Kinetic Space Plasma Turbulence

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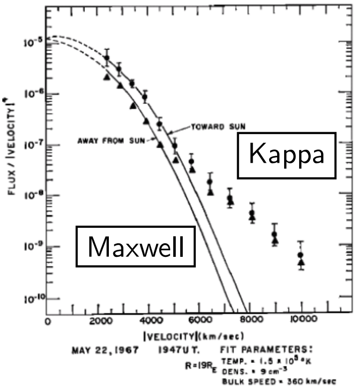
Motivation

Nonthermal charged particle distribution function in space

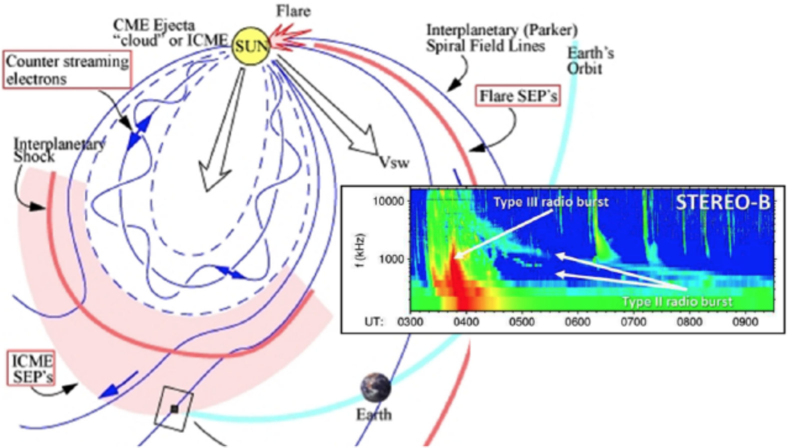


$$f_{\text{Max}} \propto e^{-v^2/v_T^2}$$

$$f_{kappa} \propto \frac{1}{(1 + v^2/\kappa v_T^2)^{\kappa+1}}$$

Motivation

Solar flare-generated radio bursts



Fundamentals of Plasma Kinetic Theory

1D, field-free electrostatic Vlasov-Poisson equation

$$\left(\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial x} + \frac{e_a E}{m_a}\frac{\partial}{\partial v}\right) f_a = 0,$$
$$\frac{\partial E}{\partial x} = 4\pi\hat{n}\sum_a e_a \int dv\ f_a.$$

Separation into Average and Fluctuation

$$f_a(x,v,t) = F_a(v,t) + \delta f_a(x,v,t), \qquad E(x,t) = \delta E(x,t).$$

Rewrite the equations

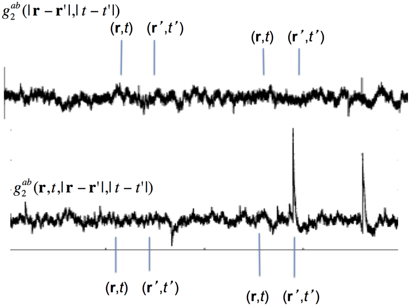
$$\left(\frac{\partial}{\partial t} + \frac{e_a}{m_a}\,\delta E\,\frac{\partial}{\partial v}\right) F_a + \left(\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial x} + \frac{e_a}{m_a}\,\delta E\,\frac{\partial}{\partial v}\right)\delta f_a = 0,$$
$$\frac{\partial}{\partial x}\,\delta E = 4\pi\hat{n}\sum_a e_a \int dv\ \delta f_a.$$

Random phase approximation

$$< \delta f_a >= 0, \qquad < \delta E >= 0.$$

Stationary and homogeneous turbulence,

$$< \delta f_a(x, v, t)\, \delta f_b(x', v', t') >= g(|x - x'|, |t - t'|, v, v').$$



$$\left(\frac{\partial}{\partial t} + \frac{e_a}{m_a}\,\delta E\,\frac{\partial}{\partial v}\right)\,F_a + \left(\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial x} + \frac{e_a}{m_a}\,\delta E\,\frac{\partial}{\partial v}\right)\delta f_a = 0,$$

Upon averaging we have the *formal* particle kinetic equation

$$\frac{\partial\,F_a}{\partial t} = -\frac{e_a}{m_a}\frac{\partial}{\partial v}\,\langle\delta f_a\,\delta E\rangle.$$

Inserting to the original equation, we obtain the equation for the perturbed distribution,

$$\left(\frac{\partial}{\partial t} + v\,\frac{\partial}{\partial x}\right)\delta f_a = -\frac{e_a}{m_a}\,\delta E\,\frac{\partial F_a}{\partial v} - \frac{e_a}{m_a}\frac{\partial}{\partial v}\,\left(\delta f_a\,\delta E - < \delta f_a\,\delta E >\right).$$

Fourier-Laplace transformation:

$$\downarrow \text{ (slow time)}$$

$$\begin{aligned}
\delta f_a(x,v,t) &= \int dk \int_L d\omega\, \delta f^a_{k\omega}(v,t)\, e^{ikx-i\omega t}, \\
\delta f^a_{k\omega}(v,t) &= \frac{1}{(2\pi)^2} \int dx \int_0^\infty dt\, \delta f_a(x,v,t)\, e^{-ikx+i\omega t}, \\
\delta E(x,t) &= \int dk \int_L d\omega\, \delta E_{k\omega}(t)\, e^{ikx-i\omega t}, \\
\delta E_{k\omega}(t) &= \frac{1}{(2\pi)^2} \int dx \int_0^\infty dt\, \delta E(x,t)\, e^{-ikx+i\omega t},
\end{aligned}$$

where $L = (-\infty + i\sigma, +\infty + i\sigma)$ ($\sigma > 0$ and $\sigma \to 0$: causality).

$$
\begin{aligned}
&\delta E_{k\omega}(t) = -i \sum_a \frac{4\pi \hat{n} e_a}{k} \int dv \, \delta f^a_{k\omega}(v,t), \\
&\frac{\partial\, F_a(v,t)}{\partial t} = -\frac{e_a}{m_a} \int dk \int d\omega \, \frac{\partial}{\partial v} < \delta E_{-k,-\omega}(t) \, \delta f^a_{k\omega}(v,t) >, \\
&\left(\omega - kv + i \, \frac{\partial}{\partial t}\right) \delta f^a_{k\omega}(v,t) = -i \, \frac{e_a}{m_a} \, \delta E_{k\omega}(t) \, \frac{\partial F_a(v,t)}{\partial v} \\
&\qquad - i \, \frac{e_a}{m_a} \int dk' \int d\omega' \, \frac{\partial}{\partial v} \, \Big[\delta E_{k'\omega'}(t) \, \delta f^a_{k-k' \, \omega-\omega'}(v,t) \\
&\qquad - < \delta E_{k'\omega'}(t) \, \delta f^a_{k-k',\omega-\omega'}(v,t) > \Big] \, .
\end{aligned}
$$

Absorb slow time into ω (short-cut trick),

$$\omega + i\,\frac{\partial}{\partial t} \to \omega.$$

Short-hand notations:

$$\begin{aligned} K &= (k,\omega), \qquad E_K = \delta E_{k\omega}, \qquad f_K = \delta f^a_{k\omega}, \\ F &= F_a, \qquad \int dK = \int dk \int d\omega, \\ g_K &= -i\,\frac{e_a}{m_a}\,\frac{1}{\omega - kv + i0}\,\frac{\partial}{\partial v}. \end{aligned}$$

Equation for the perturbed particle distribution:

$$f_K = g_K\,F\,E_K + \int dK'\,g_K\,(E_{K'}\,f_{K-K'} - < E_{K'}\,f_{K-K'} >).$$

Iterative solution:

$$f_K = f_K^{(1)} + f_K^{(2)} + \cdots\,, \qquad (f_K^{(n)} \propto E_K^n).$$

$$\begin{aligned} f_K^{(1)} &= g_K\, F\, E_K\,, \\ f_K^{(2)} &= \int dK'\, g_K\, (E_{K'}\, f^{(1)}_{K-K'} - < E_{K'}\, f^{(1)}_{K-K'} >) \\ &= \int dK'\, g_K\, g_{K-K'}\, F\, (E_{K'}\, E_{K-K'} - < E_{K'}\, E_{K-K'} >). \end{aligned}$$

