“3d Index” as an order parameter of 3d gauge theory

Dongmin Gang
Based on work with Kazuya Yonekura
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Introduction : IR Dynamics from duality

- Question in QFT

UV description

$$\left[\frac{\mathbf{T}\ (\boldsymbol{a\ CFT\ with\ flavor\ symmetry\ G})}{(H)_k} := (\text{Gauging } H \subset G\ of\ \mathbf{T}\ \text{of})\ with\ CS\ leve\ k\right]$$

$+$(deformation by relevant scalar operators)

RG →

IR fixed point

$-$ *Gapped (Trivially Gapped, Topological)*

$-$ *Flvor symmetry?*

$-$ *Spectrum of local operators*
(number of stress-energy tensor)

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- Duality

Property A (Flavor symmetry G_A)

Ex)

$$\frac{\text{free } \mathbf{\Phi}}{\mathrm{U}\ (1)_{k=-3/2}} \otimes \frac{\text{free } \mathbf{\Phi}}{\mathrm{U}\ (1)_{k=-3/2}}$$

$\cong$

$$\frac{\mathbf{T[SU(2)]}}{(\mathrm{SU}\ (2)_{\text{diag}})_{k=3}}$$

Property B (Flavor symmetry G_B)

→ **(a 3d N=4 SCFT)$^{\otimes 2}$**

Property A and B
($G_B \times G_B \subset$ Flavor symmetry)

IR Dynamics of DGG theory from duality across dim

- Main duality in my talk

$\mathbf{T}^{DGG}[N,A] = [\ \frac{(\text{free } \Phi)^{\otimes \#(T)}}{(H \subset U(1)^{\#(T)}\ )_K}]$
$+(W_{sup}$ using chiral primaries)

$\cong$

$\mathbf{T}^{6d,(irred)}[M,K]$
~(6d su(2) (2,0) theory on $\mathbb{R}^{1,2} \times M$
with co-dimenson 2 defect along $\mathbb{R}^{1,2} \times K$)

N : a 3-manifold with a torus boundary
$A \in H_1(\partial N, \mathbb{Z})$

1-1 correspondence
$N = M \setminus K, A$: *circle linking K*
$M = N_A$

M : a 3-manifold without boundary
K is a knot inside M

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Let me explain the two theories in the duality more in detail

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- LHS of duality

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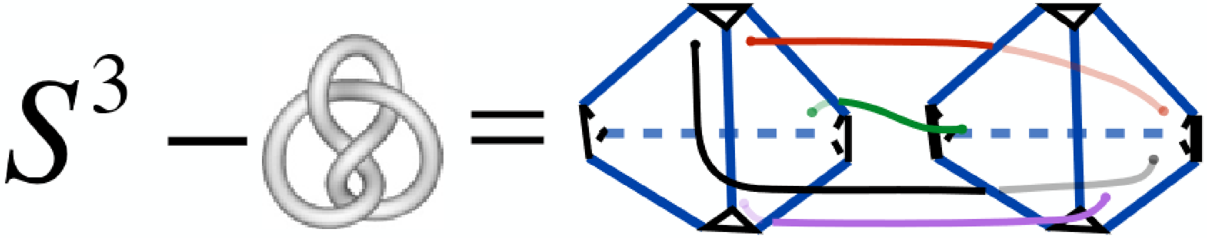

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$(A = (merdian))$

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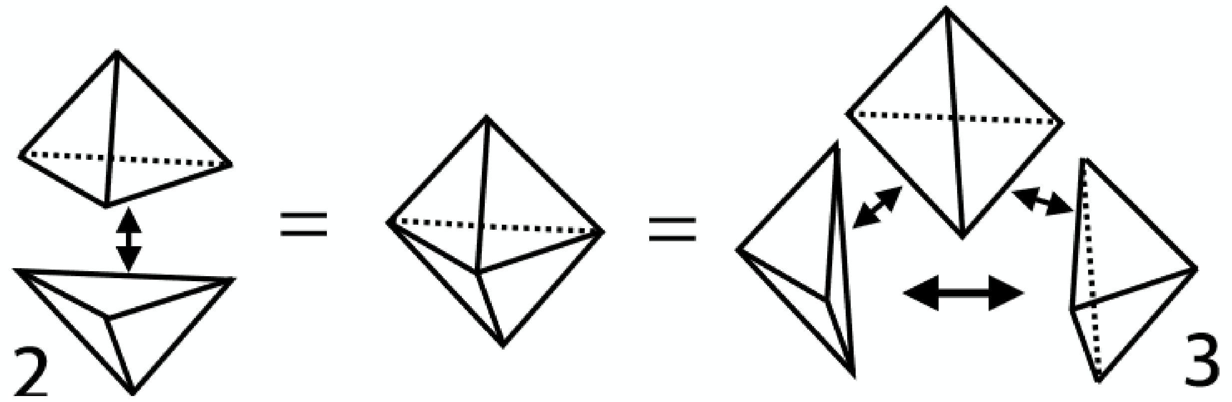

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  $U(1)_{pA+B} = U(1)_{top}$

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Question : Will $U(1)_A$ be enhanced to $SU(2)_A$/ $SO(3)_A$?

