Relativistic guiding-center motion of runaway electrons

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Outline

- Introduction
- Non-canonical Hamiltonian formulation of guiding-center motion
- Singularity free formulation
- Simulation
- Summary and future plan

Runaway electrons

- Averaged Coulomb friction force with a Maxwellian background [b]

$$\frac{m_a < \Delta v_{\parallel} >^{ab}}{\Delta t} \propto -G\left(\frac{v}{v_{T_b}}\right)$$

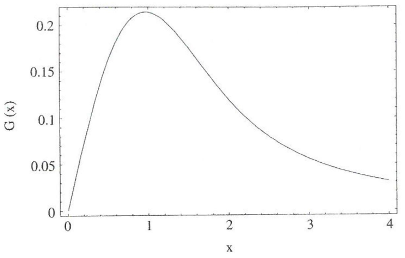


Fig. 3.6. The Chandrasekhar function $G(x)$, which describes the drag force on a particle by collisions with a Maxwellian background.

Condition of RE

- Minimum electric field force for runaway electrons

$$E_D = \frac{n_e e^3 \ln \Lambda}{4\pi\epsilon_0^2 T_e} < E$$

$$E_c = \frac{n_e e^3 \ln \Lambda}{4\pi\epsilon_0^2 m_e c^2} < E \qquad \text{Relativistic}$$

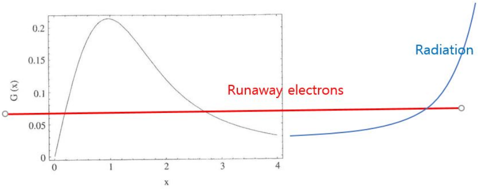


Fig. 3.6. The Chandrasekhar function $G(x)$, which describes the drag force on a particle by collisions with a Maxwellian background.

Charged particle motion

Charged particle in (external) electromagnetic fields

- Lorentz force equation

$$\frac{d\vec{p}}{dt} = e\left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right]$$

where $\vec{p} = m\vec{v}$.

Assume uniform electromagnetic fields.

- Guiding-center moving frame $\vec{u} = \vec{v} - \frac{1}{B^2}\vec{E} \times \vec{B}$
  i) $m\dot{v}_{\parallel} = eE_{\parallel}$
  ii) $m\dot{\vec{u}}_{\perp} = e\vec{u}_{\perp} \times \vec{B}$
- Guiding-center motion $\vec{u}_{gc} = v_{\parallel}\hat{b} + \frac{1}{B^2}\vec{E} \times \vec{B}$
- Gyromotion: gyrofrequency $\Omega = \frac{eB}{mc}$, gyroradius $\vec{\rho} = \hat{b} \times \frac{\vec{u}_{\perp}}{\Omega}$

Guiding-center theory

Slowly varying magnetic fields ($\frac{\rho}{L} = \frac{v_\perp}{\Omega L}, \frac{v_\parallel}{\Omega L} \ll 1$)

- ∇B drift $\frac{W_\perp \vec{B} \times \nabla B}{eB^3}$
- Curvature drift $\frac{2W_\parallel \vec{B} \times (\hat{b} \cdot \nabla) b}{eB^2}$
- Polarization drift

Hamiltonian theory of guiding-center motion

- Nöther's theorem
  i) Energy conservation law
  ii) Ignorable coordiantes
- Liouville's property
  Hamiltonian flow conserves the phase-space volume.
- Derivation from variational principle

Non-canonical Hamiltonian formulation

- Use non-canonical (physical) coordinates
- Phase-space Lagrangian
  $\mathcal{L}(\vec{q}, \vec{p}, \dot{\vec{q}}, \dot{\vec{p}}; t) \equiv \vec{p} \cdot \dot{\vec{q}} - H(\vec{q}, \vec{p}; t)$
  Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right) = \frac{\partial \mathcal{L}}{\partial q^i}, \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{p}^i}\right) = \frac{\partial \mathcal{L}}{\partial p^i}$$

