closed subspace Σ on the spatial boundary of the AdS space to define the entanglement entropy with respect to Σ, the entropy may be determined by $A(\Gamma)$, the area of minimal surface Γ, in the bulk AdS space:

$$S_\Sigma = A(\Gamma)/4G_N, \quad \partial\Sigma = \partial\Gamma.$$

S. Ryu and T. Takayanagi, Phys. Rev. Lett. **96**, 181602 (2006).
S. Ryu and T. Takayanagi, JHEP **0608**, 045 (2006).

Quantum Gravity and String Field Theory

Classical General Relativity Derived from Quantum Gravity

Boulware and Deser, Ann. Phys. 89 (1975):
"A quantum particle description of local (noncosmological) gravitational phenomena necessarily leads to a classical limit which is just a metric theory of gravity. As long as only helicity ± 2 gravitons are included, the theory is precisely Einsteins general relativity."

Closed String Field Theory

Closed string theory contains massless spin 2 particles in its spectrum. The low energy limit of the covariant interacting closed string field theory must be the Einstein's general relativity. The closed string field theory may provide a consistent framework to describe a finite quantum theory of the spin 2 particles, the gravitons. We need to examine the graviton scattering amplitudes of the covariant string field theory and compare them with those of the perturbation theory of the gravity in the low energy region.

Gravity as a low energy limit of a closed string field theory

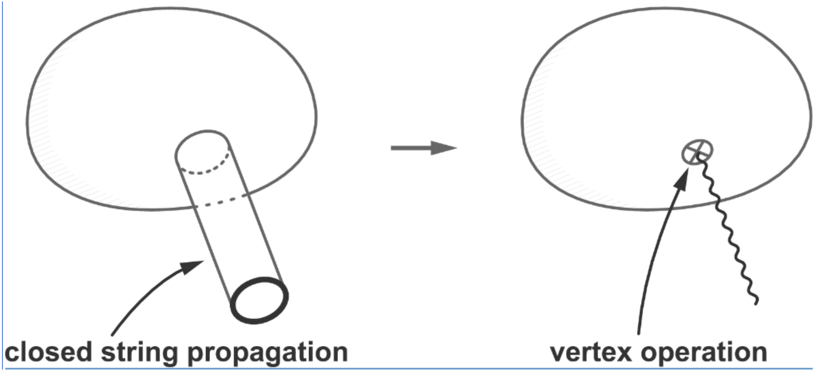
Quantum gravity and closed string field theory

1. Closed string field theory is finite: It may resolve the problem associated with non-renormalizability of the perturbative quantum gravity
2. Closed string field theory is unitary: It may provide a resolution to the problems related to the information loss and the Bekenstein-Hawking entropy of black holes
3. Closed string field theory has well-defined UV and IR behaviors: It may help us to study the dark matter physics and related cosmological problems.
4. Open/Closed string field theory as a framework to study various dualities in string theory including AdS/CFT correspondence.

UV behavior of String and SFT

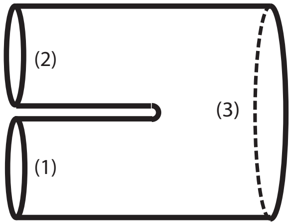
String Scattering Amplitudes with Vertex Operators

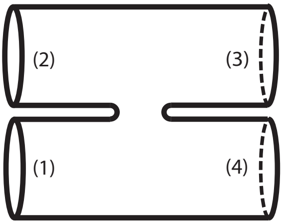
String scattering amplitudes with vertex operators may not capture correct UV behavior of string theory



II. Closed string field theory in the proper-time gauge

We will construct a covariant string field theory and calculate three-string scattering amplitude and the four-string scattering amplitude in the low energy limit by extending the previous works on the open string field theory.





Fock Space Representation of the Closed String Field Theory in the Proper-Time Gauge

Closed String Field Theory in the Proper-Time Gauge

$$S = \langle \Phi | \mathcal{K} \Phi \rangle + \frac{g}{3} \left(\langle \Phi | \Phi \circ \Phi \rangle + \langle \Phi \circ \Phi | \Phi \rangle \right).$$

The closed string field theory in the proper-time gauge generates the string scattering diagrams, which can be represented by the Polyakov string path integrals:

$$S_P = -\frac{1}{4\pi\alpha'} \int_M d\tau d\sigma \sqrt{-h} h^{\alpha\beta} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta} \eta_{\mu\nu}, \qquad \mu, \nu = 0, \dots, d-1.$$

String Scattering Amplitudes

$$\mathcal{A}_M = \int D[X] D[h] \exp \left[-i \int_M d\tau d\sigma \mathcal{L} \right].$$

String Scattering Amplitudes of Closed String Field Theory and Polyakov String Path Integral

Strategy of Calculation of String Scattering Amplitudes

1. Evaluate the scattering amplitudes by using the Polyakov string path integral
2. Re-express the Polakov string path integrals in terms of the oscillator operators
3. Identify the Fock space (operator) representations of the string field theory vertices
4. Choose appropriate external string states, corresponding to the various particle states and evaluate the scattering amplitudes.
5. Compare the resultant scattering amplitudes with those of YM theory for open string and GR for closed string in the zero-slope limit

Free String Theory

$$S = \frac{1}{4\pi\alpha'} \int d\tau d\sigma \, \partial X \cdot \partial X .$$

Decomposition of X in terms of left-movers and right-movers

$$X(\tau, \sigma) = X_L(\tau + \sigma) + X_R(\tau - \sigma).$$

Mode expansions

$$\begin{aligned} X_L(\tau, \sigma) &= x_L + \sqrt{\frac{\alpha'}{2}} p_L(\tau + \sigma) + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \alpha_n e^{-in(\tau+\sigma)}, \\ X_R(\tau, \sigma) &= x_R + \sqrt{\frac{\alpha'}{2}} p_R(\tau - \sigma) + i \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}_n e^{-in(\tau-\sigma)}, \end{aligned}$$

where $x = x_L + x_R$.

