Mean field method and beyond in dense matter

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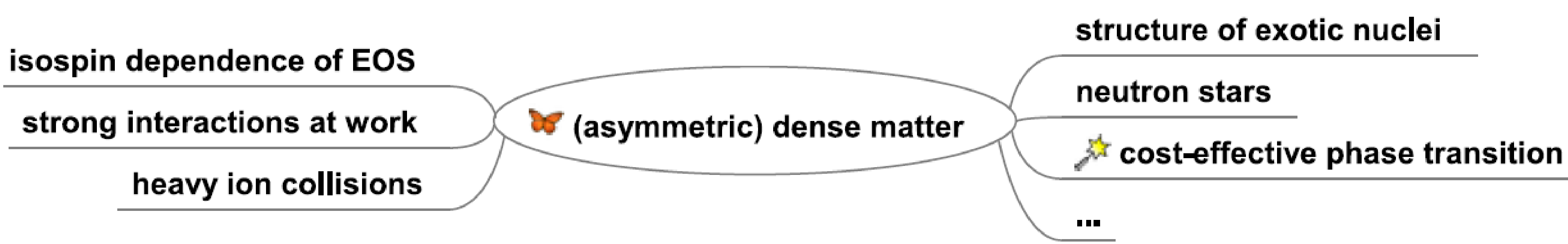


Models for low energy QCD

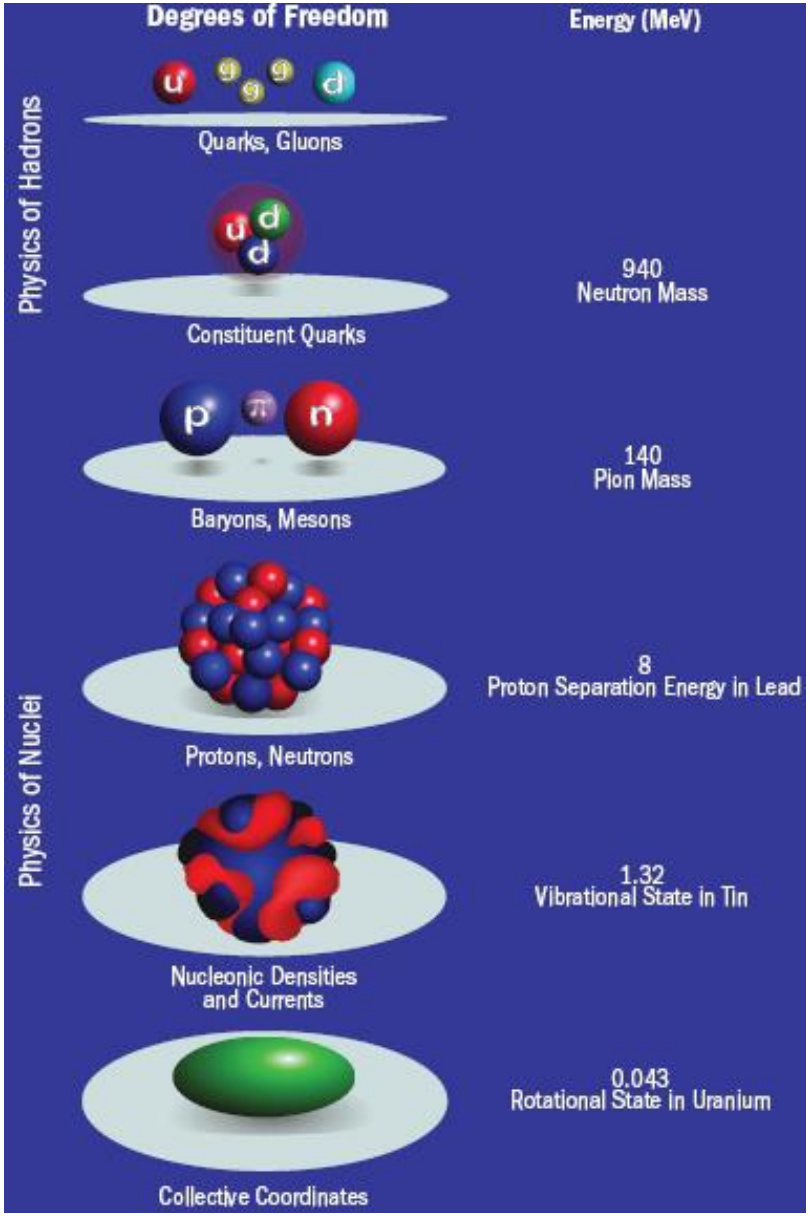
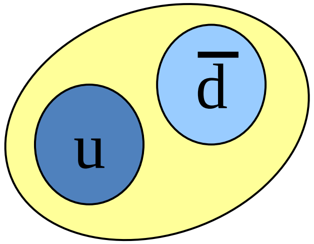
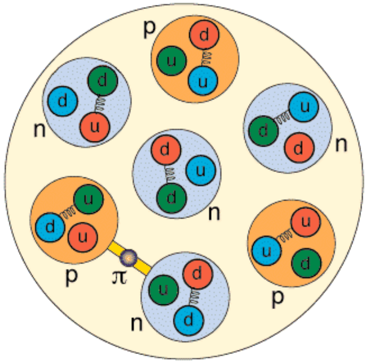
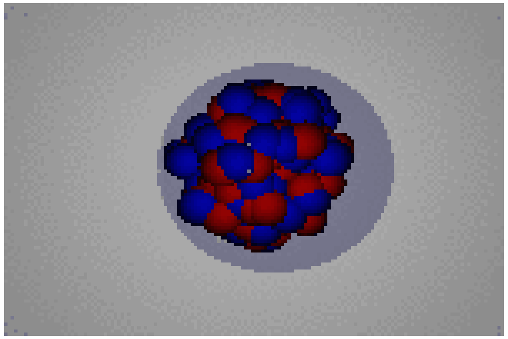


Figure: taken from G.F. Bertsch, D.J. Dean, and W. Nazarewicz, SciDAC Review 6, 42 (2007)



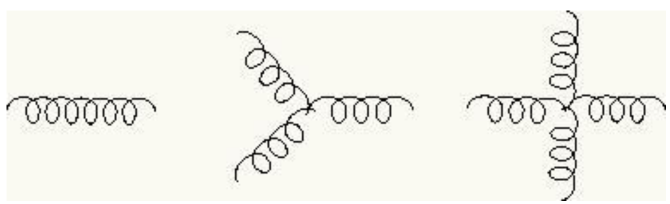




Kernphysik Homepage (H.J.Wollersheim)

After all, QCD (quantum chromodynamics) is here to describe busy things: quarks, baryons, nuclei.

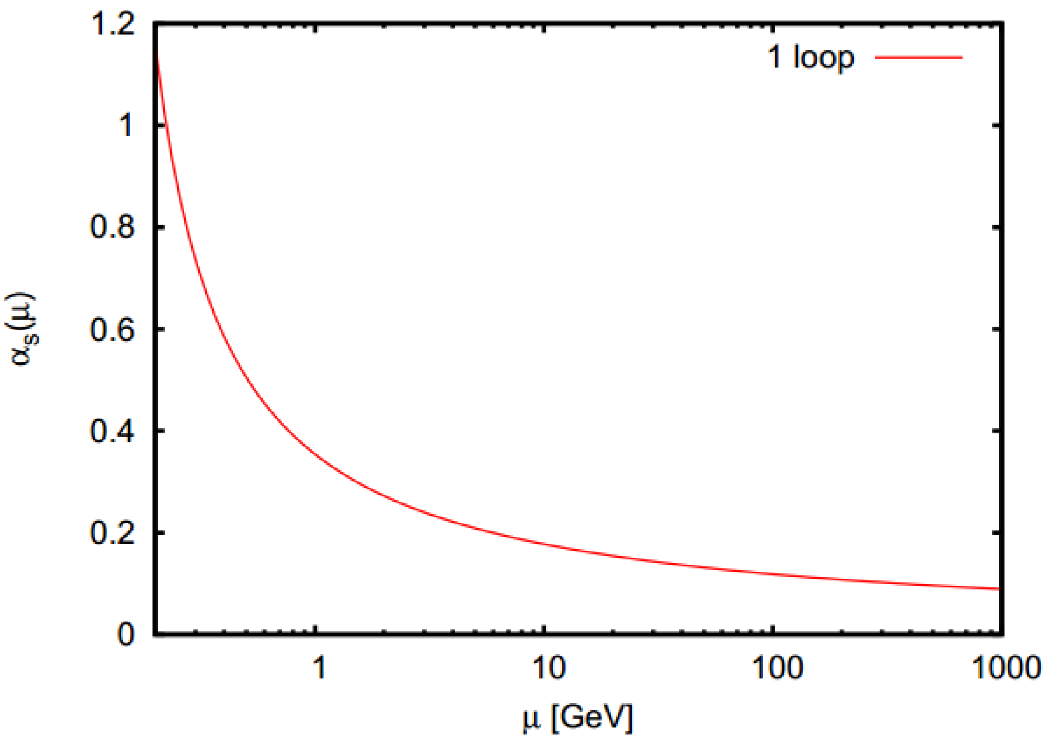
$$\mathcal{L}_{\mathrm{QCD}} = \bar{q}(i\gamma^\mu \partial_\mu - m)q - g(\bar{q}\gamma^\mu T_a q)G^a_\mu - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a$$



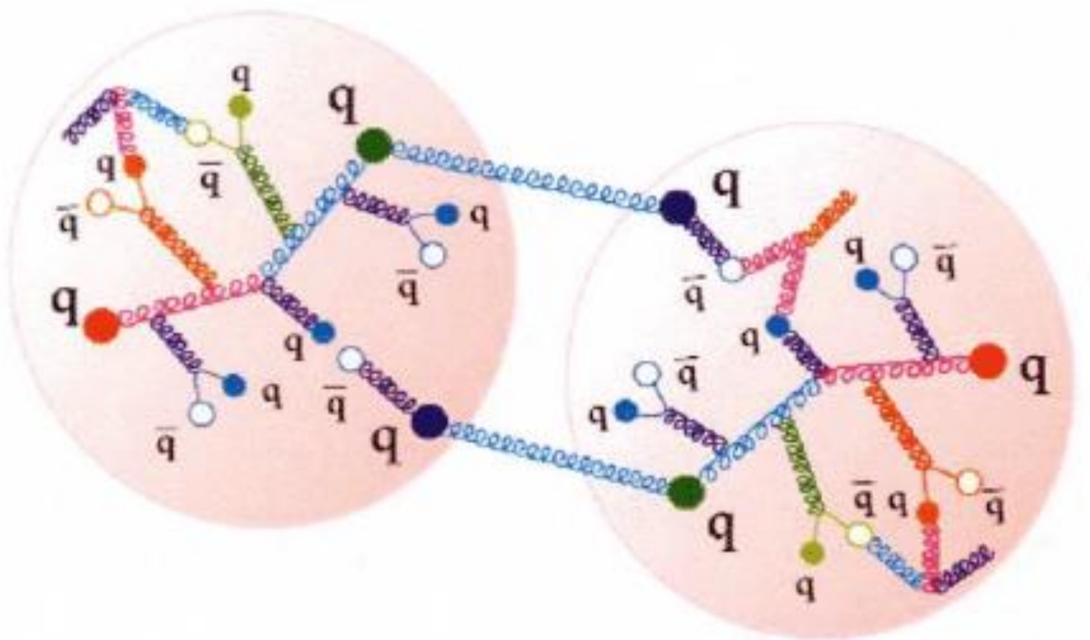
Nuclear physics is governed by strong and electroweak interaction!

As it is, life cannot be that simple!

Asymptotic freedom makes QCD messy at low energies



The force between nucleons in QCD

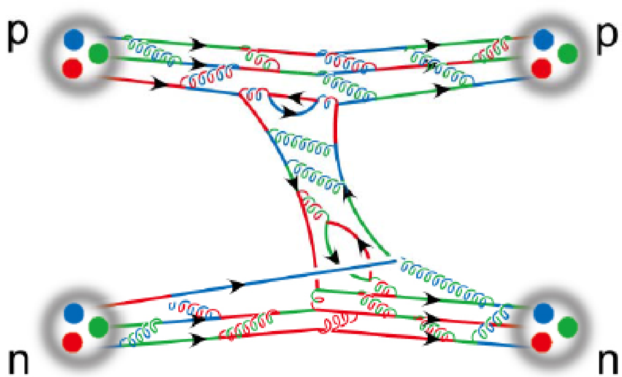


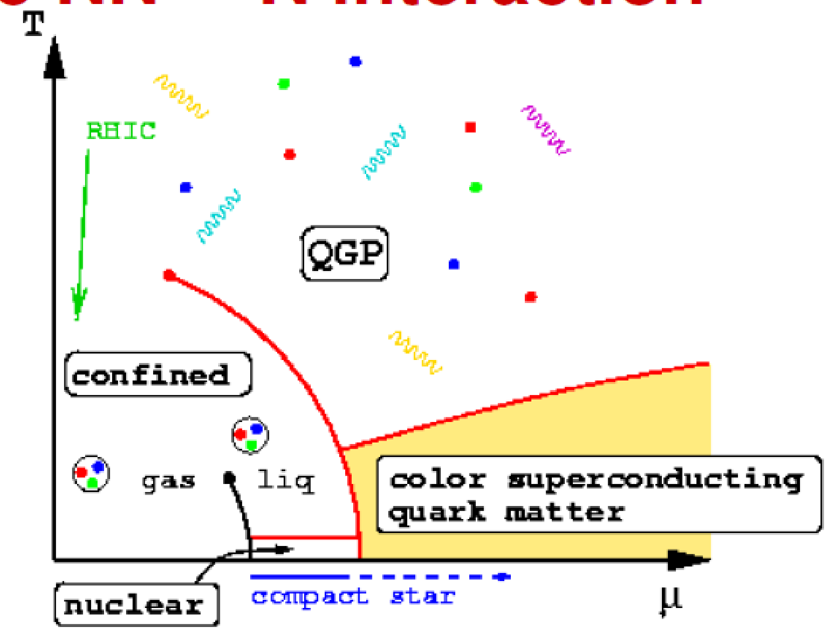
The force between nucleons in QCD. The exchange of two colored gluons causes two quarks in each nucleon to change their colors (here blue changes to green and vice versa). This process produces a force without violating the overall color neutrality of the nucleons. The strength of the force depends on the separation of the different quark colors within each nucleon.

["Nuclear Physics, The Core of Matter, the Fuel of Stars", National Research Council, 1999]

QCD-rooted effective theories (models) should come in for baryons and nuclei.

Quark (QCD) vs. hadronic NN· · · N interaction

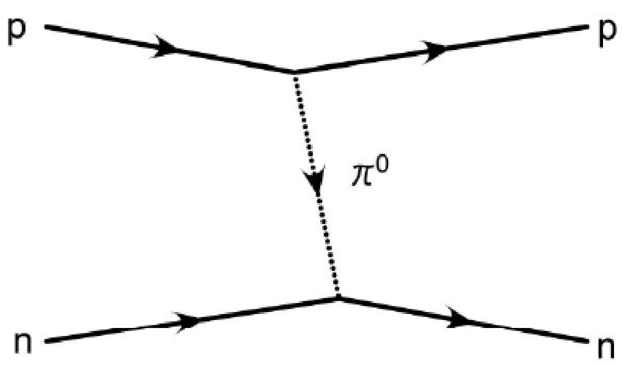


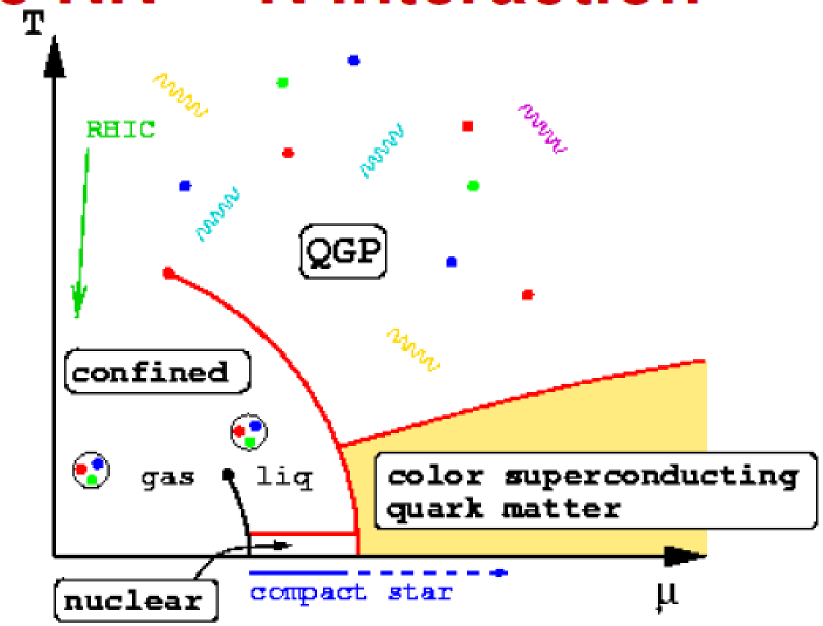


- Old goal: replace hadronic descriptions at ordinary nuclear densities with quark description (since QCD is *the* theory)

Dick Furnstahl

Quark (QCD) vs. hadronic NN· · · N interaction

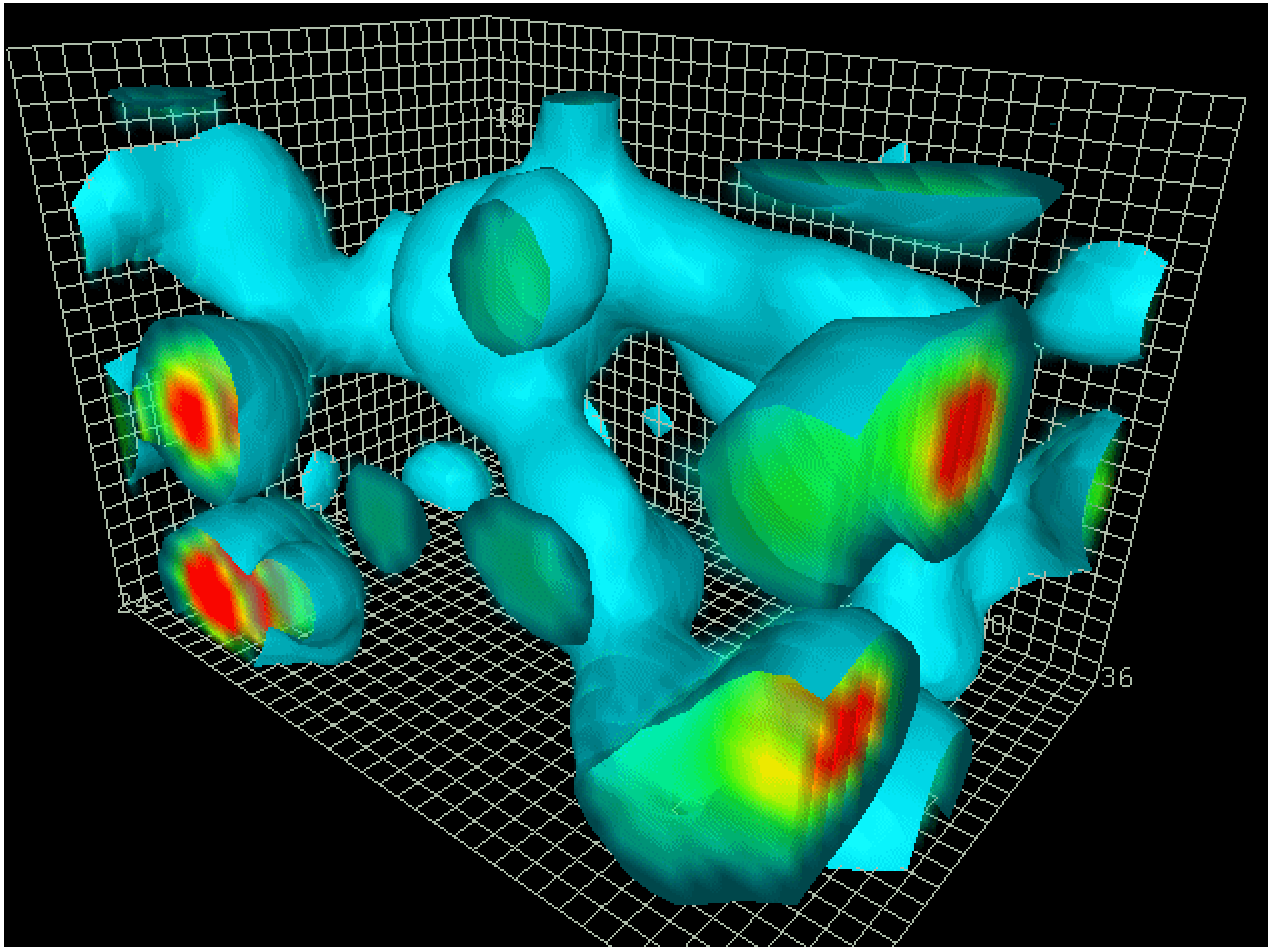




- Old goal: replace hadronic descriptions at ordinary nuclear densities with quark description (since QCD is *the* theory)
- New goal: use effective hadronic dof's *systematically*
  - Seek model independence and theory error estimates
  - Future: Use lattice QCD to match via "low-energy constants"
- Need quark dof's at higher densities (resolutions) where phase transitions happen or at high momentum transfers

Dick Furnstahl

YK: RAON and the QCD vacuum?