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Let us answer the question from 6d dual

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Comparison with $\mathrm{T}^{6d,full}[M,K]$

	$\mathrm{T}^{6d,full}[M,K]$	$\mathrm{T}^{6d,(irred)}[M,K]=\mathrm{T}^{DGG}[\mathrm{N},\mathrm{A}]$
Flavor symmetry associated to knot (puncture)	$SU(2)_K$	Generically $U(1)_A$
Vacua on $\mathbb{R}^2 \times S^1$	Contains all SL(2,C) flat-connections on $\mathbf{N}$	Only irreducible SL(2,C) flat-connections on $\mathbf{N}$

6d su(2) (2,0) theory on $\mathbb{R}^2 \times S^1 \times M$ $\rightarrow$ $5d\ N = 2\ SYM$ on $\mathbb{R}^2 \times M$

Codimension 2 defect on $\mathbb{R}^2 \times S^1 \times K$ $\rightarrow$ Coupling T[SU(2)] theory on $\mathbb{R}^2 \times K$

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6d su(2) (2,0) theory on $\mathbb{R}^2 \times \mathbf{S}^1 \times M$ → $5d\ N = 2\ SYM$ on $\mathbb{R}^2 \times M$

(SU(2) Gauge field A_μ , 5 scalars → Complex gauge field $\boldsymbol{A}$ = A_μ +i ϕ_μ , *and* 1 *complex sclar*)

Codimension 2 defect on $\mathbb{R}^2 \times \mathbf{S}^1 \times K$ → Coupling T[SU(2)] theory on $\mathbb{R}^2 \times K$

T[SU(2)] : SU(2)xSU(2) momenta operators , μ_C and μ_H ← Coupled to bulk su(2) gauge fields $\langle \mu_H \rangle = \log \oint_A \boldsymbol{A}$

Claim : $\mathbf{T}^{6d,full}[M,K]$ → $\underline{\mathbf{T}}^{6d,irred}[M,K]$ → $\mathbf{T}^{6d,irred}[M,K]$

(first arrow) Choose a particular H-saddle corresponding to the component of irreducible ones

(second arrow) Superpotential deformation W= μ_C

IR Dynamics of DGG theory from duality across dim

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It explains why 1) only irreducible flat-connections

2) Generically $U(1)_A$ instead of $SU(2)_A$

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$SU(2)_A$ **symmetry condition :**

μ_C is irrelevant

$\langle \mu_H \rangle \neq 0$

No irreducible flat connections on $M = N_A$

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From analysis of 6d dual

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IR Dynamics of DGG theory from duality across dim

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Consistent check :

Thurston's theorem : Except only finite many $A \in H_1(\partial N, \mathbb{Z})$, $M = N_A$ is hyperbolic

For hyperbolic $M = N_A$, there is always an irreducible flat connections

For only finite many $A \in H_1(\partial N, \mathbb{Z})$, **T**$^{DGG}[N,A]$ **has** $\boldsymbol{SU(2)_A}$ symmetry

Comptibile with the fact that T$^{DGG}[N, pA + B]$ = (Gauging $U(1)_A$ of **T**$^{DGG}[N,A]$ with CS level p)
can not have $\boldsymbol{SU(2)_A}$ for large p

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Example :

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$$U(1)_J \times SU(2)_\Phi \supset U(1)_J \times U(1)_\Phi \supset U(1)_{\rm diag} = \boldsymbol{U(1)_A}$$

$$SU(3) \supset \boldsymbol{SO(3)_A = SU(2)_A} \qquad \boldsymbol{SU(2)_A}$$

$(N_A = S^3, no\ irreducible\ flat\ connection)$

cf) [Gaiotto,Komargodski,Wu;'18] [Benini,Benvenuti; '18]

IR Dynamics of DGG theory from duality across dim

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→ **No irreducible flat connections on** $M = N_A$

→ $T^{DGG}[N,A]$ **has** $SU(2)_A$ **symmetry**

- 3d Index as an order parameter

$$\mathcal{I}_M(q) := \int D\mathcal{A} e^{i\frac{k+\sigma}{4\pi}\int_M \mathrm{Tr}(\mathcal{A}\wedge d\mathcal{A}+\frac{2}{3}\mathcal{A}^3)+c.c}\Big|_{k=0,\sigma=\frac{4\pi}{\log q}}$$

$$= \sum_{\alpha\in \text{flat } SL(2,C) \text{ connections on } M} \frac{1}{\mathrm{vol}(Stab_\alpha)} \mathcal{I}_M^{(\alpha)}(q)$$

$$= \sum_{\alpha\in \text{irreducible flat connections on } M} \mathcal{I}_M^{(\alpha)}(q)$$

(3d Index : Polynomial in q with integer coefficient)

→ $\mathcal{I}_M(q) = 0$ if M does not allow any irreducible flat connections

↓

$T^{DGG}[N.A]$ **has** $SU(2)_A$ **symmetry**

if $\mathcal{I}_M(q) = 0$

Summary

$\mathrm{T}^{DGG}[N,A]=[\frac{(\text{free }\Phi)^{\otimes\#(T)}}{(H\subset U(1)^{\#(T)})_K}]$
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If No irreducible flat connections on $M=N_A$

⟺ $\mathcal{I}_M(q)=0$