Closed String Theory: Review

Canonical commutation relations

$$[x_L, p_L] = [x_R, p_R] = i\sqrt{\frac{\alpha'}{2}},$$
$$[\alpha_m, \alpha_n] = [\tilde{\alpha}_m, \tilde{\alpha}_n] = m\delta(m+n).$$

Momentum eigentate with eigenvalue $P_n, n \neq 0$:

$$|P_n\rangle = \sqrt{\frac{1}{\pi n}} \exp\left\{\left(\frac{P_n \alpha_{-n}}{n} + \frac{P_{-n}\tilde{\alpha}_{-n}}{n} - \frac{\alpha_{-n}\tilde{\alpha}_{-n}}{n} - \frac{P_n \cdot P_{-n}}{4n}\right)\right\}|0\rangle.$$

Mapping from cylindrical surface onto the complex plane

$$z = e^{\rho} = e^{\xi + i\eta}, \quad -\pi \leq \eta \leq \pi.$$

Green's function on complex plane $(\xi > \xi')$, $\Delta = |\xi - \xi'|$,

$$G_C(z, z') = \ln|z - z'|$$
$$= \max(\xi, \xi') - \frac{1}{2}\sum_{n=1} \frac{e^{-n\Delta}}{n}\left(e^{in(\eta'-\eta)} + e^{-in(\eta'-\eta)}\right).$$

Closed String Interaction in the Proper Time Gauge

CS mapping from the world sheet of three closed string scattering onto the complex plane. For the three-string vertex in the proper-time gauge,

$$\rho = \ln(z-1) + \ln z.$$

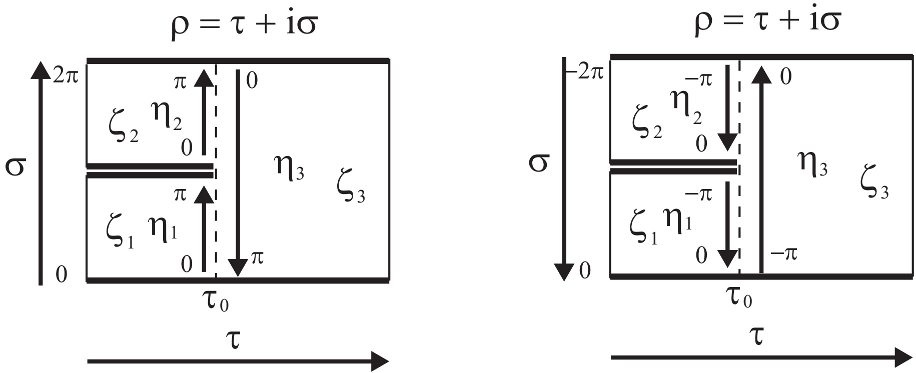
The local coordinates $\zeta_r = \xi_r + i\eta_r,\ r = 1, 2, 3$ defined on invidual string world sheet patches are related to z as follows:

$$\begin{aligned} e^{-\zeta_1} &= e^{\tau_0}\frac{1}{z(z-1)}, \\ e^{-\zeta_2} &= -e^{\tau_0}\frac{1}{z(z-1)}, \\ e^{-\zeta_3} &= -e^{-\frac{\tau_0}{2}}\sqrt{z(z-1)}. \end{aligned}$$

For convenience we choose, by using $SL(2, C)$ invariance

$$Z_1 = 0, \quad Z_2 = 1, \quad Z_3 = \infty.$$

Local Coordinates



Neumann Functions of the Closed String Vertices

Fourier components of the Green's function on complex plane

$$
\begin{aligned}
G_C(\rho_r, \rho'_s) &= \ln |z_r - z'_s| \\
&= -\delta_{rs} \left\{ \sum_{n=1} \frac{e^{-n\Delta}}{2n} \left(e^{in(\eta'_s - \eta_r)} + e^{-in(\eta'_s - \eta_r)} \right) - \max(\xi, \xi') \right\} \\
&\quad + \sum_{n,m} \bar{C}^{rs}_{nm} e^{|n|\xi_r + |m|\xi'_s} e^{in\eta_r} e^{im\eta'_s}.
\end{aligned}
$$

Integral Formulas for $\bar{C}^{rs}_{nm}$

$$
\begin{aligned}
\bar{C}^{rs}_{00} &= \ln|Z_r - Z_s|, \quad r \neq s, \\
\bar{C}^{rr}_{00} &= -\sum_{i\neq r} \frac{\alpha_i}{\alpha_r} \ln|Z_r - Z_i| + \frac{1}{\alpha_r}\tau_0^{(r)} \\
\bar{C}^{rs}_{n0} &= \bar{C}^{sr}_{0n} = \frac{1}{2n}\oint_{Z_r} \frac{dz}{2\pi i} \frac{1}{z - Z_s} e^{-n\zeta_r(z)}, \quad n \geq 1, \\
\bar{C}^{rs}_{nm} &= \frac{1}{2nm}\oint_{Z_r} \frac{dz}{2\pi i} \oint_{Z_s} \frac{dz'}{2\pi i} \frac{1}{(z - z')^2} e^{-n\zeta_r(z) - m\zeta'_s(z')}, \quad n, m \geq 1.
\end{aligned}
$$

Reality conditions of the Green's function

$$
\bar{C}^{rs}_{nm} = \bar{C}^{*rs}_{-n-m}, \quad \bar{C}^{rs}_{-nm} = 0, \quad n, m \geq 1.
$$

Scattering amplitude

$$\begin{aligned}\mathcal{W} &= \int DX \exp\left(i\sum_{r=1}^{M}\int P_r(\sigma)\cdot X(\tau_r,\sigma)d\sigma - \int d\tau d\sigma \mathcal{L}\right) \\ &= [\det\Delta]^{-d/2}\exp\Bigg\{\frac{1}{4}\Bigg\{\sum_r \xi_r P_0^2 - \sum_r\sum_{n=1}\frac{1}{n}P_n^{(r)}\cdot P_{-n}^{(r)} \\ &\quad + \sum_{n,m}\bar{C}_{nm}^{rs}e^{|n|\xi_r+|m|\xi'_s}P_{-n}^{(r)}\cdot P_{-m}^{(s)}\Bigg\} \\ &= \langle\mathbf{P}|\exp\left(\sum_r \xi_r L_0^{(r)}\right)|V[N]\rangle.\end{aligned}$$

Neumann functions of three-closed string vertex and those of three-open string vertex:

$$
\begin{aligned}
\bar{C}^{rs}_{00} &= \bar{N}^{rs}_{00} = \ln|Z_r - Z_s|, \quad r \neq s, \\
\bar{C}^{rr}_{00} &= \bar{N}^{rr}_{00} = -\sum_{i\neq r} \frac{\alpha_i}{\alpha_r} \ln|Z_r - Z_i| + \frac{1}{\alpha_r}\tau_0, \\
\bar{C}^{rs}_{n0} &= \bar{C}^{rs}_{-n0} = \frac{1}{2}\bar{N}^{rs}_{n0} = \frac{1}{2n}\oint_{Z_r} \frac{dz}{2\pi i}\frac{1}{z - Z_s} e^{-n\zeta_r(z)}, \quad n \geq 1, \\
\bar{C}^{rs}_{nm} &= \bar{C}^{rs}_{-n-m} = \frac{1}{2}\bar{N}^{rs}_{mn} \\
&= \frac{1}{2nm}\oint_{Z_r}\frac{dz}{2\pi i}\oint_{Z_s}\frac{dz'}{2\pi i}\frac{1}{(z-z')^2} e^{-n\zeta_r(z) - m\zeta'_s(z')}, \quad n, m \geq 1, \\
\bar{C}^{rs}_{n-m} &= \bar{C}^{rs}_{-nm} = 0.
\end{aligned}
$$