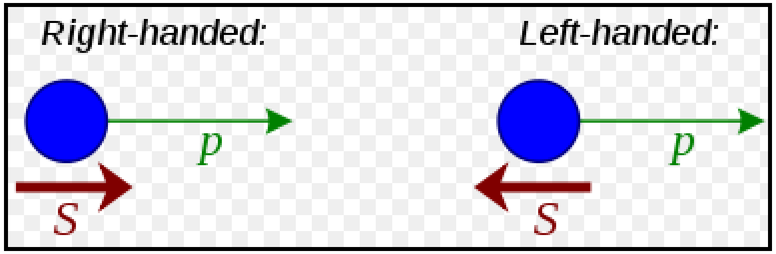
Quantum fluctuations of the vacuum of QCD
(Derek's Visual QCD - The QCD Vacuum)

Low energy QCD

- Mesons and baryons
- (spontaneous) Chiral symmetry breaking
- Condensates
- Various EFTs

(partial) chiral symmetry restoration?

- **Chiral symmetry**



wikipedia

$$\mathcal{L} = i\bar{\psi}_j \partial\!\!\!/ \psi_j$$

$$\Lambda_V : \ \psi \longrightarrow e^{-i\frac{\vec{\tau}}{2}\vec{\Theta}}\psi \simeq (1 - i\frac{\vec{\tau}}{2}\vec{\Theta})\psi$$

: vector transform

$$\begin{aligned} i\bar{\psi}\partial\!\!\!/\psi &\longrightarrow i\bar{\psi}\partial\!\!\!/\psi - i\vec{\Theta}\left(\bar{\psi}i\partial\!\!\!/\frac{\vec{\tau}}{2}\psi - \bar{\psi}\frac{\vec{\tau}}{2}i\partial\!\!\!/\psi\right) \\ &= i\bar{\psi}\partial\!\!\!/\psi \end{aligned}$$

$$V_\mu^a = \bar{\psi}\,\gamma_\mu\frac{\tau^a}{2}\,\psi$$

$$\Lambda_A : \qquad \psi \longrightarrow e^{-i\gamma_5\frac{\vec{\tau}}{2}\vec{\Theta}}\psi = (1 - i\gamma_5\frac{\vec{\tau}}{2}\vec{\Theta})\psi$$

: axial−vector transform

$$\begin{aligned} i\bar{\psi}\partial\!\!\!/\psi &\longrightarrow i\bar{\psi}\partial\!\!\!/\psi - i\vec{\Theta}\left(\bar{\psi}\,i\partial_\mu\gamma^\mu\gamma_5\frac{\vec{\tau}}{2}\psi + \bar{\psi}\,\gamma_5\frac{\vec{\tau}}{2}i\partial_\mu\gamma^\mu\,\psi\right) \\ &= i\bar{\psi}\partial\!\!\!/\psi \end{aligned}$$

$$A_\mu^a = \bar{\psi}\gamma_\mu\gamma_5\frac{\tau}{2}\psi$$

Chiral symmetry breaking

$$\delta \mathcal{L} = -m\,(\bar{\psi}\psi)$$

$$\Lambda_A:\ \ m\,(\bar{\psi}\psi) \longrightarrow m\bar{\psi}\psi - 2im\vec{\Theta}\left(\bar{\psi}\frac{\vec{\tau}}{2}\gamma_5\psi\right)$$

→ Explicit chiral symmtery breaking

$$\frac{m}{\Lambda_{\mathrm{QCD}}} \sim 0.05$$ → chiral limit: m=0

$$< \bar{q}q >^{1/3} / \Lambda_{\mathrm{QCD}} \sim 1$$ → SSB of chiral symmetry

$$m \sim (5-10)\ \mathrm{MeV},\ \ \Lambda_{\mathrm{QCD}} \sim 200\ \mathrm{MeV}, < \bar{q}q >^{1/3} \simeq -240\ \mathrm{MeV}$$

Mesons and chiral symmetry

pion-like state: $\vec{\pi} \equiv i\bar{\psi}\vec{\tau}\gamma_5\psi$; sigma-like state: $\sigma \equiv \bar{\psi}\psi$

rho-like state: $\vec{\rho}_\mu \equiv \bar{\psi}\vec{\tau}\gamma_\mu\psi$; a_1-like state: $\vec{a_1}_\mu \equiv \bar{\psi}\vec{\tau}\gamma_\mu\gamma_5\psi$

$$\begin{aligned} \pi_i: \quad i\bar{\psi}\tau_i\gamma_5\psi \quad &\longrightarrow \quad i\bar{\psi}\tau_i\gamma_5\psi + \Theta_j\left(\bar{\psi}\tau_i\gamma_5\gamma_5\frac{\tau_j}{2}\psi + \bar{\psi}\gamma_5\frac{\tau_j}{2}\tau_i\gamma_5\psi\right) \\ &= \quad i\bar{\psi}\tau_i\gamma_5\psi + \Theta_i\bar{\psi}\psi \end{aligned}$$

→ $\vec{\pi} \longrightarrow \vec{\pi} + \vec{\Theta}\sigma$

$\sigma \longrightarrow \sigma - \vec{\Theta}\vec{\pi}$ $\qquad$ $\vec{\rho}_\mu \longrightarrow \vec{\rho}_\mu + \vec{\Theta} \times \vec{a_1}_\mu$

Linear sigma-model

$$\Lambda_V: \quad \pi^2 \longrightarrow \pi^2; \qquad \sigma^2 \longrightarrow \sigma^2 \qquad \Lambda_A: \quad \vec{\pi}^2 \longrightarrow \vec{\pi}^2 + 2\sigma\Theta_i\pi_i; \qquad \sigma^2 \longrightarrow \sigma^2 - 2\sigma\Theta_i\pi_i$$

$$(\vec{\pi}^2 + \sigma^2) \stackrel{\Lambda_V, \Lambda_A}{\longrightarrow} (\vec{\pi}^2 + \sigma^2)$$

* SSB → $V = V(\pi^2 + \sigma^2) = \frac{\lambda}{4}\left((\pi^2 + \sigma^2) - f_\pi^2\right)^2$

$$\mathcal{L}_{L.S.} = \frac{1}{2}\partial_\mu \pi \partial^\mu \pi + \frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma \quad -\frac{\lambda}{4}\left((\pi^2 + \sigma^2) - f_\pi^2\right)^2$$

Quantum Hadrodynamics

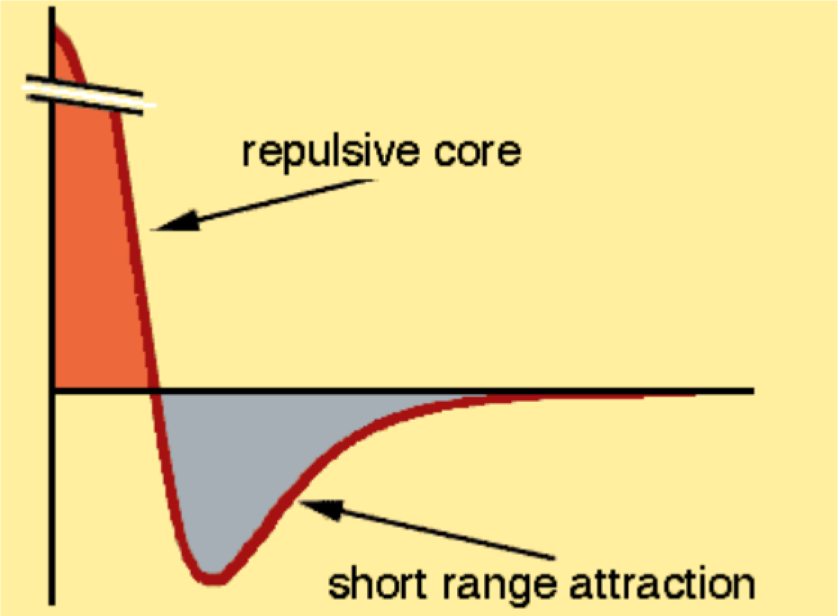
Quantum hadrodynamics (QHD) is a framework for describing the nuclear many-body problem as a relativistic system of baryons and mesons.

$$\mathcal{L} = -\frac{1}{4}V_{\mu\nu}V_{\mu\nu} - \frac{1}{2}m_v^2 V_\mu^2 - \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x_\mu}\right)^2 + m_s^2\phi^2\right]$$
$$-\bar{\psi}\left[\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - i g_v V_\mu\right) + (M - g_s\phi)\right]\psi$$

$$V_{\text{static}} = \frac{g_v^2}{4\pi}\frac{e^{-m_v r}}{r} - \frac{g_s^2}{4\pi}\frac{e^{-m_s r}}{r}$$

Recent progress in quantum hadrodynamics
Brian D. Serot, John Dirk Walecka, Int. J. Mod. Phys. E6 (1997) 515-631

Nuclear matter



Semi-Empirical Mass Formula for Nuclei

1. The nucleus consists of incompressible matter so that $R \sim A^{1/3}$.
2. The nuclear force is identical for every nucleon and in particular does not depend on whether it is a neutron or a proton.
3. The nuclear force saturates

$$E = -a_1 A + a_2 A^{2/3} + a_3 \frac{Z^2}{A^{1/3}} + a_4 \frac{(A-2Z)^2}{A} + \lambda \frac{a_5}{A^{3/4}}$$

$a_1 = 15.75 \text{ MeV}$ $a_2 = 17.8 \text{ MeV}$

$a_3 = 0.710 \text{ MeV}$ $a_4 = 23.7 \text{ MeV}$

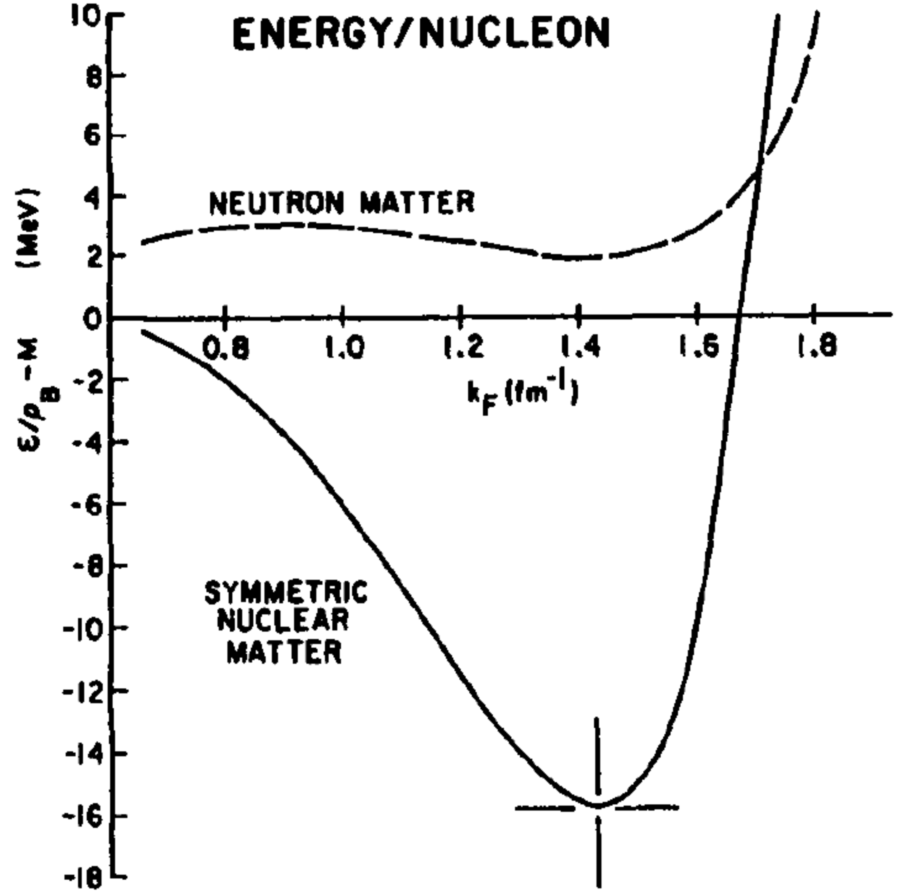
$a_5 = 34 \text{ MeV}$

(1) Let $A \to \infty$ so that surface properties are negligible with respect to bulk properties; set $N = Z$ so that the symmetry energy vanishes; and then turn off the electric charge so that there is no Coulomb interaction. The resulting extended, uniform material is known as nuclear matter. It evidently has a *binding energy/nucleon* of

$$\frac{E}{A} \approx -15.7 \text{ MeV}$$

That this expression is a constant independent of A is known as the *saturation of nuclear forces*;

E/A-M



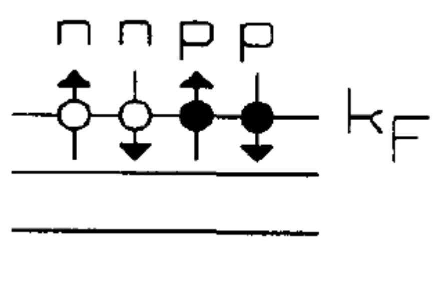
N.B.: neutron stars are gravitationally bound (not self-bound objects).

(2) Picture nuclear matter as a degenerate Fermi gas   The degeneracy factor is 4 corresponding to neutrons and protons with spin up and spin down ($n\uparrow n\downarrow p\uparrow p\downarrow$). The total number of occupied levels is A. Thus

$$A = \frac{4V}{(2\pi)^3}\int_0^{k_{\mathrm{F}}} d^3k$$

This yields

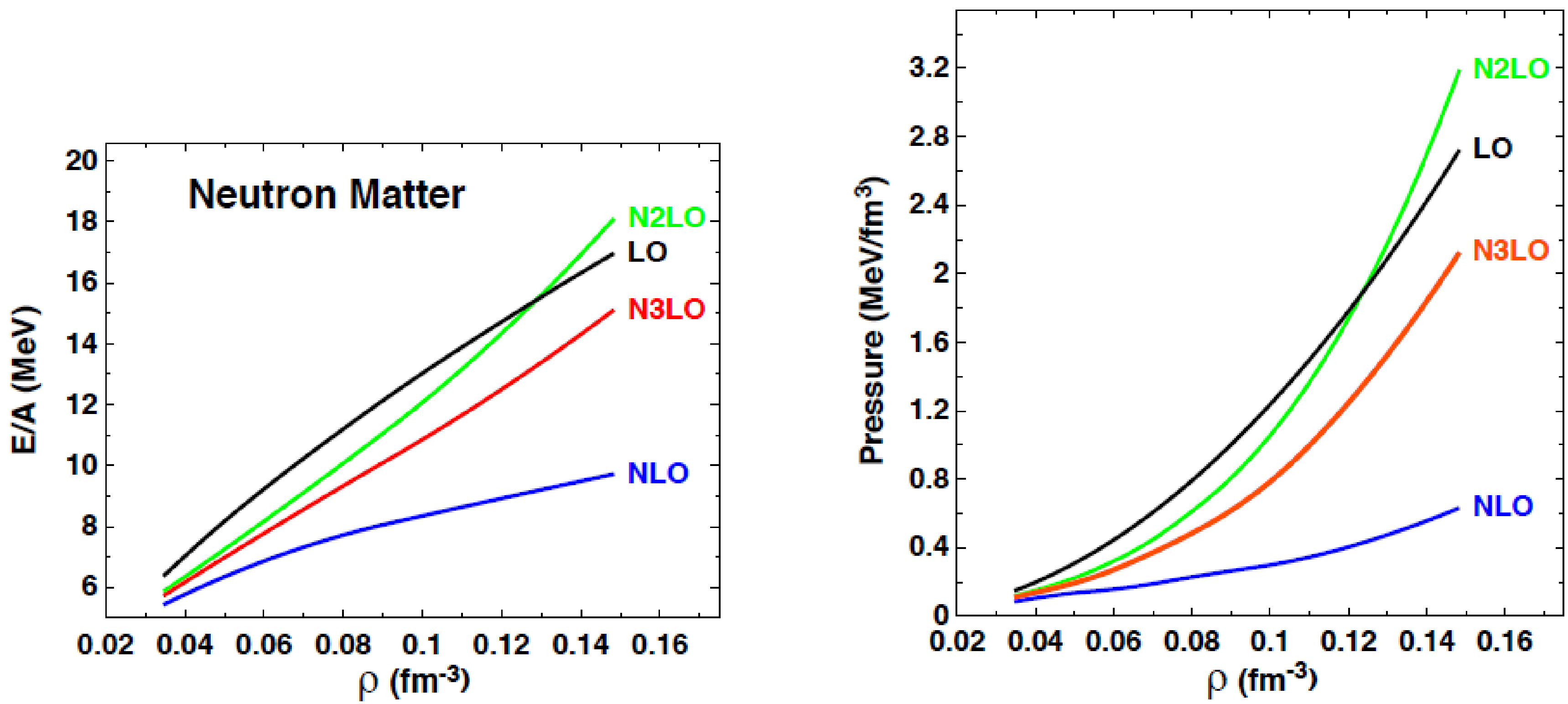
$$\frac{A}{V} = \frac{2}{3\pi^2}k_{\mathrm{F}}^3$$



