Lecture 2: Entropic c-theorem

Abstract : We discuss an entropic c-theorem that was derived by Casini and Huerta [1]. We make comparisons to Zamolodchikov's c-theorem [2].

1 Motivation

Crudely speaking, c-theorem is a statement that some kind of information is lost along the renormalization group flow. First such theorem was proved by Zamolodchikov [2] in $1+1$-dimension. An entropic version of this theorem was proved by Casini and Huerta [1]. The main motivation for us to understand the main argument in Ref. [1], so that we can apply it in higher dimensions.

But this motivation aside, it is still interesting to compare these two theorems. In both cases, there is some function that decreases along the renormalization group(RG) flow. However, behavior of the function along the RG flow, as well as the argument behind the theorem differs. Another interesting fact is that both functions become the central charge of the theory at the fixed point of the RG flow, despite these differences.

There is a catch in our discussion. Namely, entanglement entropy in quantum field theory is, strictly speaking, not a very well-defined quantity. However, we will take a point of view that any regulator-independent quantity will have a well-defined continuum meaning.

2 The ground rule

We consider a Lorentz-invariant quantum field theory. This means that entanglement entropy must obey this symmetry. If we can assign an entanglement entropy to an interval, the entanglement entropy must be invariant under not just translation but also on boosts.

$$S(A) = S(\Lambda A), \tag{1}$$

where

$$\Lambda A = \{\Lambda x | x \in A\}. \tag{2}$$

Here Λ can be any element in the Lorentz group.

But in relativistic theories, it makes more sense to ascribe entropy to not just an interval, but a (partial) Cauchy surface. Intuitively, this is because the initial condition on the surface determines the future and the past uniquely. A bit more formally, a partial Cauchy surface is a hypersurface which is intersected by any causal curve at most once.

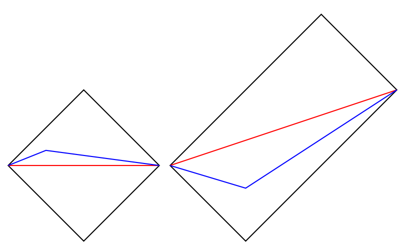

Figure 1: The blue interval is related to the red interval unitarily. Therefore, their entropies are the same. The black interval is the lightcone.

Now here comes a nontrivial assumption. We assume that, on every Cauchy surface, one can divide the degrees of freedom over different intervals, so that one can define a reduced density matrix over every interval. This cannot be true in relativistic quantum field theories, so the derivation of the c-theorem in this note is heuristic at best. However, the value of this theorem is that the core idea can be generalized to higher dimensions.

3 Derivation

Without loss of generality, consider two intervals of length x and $x' < x$; see Fig.2. There are a few things to note here. First, the interval $A \cup B \cup C$ is unitarily related to D. Therefore $S(A \cup B \cup C) = S(D)$. Second, by the strong subadditivity of entropy, we have $S(AB) + S(BC) - S(B) - S(ABC)$.

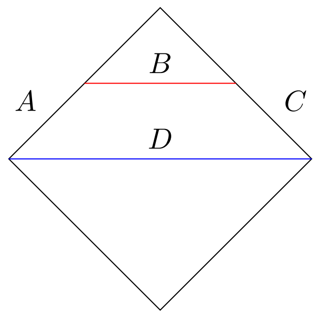

Figure 2: Choose A, B, C, and D such that the length of B is x' and the length of D is x.

Now note that the entropy of AB and BC can be related to another interval; see Fig.3. Now these new intervals can be boosted to an interval on an equal-time slice. So the entropy of both AB and BC must be determined by the length of this interval. This can be done in a simple calculation in Minkowski space. Consider the green interval in Fig.3 for example. Its length is $\sqrt{\left(x' + \frac{x-x'}{2}\right) - \left(\frac{x-x'}{2}\right)^2} = \sqrt{xx'}$.