Factorization of Three-Closed-String Scattering Amplitude

$$
\begin{aligned}
\mathcal{A}[1,2,3] \quad = \quad & g\langle\{\mathbf{k}^{(r)}\}| \\
& \exp\left\{\sum_{r,s}\Big(\sum_{n,m\geq1}\frac{1}{2}\bar{N}^{rs}_{nm}\,\frac{\alpha^{(r)\dagger}_{n}}{2}\cdot\frac{\alpha^{(r)\dagger}_{m}}{2}+\sum_{n\geq1}\bar{N}^{rs}_{n0}\frac{\alpha^{(r)\dagger}_{n}}{2}\cdot\frac{p^{(s)}}{2}\Big)\right\} \\
& \exp\left\{\tau_0\sum_{r}\frac{1}{\alpha_r}\left(\frac{1}{2}\left(\frac{p^{(r)}}{2}\right)^2-1\right)\right\} \\
& \exp\left\{\sum_{r,s}\Big(\sum_{n,m\geq1}\frac{1}{2}\bar{N}^{rs}_{nm}\,\frac{\tilde{\alpha}^{(r)\dagger}_{n}}{2}\cdot\frac{\tilde{\alpha}^{(r)\dagger}_{m}}{2}+\sum_{n\geq1}\bar{N}^{rs}_{n0}\frac{\tilde{\alpha}^{(r)\dagger}_{n}}{2}\cdot\frac{p^{(s)}}{2}\Big)\right\} \\
& \exp\left\{\tau_0\sum_{r}\frac{1}{\alpha_r}\left(\frac{1}{2}\left(\frac{p^{(r)}}{2}\right)^2-1\right)\right\}|0\rangle.
\end{aligned}
$$

Factorization of Three-Closed-String Scattering Amplitude

Factorization of three-closed-string scattering amplitude

$$\mathcal{A}_{\text{closed}}[1,2,3] = \mathcal{A}_{\text{open}}[1,2,3]\,\mathcal{A}_{\text{open}}[1,2,3].$$

Scattering amplitude of three closed strings can be completely factorized into those of three open strings except for the zero modes.

Question: Can we factorize general closed string scattering amplitudes into those of open string theory?

Realization of the Kawai-Lewellen-Tye (KLT) relations in the framework of the second quantized string theory .

Three-Graviton Scattering Amplitude

Decomposition of the spin-2 field into graviton, anti-symmetric tensor, and scalar field

$$h_{\mu\nu} = \left\{\frac{1}{2}\left(h_{\mu\nu}+h_{\nu\mu}\right)-\eta_{\mu\nu}\frac{1}{d}h^{\sigma}{}_{\sigma}\right\}+\left\{\frac{1}{2}\left(h_{\mu\nu}-h_{\nu\mu}\right)\right\}+\eta_{\mu\nu}\left\{\frac{1}{d}h^{\sigma}{}_{\sigma}\right\}.$$

We choose the covariant gauge condition

$$\partial^{\mu}h_{\mu\nu}=0,$$

which becomes de Donder gauge condition for the graviton

$$\partial^{\mu}h_{\mu\nu}-\frac{1}{d-2}\partial_{\nu}h^{\sigma}{}_{\sigma}=0.$$

For three-graviton scattering, we choose the external string state as

$$|\Psi_{3G}\rangle=\prod_{r=1}^{3}\left\{h_{\mu\nu}(p^{r})\alpha_{-1}^{(r)\mu}\tilde{\alpha}_{-1}^{(r)\nu}\right\}|0\rangle.$$

Three-graviton scattering amplitude

$$\begin{aligned}
\mathcal{A}_{[3-graviton]} &= \int \prod_{r=1}^{3} dp^{(r)} \delta\left(\sum_{r=1}^{3} p^{(r)}\right) \frac{2g}{3} \langle \Psi_{3G} | E^{\text{Closed}}_{[3]}[1,2,3] | 0 \rangle \\
&= \left(\frac{2g}{3}\right) e^{-2\tau_0 \sum_{r=1}^{3} \frac{1}{\alpha_r}} \int \prod_{i=1}^{3} dp^{(i)} \delta\left(\sum_{i=1}^{3} p^{(i)}\right) \\
& \langle 0 | \left\{ \prod_{i=1}^{3} h_{\mu\nu}(p^{(i)}) a_1^{(i)\mu} \cdot \tilde{a}_1^{(i)\nu} \right\} \frac{1}{2^5} \left(\sum_{r,s=1}^{3} \bar{N}^{rs}_{11} a_1^{(r)\dagger} \cdot a_1^{(s)\dagger} \right) \\
& \left(\sum_{t=1}^{3} \bar{N}^{t}_{1} a_1^{(t)\dagger} \cdot \mathbf{p} \right) \frac{1}{2^5} \left(\sum_{l,m=1}^{3} \bar{N}^{lm}_{11} \tilde{a}_1^{(l)\dagger} \cdot \tilde{a}_1^{(m)\dagger} \right) \\
& \left(\sum_{n=1}^{3} \bar{N}^{n}_{1} \tilde{a}_1^{(n)\dagger} \cdot \mathbf{p} \right) |0\rangle .
\end{aligned}$$

Three-Graviton Scattering Amplitude

We note that $\mathcal{A}_{[3-graviton]}$ can be written also as

$$\mathcal{A}_{[3-graviton]} = \left(\frac{2g}{3}\right)\frac{1}{2^8}\int\prod_{i=1}^{3}dp^{(i)}\delta\left(\sum_{i=1}^{3}p^{(i)}\right)$$
$$\langle 0|\left\{\prod_{i=1}^{3}h_{\mu\nu}(p^{(i)})a_1^{(i)\mu}\cdot\tilde{a}_1^{(i)\nu}\right\}E^{\text{Open}}_{[3-Gauge]}\,\tilde{E}^{\text{Open}}_{[3-Gauge]}|0\rangle.$$