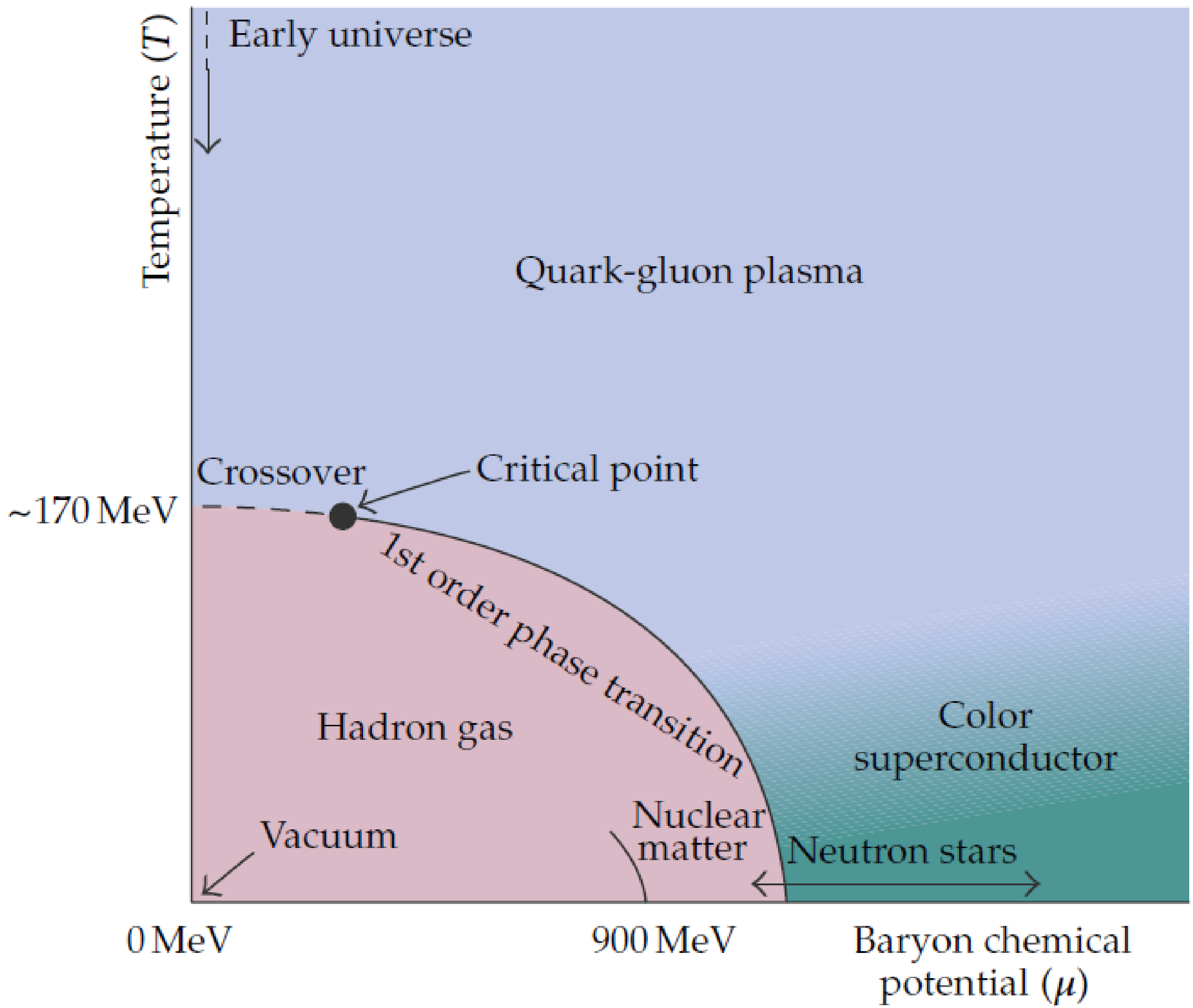
Nuclear matter as a degenerate Fermi gas.

The typical output of nuclear matter calculations is the energy per particle as a function of density, known as the equation of state (EoS).

F. Sammarruca, Modern Physics Letters A 32, (2017) 1730027



Energy per particle in neutron matter at various orders of chiral EFT.


Early universe
Temperature (T)
Quark-gluon plasma
Crossover
Critical point
~170 MeV
1st order phase transition
Hadron gas
Color
superconductor
Nuclear
matter
Vacuum
Neutron stars
0 MeV
900 MeV
Baryon chemical
potential (μ)

❖ Isoscalar parameters

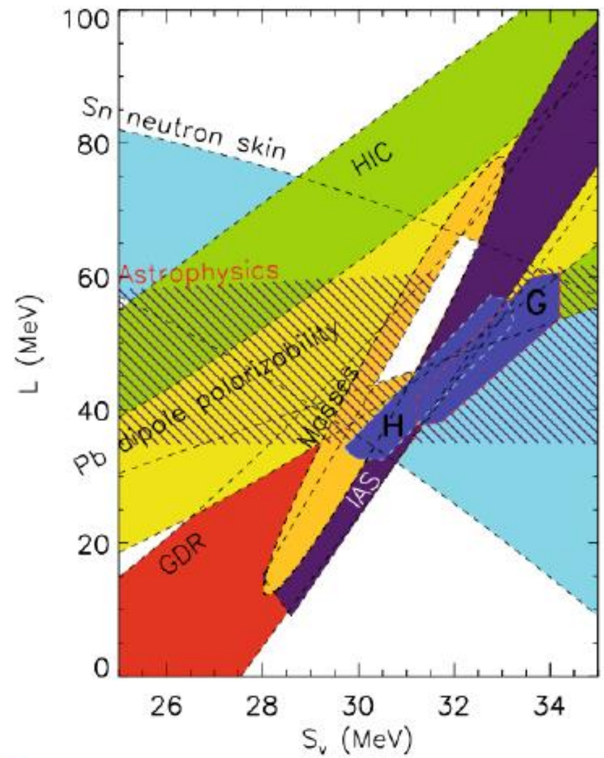
$$E_0 \approx -16\ \ MeV \quad , \quad K_0 = 9\rho_0^2 \frac{\partial^2 E_{IS}(\rho)}{\partial \rho^2}\bigg|_{\rho=\rho_0} \approx 240 \pm 20\ \ MeV \quad , \quad Q_0 = 27\rho_0^3 \frac{\partial^3 E_{IS}(\rho)}{\partial \rho^3}\bigg|_{\rho=\rho_0} \approx -500 \div 300\ \ MeV$$

❖ Isovector parameters

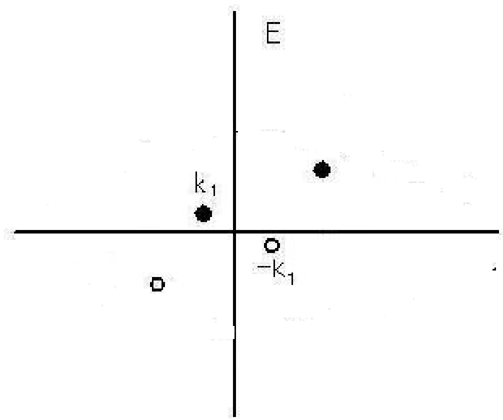
Less certain. Large variation of the prediction of the different models

$$E_{sym} = \frac{1}{2}\frac{\partial^2 E/A}{\partial \beta^2}\bigg|_{\beta=0} \quad , \quad L = 3\rho_0 \frac{\partial E_{IV}}{\partial \rho}\bigg|_{\rho=\rho_0}$$

$$K_{sym} = 9\rho_0^2 \frac{\partial^2 E_{IV}}{\partial \rho^2}\bigg|_{\rho=\rho_0} \quad , \quad Q_{sym} = 27\rho_0^3 \frac{\partial^3 E_{IV}}{\partial \rho^3}\bigg|_{\rho=\rho_0}$$



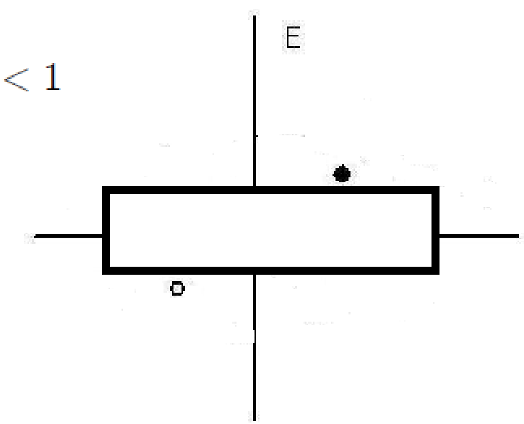
J. M. Lattimer & A. W. Steiner, EPJA 50, 40 (2014)



$$< 0|\bar{q}q|0 > \neq 0$$

A BCS-like trial ground state --> true vacuum contains chiral (quark-antiquark) condensates **in free space** [Finger & Mandula, NPB 199, 168 (1982)]

$$\frac{< F|\bar{q}q|F >}{< 0|\bar{q}q|0 >} < 1$$



In dense matter, chiral condensates will be reduced as low energy phase space is already occupied by the fermions in Fermi sea.

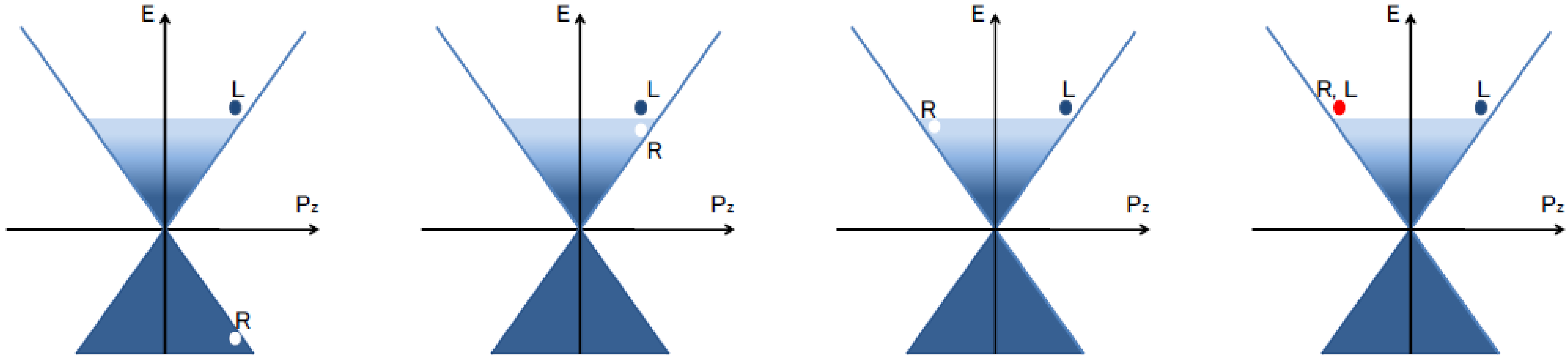


Figure 1: Different pairing mechanisms in the presence of a Fermi sea. From left to right: (a) quark-antiquark pairing, (b) quark-hole pairing with vanishing total momentum ("exciton"), (c) quark-hole pairing with nonzero total momentum, (d) quark-quark pairing generating color superconductivity.

T. Kojo et al., *Nucl.Phys. A* 843 (2010) 37

EPJ Web of Conferences **37**, 08010 (2012)
DOI: 10.1051/epjconf/20123708010

Chiral condensate in nuclear matter beyond linear density using chiral Ward identity*

Soichiro Goda[1,a] and Daisuke Jido[2]

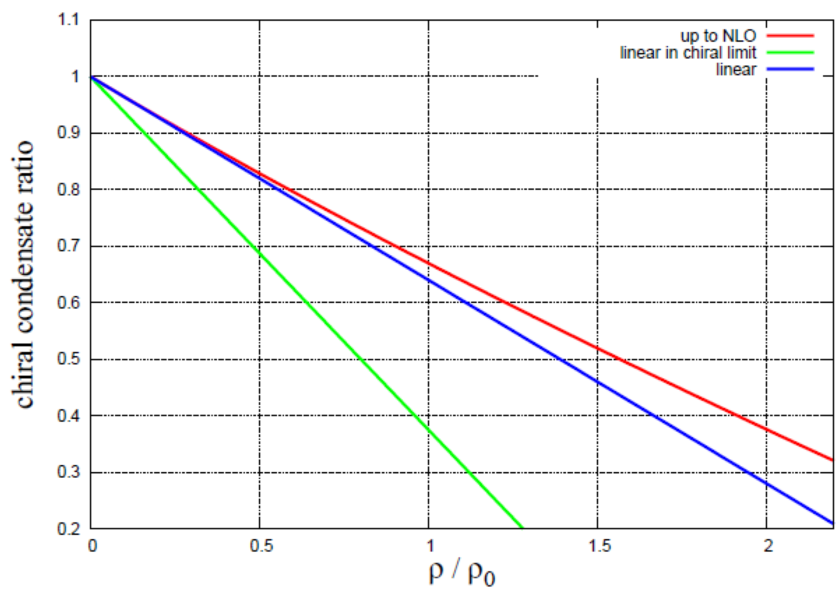
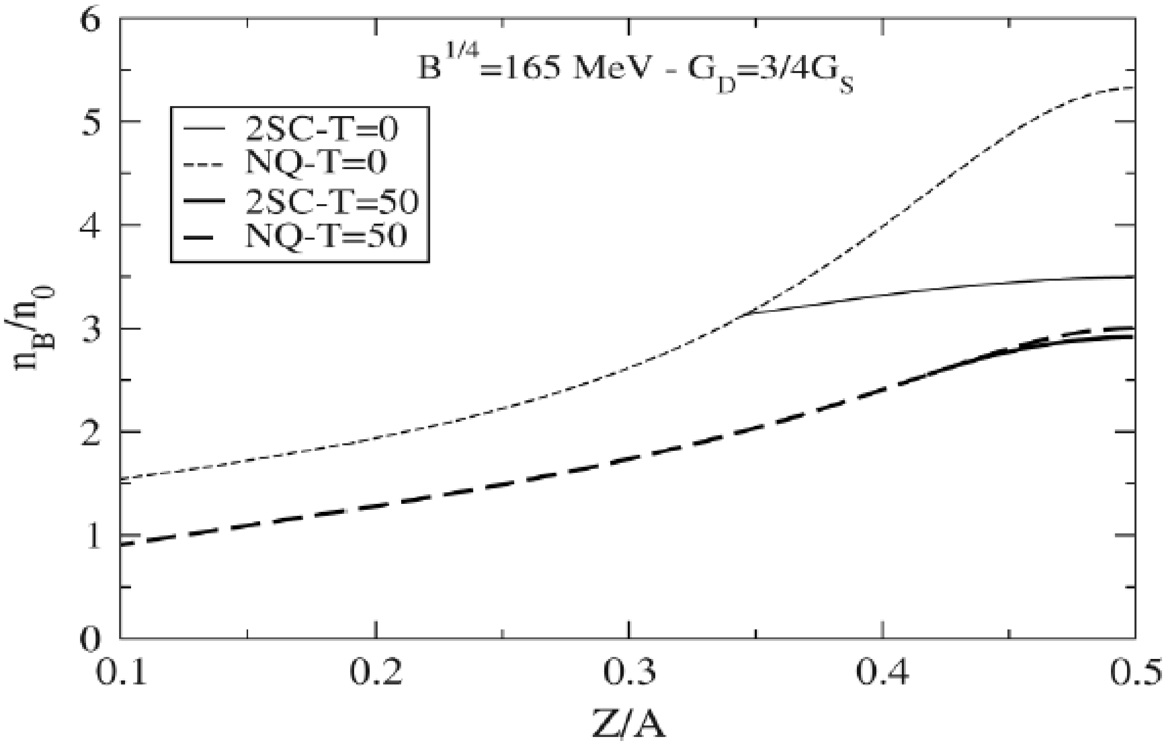


Fig. 2. The density dependence of chiral condensate in symmetric nuclear matter. The green, blue and red lines are the chiral condensates obtained by the linear density approximation in chiral limit, by the linear density approximation off the chiral limit and by the NLO corrections off the chiral limit.

PHYSICAL REVIEW D **81,** 094024 (2010)

Hadron-quark phase transition at nonzero isospin density: The effect of quark pairing

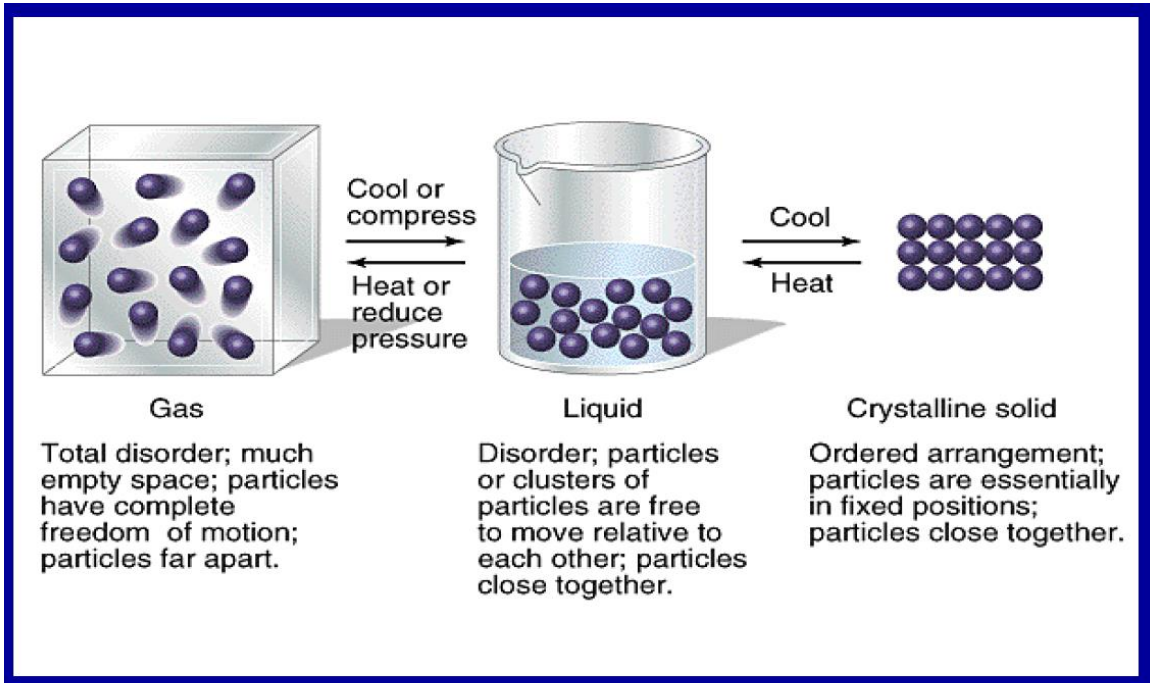
G. Pagliara and J. Schaffner-Bielich



But, results are quite model-dependent!