More simplified notations:

$$\sum_{1+2=K} = \int dK_1 \int dK_2\, \delta(K_1 + K_2 - K),$$

$$E_1 = E_{K_1}, \quad E_2 = E_{K_2}, \qquad g_1 = g_{K_1}, \quad g_2 = g_{K_2}.$$

Iterative solution for f_K

$$f_K = g_K\, F\, E_K + \sum_{1+2=K} g_K\, g_2\, F\, (E_1 E_2 - < E_1 E_2 >).$$

Symmetrized version

$$f_K = g_K\, F\, E_K + \sum_{1+2=K} \frac{1}{2}\, g_{1+2}\, (g_1 + g_2)\, F\, (E_1\, E_2 - < E_1\, E_2 >).$$

Insert f_K to Poisson equation,

$$E_K = -i \sum_a \frac{4\pi \hat{n} e_a}{k} \int dv\, f_K.$$

Linear and nonlinear susceptibility response functions,

$$\begin{aligned}\epsilon(K) &= 1 + \sum_a i\,\frac{4\pi e_a \hat{n}}{k}\int dv\; g_K\, F,\\ \chi_a^{(2)}(1|2) &= \frac{i}{2}\frac{4\pi e_a \hat{n}}{k_1+k_2}\int dv\; g_{1+2}\,(g_1+g_2)\,F.\end{aligned}$$

or explicitly

$$\begin{aligned}\epsilon(K) &= 1 + \sum_a \frac{\omega_{pa}^2}{k}\int dv\;\frac{\partial F_a/\partial v}{\omega - kv + i0},\\ \chi^{(2)}(1|2) &= -\frac{i}{2}\sum_a \frac{e_a}{m_a}\frac{\omega_{pa}^2}{k_1+k_2}\int dv\;\frac{1}{\omega_1+\omega_2-(k_1+k_2)\,v+i0}\\ &\quad\times\frac{\partial}{\partial v}\left[\left(\frac{1}{\omega_1-k_1v+i0}+\frac{1}{\omega_2-k_2v+i0}\right)\frac{\partial F_a}{\partial v}\right].\end{aligned}$$

$$0 = \epsilon(K)\, E_K + \sum_{1+2=K} \chi^{(2)}(1|2)\ (E_1 E_2 - < E_1 E_2 >)\,.$$

Multiply $E_{K'}$ and take ensemble average

$$0 = \epsilon(K) < E_K E_{K'} > + \sum_{1+2=K} \chi^{(2)}(1|2) < E_1 E_2 E_{K'} > .$$

Homogeneous and stationary turbulence (and spectral representation),

$$< E(x,t)\, E(x',t') >=< E^2 >_{x-x',t-t'},$$

$$< E_{k\omega}\, E_{k'\omega'} >= \delta(k+k')\, \delta(\omega+\omega') < E^2 >_{k\omega} .$$

$$\rightarrow \qquad 0 = \epsilon(K) < E^2 >_K + \sum_{1+2=K} \chi^{(2)}(1|2) < E_1 E_2 E_{-K} > .$$

Closure of Hierarchy

If E_K is linear eigenmode, then

$$\epsilon(K)\, E_K = 0,$$

and $< E_1 E_2 E_{-K} >= 0$, but for nonlinear system we write $E_K = E_K^{(0)} + E_K^{(1)}$, where $\epsilon(K)\, E_K^{(0)} = 0$. Then

$$E_K^{(1)} \approx -\frac{1}{\epsilon(K)} \int dK' \chi^{(2)}(K'|K-K') \left(E_{K'}^{(0)} E_{K-K'}^{(0)} - < E_{K'}^{(0)} E_{K-K'}^{(0)} > \right).$$

Three-body correlation,

$$\begin{aligned} < E_{K'} E_{K-K'} E_{-K} > \quad &\approx \quad \underbrace{< E_{K'}^{(0)} E_{K-K'}^{(0)} E_{-K}^{(0)} >}_{0} + < E_{K'}^{(1)} E_{K-K'}^{(0)} E_{-K}^{(0)} > \\ &+ < E_{K'}^{(0)} E_{K-K'}^{(1)} E_{-K}^{(0)} > + < E_{K'}^{(0)} E_{K-K'}^{(0)} E_{-K}^{(1)} > . \end{aligned}$$

$$
\begin{aligned}
< E_{K'} E_{K-K'} E_{-K} >= & -\frac{1}{\epsilon(K')} \int dK'' \chi^{(2)}(K''|K' - K'') \\
& \times \left(\langle E_{K''} E_{K'-K''} E_{K-K'} E_{-K} \rangle - \langle E_{K''} E_{K'-K''} \rangle \langle E_{K-K'} E_{-K} \rangle \right) \\
& - \frac{1}{\epsilon(K-K')} \int dK'' \chi^{(2)}(K''|K - K' - K'') \\
& \times \left(\langle E_{K''} E_{K-K'-K''} E_{K'} E_{-K} \rangle - \langle E_{K''} E_{K-K'-K''} \rangle \langle E_{K'} E_{-K} \rangle \right) \\
& - \frac{1}{\epsilon(-K)} \int dK'' \chi^{(2)}(-K''| - K + K'') \\
& \times \left(\langle E_{K'} E_{K-K'} E_{-K''} E_{-K+K''} \rangle - \langle E_{K'} E_{K-K'} \rangle \langle E_{-K''} E_{-K+K''} \rangle \right) .
\end{aligned}
$$

where we dropped the superscript (0). To obtain $< E^3 >$ one needs $< E^4 >$.

For homogeneous and stationary turbulence,

$$
\begin{aligned}
< E_K E_{K'} E_{K''} E_{K'''} >= \delta(K + K' + K'' + K''')\\
\times \Big[\delta(K + K') < E^2 >_K < E^2 >_{K''}\\
+\delta(K + K'') < E^2 >_K < E^2 >_{K'}\\
+ \delta(K' + K'') < E^2 >_K < E^2 >_{K'}\\
+ < E^4 >_{K;K+K';K+K'+K''}\Big].
\end{aligned}
$$

Useful symmetry relations:

$$
\begin{aligned}
&\chi^{(2)}(-1|-2) = \chi^{(2)*}(1|2),\\
&\chi^{(2)}(1|2) = \chi^{(2)}(2|1),\\
&\chi^{(2)}(1|2) = \chi^{(2)}(2|1) = -\chi^{(2)}(1+2|-2).
\end{aligned}
$$

Three-body cumulants:

$$
\begin{aligned}
< E_{K'} E_{K-K'} E_{-K} > \quad = \quad & \frac{2\chi^{(2)}(K'|K-K')}{\epsilon(K')} < E^2 >_{K-K'} < E^2 >_K \\
& + \frac{2\chi^{(2)}(K'|K-K')}{\epsilon(K-K')} < E^2 >_{K'} < E^2 >_K \\
& - \frac{2\chi^{(2)*}(K'|K-K')}{\epsilon^*(K)} < E^2 >_{K'} < E^2 >_{K-K'} \ .
\end{aligned}
$$

Spectral balance equation

$$
\begin{aligned}
0 \quad = \quad & \epsilon(K) < E^2 >_K \\
& +2 \int dK' \left(\frac{\{\chi^{(2)}(K'|K-K')\}^2}{\epsilon(K')} < E^2 >_{K-K'} < E^2 >_K \right. \\
& + \frac{\{\chi^{(2)}(K'|K-K')\}^2}{\epsilon(K-K')} < E^2 >_{K'} < E^2 >_K \\
& \left. - \frac{|\chi^{(2)}(K'|K-K')|^2}{\epsilon^*(K)} < E^2 >_{K'} < E^2 >_{K-K'} \right) .
\end{aligned}
$$