Making use of the Neumann functions of the open string

$$\bar{N}^{11}_{11} = \frac{1}{2^4},\quad \bar{N}^{22}_{11} = \frac{1}{2^4},\quad \bar{N}^{33}_{11} = 2^2,$$
$$\bar{N}^{12}_{11} = \bar{N}^{21}_{11} = \frac{1}{2^4},\quad \bar{N}^{23}_{11} = \bar{N}^{32}_{11} = \frac{1}{2},\quad \bar{N}^{31}_{11} = \bar{N}^{13}_{11} = \frac{1}{2},$$
$$\bar{N}^{1}_{1} = \bar{N}^{2}_{1} = \frac{1}{4},\quad \bar{N}^{3}_{1} = -1,$$

Three-Graviton Scattering Amplitude

We are able evaluate the three-graviton interaction term as follows

$$\begin{aligned}\mathcal{A}_{[3-graviton]} \quad = \quad & \left(\frac{2g}{3}\right) 2^6 \left(\frac{1}{2^5}\right)^2 \int \prod_{i=1}^{3} dp^{(i)} \delta\left(\sum_{i=1}^{3} p^{(i)}\right) \\ & h_{\mu_1\nu_1}(p^{(1)}) h_{\mu_2\nu_2}(p^{(2)}) h_{\mu_3\nu_3}(p^{(3)}) \\ & \Big\{ -\frac{1}{2^4}\eta^{\mu_1\mu_2}\mathbf{p}^{\mu_3} + \frac{1}{2^3}\eta^{\mu_1\mu_3}\mathbf{p}^{\mu_2} + \frac{1}{2^3}\eta^{\mu_2\mu_3}\mathbf{p}^{\mu_1} \\ & -\frac{1}{2^4}\eta^{\mu_2\mu_1}\mathbf{p}^{\mu_3} + \frac{1}{2^3}\eta^{\mu_3\mu_1}\mathbf{p}^{\mu_2} + \frac{1}{2^3}\eta^{\mu_3\mu_2}\mathbf{p}^{\mu_1} \Big\} \\ & \Big\{ -\frac{1}{2^4}\eta^{\nu_1\nu_2}\mathbf{p}^{\nu_3} + \frac{1}{2^3}\eta^{\nu_1\nu_3}\mathbf{p}^{\nu_2} + \frac{1}{2^3}\eta^{\nu_2\nu_3}\mathbf{p}^{\nu_1} \\ & -\frac{1}{2^4}\eta^{\nu_2\nu_1}\mathbf{p}^{\nu_3} + \frac{1}{2^3}\eta^{\nu_3\nu_1}\mathbf{p}^{\nu_2} + \frac{1}{2^3}\eta^{\nu_3\nu_2}\mathbf{p}^{\nu_1} \Big\}.\end{aligned}$$

Three-Graviton Scattering Amplitude

$\mathcal{A}_{[3-graviton]}$ is precisely the three-graviton interaction term which may be obtained from the Einstein's gravity action.

$$
\begin{aligned}
\mathcal{A}_{[3-graviton]} &= \kappa \int \prod_{i=1}^{3} dp^{(i)} \delta\left(\sum_{i=1}^{3} p^{(i)}\right) \\
& h_{\mu_1\nu_1}(p^{(1)}) h_{\mu_2\nu_2}(p^{(2)}) h_{\mu_3\nu_3}(p^{(3)}) \\
& \left\{\eta^{\mu_1\mu_2} p^{(1)\mu_3} + \eta^{\mu_2\mu_3} p^{(2)\mu_1} + \eta^{\mu_3\mu_1} p^{(3)\mu_2}\right\} \\
& \left\{\eta^{\nu_1\nu_2} p^{(1)\nu_3} + \eta^{\nu_2\nu_3} p^{(2)\nu_1} + \eta^{\nu_3\nu_1} p^{(3)\nu_2}\right\}
\end{aligned}
$$

where $\kappa = \frac{g}{2^7 \cdot 3} = \sqrt{32\pi G_{10}}$.

Using the Cremmer-Gervais identity, we may write the scattering amplitude of four closed strings as follows

$$\mathcal{A}[1,2,3,4] = g^2 \int \prod_r dZ_r^2 \frac{|Z_a - Z_b|^2 |Z_b - Z_c|^2 |Z_c - Z_a|^2}{d^2 Z_a d^2 Z_b d^2 Z_c}$$

$$\prod_{r<s} |Z_r - Z_s|^{2\frac{p^{(r)}}{2}\cdot\frac{p^{(s)}}{2}} \prod_{r<s} |Z_r - Z_s|^{2\left\{\frac{\alpha_s}{\alpha_r}\left(1-\frac{1}{2}\left(\frac{p^{(r)}}{2}\right)^2\right)+\frac{\alpha_r}{\alpha_s}\left(1-\frac{1}{2}\left(\frac{p^{(s)}}{2}\right)^2\right)\right\}}$$

$$\langle\{\mathbf{k}^{(r)}\}| \exp\left\{\frac{1}{4}\sum_{r,s}\sum_{n,m\geq 1}\left(\bar{C}^{rs}_{nm}\,\tilde{\alpha}^{(r)\dagger}_n\cdot\tilde{\alpha}^{(s)\dagger}_m + \bar{C}^{rs*}_{nm}\,\alpha^{(r)\dagger}_n\cdot\alpha^{(r)\dagger}_m\right)\right.$$

$$+\sum_{r,s}\left(\sum_{n\geq 1}\left(\bar{C}^{rs}_{n0}\tilde{\alpha}^{(r)\dagger}_n\cdot\frac{p^{(s)}}{2} + \bar{C}^{rs*}_{n0}\,\alpha^{(r)\dagger}_n\cdot\frac{p^{(s)}}{2}\right)\right)$$

$$\left.+2\tau^{(r)}\sum_r \frac{1}{\alpha_r}\left(\frac{1}{2}\left(\frac{p^{(r)}}{2}\right)^2 - 1\right)\right\}|0\rangle.$$

Scattering of four closed string tachyons

$$\begin{aligned}\mathcal{A}_{\rm Tachyon}[1,2,3,4] &= g^2 \int d^2 Z\, |Z|^{2\,\frac{p^{(1)}}{2}\cdot\frac{p^{(2)}}{2}} |1-Z|^{2\,\frac{p^{(2)}}{2}\cdot\frac{p^{(3)}}{2}} \\ &= g^2 \int d^2 Z\, |Z|^{2\left(-\frac{s}{8}-2\right)} |1-Z|^{2\left(-\frac{u}{8}-2\right)} \\ &= 2\pi g^2 \frac{\Gamma\left(-1-\frac{s}{8}\right)\Gamma\left(-1-\frac{t}{8}\right)\Gamma\left(-1-\frac{u}{8}\right)}{\Gamma\left(2+\frac{s}{8}\right)\Gamma\left(2+\frac{t}{8}\right)\Gamma\left(2+\frac{u}{8}\right)}.\end{aligned}$$