Liquid-gas transition

- Liquid-gas transition: phase transition between the nuclear liquid and a gas of nucleons, T_c were found to be in the range 10-20 MeV .
- Number density can be served as an order parameter for the LGT in n uclear matter.



Liquid-gas phase transition of nuclear matter: Since the nucleon mass is $m_N \simeq 939\,\mathrm{MeV}$ and the binding energy in isospin-symmetric nuclear matter is around $16\,\mathrm{MeV}$, a non-vanishing baryon density of nuclear matter starts arising at $\mu_\mathrm{B} = \mu_\mathrm{NM} \simeq 924\,\mathrm{MeV}$ at $T = 0$. At the threshold $\mu_\mathrm{B} = \mu_\mathrm{NM}$, the density n_B varies from zero to the normal nuclear density $n_0 = 0.17\,\mathrm{fm}^{-3}$. For $0 < n_\mathrm{B} < n_0$ the nuclear matter is fragmented into droplets with $n_\mathrm{B} = n_0$, so that $n_\mathrm{B} < n_0$ is achieved on spatial average. This is a typical first-order phase transition of the liquid-gas type. The first-order transition weakens as T grows and eventually ends up with a second-order critical point at $(\mu_\mathrm{G}, T_\mathrm{G})$ as indicated by the point G in figure 2. Low energy HIC experiments indicate that $\mu_\mathrm{G} \sim \mu_\mathrm{NM}$ and $T_\mathrm{G} = 15 \sim 20\,\mathrm{MeV}$ [51].

Finite size effects!!!

PHYSICAL REVIEW C **83**, 045201 (2011)

Hadron-quark phase transition in asymmetric matter with boson condensation

TABLE II. Some ions used in collision experiments and the respective asymmetry parameter (α) of the system.

	$^{12}C + ^{12}C$	$^{20}Ne + ^{20}Ne$	$^{58}Ni + ^{58}Ni$
α	0	0	0.034
	$^{20}Ne + ^{63}Cu$	$^{20}Ne + ^{118}Sn$	$^{118}Sn + ^{118}Sn$
α	0.060	0.130	0.150
	$^{20}Ne + ^{209}Bi$	$^{197}Au + ^{197}Au$	$^{20}Ne + ^{238}U$
α	0.188	0.198	0.201
	$^{197}Au + ^{208}Pb$	$^{208}Pb + ^{208}Pb$	$^{238}U + ^{238}U$
α	0.205	0.211	0.227

It is shown that the phase transition is very sensitive to the density dependence of the equation of state and the symmetry energy. For an isospin asymmetry of 0.2 and a mixed phase with a fraction of 20% of quarks, a transition density in the interval $2\rho_0 < \rho < 4\rho_0$ was obtained for temperatures $30 < T < 65$ MeV.

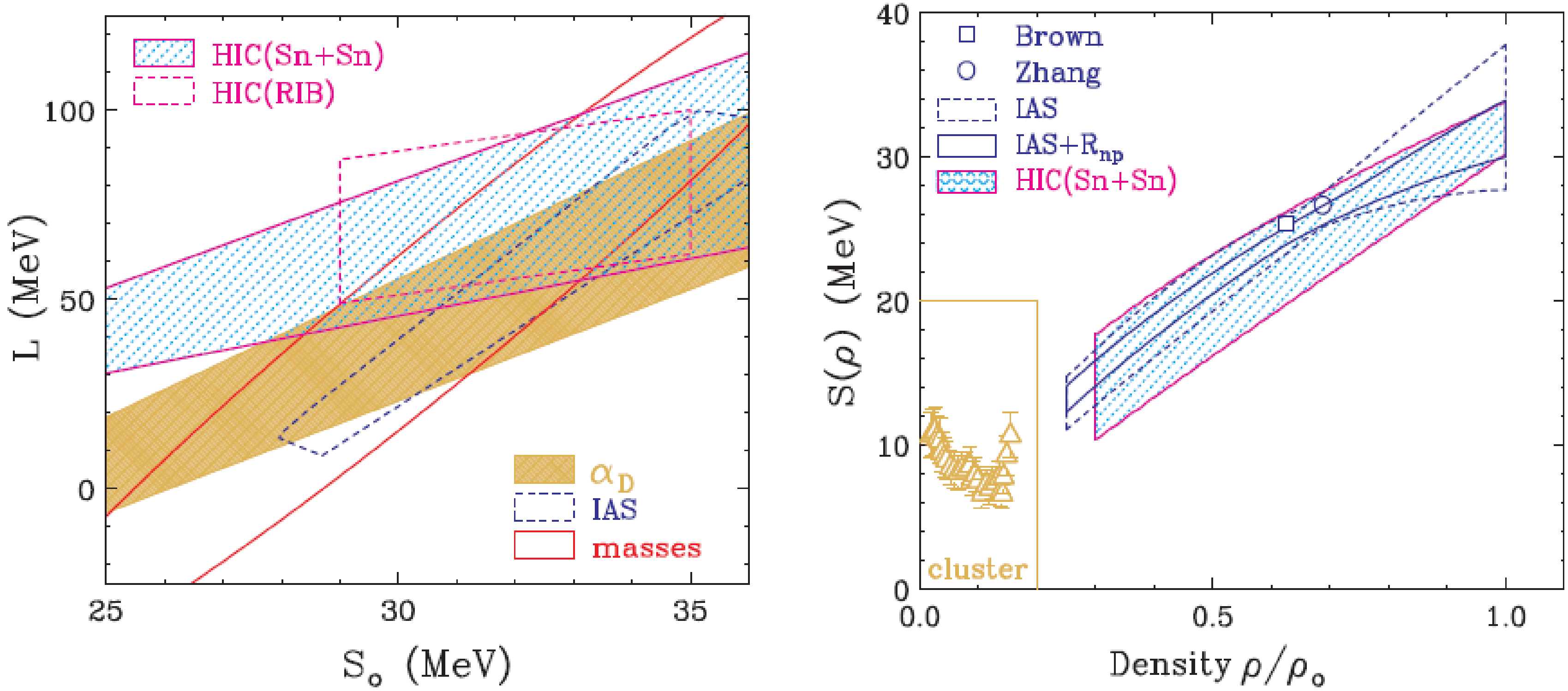
Symmetry energy

$$\mathcal{E}(\rho,\alpha) = \mathcal{E}(\rho,\alpha=0) + S(\rho)\alpha^2 + \cdots \qquad \alpha = (N-Z)/A$$

$$S(\rho) \equiv \frac{1}{2}\left(\frac{\partial^2 \mathcal{E}(\rho,\alpha)}{\partial \alpha^2}\right)_{\alpha=0} \approx \mathcal{E}(\rho,\alpha=1) - \mathcal{E}(\rho,\alpha=0)$$

$$S(\rho) = J + Lx + \tfrac{1}{2}K_{\mathrm{sym}}x^2 + \cdots \qquad x = (\rho-\rho_0)/3\rho_0$$

- **At low densities, uniform nuclear matter becomes unstable against cluster formation. Indeed, at densities of $\rho < \rho_0/2$ the inter-nucleon separation becomes comparable to the range of the NN interaction, so it becomes energetically favorable for the system to fragment into neutron-rich clusters. Cluster formation significantly increases the symmetry energy at very low densities**
- **Heavy-ion collisions at incident energies from about 35 to 150 MeV per nucleon give access to the symmetry energy at densities from about 50% above ρ_0 down to about $0.1\rho_0$.**
- **Specifically, the region at about twice saturation density is critical for the determination of ne utron-star radii.**



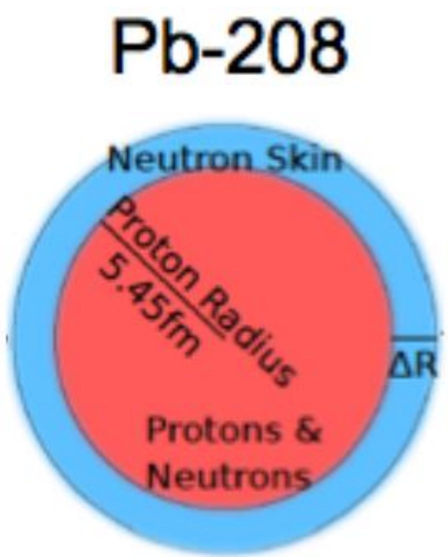
J. Phys. G: Nucl. Part. Phys. **41** (2014) 093001

A way forward in the study of the symmetry energy: experiment, theory, and observation

C J Horowitz , E F Brown , Y Kim , W G Lynch , R Michaels , A Ono , J Piekarewicz , M B Tsang and H H Wolter

Symmetry energy: nuclear structure

- **Neutron skin**



Z=82

* Heavy nuclei are expected to have a neutron-rich skin due to the large neutron excess and the Coulomb barrier (which reduces the proton density at the surface).

* The stiffer the equation of state the thicker the neutron skin.

$$R^{208}_{\text{skin}} = \frac{r_s}{2}\left(\frac{L + L_s \pm \delta L_s}{L_s}\right)$$

$$R^{208}_{\text{skin}} = R^{208}_n - R^{208}_p = 0.33^{+0.16}_{-0.18}\ \text{fm}.$$

PREX

$$r_s = 0.2\ \text{fm},\ L_s = 68.7\ \text{MeV},$$

$$\delta L_s = 6.8\ \text{MeV}$$

Symmetry energy: neutron stars

The thickness of the neutron skin depends on the pressure of neutron-rich matter: the greater the pressure, the thicker the skin as neutrons are pushed out against surface tension. The same pressure supports a neutron star against gravity. Thus models with thicker neutron skins often produce neutron stars with larger radii [C. J. Horowitz and J. Piekarewicz, Phys. Rev. Lett. 86, 5647 (2001)].

We note, however, that the neutron star radius reflects the pressure due to the symmetry energy at a range of densities and is also highly sensitive to its pressure at 2-3 times saturation density [Lattimer, J.M. et al, Science 23, 536 (2004).]

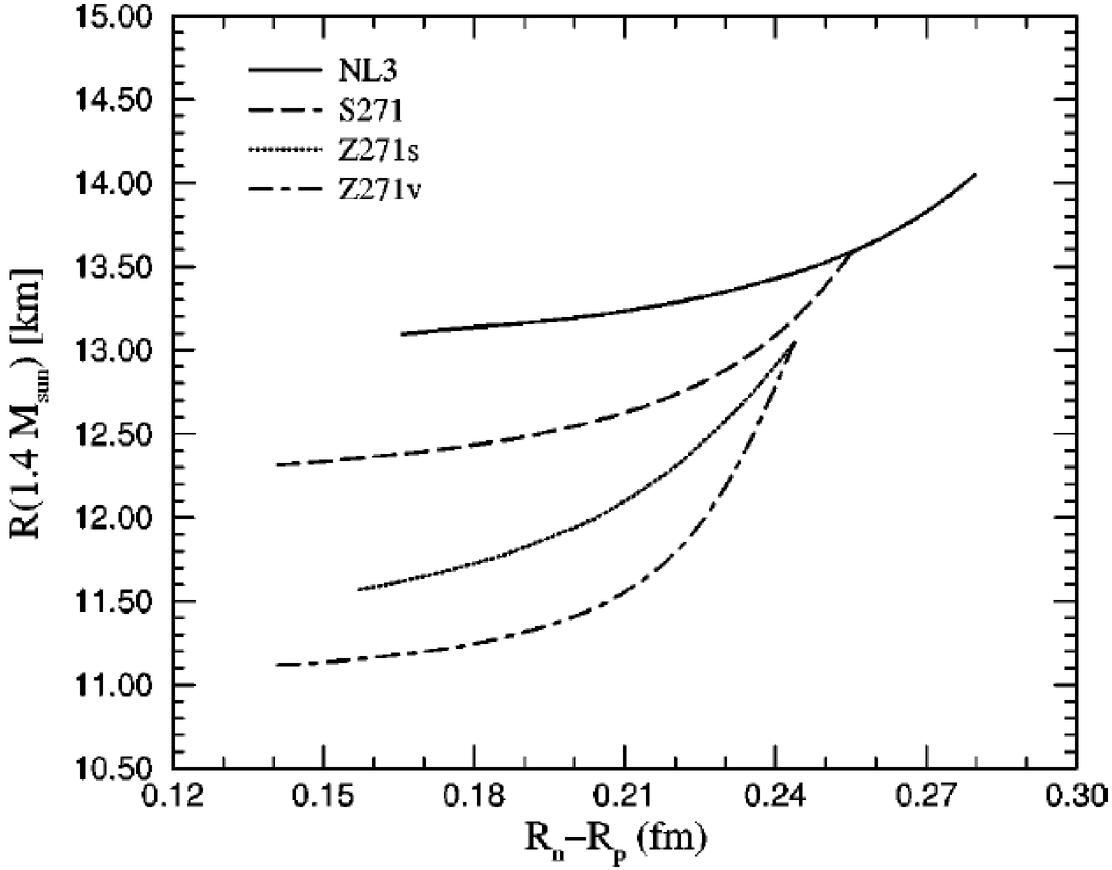


FIG. 4. Radius of a $M=1.4M_{\odot}$ neutron star as a function of the neutron-minus-proton radius in ^{208}Pb for the four parameter sets described in the text.

C. J. Horowitz J. Piekarewicz PHYSICAL REVIEW C, VOLUME 64, 062802(R)

Mean field approximation and beyond: overview

Approaches to the Nuclear EoS

Phenomenological approaches

Based on effective density-dependent interactions with parameters adjusted to reproduce nuclear observables and compact star properties

- Liquid drop type: BPS, BBP, LS, OFN
- Thomas-Fermi: Shen
- ETFSI: BSk
- HF: NV, Sk, PAL, RMF, RHF, QMC
- Statistical models: HWN, RG, HS

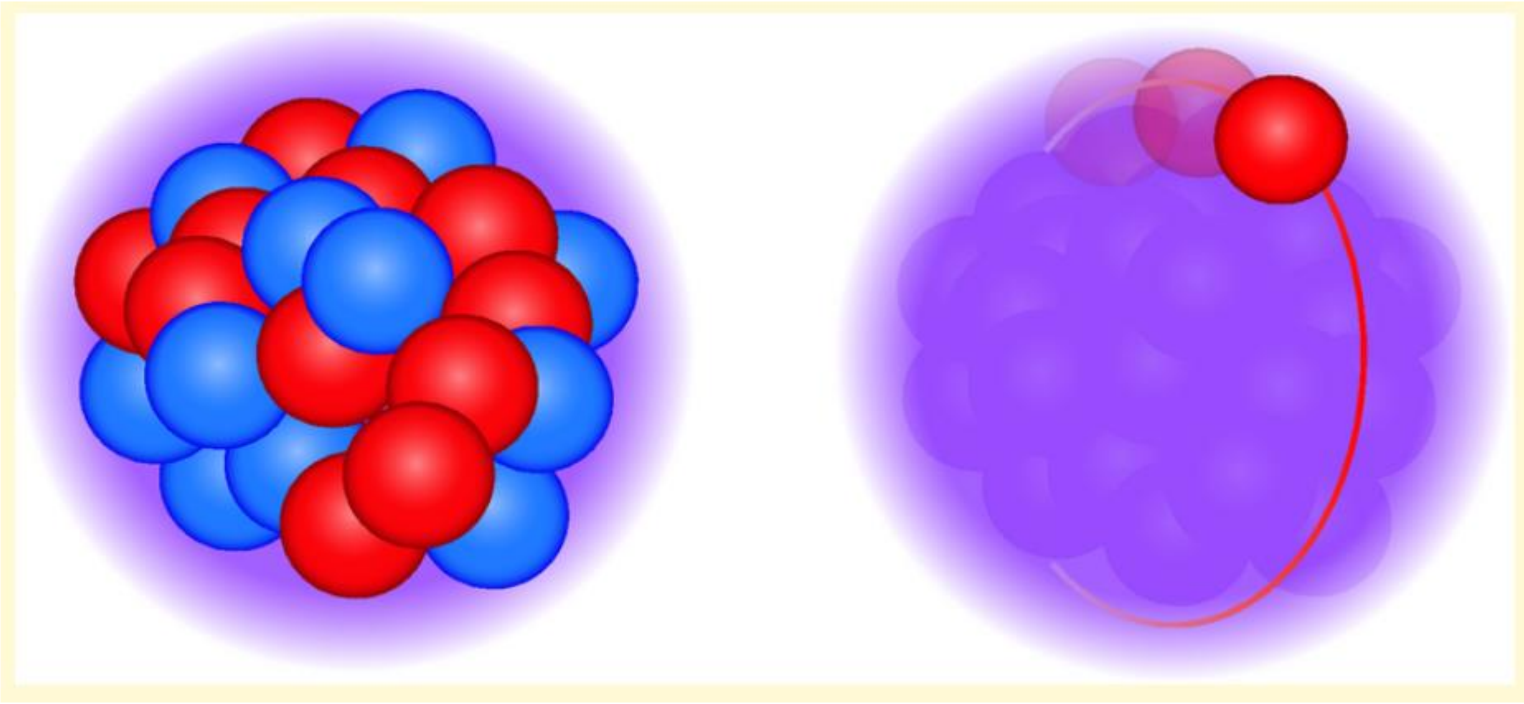
I apologize for all those approaches I have missed