Formal Wave Kinetic Equation

Reintroduce the slow time dependence,

$$\omega \to \omega + i\,\frac{\partial}{\partial t}.$$

This leads to

$$\epsilon(k,\omega) < E^2 >_{k\omega} \to \epsilon\left(k, \omega + i\frac{\partial}{\partial t}\right) < E^2 >_{k\omega}$$

$$\to \left(\epsilon(k,\omega) + \frac{i}{2}\frac{\partial \epsilon(k,\omega)}{\partial \omega}\frac{\partial}{\partial t}\right) < E^2 >_{k\omega} \; .$$

$$
\begin{aligned}
0 \quad = \quad & \frac{i}{2}\frac{\partial\epsilon(k,\omega)}{\partial\omega}\frac{\partial}{\partial t} < E^2 >_{k\omega} + \epsilon(k,\omega) < E^2 >_{k\omega} \\
& + 2\int dk' \int d\omega' \; \left(\frac{\{\chi^{(2)}(k',\omega'|k-k',\omega-\omega')\}^2}{\epsilon(k',\omega')} \right. \\
& \times \left\langle E^2 \right\rangle_{k-k',\omega-\omega'} \left\langle E^2 \right\rangle_{k\omega} \\
& + \frac{\{\chi^{(2)}(k',\omega'|k-k',\omega-\omega')\}^2}{\epsilon(k-k',\omega-\omega')} \left\langle E^2 \right\rangle_{k'\omega'} \left\langle E^2 \right\rangle_{k\omega} \\
& \left. - \frac{|\chi^{(2)}(k',\omega'|k-k',\omega-\omega')|^2}{\epsilon^*(k,\omega)} \left\langle E^2 \right\rangle_{k'\omega'} \left\langle E^2 \right\rangle_{k-k',\omega-\omega'} \right) .
\end{aligned}
$$

Real part leads to the dispersion relation, and imaginary part leads to the wave kinetic equation.

Particle kinetic equation

For particles, leading order perturbed distribution is sufficient,

$$\frac{\partial F_a}{\partial t} = -\frac{e_a}{m_a} \int dK \, \frac{\partial}{\partial v} < E_{-K} \, f_K^a >,$$

where

$$f_K = f_K^{(1)} + \cancel{f_K^{(2)} + \cdots}.$$

$$\frac{\partial F_a}{\partial t} = \mathrm{Re}\, i \, \frac{e_a^2}{m_a^2} \frac{\partial}{\partial v} \int dk \int d\omega \, \frac{< E^2 >_{k\omega}}{\omega - kv + i0} \frac{\partial F_a}{\partial v}.$$

Particle Kinetic Equation

$$\frac{\partial F_a}{\partial t} = \mathrm{Re}\, i \, \frac{e_a^2}{m_a^2} \frac{\partial}{\partial v} \int dk \int d\omega \; \frac{< E^2 >_{k\omega}}{\omega - kv + i0} \frac{\partial F_a}{\partial v}.$$

Dispersion Relation

$$\mathrm{Re}\, \epsilon(k,\omega) < E^2 >_{k\omega} = 0.$$

For dispersion relation nonlinear terms are ignored.

Wave Kinetic Equation

$$
\begin{aligned}
&\frac{\partial}{\partial t} < E^2 >_{k\omega} = -\frac{2\,\mathrm{Im}\,\epsilon(k,\omega)}{\partial\,\mathrm{Re}\,\epsilon(k,\omega)/\partial\omega} < E^2 >_{k\omega} \\
&- \frac{4}{\partial\,\mathrm{Re}\,\epsilon(k,\omega)/\partial\omega}\,\mathrm{Im}\int dk' \int d\omega' \\
&\times \left[\{\chi^{(2)}(k',\omega'|k-k',\omega-\omega')\}^2 \left(\frac{< E^2 >_{k-k',\omega-\omega'}}{\epsilon(k',\omega')} \right.\right. \\
&\left. + \frac{< E^2 >_{k'\omega'}}{\epsilon(k-k',\omega-\omega')} \right) < E^2 >_{k\omega} \\
&\left. - \frac{|\chi^{(2)}(k',\omega'|k-k',\omega-\omega')|^2}{\epsilon^*(k,\omega)} < E^2 >_{k'\omega'} < E^2 >_{k-k',\omega-\omega'} \right].
\end{aligned}
$$

Linear and Nonlinear Suceptibilities

$$\chi_a(k,\omega) = \frac{\omega_{pa}^2}{k}\int dv\,\frac{\partial F_a/\partial v}{\omega - kv + i0},$$

$$\chi_a^{(2)}(k_1,\omega_1|k_2,\omega_2) = \frac{-i}{2}\frac{e_a}{m_a}\frac{\omega_{pa}^2}{|k_1+k_2|}\int dv$$

$$\times\frac{1}{\omega_1+\omega_2-(k_1+k_2)v+i0}\frac{\partial}{\partial v}$$

$$\times\left(\frac{\partial F_a/\partial v}{\omega_2 - k_2 v + i0} + \frac{\partial F_a/\partial v}{\omega_1 - k_1 v + i0}\right).$$

Partial integrations,

$$\chi_a(\mathbf{k},\omega) = -\omega_{pa}^2 \int dv\, \frac{F_a}{(\omega - kv + i0)^2}.$$

$$\chi_a^{(2)}(k_1,\omega_1|k_2,\omega_2) = -\int dv\, F_a\, \frac{k_1k_2(k_1+k_2)}{(\omega_1 - k_1v)(\omega_2 - k_2v)}$$
$$\times \frac{1}{\omega_1+\omega_2-(k_1+k_2)v}\left(\frac{k_1}{\omega_1-k_1v}\right.$$
$$\left.+\frac{k_2}{\omega_2-k_2v}+\frac{k_1+k_2}{\omega_1+\omega_2-(k_1+k_2)v}\right),$$

where the small positive imaginary parts $+i0$ are implicit in the resonance denominators.

Symmetry Relations

$$\chi^{(2)}(k_2, \omega_2 | k_1, \omega_1) = \chi^{(2)}(k_1, \omega_1 | k_2, \omega_2).$$

$$\epsilon(-k, -\omega) = \epsilon^*(k, \omega),$$
$$\chi^{(2)}(-k_1, -\omega_1 | -k_2, -\omega_2) = \chi^{(2)*}(k_1, \omega_1 | k_2, \omega_2).$$

$$\chi^{(2)}(k_1, \omega_1 | k_2, \omega_2) = \chi^{(2)}(k_2, \omega_2 | k_1, \omega_1)$$
$$= \chi^{(2)}(k_1 + k_2, \omega_1 + \omega_2 | -k_2, -\omega_2).$$

Linear Dielectric Susceptibility

$$\chi^a(0,\omega) = -\omega_{pa}^2/\omega^2.$$

For fast wave $\omega \gg kv_{th}^a$,

$$\chi^a(k,\omega) = -\frac{\omega_{pa}^2}{\omega^2}\left(1 + \frac{3k^2 T_a}{m_a \omega^2}\right) - i\pi\frac{\omega_{pa}^2}{k}\int dv\, \frac{\partial F_a}{\partial v}\,\delta(\omega - kv).$$

For slow wave $\omega \ll kv_{th}^a$

$$\chi^a(k,\omega) = \frac{2\omega_{pa}^2}{k^2 v_{th}^{a2}} - i\pi\frac{\omega_{pa}^2}{k}\int dv\, \frac{\partial F_a}{\partial v}\,\delta(\omega - kv).$$

Nonlinear Susceptibility

$$\chi_a^{(2)}(0,\omega_1|0,\omega_2) = 0, \quad \chi_a^{(2)}(k_1,0|k_2,0) = -\frac{ie_a}{T_a}\frac{\omega_{pa}^2}{k_1\,k_2\,|k_1+k_2|}\frac{1}{v_{th}^{a2}}.$$

Fast-Wave Approximation

$$\omega_1 \gg k_1 v_{th}^a, \quad \omega_2 \gg k_2 v_{th}^a, \quad \omega_1+\omega_2 \gg |k_1+k_2| v_{th}^a,$$

$$\chi_a^{(2)}(k_1,\omega_1|k_2,\omega_2) = \frac{-i}{2}\frac{e_a}{m_a}\frac{\omega_{pa}^2}{\omega_1\,\omega_2\,(\omega_1+\omega_2)} \times\left(\frac{k_1}{\omega_1}+\frac{k_2}{\omega_2}+\frac{k_1+k_2}{\omega_1+\omega_2}\right).$$