Mandelstam variables:

$$s = -(p_1+p_2)^2, \quad t = -(p_1+p_3)^2, \quad u = -(p_1+p_4)^2.$$

The Koba-Nielsen variables:

$$Z_1 = 0, \quad Z_2 = Z, \quad Z_3 = 1, \quad Z_4 = \infty.$$

Local coordinates for four-closed-string scattering

Schwarz-Christoffel transformation

$$\rho = \sum_{r=1}^{4} \alpha_r \ln(z - Z_r) = \ln z + \ln(z - Z) - \ln(z - 1) + i\pi.$$

Equivalently, $e^{\rho} = -\frac{z(z-Z)}{z-1}$. Interaction points on the complex plane are determined by

$$\frac{\partial \rho}{\partial z} = \sum_{r=1}^{4} \frac{\alpha_r}{z - Z_r} = 0$$

which has two solutions:

$$z_{\pm} = 1 \pm \sqrt{1 - Z}.$$

These two solutions define two interaction points

$$\rho(z_{\pm}) = \ln \frac{z_{\pm}(z_{\pm} - Z)}{z_{\pm} - 1} + i\pi$$

Local coordinate patches

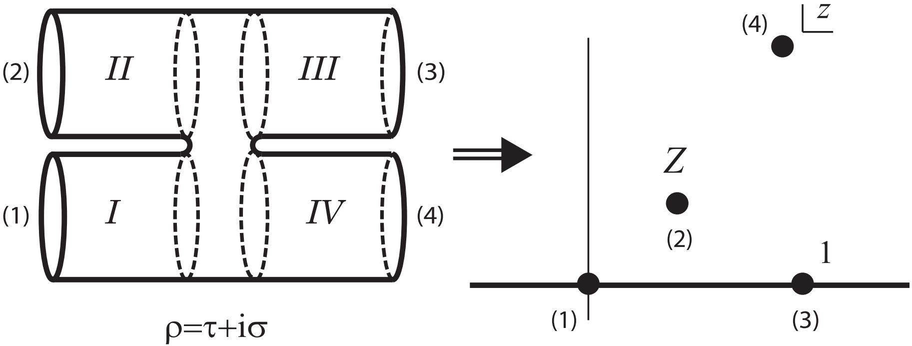
Two interaction points on the world sheet:

$$\tau_1 = 2\,\mathrm{Re}\,\ln\left(1-\sqrt{1-Z}\right), \quad \tau_2 = 2\,\mathrm{Re}\,\ln\left(1+\sqrt{1-Z}\right),$$
$$\sigma_1 = 2\,\mathrm{Im}\,\ln\left(1-\sqrt{1-Z}\right)+\pi, \quad \sigma_2 = 2\,\mathrm{Im}\,\ln\left(1+\sqrt{1-Z}\right)+\pi.$$

Global coordinates on the local patches:

$$\rho = \begin{cases} \zeta_1+\tau_1+i\sigma_1-i\pi, & \text{on } I \\ \zeta_2+\tau_1+i\sigma_1, & \text{on } II \\ -\zeta_3+\tau_2+i\sigma_2+i\pi, & \text{on } III \\ -\zeta_4+\tau_2+i\sigma_2, & \text{on } IV. \end{cases}$$

Local coordinate patches



Conformal mapping from the string world-sheet to the complex plane

The conformal mapping from the local string patches to the complex plane may be written as follows :

$$e^{-\zeta_1} = e^{\tau_1 + i\sigma_1} \frac{(z-1)}{z(z-Z)}, \qquad \text{on the 1st string patch,}$$

$$e^{-\zeta_2} = -e^{\tau_1 + i\sigma_1} \frac{(z-1)}{z(z-Z)}, \qquad \text{on the 2nd string patch,}$$

$$e^{-\zeta_3} = e^{-\tau_2 - i\sigma_2} \frac{z(z-Z)}{z-1}, \qquad \text{on the 3rd string patch,}$$

$$e^{-\zeta_4} = -e^{-\tau_2 - i\sigma_2} \frac{z(z-Z)}{z-1}, \qquad \text{on the 4th string patch.}$$

Neuumann functions for four-closed-string scattering

$$\bar{C}^{rs}_{n0} = \bar{C}^{sr}_{0n} = \frac{1}{2n}\oint_{Z_r}\frac{dz}{2\pi i}\frac{1}{z-Z_s}e^{-n\zeta_r(z)}, \quad n \geq 1,$$

$$\bar{C}^{11}_{10} = \frac{e^{\tau_1+i\sigma_1}}{2}\frac{1-Z}{Z^2}, \quad \bar{C}^{12}_{10} = -\frac{e^{\tau_1+i\sigma_1}}{2}\frac{1}{Z^2}, \quad \bar{C}^{13}_{10} = -\frac{e^{\tau_1+i\sigma_1}}{2}\frac{1}{Z},$$

$$\bar{C}^{14}_{10} = 0, \quad \bar{C}^{21}_{10} = \frac{e^{\tau_1+i\sigma_1}}{2}\frac{1-Z}{Z^2}, \quad \bar{C}^{22}_{10} = -\frac{e^{\tau_1+i\sigma_1}}{2}\frac{1}{Z^2},$$

$$\bar{C}^{23}_{10} = -\frac{e^{\tau_1+i\sigma_1}}{2}\frac{1}{Z}, \quad \bar{C}^{24}_{10} = 0, \quad \bar{C}^{31}_{10} = \frac{e^{-\tau_2-i\sigma_2}}{2}(1-Z),$$

$$\bar{C}^{32}_{10} = \frac{e^{-\tau_2-i\sigma_2}}{2}, \quad \bar{C}^{33}_{10} = \frac{e^{-\tau_2-i\sigma_2}}{2}(2-Z), \quad \bar{C}^{34}_{10} = \bar{C}^{44}_{10} = 0,$$

$$\bar{C}^{41}_{10} = \frac{e^{-\tau_2-i\sigma_2}}{2}(1-Z), \quad \bar{C}^{42}_{10} = \frac{e^{-\tau_2-i\sigma_2}}{2}, \quad \bar{C}^{43}_{10} = \frac{e^{-\tau_2-i\sigma_2}}{2}(2-Z).$$