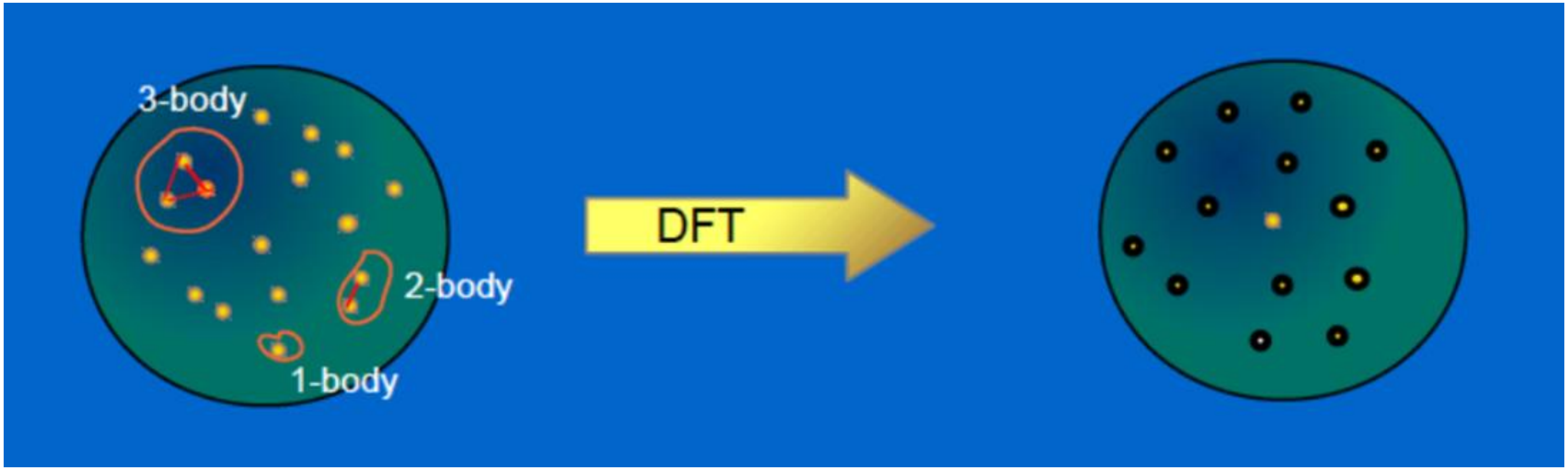
Microscopic ab-initio approaches

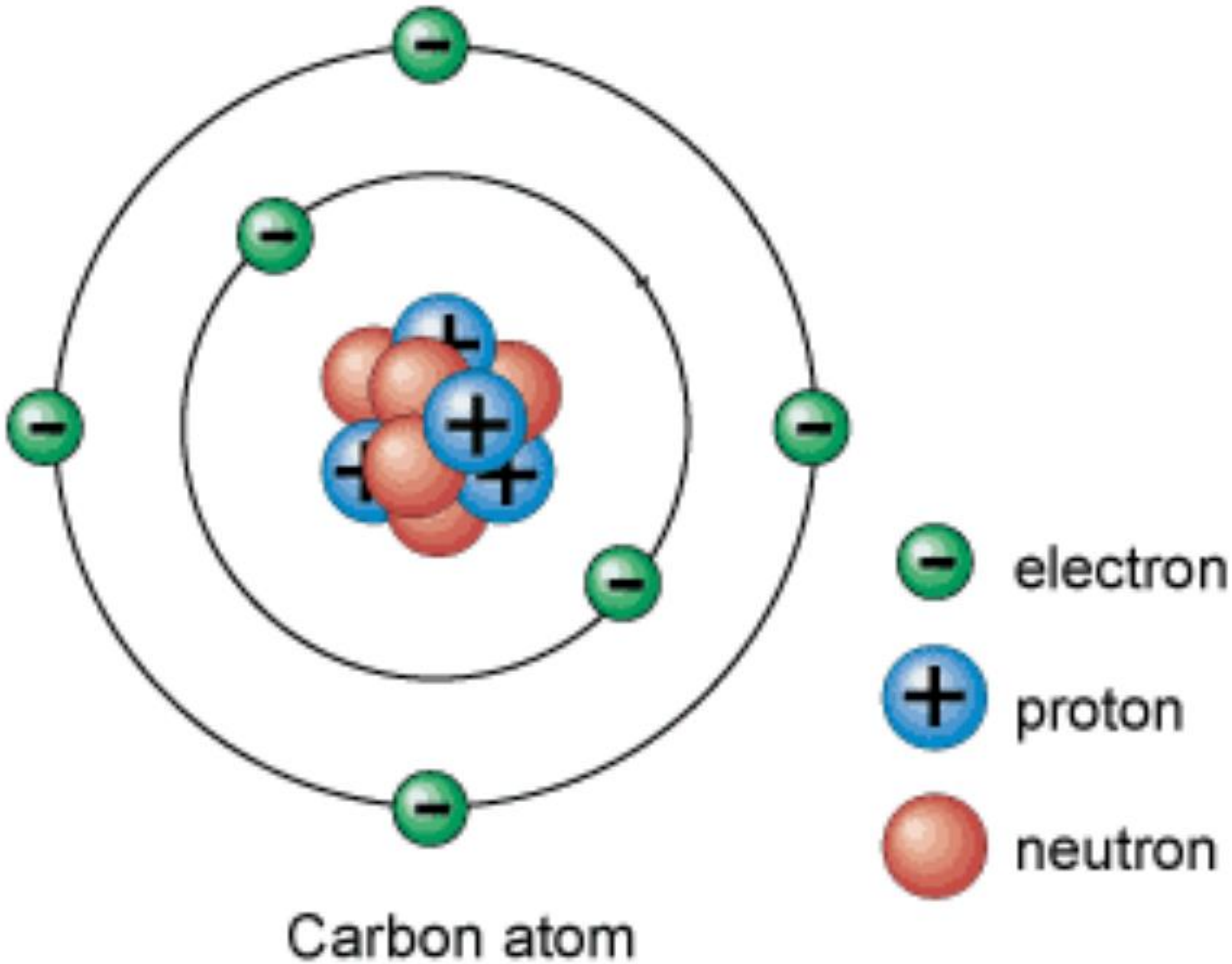
Based on two- & three-body realistic interactions. The EoS is obtained by "solving" the complicated many-body problem

- Variational: APS, CBF, FHNC, LOVC
- Monte-Carlo: VMC, DMC, GFMC, AFDMC
- Diagrammatic: BBG (BHF), SCGF
- RG methods: $V_{low\,k}$ & SRG from χEFT potentials
- DBHF

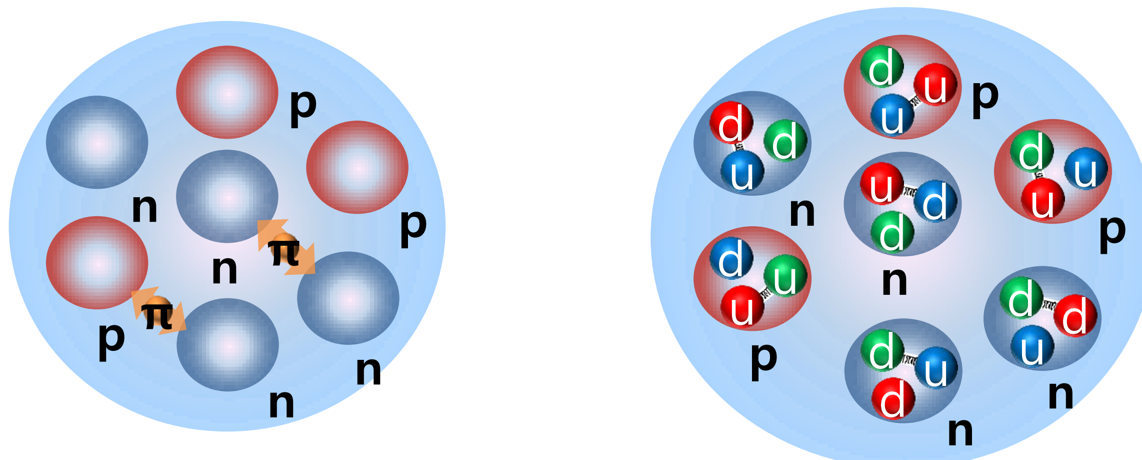
Mean field







Clusters of levels → shell structure



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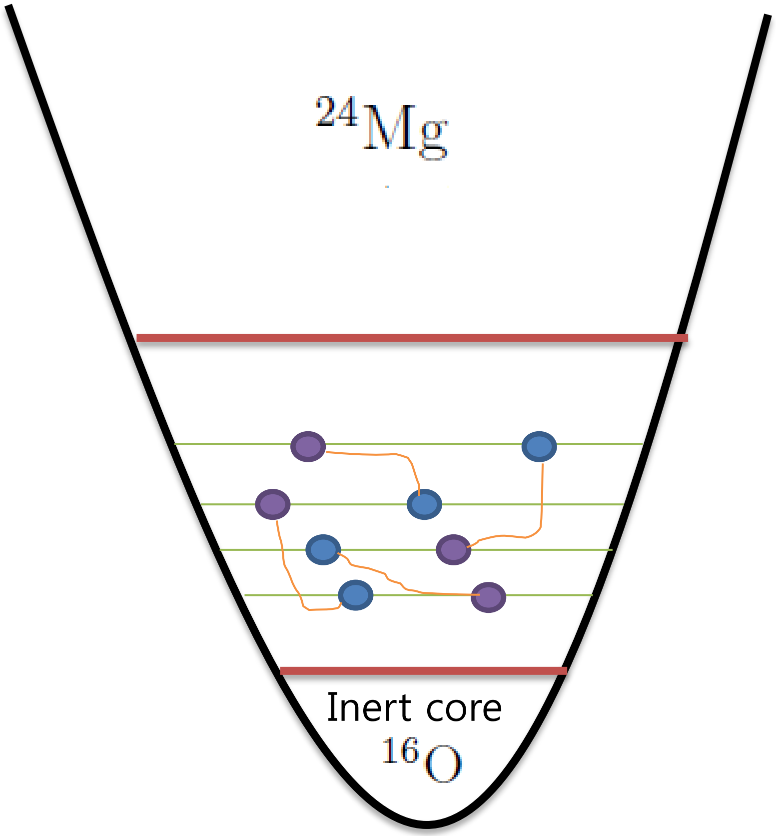
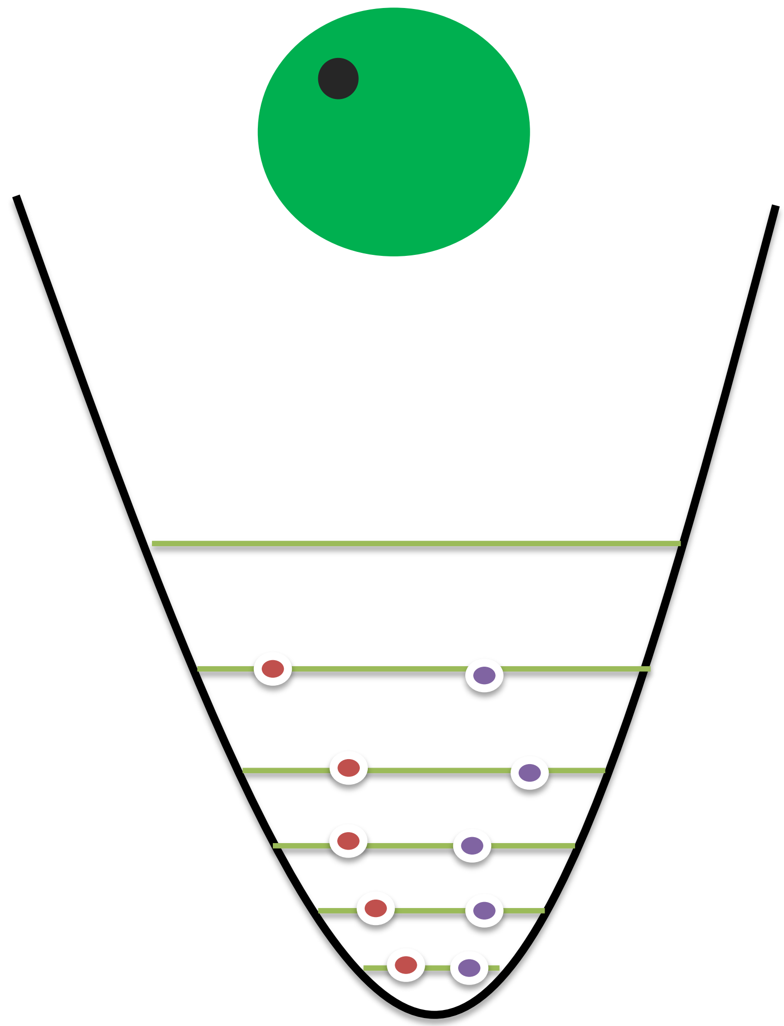
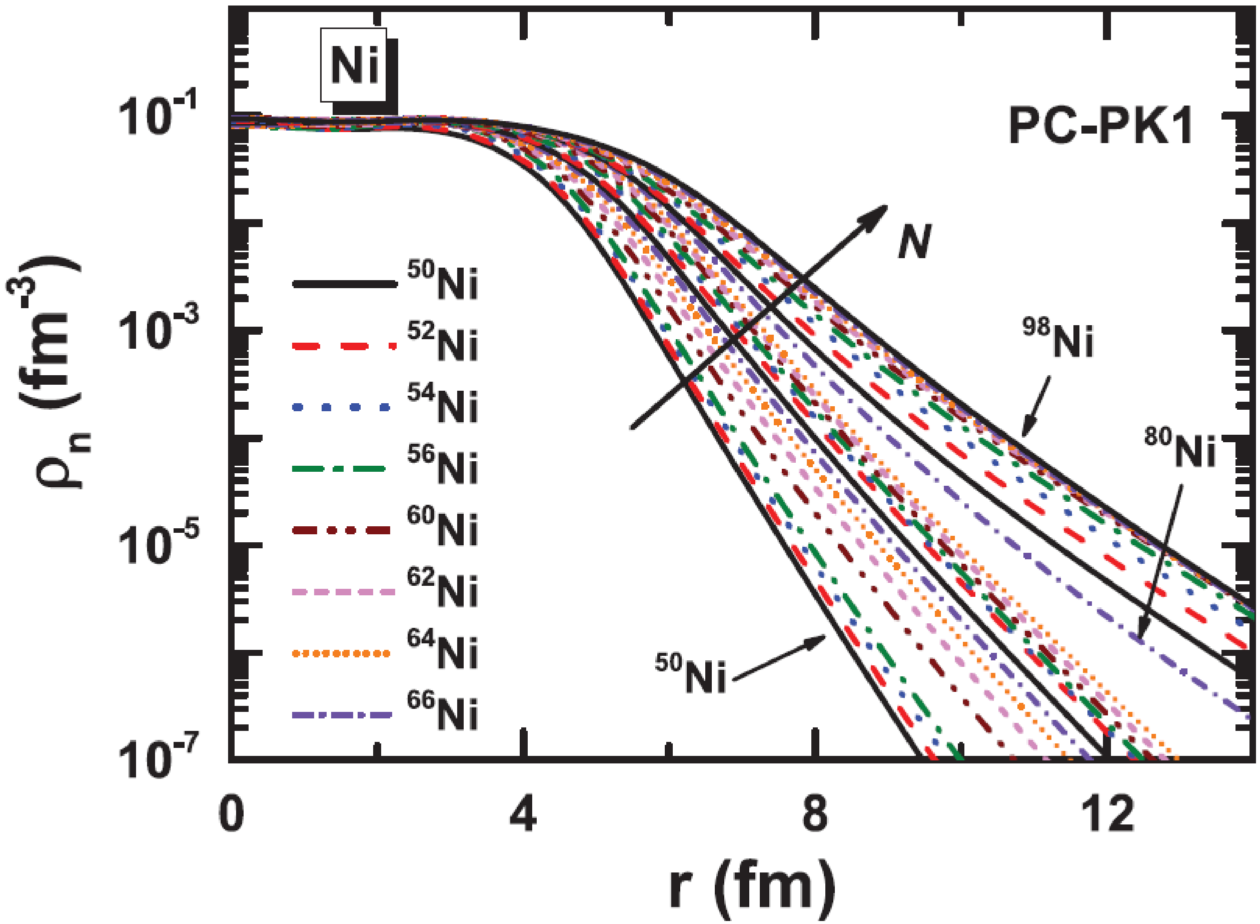


Interacting SM



HF (RMF)



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Atomic Data and Nuclear Data Tables

journal homepage: www.elsevier.com/locate/adt

The limits of the nuclear landscape explored by the relativistic continuum Hartree–Bogoliubov theory

X.W. Xia [a], Y. Lim [b,c], P.W. Zhao [d,e], H.Z. Liang [f], X.Y. Qu [a,g], Y. Chen [d,h], H. Liu [d], L.F. Zhang [d], S.Q. Zhang [d], Y. Kim [c], J. Meng [d,a,i,*]

Effective interactions in the *sd* shell

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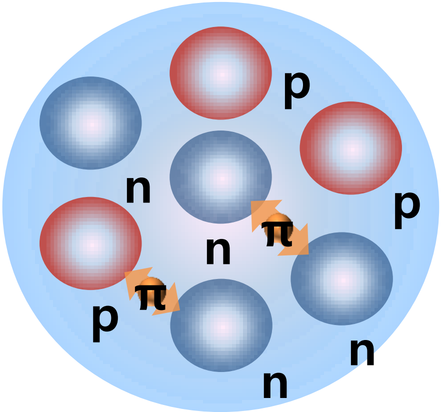
(Dated: November 20, 2018)

We perform a quantitative study of the microscopic effective shell-model interactions in the valence *sd* shell, obtained from modern nucleon-nucleon potentials, chiral N3LO, JISP16 and Daejeon16, using no-core shell-model wave functions and the Okubo-Lee-Suzuki transformation. We investigate the monopole properties of those interactions in comparison with the phenomenological universal *sd*-shell interaction, USDB. Theoretical binding energies and low-energy spectra of O-isotopes and of selected *sd*-shell nuclei, are presented. In general, we conclude that there is a noticeable improvement in the quality of the effective interaction derived from the Daejeon16 potential. We show that its proton-neutron centroids are consistent with those from USDB. We then propose monopole modifications of the centroids in order to provide an adjusted interaction yielding significantly improved agreement with the experiment. A spin-tensor decomposition of two-body effective interactions is applied in order to extract more information on the structure of the centroids and to understand the reason for deficiences arising from the present level of theoretical approximations. The issue of the possible role of the three-nucleon forces is addressed.

