Lecture 3: Monotonicity theorem in higher dimensions

Abstract : We discuss the entropic F- and a-theorem, which can be thought as that was derived by Casini and Huerta [1].

1 Motivation

Now we are getting the real meat. The entropy c-theorem is nice, but a similar result was already known long time ago [2]. In this note, we describe a proof of F-theorem due to Casini and Huerta [1]. This is a monotinicity theorem in (2+1)D.

2 Derivation

As in the $1+1$D case, the derivation of the monotonicity theorem can be broken down into two pieces. The first piece is concerns inequalities for entropies. The second piece concerns the geometry in Minkowski space.

2.1 Entropy Inequality

The strong subadditivity of entropy(SSA) implies that

$$S(A) + S(B) \geq S(A \cap B) + S(A \cup B). \tag{1}$$

Thus we have

$$\begin{aligned} S(A) + S(B) + S(C) &\geq S(A \cap B) + S(A \cup B) + S(C) \\ &\geq S(A \cup B \cup C) + S(A \cap B) + S((A \cup B) \cap C) \\ &\geq S(A \cup B \cup C) + S(A \cap B \cap C) + S((A \cap B) \cup ((A \cup B) \cap C)) \\ &= S(A \cup B \cup C) + S((A \cap B) \cup (B \cap C) \cup (A \cap C)) + S(A \cap B \cap C). \end{aligned} \tag{2}$$

We can prove a more general inequality:

$$\sum_i S(X_i) \geq S(\cup_i X_i) + S(\cup_{\{ij\}}(X_i \cap X_j)) + S(\cup_{\{ijk\}}(X_i \cap X_j \cap X_k)) + \ldots + S(\cap_i X_i). \tag{3}$$

This can be proved from induction. Suppose Eq.3 is correct for $\{X_i | i = 1, \ldots, n\}$. Then

$$\begin{aligned} \sum_{i=1}^{n+1} S(X_i) &\geq S(X_{n+1}) + S(\cup_i X_i) + S(\cup_{\{ij\}}(X_i \cap X_j)) + S(\cup_{\{ijk\}}(X_i \cap X_j \cap X_k)) + \ldots + S(\cap_i X_i) \\ &\geq S(\cup'_i X_i) + S(X_{n+1} \cap (\cup_i X_i)) + S(\cup_{\{ij\}}(X_i \cap X_j)) + S(\cup_{\{ijk\}}(X_i \cap X_j \cap X_k)) + \ldots + S(\cap_i X_i), \end{aligned} \tag{4}$$

where $\cup'_i$ is a union over $i = 1$ to $n+1$ whereas the union/interesection without hte primes are over $i = 1$ to n. Then one can SSA to the second and the third term, and then to the third and the fourth term, etc.

2.2 Idea

The main difference between the $1+1D$ and $2+1D$ case concerns the geometry. In $1+1D$ we only needed to consider intervals, but in $2+1$, we need to consider circles. The idea is this. We want to consider the continuum limit of Eq.3 and then choose X_i to be spheres in a Minkowski space. An important subtlety is whether the union/intersection of different sphere becomes a sphere. Obviously, those constructions will generally have singular points. But the hope is that these singular contributions become negligible in an appropriate limit, so that the union/intersection of different sphere can be really approximated by a sphere. It will be important to understand why this approximation is valid or not.

2.3 Geometry

Let us draw an analogy. In 1D, we considered boosted intervals and their intersections and unions. Here, we are considering boosted spheres and their intersections and unions. We want to choose the boosted spheres so that their intersection and union becomes spheres on an equal-time slice. Specifically, we want to choose X_is in Eq.3 to be the booster spheres, so that the right hand side of Eq.3 becomes spheres on an equal-time slice. In particular, we want the intersections of all the boosted spheres to be a sphere of radius r and their union to become a sphere of radius R; see Fig.1

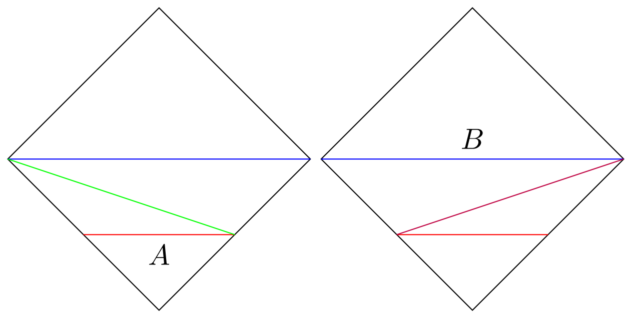

Figure 1: Configurations for the spheres, projected onto a 2D slice. The y axis is time. A has radius r, and B has radius R. The green/purple lines are spheres projected onto a 2D slice.