Neuumann functions for four-closed-string scattering

$$\bar{C}^{rs}_{nm} = \frac{1}{2nm}\oint_{Z_r}\frac{dz}{2\pi i}\oint_{Z_s}\frac{dz'}{2\pi i}\frac{1}{(z-z')^2}e^{-n\zeta_r(z)-m\zeta_s(z')}, \quad n,m \geq 1.$$

$$\bar{C}^{12}_{11} = \frac{e^{2\tau_1+i2\sigma_1}}{2}\frac{(1-Z)}{Z^4}, \quad \bar{C}^{13}_{11} = \frac{e^{\tau_1+i\sigma_1-\tau_2-i\sigma_2}}{2}\frac{(1-Z)}{Z},$$

$$\bar{C}^{14}_{11} = \frac{e^{\tau_1+i\sigma_1-\tau_2-i\sigma_2}}{2}\frac{1}{Z}, \quad \bar{C}^{23}_{11} = \frac{e^{\tau_1+i\sigma_1-\tau_2-i\sigma_2}}{2}\frac{1}{Z},$$

$$\bar{C}^{24}_{11} = \frac{e^{\tau_1+i\sigma_1-\tau_2-i\sigma_2}}{2}\frac{(1-Z)}{Z}. \quad \bar{C}^{34}_{11} = \frac{e^{-2\tau_2-i2\sigma_2}}{2}(1-Z).$$

$$
\begin{aligned}
\mathcal{A}_{[4]} \;=\; & g^2 \int |Z_4|^4 d^2 Z \prod_{r<s} |Z_r - Z_s|^{2\,\frac{p^{(r)}}{2}\cdot\frac{p^{(s)}}{2}} e^{-2\sum_{r=1}^4 \bar{C}^{[4]rr}_{00}} \\
& \langle 0| \left\{ \prod_{i=1}^{4} h_{\mu\nu}(p^{(i)}) a_1^{(i)\mu} \tilde{a}_1^{(i)\nu} \right\} \frac{1}{2!}\cdot\frac{1}{2^4} \left\{ \left(\sum_{r,s} \bar{C}^{rs*}_{11} a_1^{(r)\dagger}\cdot a_1^{(s)\dagger} \right)^2 \right. \\
& \left. + \left(\sum_{r,s} \bar{C}^{rs*}_{11} a_1^{(r)\dagger}\cdot a_1^{(s)\dagger} \right) \left(\sum_{r,s} \bar{C}^{rs*}_{10} a_1^{(r)\dagger}\cdot p^{(s)} \right)^2 \right\} \\
& \frac{1}{2!}\cdot\frac{1}{2^4} \left\{ \left(\sum_{r,s} \bar{C}^{rs}_{11} \tilde{a}_1^{(r)\dagger}\cdot \tilde{a}_1^{(s)\dagger} \right)^2 + \left(\sum_{r,s} \bar{C}^{rs}_{11} \tilde{a}_1^{(r)\dagger}\cdot \tilde{a}_1^{(s)\dagger} \right) \right. \\
& \left. \left(\sum_{r,s} \bar{C}^{rs}_{10} \tilde{a}_1^{(r)\dagger}\cdot p^{(s)} \right)^2 \right\} |0\rangle .
\end{aligned}
$$

$$\mathcal{A}_{[4]} = \frac{g^2}{2^{10}} \int d^2 Z \, |Z|^{-\frac{s}{4}} |1 - Z|^{-\frac{t}{4}} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4}$$

$$\Bigg\{ \frac{4}{Z^{*2}} \eta^{\mu_1\mu_2} \eta^{\mu_3\mu_4} + 4\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + \frac{4}{(1-Z^*)^2} \eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}$$

$$-\eta^{\mu_1\mu_2} \frac{1}{(1-Z^*)Z^{*2}} \left(Z^* p^{(1)\mu_3} + p^{(4)\mu_3} \right) \left(Z^* p^{(2)\mu_4} + p^{(3)\mu_4} \right)$$

$$+\eta^{\mu_1\mu_3} \frac{1}{(1-Z^*)Z^*} \left(p^{(1)\mu_2} + Z^* p^{(4)\mu_2} \right) \left(Z^* p^{(2)\mu_4} + p^{(3)\mu_4} \right)$$

$$-\eta^{\mu_1\mu_4} \frac{1}{(1-Z^*)^2 Z^*} \left((1-Z^*) p^{(4)\mu_2} + p^{(3)\mu_2} \right) \left((1-Z^*) p^{(1)\mu_3} + p^{(2)\mu_3} \right)$$

$$-\eta^{\mu_2\mu_3} \frac{1}{(1-Z^*)^2 Z^*} \left((1-Z^*) p^{(3)\mu_1} + p^{(4)\mu_1} \right) \left(p^{(1)\mu_4} + (1-Z^*) p^{(2)\mu_4} \right)$$

$$+\eta^{\mu_2\mu_4} \frac{1}{(1-Z^*)Z^*} \left(Z^* p^{(3)\mu_1} + p^{(2)\mu_1} \right) \left(Z^* p^{(1)\mu_3} + p^{(4)\mu_3} \right)$$

$$-\eta^{\mu_3\mu_4} \frac{1}{(1-Z^*)Z^{*2}} \left(Z^* p^{(3)\mu_1} + p^{(2)\mu_1} \right) \left(p^{(1)\mu_2} + Z^* p^{(4)\mu_2} \right) \Bigg\} \Bigg\{ Z^* \Rightarrow Z$$

Kawai-Lewellen-Tye (KLT) Relations

$$
\begin{aligned}
I &= \int d^2 Z \prod_{r<s} |Z_r - Z_s|^{2\,\frac{p^{(r)}}{2}\cdot\frac{p^{(s)}}{2}} (Z^*)^{-2n} Z^{-2m} (1-Z^*)^{-2p} (1-Z)^{-2q} \\
&= \int d^2 Z \, Z^{-2m-\frac{s}{8}} (1-Z)^{-2q-\frac{t}{8}} (Z^*)^{-2n-\frac{s}{8}} (1-Z^*)^{-2p-\frac{t}{8}} .
\end{aligned}
$$

If we write $Z = x + iy$, then we treat x and y as independent two complex variables. The integrand is an analytic function of y with four branch points, $\pm x$, $\pm i(1-x)$. We may deform the contour of y which is along the real line to the contour along the imaginary axis.

$$
\begin{aligned}
I &= \sin\left(\frac{\pi}{8} t\right) \int_0^1 d\xi \, |\xi|^{-\frac{s}{8}} |1-\xi|^{-\frac{t}{8}} \xi^{-2m} (1-\xi)^{-2q} \\
&\quad \int_1^\infty d\eta \, |\eta|^{-\frac{s}{8}} |1-\eta|^{-\frac{t}{8}} \eta^{-2n} (1-\eta)^{-2p} .
\end{aligned}
$$

Kawai-Lewellen-Tye (KLT) Relations

Factorization

$$
\begin{aligned}
I &= \sin\left(\frac{\pi t}{8}\right) I_1(n,p) I_2(m,q), \\
I_1(n,p) &= (-1)^{2p} \frac{\Gamma\left(-\frac{u}{8}+2n+2p-1\right)\Gamma\left(-\frac{t}{8}-2p+1\right)}{\Gamma\left(\frac{s}{8}+2n\right)}, \\
I_2(m,q) &= \frac{\Gamma\left(-\frac{s}{8}-2m+1\right)\Gamma\left(-\frac{t}{8}-2q+1\right)}{\Gamma\left(\frac{u}{8}-2m-2q+2\right)}
\end{aligned}
$$

where n, p, m, q are integers or half-integers.