Ab initio No Core Shell Model for nuclear structure

- Ab initio: nuclei from first principles using **fundamental interactions without uncontrolled approximations.**
- No core: all nucleons are active, **no inert core**.
- Shell model: harmonic oscillator basis
- Point nucleons



- A-nucleon Schrödinger equation

$$\hat{H}\,\Psi(r_1, \cdots, r_A) = E\,\Psi(r_1, \cdots, r_A)$$

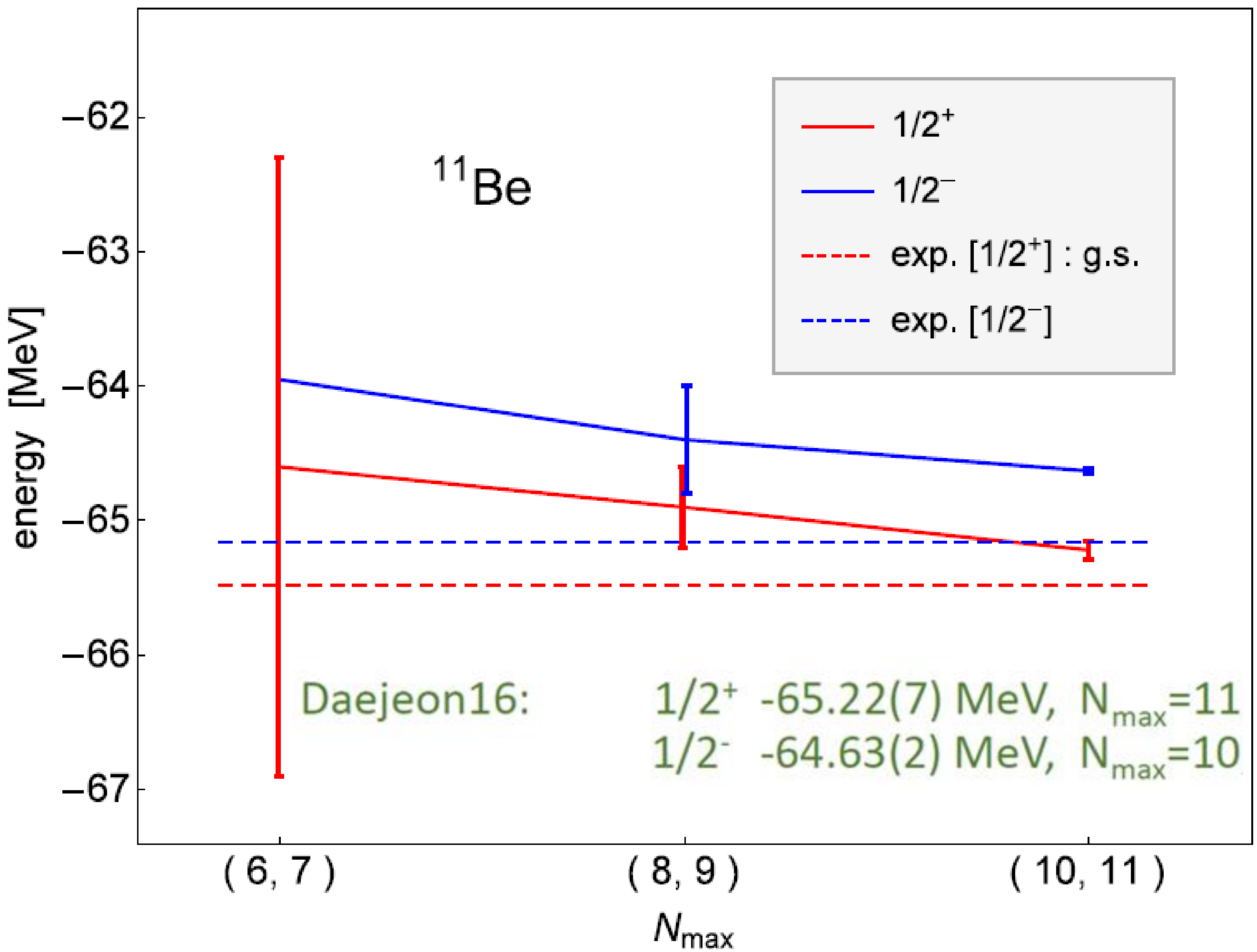
  - Hamiltonian with $NN(+NNN)$ interactions

$$\hat{H} = \frac{1}{A}\sum_{i<j}\frac{(\vec{p_i}-\vec{p_j})^2}{2m} + \sum_{i<j}V_{ij} + \sum_{i<j<k}V_{ijk} + \cdots$$

  - Wave functions are expanded in basis states

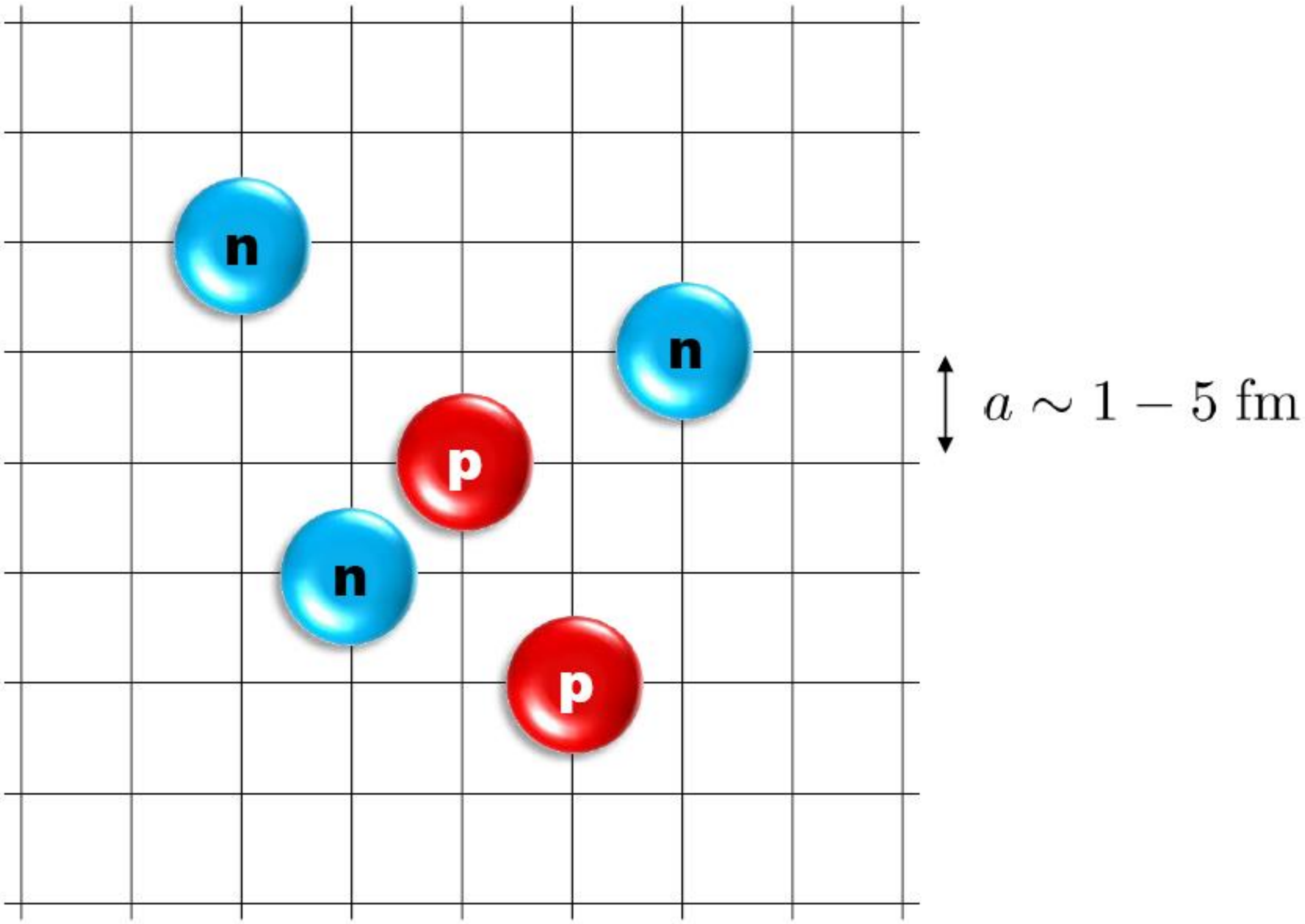
$$\Psi(r_1, \cdots, r_A) = \sum a_i \Phi_i(r_1, \cdots, r_A)$$

    basis states Φ_i : Slater determinants of single particle states



Y. Kim, I. J. Shin, A.M. Shirokov, M. Sosonkina, P. Maris, J.P. Vary, Proc. Int Conference Nuclear Theory in the Supercomputing Era, IBS Headquarters, Daejeon, Korea 29 October – 2 November 2018.

Lattice effective field theory



Nuclear Lattice EFT Collaboration

Quantum Monte Carlo Methods in Nuclear Physics: Recent Advances

J. E. Lynn,[1,2] I. Tews,[3] S. Gandolfi,[3] and A. Lovato[4,5]

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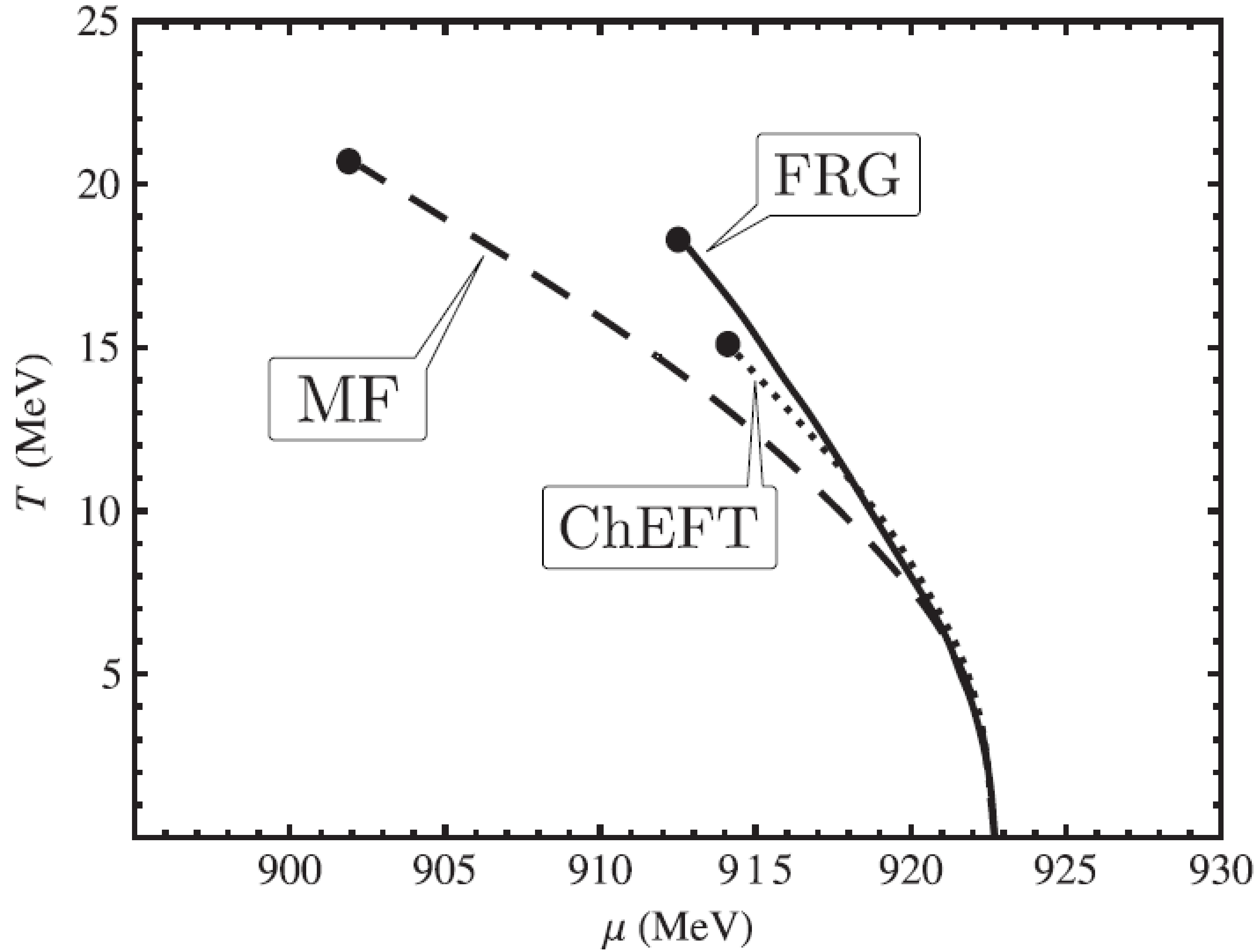
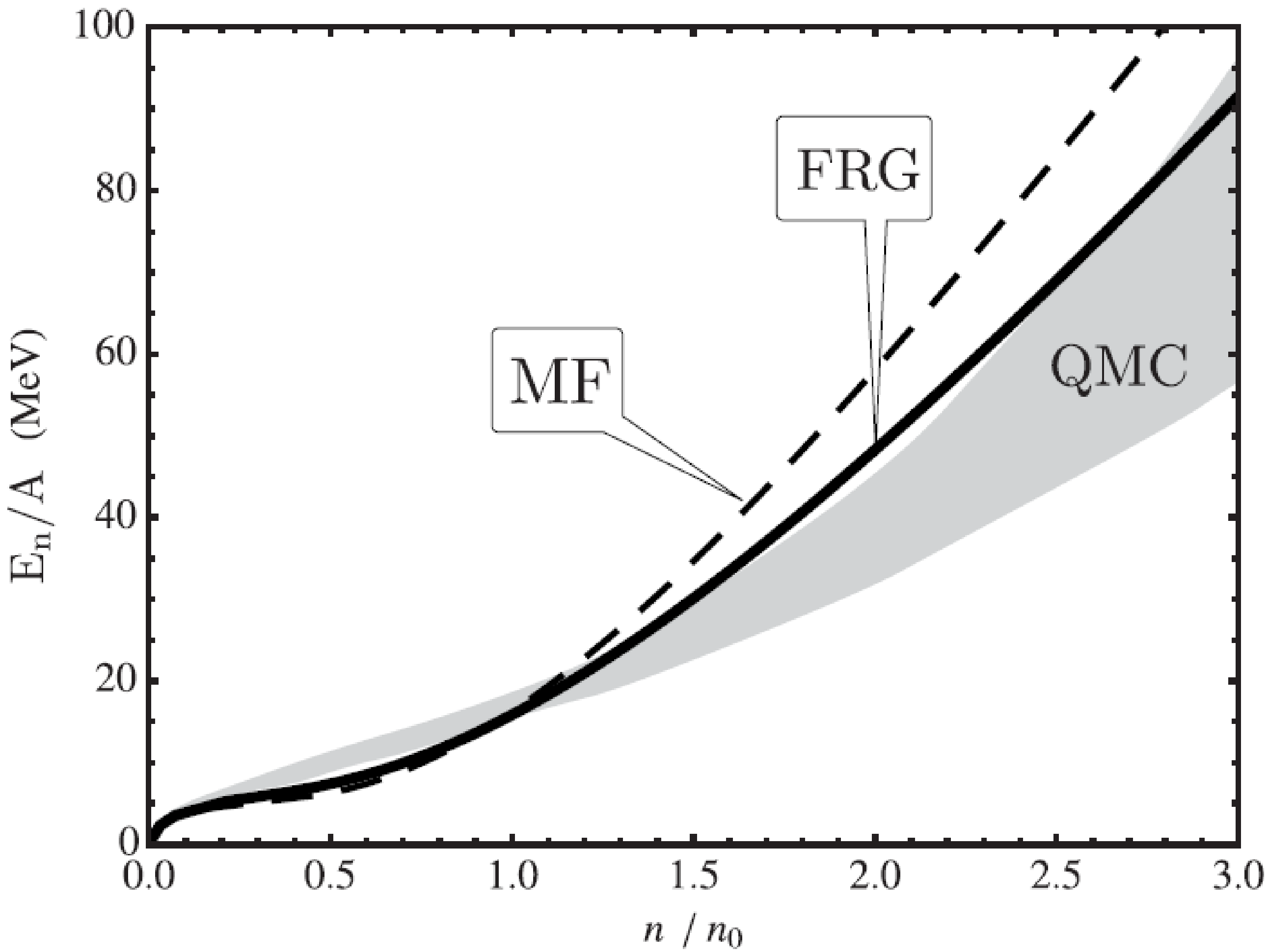
[4]Physics Division, Argonne National Laboratory, Argonne, Illinois 60439, USA; email: lovato@anl.gov
[5]INFN-TIFPA Trento Institute of Fundamental Physics and Applications, Via Sommarive, 14, 38123 Trento, Italy

e-Print: arXiv:1906.01674 [nucl-th]

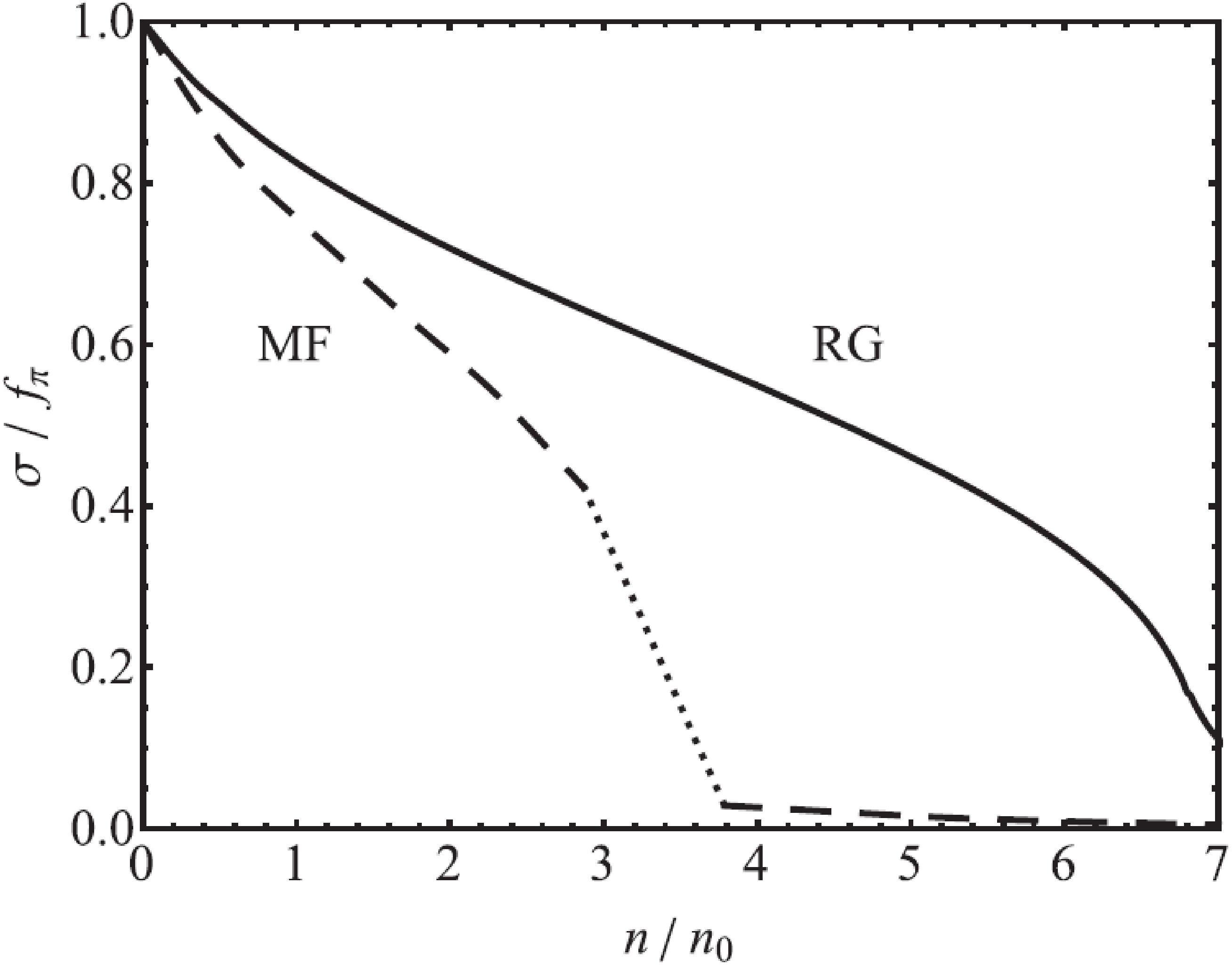
Liquid–gas phase transition in a $T - \mu$ diagram.