- Canonical $(\vec{q}, \vec{p})$ to non-canonical z^α, general pase-space Lagrangian

$$\mathcal{L} \equiv \Lambda_\alpha \dot{z}^\alpha - \mathcal{H}$$

  where $\Lambda_\alpha \equiv \vec{p} \cdot \frac{\partial \vec{q}}{\partial z^\alpha}, \quad \mathcal{H} = H - \vec{p} \cdot \frac{\partial \vec{q}}{\partial t}$
- Equations of motion
  Lagrange matrix $\omega_{\alpha\beta} = \frac{\partial \Lambda_\beta}{\partial z^\alpha} - \frac{\partial \Lambda_\alpha}{\partial z^\beta}$, Poisson tensor $\Pi = \omega^{-1}$

$$\frac{dz^\alpha}{dt} = \Pi^{\alpha\beta}\left[\frac{\partial \mathcal{H}}{\partial z^\beta} + \frac{\partial \Lambda_\beta}{\partial t}\right]$$

Poisson structure of non-canonical Hamiltonian formulation

- Energy equation

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} - \frac{dz^\beta}{dt}\frac{\partial \Lambda_\beta}{\partial t}$$

- Poisson tensor
  Jacobian $D^\beta_\alpha = \frac{\partial z^\beta}{\partial Z^\alpha}$ where $\vec{Z} = (\vec{q}, \vec{p})$

$$\Pi = \mathbf{D}\sigma\mathbf{D}^T$$

For time-independent transformations to non-canonical coordinates

- Equations of motion

$$\frac{dz^\alpha}{dt} = \Pi^{\alpha\beta}\frac{\partial H}{\partial z^\beta} \equiv \{z^\alpha, H\}$$

- Generalized Poisson bracket

$$\{f, g\} \equiv \frac{\partial f}{\partial z^\beta}\Pi^{\alpha\beta}\frac{\partial g}{\partial z^\beta}$$

Nöther's theorem and Liouville's property

- Nöther's theorem
  z^β is cyclic, i.e., $\frac{\partial \mathcal{L}}{\partial z^\beta} = 0$
  Euler-Lagrange equations $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{z}^\alpha}\right) = \frac{\partial \mathcal{L}}{\partial z^\alpha}$
  $\frac{\partial \mathcal{L}}{\partial \dot{z}^\alpha} = \Lambda_\alpha$, so $\frac{d}{dt}\Lambda_\beta = 0$

- Liouville's theorem
  If the transformation Jacobian $\mathcal{J} = det(\mathbf{D}^{-1})$ satisfies

$$\frac{\partial \mathcal{J}}{\partial t} + \frac{\partial}{\partial z^\alpha}\left(\mathcal{J}\frac{dz^\alpha}{dt}\right) = 0$$

  then the Hamiltonian flow conserves the phase-space volume $d^3q d^3p = \mathcal{J} d^6 z$.

Derivation of the non-canonical guiding-center Lagrangian

- Perturbation

$$\frac{1}{m}\mathcal{L}(\vec{x},\dot{\vec{x}},\vec{v};t) = \left[\epsilon^{-1}\vec{A}(\vec{x},t) + \vec{v}\right]\cdot\dot{\vec{x}} - \left[\frac{1}{2}|\vec{v}|^2 + \epsilon^{-1}\Phi(\vec{x},t)\right]$$

where $c = 1$ and $\frac{e}{m} = \epsilon^{-1}$
Guiding-center position $\vec{X}$

$$\vec{x} = \vec{X} + \epsilon\vec{\rho}$$

Non-canonical guiding-center Lagrangian

- Guiding-center phase-space Lagrangian

$$\mathcal{L}_{gc}(\vec{X}, u, \mu, \zeta; t) = \left[\frac{e}{c}\vec{A}(\vec{X}, t) + mu\hat{b}(\vec{X}, t)\right] \cdot \dot{\vec{X}} - J\dot{\zeta} - H_{gc}$$

- Guiding-center Hamiltonian

$$H_{gc}(\vec{X}, u, \mu; t) = \frac{m}{2}u^2 + \mu B(\vec{X}, t) + e\Phi(\vec{X}, t) - \frac{m}{2}|\vec{v}_E(\vec{X}, t)|^2$$

from Cary & Brizard (2009) Rev. Mod. Phys. 81, 693

Properties

- Energy is conserved in time-independent Hamiltonian system.
- For azimuthally symmetric system, the toroidal angle ϕ is cyclic.
  Canonically conjugate momentum p_ϕ is conserved.

$$p_\phi = \frac{e}{c}A_\phi + mub_\phi$$

- $\mathcal{J} = m^2 B_\parallel^*$ satisfies the divergence equation.
  Guiding-center phase-space volume is conserved.