Two Fast Waves and a Slow Wave ω_1 is arbitrary and

$$\omega_2 \gg k_2 v_{th}^a, \quad \omega_1 + \omega_2 \gg |k_1 + k_2| v_{th}^a,$$

$$\chi_a^{(2)}(k_1, \omega_1 | k_2, \omega_2) = \frac{i}{2} \frac{e_a}{m_a} \frac{\omega_{pa}^2}{\omega_2 (\omega_1 + \omega_2)} \times \left[k_2 \int dv \, \frac{\partial F_a / \partial v}{\omega_1 - k_1 v} - \left(\frac{k_2}{\omega_2} + \frac{k_1 + k_2}{\omega_1 + \omega_2} \right) \int dv \, \frac{F_a}{\omega_1 - k_1 v} \right],$$

where we have used the identity

$$\int dv \, \frac{F_a}{(\omega_1 - k_1 v)^2} = -\frac{1}{k_1} \int dv \, \frac{\partial F_a / \partial v}{\omega_1 - k_1 v}.$$

$$
\begin{aligned}
\chi_a^{(2)}(k_1,\omega_1|k_2,\omega_2) &= \frac{i}{2}\frac{e_a}{m_a}\frac{k_1}{\omega_2\,(\omega_1+\omega_2)}\,\chi_a(k_1,\omega_1) \\
&\approx \frac{i}{2}\frac{e_a}{m_a}\frac{\omega_{pa}^2}{\omega_2\,(\omega_1+\omega_2)} \\
&\quad \times\left(\frac{2}{k_1 v_{th}^{a2}} - i\pi\int dv\,\frac{\partial F_a}{\partial v}\,\delta(\omega_1-k_1 v)\right).
\end{aligned}
$$

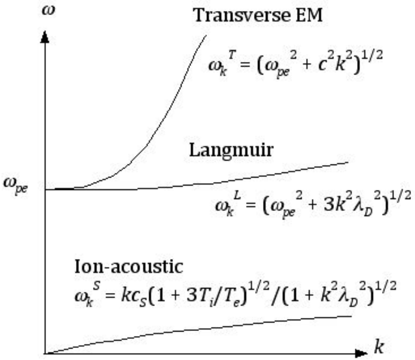
- $$\mathrm{Re}\,\epsilon(k,\omega) < E^2 >_{k\omega} = 0,$$
- $$\frac{\partial\, F_a}{\partial t} = \mathrm{Re}\, i\, \frac{e_a^2}{m_a^2} \frac{\partial}{\partial v} \int dk \int d\omega\, \frac{< E^2 >_{k\omega}}{\omega - kv + i0} \frac{\partial F_a}{\partial v},$$
- $$\frac{\partial}{\partial t} < E^2 >_{k\omega} = -\frac{2\,\mathrm{Im}\,\epsilon(k,\omega)}{\partial\,\mathrm{Re}\,\epsilon(k,\omega)/\partial\omega} < E^2 >_{k\omega}$$

$$- \frac{4}{\partial\,\mathrm{Re}\,\epsilon(k,\omega)/\partial\omega}\,\mathrm{Im} \int dk' \int d\omega'$$

$$\times \Bigg[\{\chi^{(2)}(k',\omega'|k-k',\omega-\omega')\}^2 \left(\frac{< E^2 >_{k-k',\omega-\omega'}}{\epsilon(k',\omega')} \right.$$

$$\left. + \frac{< E^2 >_{k'\omega'}}{\epsilon(k-k',\omega-\omega')} \right) < E^2 >_{k\omega}$$

$$- \frac{|\chi^{(2)}(k',\omega'|k-k',\omega-\omega')|^2}{\epsilon^*(k,\omega)} < E^2 >_{k'\omega'} < E^2 >_{k-k',\omega-\omega'} \Bigg].$$



- $\mathrm{Re}\,\epsilon(k,\omega) < E^2 >_{k\omega} = 0, \qquad \Rightarrow \qquad \omega = \omega_k^\alpha \quad (\alpha = L, S).$
- $$< E^2 >_{k\omega} = \sum_\alpha \left[I_k^{+\alpha}\,\delta(\omega - \omega_k^\alpha) + I_k^{-\alpha}\,\delta(\omega + \omega_k^\alpha) \right]$$
  $$= \sum_{\sigma=\pm 1} \sum_\alpha I_k^{\sigma\alpha}\,\delta(\omega - \sigma\omega_k^\alpha).$$

Langmuir ($\alpha = L$) and ion-sound wave ($\alpha = S$)

$$\omega_k^L = \omega_{pe}\left(1 + \frac{3}{2}\,k^2\lambda_{De}^2\right) = \omega_{pe}\left(1 + \frac{3}{4}\,\frac{k^2 v_{Te}^2}{\omega_{pe}^2}\right), \qquad \omega_{-k}^L = -\omega_k^L,$$

$$\omega_k^S = \frac{k c_S\,(1 + 3T_i/T_e)^{1/2}}{(1 + k^2\lambda_{De}^2)^{1/2}}, \qquad \omega_{-k}^S = -\omega_k^S.$$

Particle kinetic equation,

$$\frac{\partial\, F_a}{\partial t} = \frac{\partial}{\partial v}\left(D\,\frac{\partial F_a}{\partial v}\right),$$

$$D = \frac{\pi e_a^2}{m_a^2}\sum_{\sigma=\pm 1}\sum_{\alpha}\int dk\; I_k^{\sigma\alpha}\delta(\sigma\omega_k^\alpha - kv).$$

$$< E^2 >_{k\omega} = \sum_{\sigma=\pm 1} \sum_{\alpha=L,S} I_k^{\sigma\alpha} \, \delta(\omega - \sigma\omega_k^\alpha),$$

$$< E^2 >_{k'\omega'} = \sum_{\sigma'=\pm 1} \sum_{\beta=L,S} I_{k'}^{\sigma'\beta} \, \delta(\omega' - \sigma'\omega_{k'}^\beta),$$

$$< E^2 >_{k-k',\omega-\omega'} = \sum_{\sigma''=\pm 1} \sum_{\gamma=L,S} I_{k-k'}^{\sigma''\gamma} \, \delta(\omega - \omega' - \sigma''\omega_{k-k'}^\gamma).$$

$$\frac{1}{\epsilon(k,\omega)} = \mathcal{P}\,\frac{1}{\epsilon(k,\omega)} - \sum_{\alpha=L,S} \sum_{\sigma=\pm 1} \frac{i\pi\,\delta(\omega - \sigma\omega_k^\alpha)}{\partial \mathrm{Re}\,\epsilon(k,\sigma\omega_k^\alpha)/\partial\sigma\omega_k^\alpha},$$

$$\frac{1}{\epsilon^*(k,\omega)} = \mathcal{P}\,\frac{1}{\epsilon^*(k,\omega)} + \sum_{\alpha=L,S} \sum_{\sigma=\pm 1} \frac{i\pi\,\delta(\omega - \sigma\omega_k^\alpha)}{\partial \mathrm{Re}\,\epsilon(k,\sigma\omega_k^\alpha)/\partial\sigma\omega_k^\alpha}.$$