The equation of state for pure neutron matter at $T = 0$ with $E_{sym} = 32$ MeV



M. Drews, W. Weise / Progress in Particle and Nuclear Physics 93 (2017) 69–107


1.0
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6
7
MF
RG
σ / f_π
n / n_0

Mean field approximation: some details

Reviews

Euler - Lagrange Eq. in E.M.T

$$S = \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi)$$

$\delta S = 0$ for arbitrary $\delta\phi$

$$\delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta(\partial_\mu \phi) \right\}$$

$\hookrightarrow = \partial_\mu (\delta\phi)$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \delta\phi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) \right\}$$

↑ surface term ($\delta\phi = 0$)

$$\Rightarrow \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

Note that x does not change in this Variation

$$\Rightarrow \delta(\partial_\mu \phi(x)) = \partial_\mu \delta\phi(x)$$

Ex 1 $\mathcal{L} = \bar{\psi}\,(i\not{\partial} - m)\,\psi$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = 0, \quad \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = (i\not{\partial} - m)\psi$$

$$\Rightarrow (i\not{\partial} - m)\psi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = -i\,\bar{\psi}\gamma^\mu$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = m\,\bar{\psi}$$

$$\Rightarrow -i\,\partial_\mu \bar{\psi}\gamma^\mu - m\,\bar{\psi} = 0$$

Note that

$$\frac{\partial}{\partial \psi_b}(\bar{\psi}_a \psi_a) = (-)\,\bar{\psi}_b$$

$$\frac{\partial}{\partial \bar{\psi}_b}(\bar{\psi}_a \psi_a) = \psi_b$$

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Ex 2

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\delta S = \delta \int_{\sigma_1}^{\sigma_2} d^4x \, \mathcal{L}$$

$$= \int \left(-F_{\mu\nu} \partial^\mu \delta A^\nu + m^2 A_\mu \delta A^\mu \right) d^4x$$

(ignoring the surface term)

$$= \int d^4x \left[\left(\partial^\mu F_{\mu\nu} + m^2 A_\nu \right) \delta A^\nu \right] \to 0$$

$$\therefore \quad \underline{\partial^\mu F_{\mu\nu} + m^2 A_\nu = 0}$$

Note $\delta F^2 = 2 F_{\alpha\beta} \, \delta F^{\alpha\beta}$

$$= 2 F_{\alpha\beta} \left(\partial^\alpha \delta A^\beta - \partial^\beta \delta A^\alpha \right)$$

$$= 4 F_{\alpha\beta} \, \partial^\alpha \delta A^\beta$$

Noether's Theorem

$$\phi(x) \to \phi'(x) = \phi(x) + \alpha \Delta\phi(x)$$

symmetric : if it leaves the E.O.M invariant.

Expected: $\mathcal{L}(x) \to \mathcal{L}(x) + \alpha \partial_\mu J^\mu(x)$

↳ does not affect the derivation of E·L eq.

$$\alpha \Delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}(\alpha \Delta\phi) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\mu (\alpha \Delta \phi)$$

($\sim \delta$) by varying the fields.

$$= \alpha \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi \right) + \alpha \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] \Delta\phi$$

$= 0$ by E.O.M

$$\therefore \partial_\mu J^\mu = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi$$

$\partial_\mu j^\mu(x) = 0$ for $j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \Delta\phi - J^\mu$

Ex E.M.T

$x^\mu \to x^\mu - a^\mu$ infinitesimal translation

$\phi(x) \to \phi(x+a) = \phi(x) + \underbrace{a^\mu \partial_\mu \phi(x)}_{\Delta\phi}$ μ: dummy index

$\mathcal{L}$ is also a scalar.

$$\mathcal{L} \to \mathcal{L} + a^\mu \partial_\mu \mathcal{L} = \mathcal{L} + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L})$$

$$T^\mu{}_\nu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} \delta^\mu_\nu$$

energy-momentum-stress (pressure)

$$T^{\mu\nu} = \begin{pmatrix} \rho & f_i \\ f_i & p\delta_{ij} + \Sigma_{ij} \end{pmatrix}$$

in a local Lorentz frame

ρ = mass-energy density

f_i = momentum density [$f_i = (\rho+p)v_i$ for a perfect fluid]

p = pressure

Σ_{ij} = shear stress [$\Sigma_{ij} = 0$ for a perfect fluid]

$$T^{(I)}_{\mu\nu} = i\bar{\psi}\gamma_\mu \partial_\nu \psi$$

No need to symmetrize $T_{\mu\nu}$ since the additional terms in symmetrized tensor enter as a total four-divergence, whose diagonal elements vanish in a uniform system.

$$\hat{H} = \int d^3x \, T^{00}$$

$$= \int d^3x \; i\psi^\dagger \partial_0 \psi$$

$u^\dagger u = 2E_p$

$$\left[\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(a^s_p u^s(p) e^{-ip\cdot x} + b^{s\dagger}_p v^s(p) e^{ip\cdot x} \right) \right]$$

from ∂_0

$$\sim \int d^3x \int d^3p \, d^3p' \; E_{p'} \, a^\dagger_p a_{p'} \, u^\dagger(p) u(p') \cdot e^{i(p-p')\cdot x}$$

$$= \int \frac{d^3p}{(2\pi)^3} \sum_s E_{\vec{p}} \, a^\dagger_{s\vec{p}} a_{s\vec{p}} \quad : \quad E_p = \sqrt{\vec{p}^{\,2} + M_*^2}$$

$\hookrightarrow \times 2$

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$$\langle F|\hat{H}|F\rangle = \gamma \int \frac{d^3 p}{(2\pi)^3} \sqrt{M^{*2}+\vec{p}^{\,2}}$$

$$\langle F|a^\dagger a|F\rangle \to \int_0^{k_F}$$

Pressure

$\gamma_1 \beta_1$ or $\gamma_2 \beta_2$ or $\gamma_3 \beta_3$

$\langle T_{zz} \rangle$

$$\sim \int d^3x \int d^3p \int d^3p' \frac{1}{\sqrt{E_p}\sqrt{E_{p'}}} a^\dagger \bar{u}(p) e^{i\vec{p}\cdot\vec{x}} \cdot \vec{\gamma}\cdot\vec{p}'$$

$$\cdot \left(a_{p'} u(p') e^{-i\vec{p}'\cdot\vec{x}}\right)$$

$$\approx \int d^3p \frac{1}{E_p} a^\dagger a \, \bar{u} \vec{\gamma}\cdot\vec{p} \, u$$

$$\bar{u}(p) \gamma u(p) = 2m \delta^{rs}$$

Gordon identity

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu + p^\mu}{2m} + \frac{i\sigma^{\mu\nu} q_\nu}{2m}\right] u(p)$$

$$\underline{p'=p} \quad \to \frac{p^\mu}{2m} \bar{u}(p) u(p)$$

$$= p^\mu$$

$$\Rightarrow \frac{1}{3}\int d^3p \, \frac{p^2}{E(p)} a^\dagger a$$

$\hookrightarrow p^2 = p_1^2 + p_2^2 + p_3^2$

T_{ii} = Pressure !

$$
\begin{aligned}
\mathcal{L} = {} & \bar{\psi}[i\gamma_\mu \partial^\mu - (M_N - g_\sigma \phi - g_\delta \vec{\tau} \cdot \vec{\delta}) - g_\omega \gamma_\mu \omega^\mu \\
& - g_\rho \gamma^\mu \vec{\tau} \cdot \vec{b}_\mu]\psi + \tfrac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m_\sigma^2 \phi^2) - U(\phi) \\
& + \tfrac{1}{2} m_\omega^2 \omega_\mu \omega^\mu + \tfrac{1}{2} m_\rho^2 \vec{b}_\mu \cdot \vec{b}^\mu + \tfrac{1}{2}(\partial_\mu \vec{\delta} \cdot \partial^\mu \vec{\delta} - m_\delta^2 \vec{\delta}^2) \\
& - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{4} \vec{G}_{\mu\nu} \vec{G}^{\mu\nu}.
\end{aligned}
$$

$$
U(\phi)' = \tfrac{1}{3} a \phi^3 + \tfrac{1}{4} b \phi^4
$$

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$$[i\gamma_\mu \partial^\mu - (M_N - g_\sigma \phi - g_\delta \tau_3 \delta_3) - g_\omega \gamma^0 \omega_0 - g_\rho \gamma^0 \tau_3 b_0]\psi$$

$$= 0,$$

$$m_\sigma^2 \phi + a\phi^2 + b\phi^3 = \overline{\psi}\psi = g_\sigma \rho_S,$$

$$m_\omega^2 \omega_0 = g_\omega \overline{\psi}\gamma^0 \psi = g_\omega \rho_B,$$

$$m_\rho^2 b_0 = g_\rho \overline{\psi}\gamma^0 \tau_3 \psi = g_\rho \rho_{B3},$$

$$m_\delta^2 \delta_3 = g_\delta \overline{\psi}\tau_3 \psi = g_\delta \rho_{S3},$$

$$\epsilon = \sum_{i=n,p} 2\int \frac{d^3k}{(2\pi)^3} E_i^\star(k)[n_i(k)+\overline{n_i}(k)] + \frac{1}{2}m_\sigma^2\phi^2 + U(\phi)$$

$$+\frac{1}{2}m_\omega^2\omega_0^2 + \frac{1}{2}m_\rho^2 b_0^2 + \frac{1}{2}m_\delta^2\delta_3^2 , \qquad \frac{g_\omega^2}{2m_\omega^2}\rho_B^2 + \frac{g_\rho^2}{2m_\rho^2}\rho_{B3}^2 + \frac{g_\delta^2}{2m_\delta^2}\rho_{S3}^2$$

$$P = \sum_{i=n,p} \frac{2}{3}\int \frac{d^3k}{(2\pi)^3} \frac{k^2}{E_i^\star(k)}[n_i(k)+\overline{n_i}(k)] - \frac{1}{2}m_\sigma^2\phi^2$$

$$-U(\phi) + \frac{1}{2}m_\omega^2\omega_0^2 + \frac{1}{2}m_\rho^2 b_0^2 - \frac{1}{2}m_\delta^2\delta_3^2 , \qquad \frac{g_\omega^2}{2m_\omega^2}\rho_B^2 + \frac{g_\rho^2}{2m_\rho^2}\rho_{B3}^2 - \frac{g_\delta^2}{2m_\delta^2}\rho_{S3}^2$$

$$n_i(k) = \frac{1}{1+\exp\{(E_i^\star(k)-\mu_i^\star)/T\}}$$

$$M_i^\star = M_N - g_\sigma\phi \mp g_\delta\delta_3 \quad (-\,\text{proton}, +\,\text{neutron}).$$

$$\mu_i = \mu_i^\star - g_\omega\omega_0 \mp g_\rho b_0 \quad (-\,\text{proton}, +\,\text{neutron})$$

$$E_{sym} \equiv \frac{1}{2} \frac{\partial^2 E(\rho_B, \alpha)}{\partial \alpha^2}\bigg|_{\alpha=0} = \frac{1}{2} \rho_B \frac{\partial^2 \epsilon}{\partial \rho_{B3}^2}\bigg|_{\rho_{B3}=0}$$

$$\rho_{B3} = \rho_{Bp} - \rho_{Bn}$$

$$E_{sym}(\rho_B) = \frac{1}{6} \frac{k_F^2}{E_F} + \frac{1}{2} f_\rho \rho_B$$

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$$\rho = \frac{2k_F^2}{3\pi^2}$$

$$k_F^n = k_F (1+\alpha)^{1/3}$$

$$k_F^p = k_F (1-\alpha)^{1/3}$$

$$\frac{\partial^2}{\partial \alpha^2} = \frac{\partial}{\partial \alpha} \cdot \left(\frac{\partial k_F^n}{\partial \alpha} \frac{\partial \alpha}{\partial k_F^n} \right)$$

Inputs to fix the parameters in the model

1. Saturation density: 0.16 fm^{-3}
2. Binding energy: E/A= -16 MeV
3. Incompressibility: K=240 MeV
4. Nucleon effective mass: M* = 0.75M
5. Symmetry energy at the saturation density: 30.5 MeV

$$E_{sym}(\rho_B) = \frac{1}{6}\frac{k_F^2}{E_F} + \frac{1}{2} f_\rho \rho_B$$

TABLE I. Parameter sets.

Parameter	Set I	Set II	NL3
f_σ (fm^2)	10.33	same	15.73
f_ω (fm^2)	5.42	same	10.53
f_ρ (fm^2)	0.95	3.15	1.34
f_δ (fm^2)	0.00	2.50	0.00
A (fm^{-1})	0.033	same	-0.01
B	-0.0048	same	-0.003

$A \equiv a/g_\sigma^3$ and $B \equiv b/g_\sigma^4$

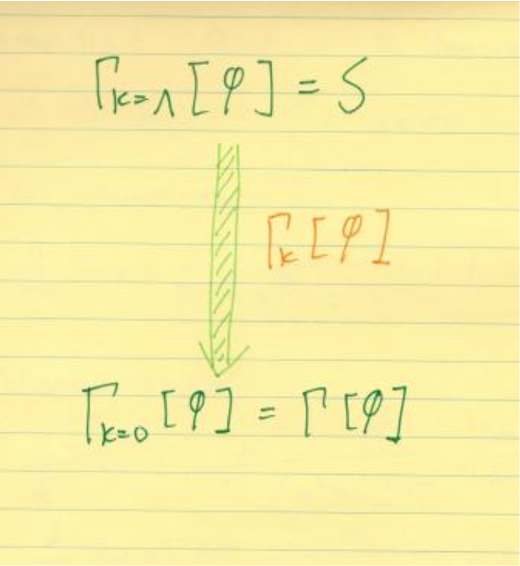
Beyond the mean field approximation: FRG

Fluctuations

- Nucleons, though massive, can fluctuate near the Fermi surface as p-h excitations.
- The pion and sigma mesons can fluctuate.
- Vector mesons such as omega mesons may not fluctuate because they are massive. Only as mean field.
- FRG is a good way to handle (above-mentioned) fluctuations.

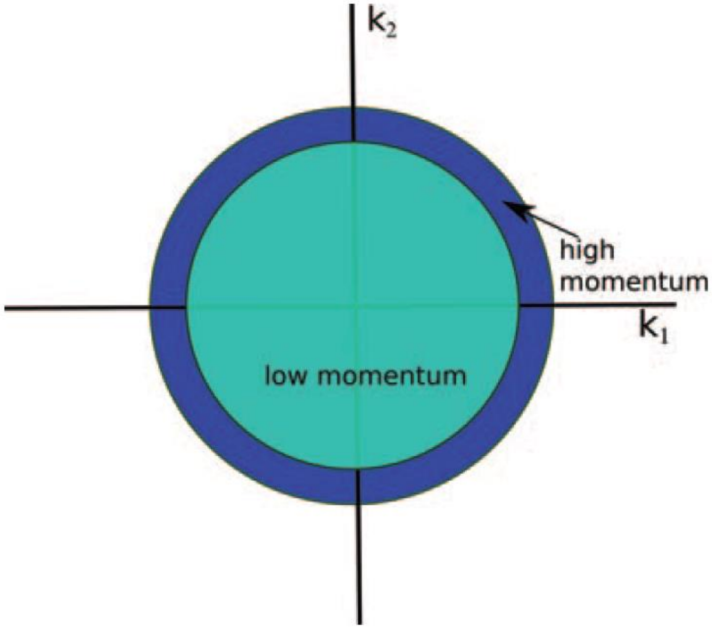
Renormalization group method with different goals

- To remove infinities (UV divergences)
- To describe the scale dependence of physical parameters
- To re-sum the perturbation expansion in QFT
- To solve strongly coupled theories
- …



Effective action in QFT:

- The generating functional of the 1PI Green functions.
- The field equations derived from the effective action include all quantum effects.
- In thermal and chemical equilibrium the effective action includes in addition the thermal fluctuations and depends on the temperature and chemical potential.
- In statistical physics it corresponds to the free energy as a functional of some (space dependent) order parameter.



- flow of Schwinger functional $W_k[j]$: Polchinski equation
- flow of effective action $\Gamma_k[\varphi]$: Wetterich equation
- flow from classical action $S[\varphi]$ to effective action $\Gamma[\varphi]$
- applied to variety of physical systems
  - strong interaction
  - electroweak phase transition
  - asymptotic safety scenario
  - condensed matter systen
    e.g. Hubbard model, liquid He^4, frustrated magnets, superconductivity ...
  - effective models in nuclear physics
  - ultra-cold atoms

The average action Γ_k is a simple generalization of the effective action, with the distinction that only fluctuations with momenta $q^2 \gtrsim k^2$ are included.

Γ_k interpolates between the classical action S and the effective action Γ as k is lowered from the ultraviolet cutoff Λ to zero: $\lim_{k\to\Lambda} \Gamma_k = S$, $\lim_{k\to 0} \Gamma_k = \Gamma$.

Wetterich Equation

$$Z[J] = \int D\phi \, e^{-S[\phi] + J\cdot\phi}$$

$$J\cdot\phi = \int d^4x \, J_a(x) \, \phi_a(x)$$

$$\langle \phi^n \rangle = \frac{1}{Z}\frac{\delta^n Z}{\delta^n J} = \frac{1}{Z}\int D\phi \, \phi^n e^{-S+\phi\cdot J}$$

$$W[J] = \ln Z[J]$$

↳ Schwinger functional

$$G = \frac{\delta^2 W}{\delta^2 J} = \frac{\delta}{\delta J}\left(\frac{1}{Z}\frac{\delta Z}{\delta J}\right)$$

$$= \frac{1}{Z}\frac{\delta^2 Z}{\delta^2 J} - \frac{1}{Z^2}\frac{\delta Z}{\delta J}\frac{\delta Z}{\delta J}$$

$$= \langle\phi\phi\rangle - \langle\phi\rangle\langle\phi\rangle$$

$$\equiv \langle\phi\phi\rangle_c$$

Introduce a cutoff ΔS_k that vanishes in the IR.

$$W_k[J] = \ln Z_k[J]$$
$$= \ln \int D\phi \, e^{-S[\phi] + J\cdot\phi - \Delta S_k[\phi]}$$

k: renormalization scale, we are probing.

$$\Delta S_k[\phi] = \frac{1}{2} \phi \cdot R_k \cdot \phi$$

$$= \frac{1}{2} \int_{xy} \phi_a(x) R_{k,ab}(x,y) \phi_b(y)$$

↳ momentum dependent mass
UV and IR regulator!