Equations of motion

- $\delta_u \mathcal{L} = 0,\ \vec{B}^* \cdot \delta_{\vec{X}} \mathcal{L} = 0,\ \hat{b} \times \delta_{\vec{X}} \mathcal{L} = 0$
- Using effective electromagnetic potentials
  $e\Phi^* = e\Phi + \mu B - \frac{m}{2}|\vec{v}_E(\vec{X}, t)|^2$
  $\vec{A}^* = \vec{A} + \frac{mc}{e} u \hat{b}$

$$\dot{u} = \frac{e}{m} \frac{\vec{B}^*}{B_\parallel^*} \cdot \vec{E}^*$$

$$\dot{\vec{X}} = u \frac{\vec{B}^*}{B_\parallel^*} + \vec{E}^* \times \frac{c\hat{b}}{B_\parallel^*}$$

- $B_\parallel^*$

$$B_\parallel^* = B(1 + \frac{u}{\Omega} \hat{b} \cdot \nabla \times \hat{b})$$

- $\hat{b} \cdot \nabla \times \hat{b} = \frac{1}{L_t}$

Relativistic motion

Relativistic charged particle in (external) electromagnetic fields

- Momentum equation

$$\frac{d\vec{p}}{dt} = e\left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right]$$

where $\vec{p} = \gamma m \vec{v}$.

- Energy equation

$$\frac{dE_{kin}}{dt} = e\vec{v} \cdot \vec{E}$$

- Covariant form

$$\frac{dp^{\mu}}{d\tau} = \frac{e}{c} F^{\mu\nu} u_{\nu}$$

where $p^{\mu} = (\gamma mc, \vec{p})$,
$F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$,
$A^{\mu} = (\Phi, \vec{A})$.

Relativistic guiding-center Lagrangian

- Covariant formulation
  Boghosian(1987) Ph.D. thesis, University of California

- Non-covariant formulation
  Cary & Brizard (2009) Rev. Mod. Phys. 81, 693
  White & Gobbin (2014) PPPL-5078 in Boozer coordinates

$$\mathcal{L}_{rgc} = \left[\frac{e}{c}\vec{A}(\vec{X},t) + p_{\|}\hat{b}(\vec{X},t)\right]\cdot\dot{\vec{X}} - J\dot{\zeta} - w\dot{t} - H_{rgc}$$

$$H_{rgc}(\vec{X}, p_{\|}, \mu; t) = \gamma mc^2 + e\Phi(\vec{X},t) - w$$

where $\gamma = \sqrt{1 + \frac{2}{mc^2}\mu B(\vec{X},t) + \frac{p_{\|}^2}{(mc)^2}}$.

Equations of motion

- Using effective electromagnetic potentials
  $e\Phi^* = e\Phi + \gamma mc^2$
  $\vec{A}^* = \vec{A} + \frac{mc}{e} p_\parallel \hat{b}$

$$\frac{dp_\parallel}{dt} = e\frac{\vec{B}^*}{B_\parallel^*} \cdot \vec{E}^*$$

$$\frac{d\vec{X}}{dt} = \frac{p_\parallel}{m\gamma}\frac{\vec{B}^*}{B_\parallel^*} + \vec{E}^* \times \frac{c\hat{b}}{B_\parallel^*}$$

- $B_\parallel^*$

$$B_\parallel^* = B + \frac{p_\parallel c}{e}\hat{b} \cdot \nabla \times \hat{b}$$

Toroidal regularization of guiding-center Lagrangian

Burby & Ellison (2017) PoP 24, 110703

- To replace $B_\parallel^*$ with non-zero quantity $R_0 B^\phi$
- Near-identity Lie transform

$$(\vec{\mathrm{X}}, \mathrm{v}_\parallel) = e^G(\vec{X}, v_\parallel)$$

where $G^{v_\parallel} = 0$

$$G^{\vec{X}} = -\frac{mc}{eB^\phi(\vec{X})} v_\parallel (\nabla\phi \times \hat{b})$$

- Non-perturbative transformation

$$\mathrm{v}_\parallel^* = \mathrm{v}_\parallel \frac{B(\mathrm{X})}{R_0 B^\phi(\mathrm{X})}$$