One of the boosted spheres, once projected onto the $x-t$ plane, looks like Fig.1. The sphere is located on a plane that passes through $(x,y,t)=(-r,y,r)$ and $(x,y,t)=(R,y,R)$. This plane can be parametrized in the following way:

$$t=\frac{R-r}{R+r}x+\frac{2rR}{R+r}. \tag{5}$$

The center of the circle is located at $(x,y,t)=\left(\frac{R-r}{2},0,\frac{R+r}{2}\right)$. Because the radius of the circle is $\sqrt{Rr}$, the circle is parametrized as follows:

$$\begin{aligned} t&=\frac{R-r}{R+r}x+\frac{2rR}{R+r} \\ Rr&=\left(x-\frac{R-r}{2}\right)^2+y^2-\left(t-\frac{R+r}{2}\right)^2. \end{aligned} \tag{6}$$

Now let us analyze the (approximate) spheres that result from the intersection and union of spheres. Suppose we have N boosted spheres that are rotated from each other around the $x=y=0$ axis by an angle of $\theta=\frac{2\pi}{N}$. We want to understand what the union of all intersections involving k spheres is. In the $N\to\infty$ limit, the unions become a sphere; see Fig.3. This is because the boundary of the union converges to the boundary of a sphere on a single time slice.

Each of these (approximate spheres) can be parametrized by an angle θ. This would be the smallest angular difference between two boosted spheres that appear in the intersection. When there is no intersection, $\theta=0$. When everything intersects with each other, $\theta=\pi$. The intersection between two such spheres must pass through the middle. So the intersection point must lie on the following plane:

$$y=x\tan\frac{\theta}{2}. \tag{7}$$

Now let us see where the intersection of the two spheres that are roated from each other by θ. It will be convenient to consider the point that lies on the lightcone. Setting the distance from the $x=y=0$ line to the point to be l, we have $(x,y,t)=(l\cos\frac{\theta}{2},l\sin\frac{\theta}{2},l)$. Plugging in this expression to the equation for the sphere, we have

$$Rr=\left(\frac{R-r}{2}\right)^2-(R-r)l\cos\frac{\theta}{2}-\left(\frac{R+r}{2}\right)^2-(R-r)l. \tag{8}$$

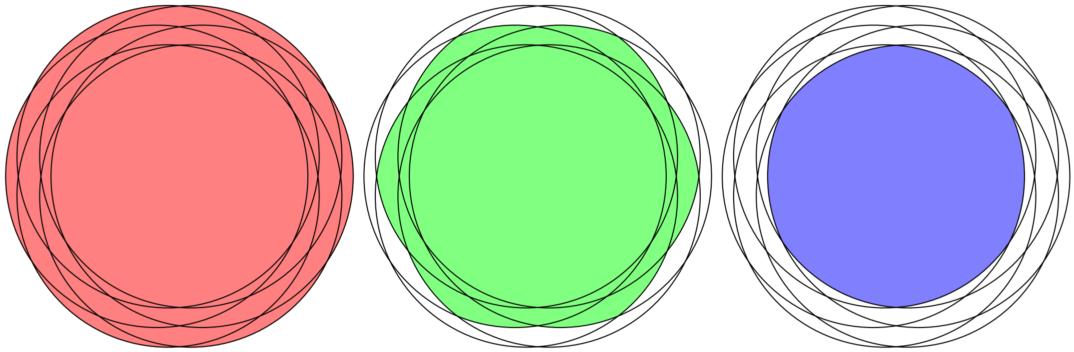

Figure 2: Unions of intersections of spheres, projecte onto the $x-y$ plane. The first diagram is the union of all the spheres. The second diagram is the union of all intersections involving three spheres. The third diagram is the intersection of all the spheres.