Four-graviton scattering

$$
\begin{aligned}
\mathcal{A}_{[4G]} &= g^2 c_{[4]} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4} \sin\left(\frac{\pi t}{8}\right) \\
& \Big\{ I_1(2,0)\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + I_1(0,0)\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + I_1(0,2)\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3} \\
& -\frac{1}{4}\eta^{\mu_1\mu_2}\left(I_1(0,1)p^{(1)\mu_3}p^{(2)\mu_4} + I_1(1,1)p^{(1)\mu_3}p^{(3)\mu_4} + I_1(1,1)p^{(4)\mu_3}p^{(2)\mu_4} + I_1(2,1)p^{(4)\mu_3}p^{(3)\mu_4}\right) \\
& +\frac{1}{4}\eta^{\mu_1\mu_3}\left(I_1(0,1)p^{(1)\mu_2}p^{(2)\mu_4} + I_1(1,1)p^{(1)\mu_2}p^{(3)\mu_4} + I_1(-1,1)p^{(4)\mu_2}p^{(2)\mu_4} + I_1(0,1)p^{(4)\mu_2}p^{(3)\mu_4}\right) \\
& -\frac{1}{4}\eta^{\mu_1\mu_4}\left(I_1(1,0)p^{(4)\mu_2}p^{(1)\mu_3} + I_1(1,1)p^{(4)\mu_2}p^{(2)\mu_3} + I_1(1,1)p^{(3)\mu_2}p^{(1)\mu_3} + I_1(1,2)p^{(3)\mu_2}p^{(2)\mu_3}\right) \\
& -\frac{1}{4}\eta^{\mu_2\mu_3}\left(I_1(1,0)p^{(3)\mu_1}p^{(2)\mu_4} + I_1(1,1)p^{(3)\mu_1}p^{(1)\mu_4} + I_1(1,1)p^{(4)\mu_1}p^{(2)\mu_4} + I_1(1,2)p^{(4)\mu_1}p^{(1)\mu_4}\right) \\
& +\frac{1}{4}\eta^{\mu_2\mu_4}\left(I_1(-1,1)p^{(3)\mu_1}p^{(1)\mu_3} + I_1(0,1)p^{(3)\mu_1}p^{(4)\mu_3} + I_1(0,1)p^{(2)\mu_1}p^{(1)\mu_3} + I_1(1,1)p^{(2)\mu_1}p^{(4)\mu_3}\right) \\
& -\frac{1}{4}\eta^{\mu_3\mu_4}\left(I_1(0,1)p^{(3)\mu_1}p^{(4)\mu_2} + I_1(1,1)p^{(3)\mu_1}p^{(1)\mu_2} + I_1(1,1)p^{(2)\mu_1}p^{(4)\mu_2} + I_1(2,1)p^{(2)\mu_1}p^{(1)\mu_2}\right)\Big\} \\
& \Big\{ I_1(n,p) \Rightarrow I_2(n,p),\ \mu \Rightarrow \nu \Big\}
\end{aligned}
$$

Using the momentum conservation, on-shell consition $s + t + u = 0$, and the gauge condition, in the zero-slope limit

$$\begin{aligned}
\mathcal{A}_{[4]} &= \frac{g^2\pi}{2^{10}} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4} \frac{1}{stu} \\
&\Bigg\{ \frac{ut}{2}\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + \frac{st}{2}\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + \frac{su}{2}\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3} \\
&-\eta^{\mu_1\mu_2}\left(tp^{(1)\mu_3}p^{(2)\mu_4} + up^{(2)\mu_3}p^{(1)\mu_4}\right) \\
&-\eta^{\mu_1\mu_3}\left(tp^{(1)\mu_2}p^{(3)\mu_4} + sp^{(3)\mu_2}p^{(1)\mu_4}\right) \\
&-\eta^{\mu_1\mu_4}\left(up^{(1)\mu_2}p^{(4)\mu_3} + sp^{(4)\mu_2}p^{(1)\mu_3}\right) \\
&-\eta^{\mu_2\mu_3}\left(up^{(2)\mu_1}p^{(3)\mu_4} + sp^{(3)\mu_1}p^{(2)\mu_4}\right) \\
&-\eta^{\mu_2\mu_4}\left(sp^{(4)\mu_1}p^{(2)\mu_3} + tp^{(2)\mu_1}p^{(4)\mu_3}\right) \\
&-\eta^{\mu_3\mu_4}\left(tp^{(3)\mu_1}p^{(4)\mu_2} + up^{(4)\mu_1}p^{(3)\mu_2}\right) \Bigg\} \Bigg\{ \mu \Rightarrow \nu \Bigg\}.
\end{aligned}$$

Four-Graviton Scattering Amplitudes

1. B. S. DeWitt, PRD **162**, 1239 (1967): Calculated the four-graviton scattering amplitude by perturbatively expanding the Einstein GR.
2. J. Schwarz, Phys. Rep. **89**, 223 (1982): Four-graviton scattering amplitude in the type-II super-string theory (by using graviton vertex operator).
3. S. Sannan, PRD **34**, 1749 (1986): Equivalence between the DeWitt's four-graviton scattering amplitude and that of string theory calculation.