$$
\begin{aligned}
&\frac{\partial I_k^{\sigma\alpha}}{\partial t} = -\frac{2\,\mathrm{Im}\,\epsilon(k,\sigma\omega_k^\alpha)}{\partial\,\mathrm{Re}\,\epsilon(k,\sigma\omega_k^\alpha)/\partial\sigma\omega_k^\alpha}\, I_k^{\sigma\alpha} \\
&-\frac{4}{\partial\,\mathrm{Re}\,\epsilon(k,\sigma\omega_k^\alpha)/\partial\sigma\omega_k^\alpha}\,\mathrm{Im}\int dk' \\
&\times\left[\sum_{\sigma''=\pm1}\sum_\gamma \mathcal{P}\,\frac{\{\chi^{(2)}(k',\sigma\omega_k^\alpha-\sigma''\omega_{k-k'}^\gamma|k-k',\sigma''\omega_{k-k'}^\gamma)\}^2}{\epsilon(k',\sigma\omega_k^\alpha-\sigma''\omega_{k-k'}^\gamma)}\; I_{k-k'}^{\sigma''\gamma}\, I_k^{\sigma\alpha}\right. \\
&\left.+\sum_{\sigma'=\pm1}\sum_\beta \mathcal{P}\,\frac{\{\chi^{(2)}(k',\sigma'\omega_{k'}^\beta|k-k',\sigma\omega_k^\alpha-\sigma'\omega_{k'}^\beta)\}^2}{\epsilon(k-k',\sigma\omega_k^\alpha-\sigma'\omega_{k'}^\beta)}\; I_{k'}^{\sigma'\beta}\, I_k^{\sigma\alpha}\right]
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{4\pi}{\partial \operatorname{Re} \epsilon(k, \sigma\omega_k^\alpha)/\partial\sigma\omega_k^\alpha} \sum_{\sigma', \sigma'' = \pm 1} \sum_{\beta, \gamma} \\
&\times \operatorname{Im} \int dk' \, \Big[\{\chi^{(2)}(k', \sigma'\omega_{k'}^\beta | k - k', \sigma''\omega_{k-k'}^\gamma)\}^2 \\
&\times \left(\frac{I_{k-k'}^{\sigma''\gamma} \, I_k^{\sigma\alpha}}{\partial \operatorname{Re} \epsilon(k', \sigma'\omega_{k'}^\beta)/\partial\sigma'\omega_{k'}^\beta} + \frac{I_{k'}^{\sigma'\beta} \, I_k^{\sigma\alpha}}{\partial \operatorname{Re} \epsilon(k - k', \sigma''\omega_{k-k'}^\gamma)/\partial\sigma''\omega_{k-k'}^\gamma} \right) \\
&+ \frac{|\chi^{(2)}(k', \sigma'\omega_{k'}^\beta | k - k', \sigma''\omega_{k-k'}^\gamma)|^2}{\partial \operatorname{Re} \epsilon(k, \sigma\omega_k^\alpha)/\partial\sigma\omega_k^\alpha} \, I_{k'}^{\sigma'\beta} I_{k-k'}^{\sigma''\gamma} \Big] \\
&\times \delta(\sigma\omega_k^\alpha - \sigma'\omega_{k'}^\beta - \sigma''\omega_{k-k'}^\gamma).
\end{aligned}
$$

$$\frac{\partial I^{\sigma L}_{\mathbf{k}}}{\partial t} = -\frac{2\,\mathsf{Im}\,\epsilon(\mathbf{k},\sigma\omega^L_{\mathbf{k}})}{\epsilon'(\mathbf{k},\sigma\omega^L_{\mathbf{k}})}\,I^{\sigma L}_{\mathbf{k}}$$

$$-\frac{4}{\epsilon'(\mathbf{k},\sigma\omega^L_{\mathbf{k}})}\,\mathrm{Im}\sum_{\sigma'=\pm1}\int d\mathbf{k}'\,\left[A^{\sigma,\sigma'}_{L,L}(\mathbf{k},\mathbf{k}')\,I^{\sigma' L}_{\mathbf{k}'}+A^{\sigma,\sigma'}_{L,S}(\mathbf{k},\mathbf{k}')\,I^{\sigma' S}_{\mathbf{k}'}\right]I^{\sigma L}_{\mathbf{k}}$$

$$+\frac{8\pi}{\epsilon'(\mathbf{k},\sigma\omega^L_{\mathbf{k}})}\sum_{\sigma',\sigma''=\pm1}\int d\mathbf{k}'\Bigg[|\chi^{(2)}(\mathbf{k}',\sigma'\omega^L_{\mathbf{k}'}|\mathbf{k}-\mathbf{k}',\sigma\omega^L_{\mathbf{k}}-\sigma'\omega^L_{\mathbf{k}'})|^2$$

$$\times\left(\frac{I^{\sigma' L}_{\mathbf{k}'}\,I^{\sigma'' S}_{\mathbf{k}-\mathbf{k}'}}{\epsilon'(\mathbf{k},\sigma\omega^L_{\mathbf{k}})}-\frac{I^{\sigma'' S}_{\mathbf{k}-\mathbf{k}'}\,I^{\sigma L}_{\mathbf{k}}}{\epsilon'(\mathbf{k}',\sigma'\omega^L_{\mathbf{k}'})}-\frac{I^{\sigma' L}_{\mathbf{k}'}\,I^{\sigma L}_{\mathbf{k}}}{\epsilon'(\mathbf{k}-\mathbf{k}',\sigma''\omega^S_{\mathbf{k}-\mathbf{k}'})}\right)$$

$$\times\,\delta(\sigma\omega^L_{\mathbf{k}}-\sigma'\omega^L_{\mathbf{k}'}-\sigma''\omega^S_{\mathbf{k}-\mathbf{k}'})\Bigg],$$

$$
\begin{aligned}
&\frac{\partial I_{\mathbf{k}}^{\sigma S}}{\partial t} = -\frac{2\,\mathsf{Im}\,\epsilon(\mathbf{k},\sigma\omega_{\mathbf{k}}^{S})}{\epsilon'(\mathbf{k},\sigma\omega_{\mathbf{k}}^{S})}\, I_{\mathbf{k}}^{\sigma S} \\
&-\frac{4}{\epsilon'(\mathbf{k},\sigma\omega_{\mathbf{k}}^{S})}\,\mathrm{Im}\sum_{\sigma'=\pm1}\int d\mathbf{k}'\,\left[A_{S,L}^{\sigma,\sigma'}(\mathbf{k},\mathbf{k}')\,I_{\mathbf{k}'}^{\sigma' L} + A_{S,S}^{\sigma,\sigma'}(\mathbf{k},\mathbf{k}')\,I_{\mathbf{k}'}^{\sigma' S}\right] I_{\mathbf{k}}^{\sigma S} \\
&+\frac{4\pi}{\epsilon'(\mathbf{k},\sigma\omega_{\mathbf{k}}^{S})}\sum_{\sigma',\sigma''=\pm1}\int d\mathbf{k}'\,\Bigg[|\chi^{(2)}(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{L}|\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^{S}-\sigma'\omega_{\mathbf{k}'}^{L})|^{2} \\
&\times\left(\frac{I_{\mathbf{k}'}^{\sigma' L}\,I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L}}{\epsilon'(\mathbf{k},\sigma\omega_{\mathbf{k}}^{S})} - \frac{I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L}\,I_{\mathbf{k}}^{\sigma S}}{\epsilon'(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{L})} - \frac{I_{\mathbf{k}'}^{\sigma' L}\,I_{\mathbf{k}}^{\sigma S}}{\epsilon'(\mathbf{k}-\mathbf{k}',\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L})}\right) \\
&\times\delta(\sigma\omega_{\mathbf{k}}^{S}-\sigma'\omega_{\mathbf{k}'}^{L}-\sigma''\omega_{\mathbf{k}-\mathbf{k}'}^{L})\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
A_{\alpha,\beta}^{\sigma,\sigma'}(\mathbf{k},\mathbf{k}') \quad &= \quad 2\,\{\chi^{(2)}(\mathbf{k}',\sigma'\omega_{\mathbf{k}'}^{\beta}|\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma'\omega_{\mathbf{k}'}^{\beta})\}^{2} \\
&\quad\times\mathcal{P}\,\frac{1}{\epsilon(\mathbf{k}-\mathbf{k}',\sigma\omega_{\mathbf{k}}^{\alpha}-\sigma'\omega_{\mathbf{k}'}^{\beta})}.
\end{aligned}
$$

See the following references for details:

- P. H. Yoon, Generalized weak turbulence theory, POP, 7, 4858 (2000)
- L. F. Ziebell, R. Gaelzer, and P. H. Yoon, Nonlinear development of weak beam-plasma instability, POP, 8, 3982 (2001)
- P. H. Yoon, Effects of spontaneous fluctuations on the generalized weak turbulence theory, POP, 12, 042306 (2005)
- P. H. Yoon, T. Rhee, and C.-M. Ryu, Effects of spontaneous thermal fluctuations on nonlinear beam-plasma interaction, POP, 12, 062310 (2005)