Relativistic Lagrangian and equations of motion

- Lagrangian

$$\mathcal{L}^*_{rgc} = \left[\frac{e}{c}\vec{A}(\vec{\mathrm{X}}, t) + \mathrm{p}^*_{\parallel} R_0 \nabla\phi\right] \cdot \dot{\vec{\mathrm{X}}} - J\dot{\zeta} - H^*_{rgc}$$

$$H^*_{rgc}(\vec{\mathrm{X}}, \mathrm{p}^*_{\parallel}, \mu; t) = e\Phi^* = \gamma mc^2 + e\Phi(\vec{\mathrm{X}}, t) - p^*_{\parallel}\hat{b} \cdot \frac{\vec{E} \times (R_0\nabla\phi)}{B}$$

where $\gamma = \sqrt{1 + \frac{2}{mc^2}\mu B(\vec{\mathrm{X}}, t) + \frac{(p^*_{\parallel})^2}{(mc)^2}\frac{(R_0 B^{\phi})^2}{B^2}}$.

- Equations of motion

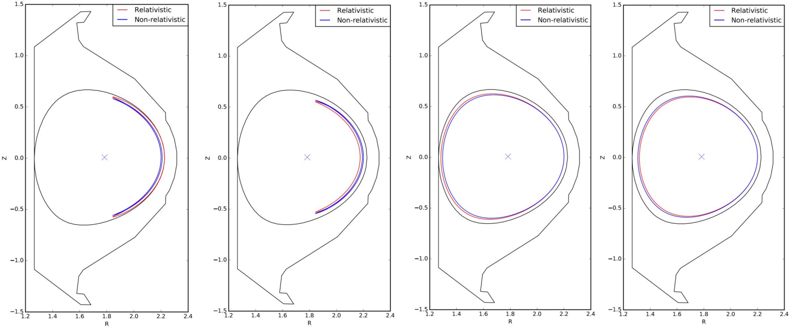
$$\frac{dp^*_{\parallel}}{dt} = e\frac{\vec{B}}{R_0 B^{\phi}_{\parallel}} \cdot \vec{E}^*$$

$$\frac{d\vec{X}}{dt} = \frac{\vec{B}}{R_0 B^{\phi}} \frac{1}{m} \frac{\partial H^*_{rgc}}{\partial p^*_{\parallel}} + \frac{c\vec{E}^* \times (R_0\nabla\phi)}{R_0 B^{\phi}}$$

where $\vec{E}^* = -\nabla\Phi^* - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$