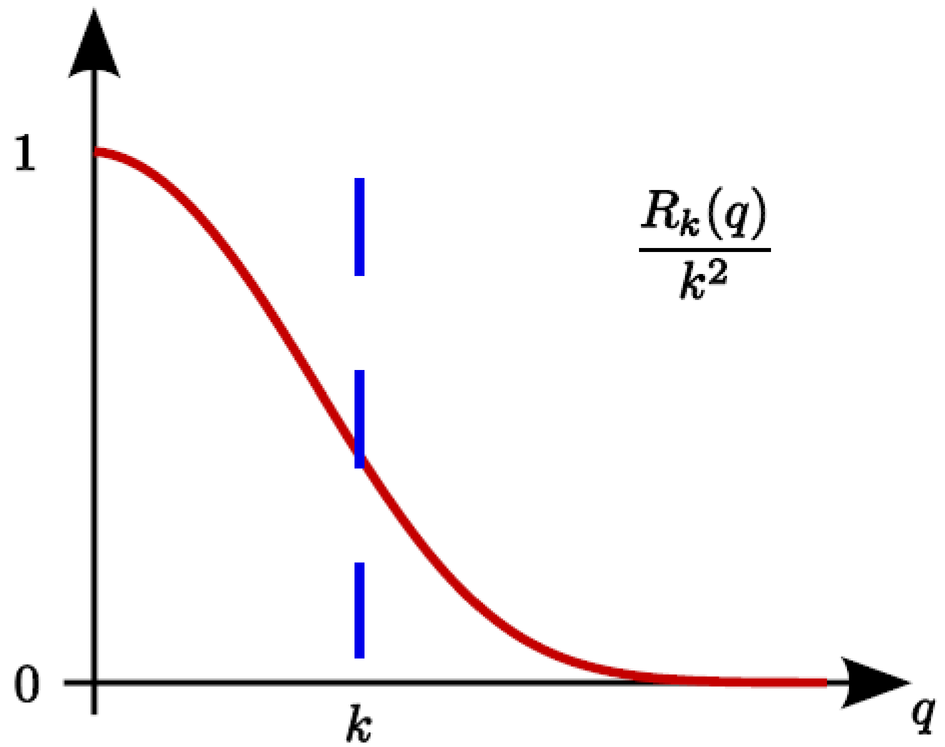
At fixed J,

$$\partial_k W_k[J] = -\frac{1}{Z_k} \int D[\phi] (\partial_k \Delta S_k[\phi]) \, e^{-S + J\cdot\phi - \Delta S_k}$$

$$= -\frac{1}{2} \langle \phi \, \partial_k R_k \, \phi \rangle$$

Using $\langle \phi\phi \rangle = \langle \phi\phi \rangle_c + \langle \phi \rangle \langle \phi \rangle$,
↳ φ

$$= -\frac{1}{2} \left(\langle \phi\phi \rangle_c + \varphi\varphi \right) \partial_k \cdot R_k$$



Typical form of the regulator function R_k

$$\langle \phi\phi \rangle_c \equiv W_k^{(2)}$$

$$= \frac{\delta^2 W_k}{\delta J} = \frac{\delta \varphi}{\delta J}$$

Now we arrive at Polchinski's equation.

$$\partial_k W_k[J] = -\frac{1}{2} \mathrm{Tr}\left[W_k^{(2)} \partial_k R_k\right]$$

$$-\frac{1}{2} \varphi (\partial_k R_k) \varphi$$

→ Integration over x (or p) and summation over a, b.

$$\mathrm{Tr}\left[(\partial_k R_k) W_k^{(2)}\right]$$

$$= \int_{x,y} W_{k,ab}^{(2)}(x,y)\, \partial_k R_{k,ab}(x,y)$$

Effective action

$(J \longleftrightarrow \varphi)$ why $\phi_{cl} \neq \phi_{cl} + \delta\phi^{1\text{-loop}}$, etc

is a function in the full QFT

gives the exact value of $\langle \phi \rangle \Rightarrow \varphi$

Analogy

Magnetic system	Q.F.T
$S(x)$	$\phi(x)$
H	$J(x)$
$\mathcal{H}(S)$	$\mathcal{L}(\phi)$
$Z(H)$	$Z[J]$
$F(H)$	$W[J]$
M	$\varphi\ (\langle\phi\rangle)$
$G(M)$	$-\Gamma[\langle\phi\rangle]$

< Digression >

Legendre transformation

$L(q, \dot{q}) \longleftrightarrow H(q, p)$

Consider $f(x, y)$: x, y are independent

① $df = \left.\frac{\partial f}{\partial x}\right|_y dx + \left.\frac{\partial f}{\partial y}\right|_x dy$

($\frac{\partial f}{\partial x} = u$, $\frac{\partial f}{\partial y} = v$)

$= u\,dx + v\,dy$ (u, x) (v, y)

Conjugate Variables

② $d(vy) = y\,dv + v\,dy$

① − ②

$d(f - vy) = u\,dx - y\,dv$

($f - vy = g$)

$g \rightarrow g(x, v)$

$\therefore g \equiv f - vy$

$$\hat{\Gamma}_k[\varphi] = J\cdot\varphi - W_k[J]$$

$$\frac{\partial \hat{\Gamma}_k}{\partial \varphi} = J_k$$

$$\frac{\partial^2 \hat{\Gamma}_k}{\partial^2 \varphi} = \frac{\partial J_k}{\partial \varphi} \left(= \tilde{\Gamma}_k^{(2)} \right)$$

↳ inverse propagator

$$? \quad \left(\tilde{\Gamma}_k^{(2)} W_k^{(2)}\right)_{ab}(x,y)$$

$$= \int_z \frac{\delta J_c(z)}{\delta \varphi_a(x)} \frac{\delta \varphi_b(y)}{\delta J_c(z)}$$

$$= \frac{\delta \varphi_b(y)}{\delta \varphi_a(x)} \rightarrow \delta_{ab}\,\delta(x-y)$$

$$\therefore \quad W_k^{(2)} = \left(\tilde{\Gamma}_k^{(2)}\right)^{-1}$$

$$= \left(\Gamma_k^{(2)} + R_k\right)^{-1}$$

for fixed φ

$$\partial_k \hat{\Gamma}_k = \varphi\, \partial_k J - \partial_k W_k[J]$$

$$- \frac{\delta W}{\delta J} \frac{\partial J}{\partial k}$$

($\frac{\delta W}{\delta J} \to \varphi$)

$$= - \partial_k W_k [J]$$

$$\Gamma_k[\varphi] = \hat{\Gamma}_k[\varphi] - \Delta S_k$$

$$\partial_k \Gamma_k[\varphi] = - \partial_k W_k \Big|_J - \frac{1}{2} \varphi (\partial_k R_k) \varphi.$$

$$\left(- \frac{1}{2} \mathrm{Tr} [W_k^{(2)} \partial_k R_k] - \frac{1}{2} \varphi (\partial_k R_k) \varphi \right)$$

$$= + \frac{1}{2} \mathrm{Tr} [W_k^{(2)} \partial_k R_k]$$

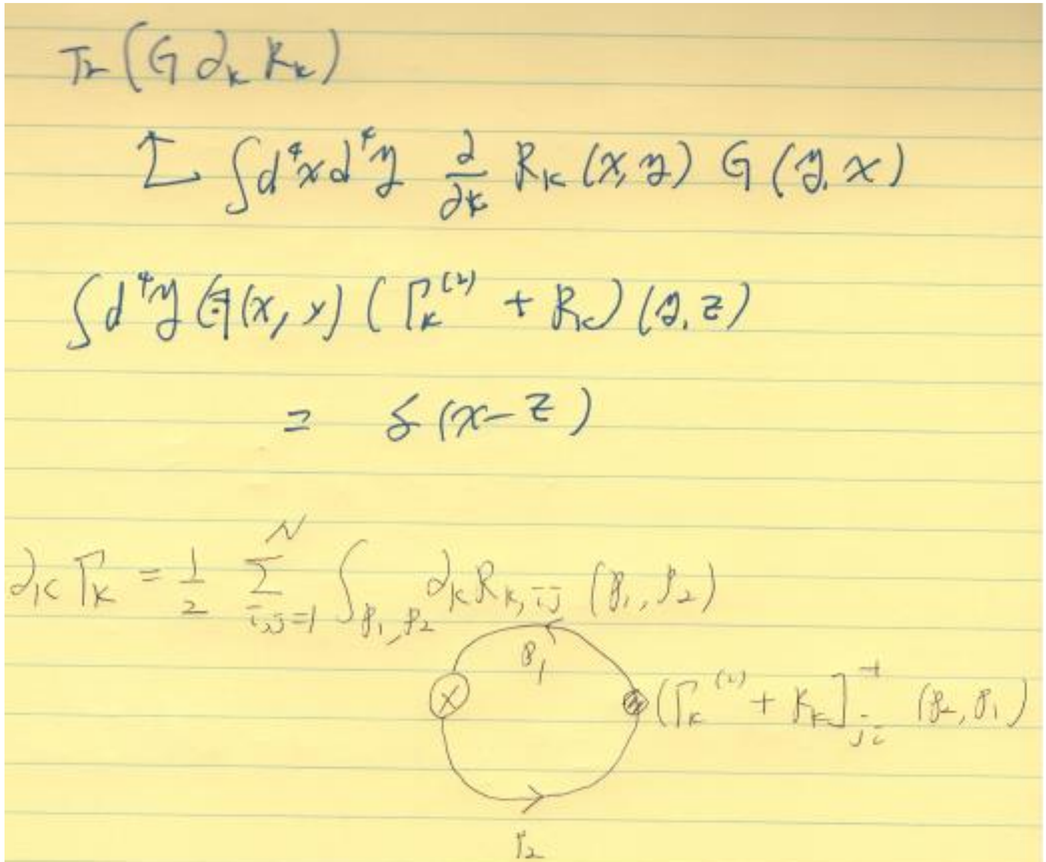
$$= \frac{1}{2} \mathrm{Tr} \left[(\Gamma_k^{(2)} + R_k)^{-1} \partial_k R_k \right]$$

< Wetterich eq. >

In terms of W_k the average action is defined via a modified Legendre transform

$$\Gamma_k[\phi] = -W_k[J] + \int d^d x J_a(x)\phi^a(x) - \Delta S_k[\phi]$$

where we have subtracted the term $\Delta S_k[\phi]$ on the r.h.s. This subtraction of the infrared cutoff term as a function of the macroscopic field ϕ is crucial for the definition of a reasonable coarse grained free energy with the property $\lim_{k\to\Lambda}\Gamma_k = S$.



$$k\,\frac{\partial \Gamma_k}{\partial k} =$$

$$= \frac{1}{2}\,\mathrm{Tr}\,\frac{k\,\frac{\partial R_k}{\partial k}}{\Gamma_k^{(2)} + R_k},$$

In practice some approximations (truncations) are required. One them is "the derivative expansion," which sounds reasonable since we are mostly interested in the long distance physics (small momentum).

A Simple example in 0+1 dim.

$$S = \int d\tau \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 + \frac{\lambda}{24} x^4 \right)$$

↳ bare action. $\omega^2, \lambda > 0$

Determine the ground state energy!

$$R_k(p) = (k^2 - p^2)\,\theta(k^2 - p^2)$$

↳ regulator

$$\partial_t R_k = 2k^2 \theta(k^2 - p^2) \; + \sim (k^2 - p^2)\,\delta(k^2 - p^2) \Rightarrow 0 \text{ integration} \qquad \left(t = \ln \frac{k}{\Lambda} \right)$$

$$\Gamma_k^{(2)} = \left(-\partial_\tau^2 + V_k''(x)\right) \delta(\tau - \tau') \qquad \left(\leftarrow \frac{\partial \Gamma_k}{\partial x^2} \right)$$

x-independent.

$$\Gamma_k[x] = \int d\tau \left(\frac{1}{2}\dot{x}^2 + V_k(x) \right)$$

partial integration: $-\frac{1}{2} x \partial_\tau^2 x$

↳ $c + \frac{1}{2} V_k''(x)\, x^2 = \frac{1}{2} x V_k''(x)\, x$

$$\partial_t V_k(x) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dp_\tau}{2\pi}\, \frac{2k^2\,\theta(k^2-p_\tau^2)}{k^2+V_k''(x)}$$

$\partial_t \to k\frac{\partial}{\partial k}$ $\qquad$ $\int \frac{dp_\tau}{2\pi}$ from "Tr"

$$\left(\Gamma_k^{(2)} + R_k(p_\tau) = \cancel{p_\tau^2} + V'' + k^2 - \cancel{p_\tau^2} \right)$$

$$\Rightarrow \frac{d}{dk} V_k(x) = \frac{1}{\pi}\, \frac{k^2}{k^2+V_k''(x)}$$

$$\left(\int_{-\infty}^{x} H(y)\,dy = xH(x) = \max\{0,x\} \right)$$

Polynomial expansion of V_k

$$V_k(x) = \frac{1}{2} m_k^2 x^2 + \frac{1}{24}\lambda_k x^4 + \cdots + \tilde{E}_k$$

$$\frac{d\hat{E}_k}{dk} = \frac{1}{\pi}\, \frac{k^2}{k^2+m_k^2}$$

In the limit $\lambda = 0$, $w = 0$, the ground ~~state~~ state

energy $E_{0,k} = 0$ is zero

To ensure that

$$\frac{d}{dk} E_{0,k} = \frac{1}{\pi} \left(\frac{k^2}{k^2 + w_k^2} - 1 \right) \qquad \underline{(1)}$$

Higher orders,

$$\frac{d}{dk} w_k^2 = -\frac{2}{\pi} \frac{k^2}{(k^2 + w_k^2)^2} \frac{\lambda_k}{2}, \qquad \underline{(2)}$$

$$\frac{d}{dk} \lambda_k = \frac{24}{\pi} \frac{k^2}{(k^2 + w_k^2)^3} \left(\frac{\lambda_k}{2} \right)^2 + \cdots \qquad \underline{(3)}$$

Let's solve Eqs (1), (2), (3) with $\lambda_k \to \lambda$.

Then we obtain w_k^2 from eq. (2)

and $E_{0,k}$ from Eq. (1).

Expanding the result in λ, we have

(to compare with PT)

$$E_0 = \frac{1}{2} w + \frac{3}{4} w \left(\frac{\lambda}{24 w^3} \right) - \frac{3(8\pi^2 + 29)}{16\pi^2} w \left(\frac{\lambda}{24 w^3} \right)^2$$

The result from 2nd-order perturbation

$$E_0^{PT} = \frac{1}{2}\omega + \frac{3}{4}\omega\left(\frac{\lambda}{24\omega^3}\right) - \frac{21}{8}\omega\left(\frac{\lambda}{24\omega^3}\right)^2 + \cdots$$

Finite temperature and density?

At nonzero temperature T and chemical potential μ our ansatz for Γ_k reads

$$\Gamma_k = \int_0^{1/T} dx^0 \int d^3x \Big\{ i\overline{\psi}^a (\gamma^\mu \partial_\mu + \mu\gamma^0)\psi_a + \overline{h}_k \overline{\psi}^a \left[\frac{1+\gamma^5}{2} \Phi_a{}^b - \frac{1-\gamma^5}{2} (\Phi^\dagger)_a{}^b \right] \psi_b$$

$$+ Z_{\Phi,k} \partial_\mu \Phi^*_{ab} \partial^\mu \Phi^{ab} + U_k(\overline{\rho}; \mu, T) \Big\} .$$

$$\mathcal{Z} = \mathrm{Tr}\, e^{-\beta \hat{H}} = \sum_n \langle n | e^{-\beta \hat{H}} | n \rangle = \sum_n e^{-\beta E_n}, \qquad Z(\beta) = \int dx \, \langle x | e^{-\beta H} | x \rangle$$

$$_H\langle x_f, t_f | x_i, t_i \rangle_H = \langle x_f | e^{-\frac{i}{\hbar} H (t_f - t_i)} | x_i \rangle = \int \mathcal{D}x \, e^{\frac{i}{\hbar} S[x]}$$

$$T = -i\beta$$

$$S_E[x] = \int_0^\beta dt \, L_E(x, \dot{x})$$

$$Z(\beta) = \int \mathcal{D}x \, e^{-S_E[x]}$$

$$x(\beta) = x(0)$$

$$\omega_n = \begin{cases} \frac{2n\pi}{\beta} & \text{for bosons} \\ \frac{(2n+1)\pi}{\beta} & \text{for fermions} \end{cases}$$

QCD in Extreme Conditions*
and the
Wilsonian 'Exact Renormalization Group'

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Abstract

This is an introduction to the use of nonperturbative flow equations in strong interaction physics at nonzero temperature and baryon density. We investigate the QCD phase diagram as a function of temperature, chemical potential for baryon number and quark mass within the linear quark meson model for two flavors. Whereas the renormalization group flow leads to spontaneous chiral symmetry breaking in vacuum, the symmetry is restored in a second order phase transition at high temperature and vanishing quark mass. We explicitly connect the physics at zero temperature and realistic quark mass with the universal behavior near the critical temperature T_c and the chiral limit. At high density we find a chiral symmetry restoring first order transition. The results imply the presence of a tricritical point with long-range correlations in the phase diagram. We end with an outlook to densities above the chiral transition, where QCD is expected to behave as a color superconductor at low temperature.

Based on five lectures presented at the 11th Summer School and Symposium on Nuclear Physics "Effective Theories of Matter", Seoul National University, June 23–27, 1998.

$$\frac{\partial}{\partial k} U_{kF}(\rho;\mu) = -\,8N_c \int\limits_{-\infty}^{\infty} \frac{d^4q}{(2\pi)^4} \frac{k\Theta(k_\Phi^2 - q^2)}{q^2 + k^2 + h_k^2\rho/2}$$

$$+\,4N_c \int\limits_{-\infty}^{\infty} \frac{d^3\vec{q}}{(2\pi)^3} \frac{k}{\sqrt{\vec{q}^2 + k^2 + h_k^2\rho/2}} \;\; \Theta\Big(\mu - \sqrt{\vec{q}^2 + k^2 + h_k^2\rho/2}\Big)\,.$$

For scales $k > \mu$ the Θ–function vanishes identically
and there is no distinction between the vacuum evolution and the $\mu \neq 0$ evolution.