Induced Emission: Linear Wave-Particle Resonance

$$\begin{aligned}
\left.\frac{\partial I_k^{\sigma L}}{\partial t}\right|_{\text{ind. emiss.}} &= \pi\,\sigma\omega_k^L\,\frac{\omega_{pe}^2}{k}\int dv\,\delta(\sigma\omega_k^L - kv)\,\frac{\partial F_e}{\partial v}\,I_k^{\sigma L}, \\
\left.\frac{\partial}{\partial t}\right|_{\text{ind. emiss.}}\frac{I_k^{\sigma S}}{\mu_k} &= \pi\,\mu_k\,\sigma\omega_k^L\,\frac{\omega_{pe}^2}{k}\int dv\,\delta(\sigma\omega_k^S - kv)\,\frac{\partial}{\partial v} \\
&\quad\times\left(F_e + \frac{m_e}{m_i}\,F_i\right)\,\frac{I_k^{\sigma S}}{\mu_k}, \\
\mu_{\mathbf{k}} &= k^3\lambda_{De}^3\sqrt{\frac{m_e}{m_i}}\sqrt{1+\frac{3T_i}{T_e}}.
\end{aligned}$$

Decay/Coalescence: Nonlinear Three-Wave Resonance

$$\left.\frac{\partial I_k^{\sigma L}}{\partial t}\right|_{\text{decay}} = \frac{\pi}{2}\frac{e^2}{T_e^2}\,\sigma\omega_k^L \sum_{\sigma',\sigma''=\pm1}\int dk'\,\frac{\mu_{k-k'}}{(k-k')^2}$$
$$\times\left[\sigma\omega_k^L\, I_{k'}^{\sigma' L}\,\frac{I_{k-k'}^{\sigma'' S}}{\mu_{k-k'}} - \left(\sigma'\omega_{k'}^L\,\frac{I_{k-k'}^{\sigma'' S}}{\mu_{k-k'}} + \sigma''\omega_{k-k'}^L\, I_{k'}^{\sigma' L}\right) I_k^{\sigma L}\right]$$
$$\times\delta(\sigma\omega_k^L - \sigma'\omega_{k'}^L - \sigma''\omega_{k-k'}^S),$$

$$\left.\frac{\partial}{\partial t}\right|_{\text{decay}} \frac{I_k^{\sigma S}}{\mu_k} = \frac{\pi}{4}\frac{e^2}{T_e^2}\,\sigma\omega_k^S \sum_{\sigma',\sigma''=\pm1}\int dk'\,\frac{\mu_k}{k^2}$$
$$\times\left[\sigma\omega_k^L\, I_{k'}^{\sigma' L}\, I_{k-k'}^{\sigma'' L} - \left(\sigma'\omega_{k'}^L\, I_{k-k'}^{\sigma'' L} + \sigma''\omega_{k-k'}^L\, I_{k'}^{\sigma' L}\right)\frac{I_k^{\sigma S}}{\mu_k}\right]$$
$$\times\delta(\sigma\omega_k^S - \sigma'\omega_{k'}^L - \sigma''\omega_{k-k'}^L).$$

Induced Scattering: Nonlinear Wave-Particle Resonance

$$\left.\frac{\partial I_k^{\sigma L}}{\partial t}\right|_{\text{scatt.}} = \sigma \omega_k^L \, \frac{\pi}{\omega_{pe}^2} \frac{e^2}{m_e \, m_i} \sum_{\sigma'=\pm 1} \int dk' \int dv$$

$$\times \, (k - k') \, \frac{\partial F_i}{\partial v} \, \delta[\sigma \omega_k^L - \sigma' \omega_{k'}^L - (k - k') \, v] \, I_{k'}^{\sigma' L} \, I_k^{\sigma L}.$$

Adding effects of spontaneous thermal fluctuation

This tutorial is too short to discuss in detail but in general, induced processes must be balanced by spontaneous processes.

$$
\begin{aligned}
\frac{\partial F_e}{\partial t} &= \frac{\partial}{\partial v_i}\left(A_i\, F_e + D_{ij}\,\frac{\partial F_e}{\partial v_j}\right), \\
A_i &= \frac{e^2}{4\pi m_e}\int d\mathbf{k}\,\frac{k_i}{k^2}\sum_{\sigma=\pm1}\sigma\omega_{\mathbf{k}}^L\,\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v}), \\
D_{ij} &= \frac{\pi e^2}{m_e^2}\int d\mathbf{k}\,\frac{k_i\,k_j}{k^2}\sum_{\sigma=\pm1}\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})\,I_{\mathbf{k}}^{\sigma L}.
\end{aligned}
$$

Langmuir Wave Kinetic Equation

$$\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} = \frac{\pi \omega_{pe}^2}{k^2} \int d\mathbf{v}\, \delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v}) \left(\underbrace{\frac{\hat{n}\, e^2}{\pi} F_e}_{\text{spont. emission}} + \underbrace{\sigma\omega_{\mathbf{k}}^L\, I_{\mathbf{k}}^{\sigma L}\, \mathbf{k}\cdot\frac{\partial F_e}{\partial \mathbf{v}}}_{\text{induced emission}} \right)$$

$$+ 2 \sum_{\sigma',\sigma''=\pm 1} \sigma\omega_{\mathbf{k}}^L \int d\mathbf{k}'\, \frac{\pi}{4}\, \frac{e^2}{T_e^2}\, \frac{\mu_{\mathbf{k}-\mathbf{k}'}\, (\mathbf{k}\cdot\mathbf{k}')^2}{k^2\, k'^2\, |\mathbf{k}-\mathbf{k}'|^2}$$

$$\times\, \delta(\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^S)$$

$$\times \left(\underbrace{\sigma\omega_{\mathbf{k}}^L\, I_{\mathbf{k}'}^{\sigma' L}\, I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S}}_{\text{spont. decay}} \underbrace{- \sigma'\omega_{\mathbf{k}'}^L\, I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S}\, I_{\mathbf{k}}^{\sigma L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L\, I_{\mathbf{k}'}^{\sigma' L}\, I_{\mathbf{k}}^{\sigma L}}_{\text{induced decay}} \right)$$

$$
-\frac{\pi e^2}{m_e^2\,\omega_{pe}^2}\sum_{\sigma'=\pm 1}\int d\mathbf{k}'\int d\mathbf{v}\,\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2\,k'^2}\,\delta[\sigma\omega_{\mathbf{k}}^L-\sigma'\omega_{\mathbf{k}'}^L-(\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]
$$

$$
\times\Bigg[\underbrace{\frac{\hat{n}\,e^2}{\pi\omega_{pe}^2}\,\sigma\omega_{\mathbf{k}}^L\left(\sigma'\omega_{\mathbf{k}'}^L\,I_{\mathbf{k}}^{\sigma L}-\sigma\omega_{\mathbf{k}}^L\,I_{\mathbf{k}'}^{\sigma' L}\right)(F_e+F_i)}_{\text{spontaneous scattering}}
$$

$$
+\underbrace{I_{\mathbf{k}'}^{\sigma' L}\,I_{\mathbf{k}}^{\sigma L}\,(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial}{\partial\mathbf{v}}\left(\left(\sigma\omega_{\mathbf{k}}^L-\sigma'\omega_{\mathbf{k}'}^L\right)F_e-\frac{m_e}{m_i}\left(\sigma\omega_{\mathbf{k}}^L\right)F_i\right)}_{\text{induced scattering}}\Bigg].
$$