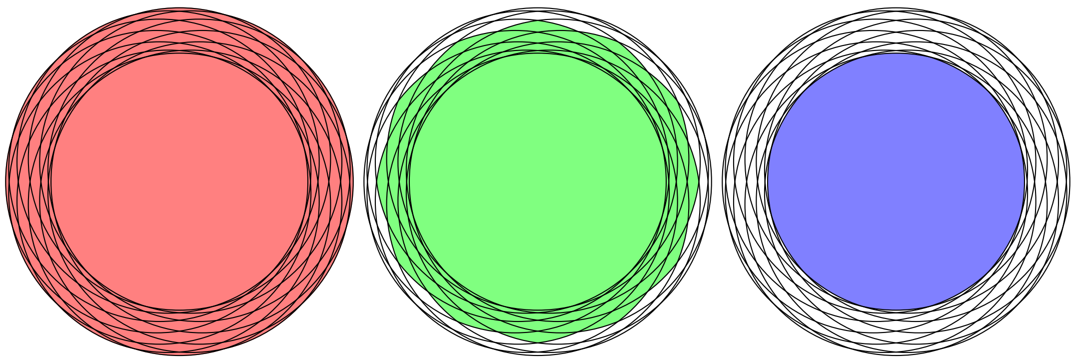

Figure 3: Unions of intersections of spheres, projecte onto the $x-y$ plane. The first diagram is the union of all the spheres. The second diagram is the union of all intersections involving three spheres. The third diagram is the intersection of all the spheres.

We can solve this equation to see that the distance from $x=y=0$ line to the intersection point is

$$\frac{2rR}{R+r-(R-r)\cos(\frac{\theta}{2})}. \tag{9}$$

So we have

$$S(\sqrt{Rr}) \geq \frac{1}{\pi}\int_0^\pi S\big(\frac{2rR}{R+r-(R-r)\cos z}\big)dz. \tag{10}$$

Note the following integral:

$$\begin{aligned}\int_0^\pi \frac{1}{a+b\cos\theta}d\theta &= \int_0^\pi \frac{1}{a+b(2\cos^2\frac{\theta}{2}-1)}d\theta \\ &= \int_0^\pi \frac{\sec^2\frac{\theta}{2}}{2b+(a-b)\sec^2\frac{\theta}{2}}d\theta \\ &= \int_0^\pi \frac{\sec^2\frac{\theta}{2}}{a+b+(a-b)\tan^2\frac{\theta}{2}}d\theta \\ &= \int_0^\infty \frac{1}{a+b+(a-b)t^2}dt\end{aligned} \tag{11}$$

By taking $R=r+\epsilon$, one can show that the entropy of S is concave in r. Define a function

$$c_0(r) = S(r) - rS'(r). \tag{12}$$

The concavity of S implies that $c_0'(r) \geq 0$. So we have a c-function that monotonically decreases under the RG flow, analogous to the 2D c-theorem.

3 Comment

In our analysis, the approximating union/intersection of spheres as a sphere in the continuum limit is a nontrivial approximation. There will be kinks in this manifold, so one should ask how these terms behave. It has been shown in prior work that entanglement entropy due to this corner term scales logarithmically with the radius and quadratically with the deviation from π. The fact that this term behaves quadratically, as opposed to linearly, follows from the fact that entanglement entropy of a subsystem over a pure state is equal to the entanglement entropy of its complement. So the hope is that the total contribution, which scales like $O(1/N^2) \times N$ vanishes in the infinite N limit. Of course this is just a heuristic argument, which makes this theorem not at the level of mathematical rigor.

4 4D

A similar analysis can be carried out in higher dimensions, but there is a subtlety. Namely, approximating the union of intersections of spheres by a sphere may not be valid. In 3D, we could argue that there is no discontiuity in this approximation, because the corner terms scales quadratically with the cusp angle and their overall contributions vanishes in the continuum limit. In higher dimensions, the same approximation is not expected to hold. What can we do? Note that the wiggly sphere actually lies on the null cone. The point is that this wiggly line, in the continuum limit, becomes a sphere lying on the null cone on a single time slice. However, the wiggly line itself is not on a single-time slice.

For general QFT, what happens to this wiggly line is unclear. However, for CFT we can say much more. For CFT, on the null cone SSA is satisfied with an equality. Therefore,

$$\Delta S(r) = S_1(r) - S_0(r) \tag{13}$$

still satisfies SSA, where the second term is the entanglement entropy of a sphere of radius r on a CFT and $S_1(r)$ is the entanglement entropy of a sphere of radius r on a perturbed theory.

References

[1] H. Casini and M. Huerta, "Renormalization group running of the entanglement entropy of a circle," *Phys. Rev. D*, vol. 85, p. 125016, Jun 2012.

[2] A. B. Zomolodchikov, "“Irreversibility” of the flux of the renormalization group in a 2D field theory," *Soviet Journal of Experimental and Theoretical Physics Letters*, vol. 43, p. 730, June 1986.