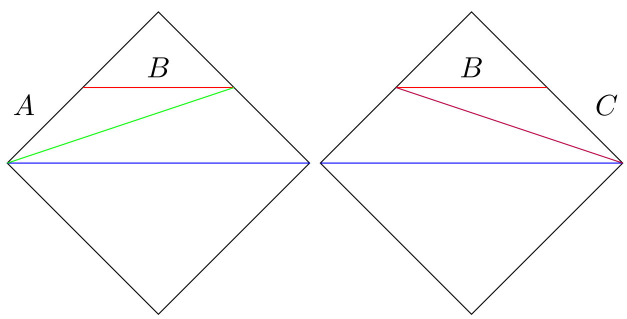

Figure 3: The entropy of AB is equal to the entropy of the green interval. The entropy of BC is equal to the entropy of the purple interval.

So we have the following inequality:

$$2S[\sqrt{xx'}] - S[x] - S[x'] \geq 0, \tag{3}$$

where $S[x]$ is an entropy of an interval of length x. Let $x' = \bar{x} - \epsilon$, $x = \bar{x} + \epsilon$. This inequality becomes

$$\begin{aligned} 2S[\bar{x} - \frac{\epsilon^2}{2\bar{x}}] - S[\bar{x} + \epsilon] - S[\bar{x} - \epsilon] &= -\epsilon^2 S[\bar{x}]'' - \frac{\epsilon^2}{\bar{x}} S[\bar{x}]' \\ &\geq 0. \end{aligned} \tag{4}$$

So we have

$$S[x]'' + \frac{1}{x} S[x]' \leq 0, \tag{5}$$

which implies that

$$\frac{d}{dx}\left(x \frac{dS[x]}{dx}\right) \leq 0. \tag{6}$$

When the inequality is satisfied with an equality, we have

$$x \frac{dS[x]}{dx} = C \tag{7}$$

for some constant C. So $S[x] = C \log x + c'$ for some constant c', recovering the famous form for the entanglement entropy at the fixed point of the RG flow.

The lesson is that there is a function $x\frac{dS[x]}{dx}$ that depends decreases monotonically under the increase of x, which is interpreted as probing the larger scale. If we are at a fixed point of this flow, then we recover the logarithmic dependence of the entanglement entropy on the size of the interval. Comparing this formula to the entanglement entropy of CFT, we conclude that $C = \frac{c}{3}$ at the fixed point, where c is the central charge of the CFT.

4 Some comments

It is important to note that the derivation of the entropic c-theorem is not rigorous in a mathematical sense. The reason is that in quantum field theory, entanglement entropy formally diverges. Of course, as usual one can introduce a regulator and compute regulator-independent quantities. In a sense that is what's going on here, because even though the entropy is not a well-defined object, its linear combination that leads to the final inequality is.

Another comment is that the function $x\frac{dS[x]}{dx}$ is different from Zamolodchikov's c-function. They coincide at the fixed point of the RG flow, but generally not along the flow.

5 More comments

A close inspection of the derivation reveals that the we did not have to choose B or D to be on an equal-time slice. For those, one will end up proving the same inequality. However, we do get something nontrivial out of the equality condition. That is, we get

$$S(AB) + S(BC) - S(B) - S(ABC) = 0 \tag{8}$$

whenever A and C are null-like. It is known that, if strong subadditivity holds with an equality, then we have an equality for the entanglement Hamiltonian:

$$H_{AB} + H_{BC} - H_B - H_{ABC} = 0, \tag{9}$$

where $H_X := -\log \rho_X$. What does this mean? This means that the entanglement Hamiltonian is invariant under the deformation in the null direction. Let

$$H_{AB} - H_B = \delta_A H_B, H_{ABC} - H_{BC} \quad = \delta_A H_{BC}. \tag{10}$$

Then we have $\delta_A \delta_C H_B = 0$, where A and C were null intervals.

References

[1] H. Casini and M. Huerta, "A c -theorem for entanglement entropy," *Journal of Physics A: Mathematical and Theoretical*, vol. 40, no. 25, p. 7031, 2007.

[2] A. B. Zomolodchikov, ""Irreversibility" of the flux of the renormalization group in a 2D field theory," *Soviet Journal of Experimental and Theoretical Physics Letters*, vol. 43, p. 730, June 1986.