Four-Graviton Scattering Amplitude in the perturbative Einstein GR

B. S. DeWitt, PRD **162**, 1239 (1967)

$$
\begin{aligned}
\mathrm{sym}[& -\tfrac{1}{4}P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}\eta_{\rho\lambda})-\tfrac{1}{4}P_{12}(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}\eta_{\rho\lambda})-\tfrac{1}{2}P_6(k_{1\nu}k_{2\mu}\eta_{\alpha\beta}\eta_{\sigma\gamma}\eta_{\rho\lambda}) \\
& +\tfrac{1}{4}P_6(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}\eta_{\rho\lambda})+\tfrac{1}{2}P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\rho}\eta_{\gamma\lambda})+\tfrac{1}{2}P_{12}(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\rho}\eta_{\gamma\lambda}) \\
& +P_6(k_{1\nu}k_{2\mu}\eta_{\alpha\beta}\eta_{\sigma\rho}\eta_{\gamma\lambda})-\tfrac{1}{2}P_6(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\rho}\eta_{\gamma\lambda})+\tfrac{1}{2}P_{24}(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}\eta_{\rho\lambda}) \\
& +\tfrac{1}{2}P_{24}(k_{1\nu}k_{1\beta}\eta_{\mu\sigma}\eta_{\alpha\gamma}\eta_{\rho\lambda})+\tfrac{1}{2}P_{12}(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\rho\lambda})+P_{24}(k_{1\nu}k_{2\sigma}\eta_{\beta\mu}\eta_{\alpha\gamma}\eta_{\rho\lambda}) \\
& -P_{12}(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu}\eta_{\rho\lambda})+P_{12}(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}\eta_{\rho\lambda})+P_{12}(k_{1\nu}k_{1\sigma}\eta_{\beta\gamma}\eta_{\mu\alpha}\eta_{\rho\lambda}) \\
& -P_{24}(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\beta\sigma}\eta_{\gamma\rho}\eta_{\lambda\nu})-2P_{12}(k_{1\nu}k_{1\beta}\eta_{\alpha\sigma}\eta_{\gamma\rho}\eta_{\lambda\mu})-2P_{12}(k_{1\sigma}k_{2\gamma}\eta_{\alpha\rho}\eta_{\lambda\nu}\eta_{\beta\mu}) \\
& -2P_{24}(k_{1\nu}k_{2\sigma}\eta_{\beta\rho}\eta_{\lambda\mu}\eta_{\alpha\gamma})-2P_{12}(k_{1\sigma}k_{2\rho}\eta_{\gamma\nu}\eta_{\beta\mu}\eta_{\alpha\lambda})+2P_6(k_1\cdot k_2\eta_{\alpha\sigma}\eta_{\gamma\nu}\eta_{\beta\rho}\eta_{\lambda\mu}) \\
& -2P_{12}(k_{1\nu}k_{1\sigma}\eta_{\mu\alpha}\eta_{\beta\rho}\eta_{\lambda\gamma})-P_{12}(k_1\cdot k_2\eta_{\mu\sigma}\eta_{\alpha\gamma}\eta_{\nu\rho}\eta_{\beta\lambda})-2P_{12}(k_{1\nu}k_{1\sigma}\eta_{\beta\gamma}\eta_{\mu\rho}\eta_{\alpha\lambda}) \\
& -P_{12}(k_{1\sigma}k_{2\rho}\eta_{\gamma\lambda}\eta_{\mu\nu}\eta_{\alpha\beta})-2P_{24}(k_{1\nu}k_{2\sigma}\eta_{\beta\mu}\eta_{\alpha\rho}\eta_{\lambda\gamma})-2P_{12}(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\rho}\eta_{\lambda\alpha}) \\
& +4P_6(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\rho}\eta_{\lambda\mu})]\ .
\end{aligned}
$$

Four-Graviton Scattering Amp by using vertex operators

$$
\mathcal{A}_{[4]} = \frac{\kappa^2}{128} \frac{\Gamma\left(-\frac{s}{8}\right)\Gamma\left(-\frac{t}{8}\right)\Gamma\left(-\frac{u}{8}\right)}{\Gamma\left(1+\frac{s}{8}\right)\Gamma\left(1+\frac{t}{8}\right)\Gamma\left(1+\frac{u}{8}\right)} h_{\mu_1\nu_1} h_{\mu_2\nu_2} h_{\mu_3\nu_3} h_{\mu_4\nu_4}
$$

$$
\Bigg\{ \frac{ut}{2}\eta^{\mu_1\mu_2}\eta^{\mu_3\mu_4} + \frac{st}{2}\eta^{\mu_1\mu_3}\eta^{\mu_2\mu_4} + \frac{su}{2}\eta^{\mu_1\mu_4}\eta^{\mu_2\mu_3}
$$

$$
-\eta^{\mu_1\mu_2}\left(tp^{(1)\mu_3}p^{(2)\mu_4} + up^{(2)\mu_3}p^{(1)\mu_4}\right) - \eta^{\mu_1\mu_3}\Big(tp^{(1)\mu_2}p^{(3)\mu_4}
$$

$$
+sp^{(3)\mu_2}p^{(1)\mu_4}\Big) - \eta^{\mu_1\mu_4}\left(up^{(1)\mu_2}p^{(4)\mu_3} + sp^{(4)\mu_2}p^{(1)\mu_3}\right)
$$

$$
-\eta^{\mu_2\mu_3}\left(up^{(2)\mu_1}p^{(3)\mu_4} + sp^{(3)\mu_1}p^{(2)\mu_4}\right)
$$

$$
-\eta^{\mu_2\mu_4}\left(sp^{(4)\mu_1}p^{(2)\mu_3} + tp^{(2)\mu_1}p^{(4)\mu_3}\right)
$$

$$
-\eta^{\mu_3\mu_4}\left(tp^{(3)\mu_1}p^{(4)\mu_2} + up^{(4)\mu_1}p^{(3)\mu_2}\right)\Bigg\}\Bigg\{\mu \Rightarrow \nu\Bigg\}.
$$

Classical Solutions with D-Brane Sources and Black p-Branes

Closed string field theory action

$$S = \int \left\{ \langle \Phi | \mathcal{K} \Phi \rangle + \frac{g}{3} \left(\langle \Phi | \Phi \circ \Phi \rangle + \langle \Phi \circ \Phi | \Phi \rangle + \langle \Phi | D_p \rangle \right\}.$$

Classical equation of motion with a D-brane source, J_D

$$\mathcal{K}\Phi + g\Phi \circ \Phi = J_D.$$

Perturbative solutions:

$$\Phi = \frac{1}{\mathcal{K} - i\epsilon} J_D - g \frac{1}{\mathcal{K} - i\epsilon} \left\{ \frac{1}{\mathcal{K} - i\epsilon} J_D \circ \frac{1}{\mathcal{K} - i\epsilon} J_D \right\} + \ldots .$$

The zero-th order by P. Di Vecchia, M. Frau, I. Pesando, S. Sciuto, A. Lerda, and R. Russo (1997).

1. M-Loop Diagrams of YM
   $\Leftrightarrow$ M-Loop Open String Diagrams with Vertex Operators
   $\Leftrightarrow$ Tree Diagrams of Closed String with M external states
2. Higher Loop Corrections to Correlators of BMN Operators

Collaborators

1. String scattering amplitudes: S. H. Lai, J. C. Lee and Y. Yang (NCTU, Taiwan)
2. Unitarity of graviton scattering amplitudes: T. Inami (SKKU, RIKEN)
3. Classical solutions: Kanghoon Lee (IBS)
4. Numerical study and DMFT : Hoonpyo Lee, H. Y. Park (KNU)