Forward/backward-Ion-sound Wave Kinetic Equation

$$\frac{\partial I_{\mathbf{k}}^{\sigma S}}{\partial t} = \frac{\pi \mu_{\mathbf{k}} \omega_{pe}^2}{k^2} \int d\mathbf{v}\, \delta(\sigma\omega_{\mathbf{k}}^S - \mathbf{k}\cdot\mathbf{v})$$

$$\times \Bigg[\underbrace{\frac{\hat{n}\, e^2}{\pi} (F_e + F_i)}_{\text{spont. emission}} + \underbrace{\sigma\omega_{\mathbf{k}}^L\, I_{\mathbf{k}}^{\sigma S} \left(\mathbf{k}\cdot\frac{\partial}{\partial\mathbf{v}}\right)\left(F_e + \frac{m_e}{m_i} F_i\right)}_{\text{induced emission}} \Bigg]$$

$$\sum_{\sigma',\sigma''=\pm 1} \sigma\omega_{\mathbf{k}}^L \int d\mathbf{k}'\, \frac{\pi}{4} \frac{e^2}{T_e^2} \frac{\mu_{\mathbf{k}}\, [\mathbf{k}'\cdot(\mathbf{k}-\mathbf{k}')]^2}{k^2\, k'^2\, |\mathbf{k}-\mathbf{k}'|^2}\, \delta(\sigma\omega_{\mathbf{k}}^S - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L)$$

$$\times \Big(\underbrace{\sigma\omega_{\mathbf{k}}^L\, I_{\mathbf{k}'}^{\sigma' L}\, I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L}}_{\text{spont. decay}} \underbrace{-\sigma'\omega_{\mathbf{k}'}^L\, I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' L}\, I_{\mathbf{k}}^{\sigma S} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L\, I_{\mathbf{k}'}^{\sigma' L}\, I_{\mathbf{k}}^{\sigma S}}_{\text{induced decay}} \Big).$$

Beam-Plasma Instability and Langmuir Turbulence

Quasi-stationary ions

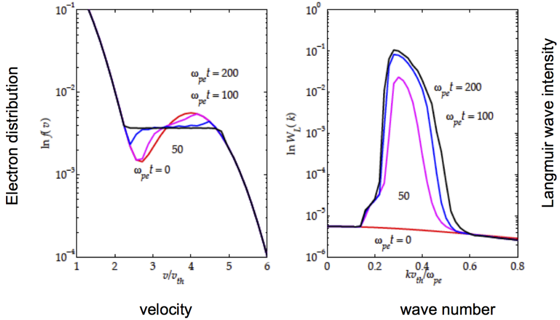
$$F_i = \frac{e^{-v^2/v_{Ti}^2}}{\pi^{1/2}\, v_{Ti}}.$$

Initial beam plus background electron distribution

$$F_e(v,0) = \frac{(1-\delta)\, e^{-v^2/v_{Te}^2}}{\pi^{1/2}\, v_{Te}} + \frac{\delta\ e^{-(v-v_0)^2/v_{Te}^2}}{\pi^{1/2}\, v_{Te}}.$$

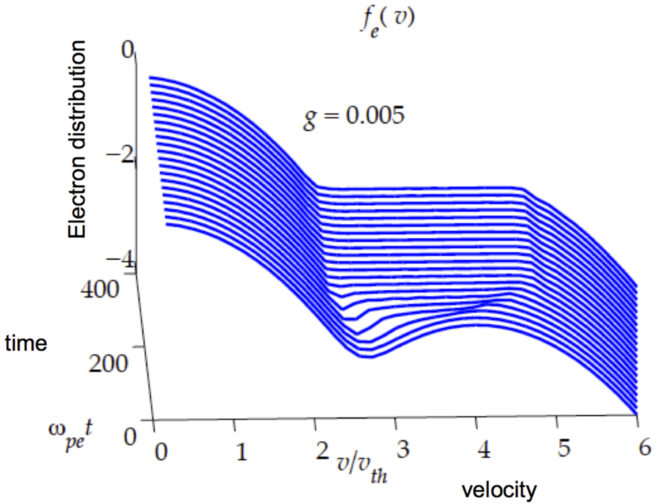
Beam-Plasma Instability and Langmuir Turbulence

Beam-Plasma Instability: Quasi-Linear Saturation Regime



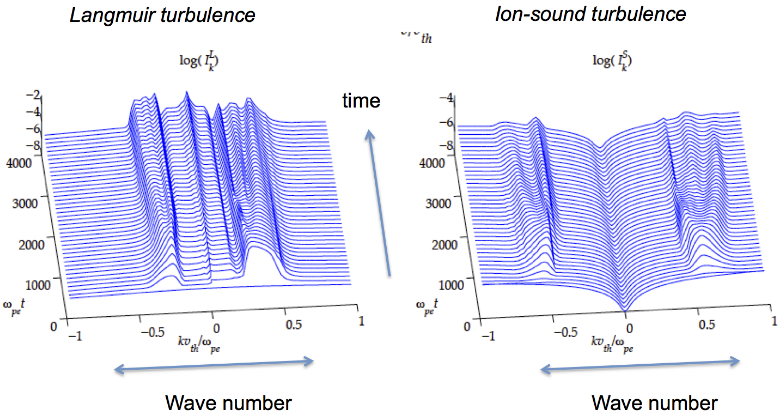
Beam-Plasma Instability and Langmuir Turbulence