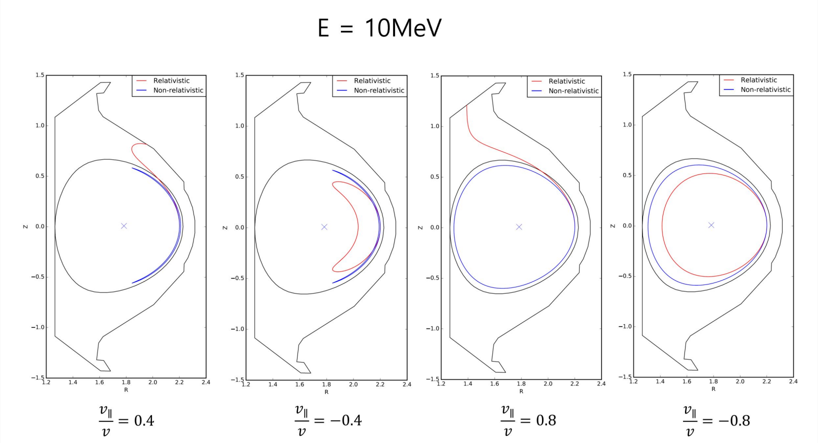
Non-relativistic vs. relativistic

E = 1MeV

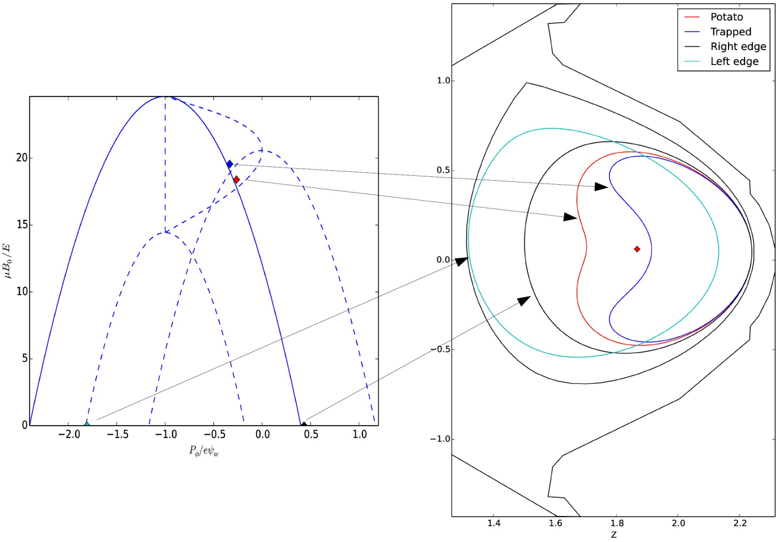


$\frac{v_\parallel}{v} = 0.4$ $\frac{v_\parallel}{v} = -0.4$ $\frac{v_\parallel}{v} = 0.8$ $\frac{v_\parallel}{v} = -0.8$

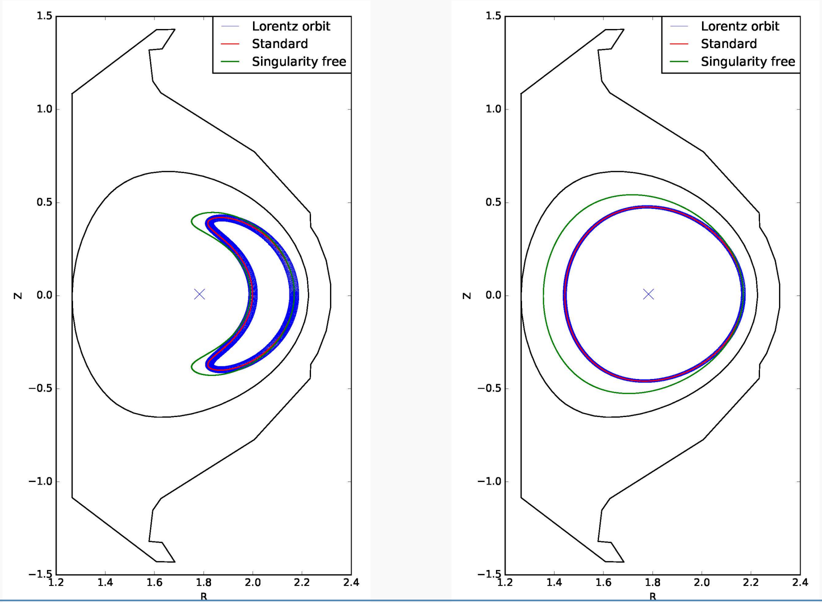
Non-relativistic vs. relativistic



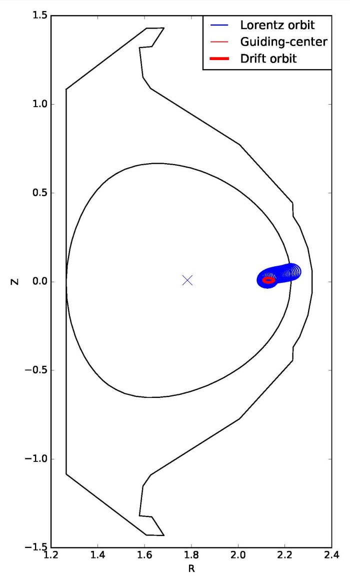
Relativistic orbit classification



Standard vs. singularity free with Lorentz orbit



High velocity (E=100MeV, $v_{\parallel}/v$=-0.99)

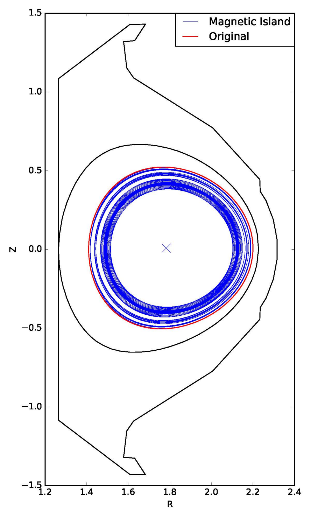


Summary and future plan

- We drived toroidally regularized relativisitic guiding-center equations of motion.
- We developed relativistic guiding-center simulation code.
- In relativistic case, orbits are drifted more.

- Investigate high velocity regime more carefully
- Address confinement property of RE in the presence of magnetic islands

Magnetic island effect (preliminary)



$$\tilde{\vec{B}} = -\nabla\alpha \times \nabla\phi$$