Beam-Plasma Instability: Nonlinear Mode-Coupling Regime

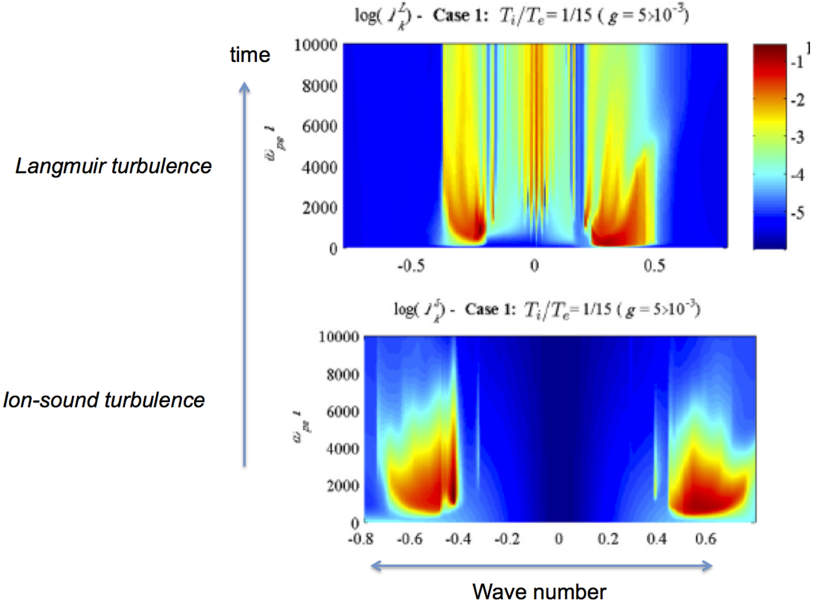


Beam-Plasma Instability and Langmuir Turbulence

Beam-Plasma Instability: Nonlinear Mode-Coupling Regime

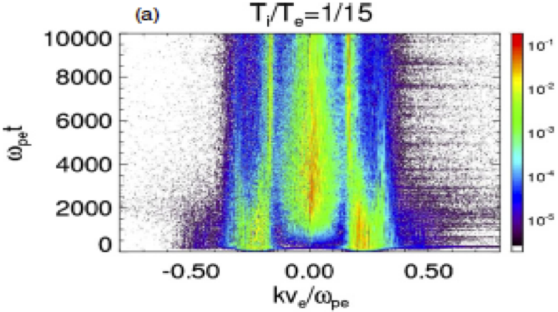


Beam-Plasma Instability and Langmuir Turbulence

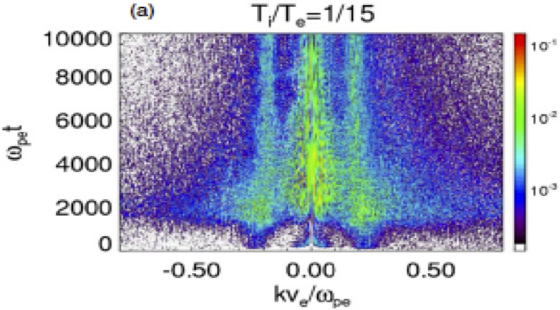


Beam-Plasma Instability and Langmuir Turbulence

Langmuir turbulence

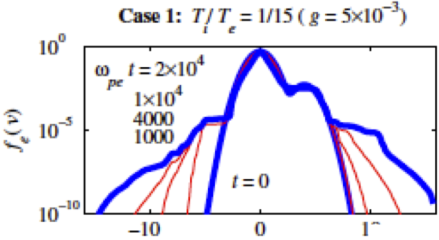


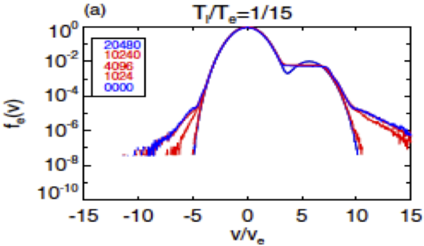
Ion-sound turbulence



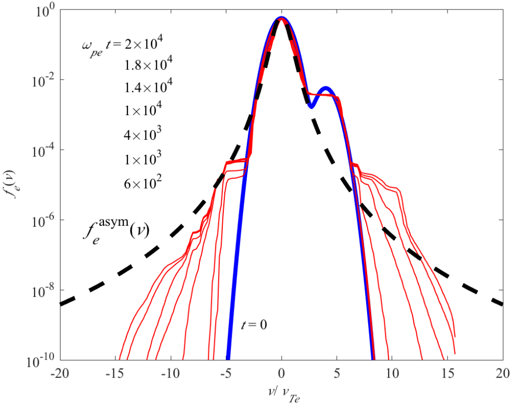
Beam-Plasma Instability and Langmuir Turbulence

Formation of energetic tail in the near the end of nonlinear regime (kappa VDF?)





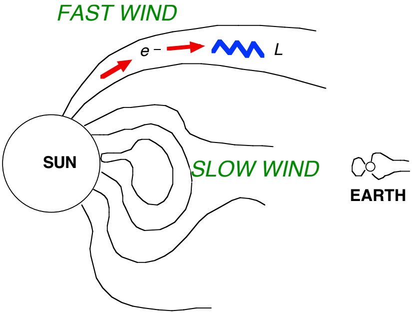
Beam-Plasma Instability and Langmuir Turbulence

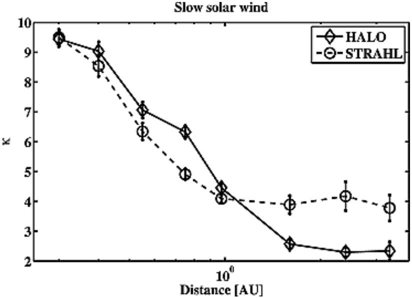


$$f_{kappa} \propto \frac{1}{(1 + v^2/\kappa v_T^2)^{\kappa+1}}, \qquad \kappa = 2.25$$

Quiet Time Solar Wind Electrons

$$f_{kappa} \propto \frac{1}{(1 + v^2/\kappa v_T^2)^{\kappa+1}},$$





As solar wind expands, f_e approaches f_κ with κ approaching $\kappa \sim 2.25$.

Turbulent Equilibrium and Kappa Distribution

Asymptotic solution ($\partial/\partial t = 0$)

$$\frac{\partial f_e}{\partial t} = \frac{\partial}{\partial v_i}\left(A_i f_e + D_{ij}\frac{\partial f_e}{\partial v_j}\right),$$

$$A_i = \frac{e^2}{4\pi m_e}\int d\mathbf{k}\,\frac{k_i}{k^2}\sum_{\sigma=\pm1}\sigma\omega_{\mathbf{k}}^L\,\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v}),$$

$$D_{ij} = \frac{\pi e^2}{m_e^2}\int d\mathbf{k}\,\frac{k_i k_j}{k^2}\sum_{\sigma=\pm1}\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})\,I_{\mathbf{k}}^{\sigma L}.$$

Linear wave-particle

$$\frac{\partial I_{\mathbf{k}}^{\sigma L}}{\partial t} = \frac{\pi\omega_{pe}^2}{k^2}\int d\mathbf{v}\,\delta(\sigma\omega_{\mathbf{k}}^L - \mathbf{k}\cdot\mathbf{v})\left(\frac{ne^2}{\pi}f_e + \sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}}^{\sigma L}\,\mathbf{k}\cdot\frac{\partial f_e}{\partial \mathbf{v}}\right)$$

Wave-wave

$$+2\sum_{\sigma',\sigma''=\pm1}\sigma\omega_{\mathbf{k}}^L\int d\mathbf{k}'\,V_{\mathbf{k},\mathbf{k}'}^L\,\delta(\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^S)$$

$$\times\left(\sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} - \sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}-\mathbf{k}'}^{\sigma'' S} I_{\mathbf{k}}^{\sigma L} - \sigma''\omega_{\mathbf{k}-\mathbf{k}'}^L I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L}\right)$$

Nonlinear wave-particle

$$-\frac{\pi e^2}{m_e^2\omega_{pe}^2}\sigma\omega_{\mathbf{k}}^L\sum_{\sigma'=\pm1}\int d\mathbf{k}'\int d\mathbf{v}\,\frac{(\mathbf{k}\cdot\mathbf{k}')^2}{k^2k'^2}\,\delta[\sigma\omega_{\mathbf{k}}^L - \sigma'\omega_{\mathbf{k}'}^L - (\mathbf{k}-\mathbf{k}')\cdot\mathbf{v}]$$

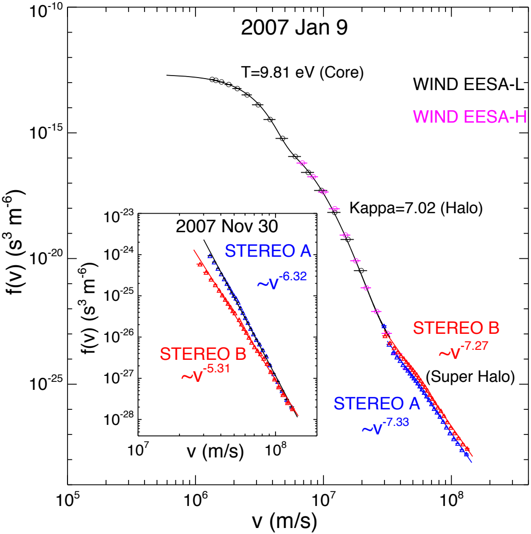
$$\times\left(\frac{ne^2}{\pi\omega_{pe}^2}(\sigma'\omega_{\mathbf{k}'}^L I_{\mathbf{k}}^{\sigma L} - \sigma\omega_{\mathbf{k}}^L I_{\mathbf{k}'}^{\sigma' L})f_i - \frac{m_e}{m_i}I_{\mathbf{k}'}^{\sigma' L} I_{\mathbf{k}}^{\sigma L}(\mathbf{k}-\mathbf{k}')\cdot\frac{\partial f_i}{\partial \mathbf{v}}\right)$$

Turbulent Equilibrium and Kappa Distribution

$$
\begin{aligned}
f_e(v) &\propto \left[1+\frac{v^2}{\left(\kappa-\frac{3}{2}\right)v_{Te}^2}\right]^{-\kappa-1}, \\
I(k) &= \frac{T_e}{4\pi^2}\frac{\kappa-\frac{3}{2}}{\kappa+1}\left(1+\frac{\omega_{pe}^2}{\left(\kappa-\frac{3}{2}\right)(kv_{Te})^2}\right).
\end{aligned}
$$

$$
\frac{T_e}{T_i}\frac{\kappa-3/2}{\kappa+1}, \qquad \text{and} \qquad \kappa=\frac{9}{4}=2.25.
$$

Quiet Time Solar Wind Electron Distribution



- Theory:

$$f_e(v) \approx \frac{1}{v^{2\kappa+2}} \sim v^{-6.5}.$$

- Observation:

$$f_e(v) \approx v^{-5.5} - v^{-7.5}$$

Conclusion

- QUiet-time solar wind electrons feature inverse power-law tail population $f(v) \propto v^{-\alpha}$, where α is close to 6.5.
- This is explained by kappa distribution with index $\kappa = 2.25$, which corresponds to an asymptotic state of Langmuir turbulence.
- Kinetic plasma turbulence theory successfully explains a long standing problem of the physical origin of nonthermal electron velocity distribution.