NPS2019@JejuIsland

Astrophysics of Compact Stars

*White Dwarfs, **Neutron Stars**, and Black Holes*
Gravitational Waves, X-ray Binaries, Dense Stellar Matter

Chang-Hwan Lee
Pusan National University

Motivation

- **Detections of gravitational waves**
- **Observations of x-ray binaries**
- **Heavy ion experiments**

RAON

LIGO Laser Interferometer Gravitational-Wave Observatory

NICER Neutron star Interior Composition ExploreR

Plan

1. Gravitational Waves from Compact Stars
   - Tidal Love number (tidal deformability) of neutron stars

2. Neutron Star Structure & Observations
   - White dwarf, neutron stars, and black holes
   - Low-mass X-ray binaries
   - Physics of Dense Stellar Matter

3. Workout Problem
   - Polytropic structure of compact stars

Part 1

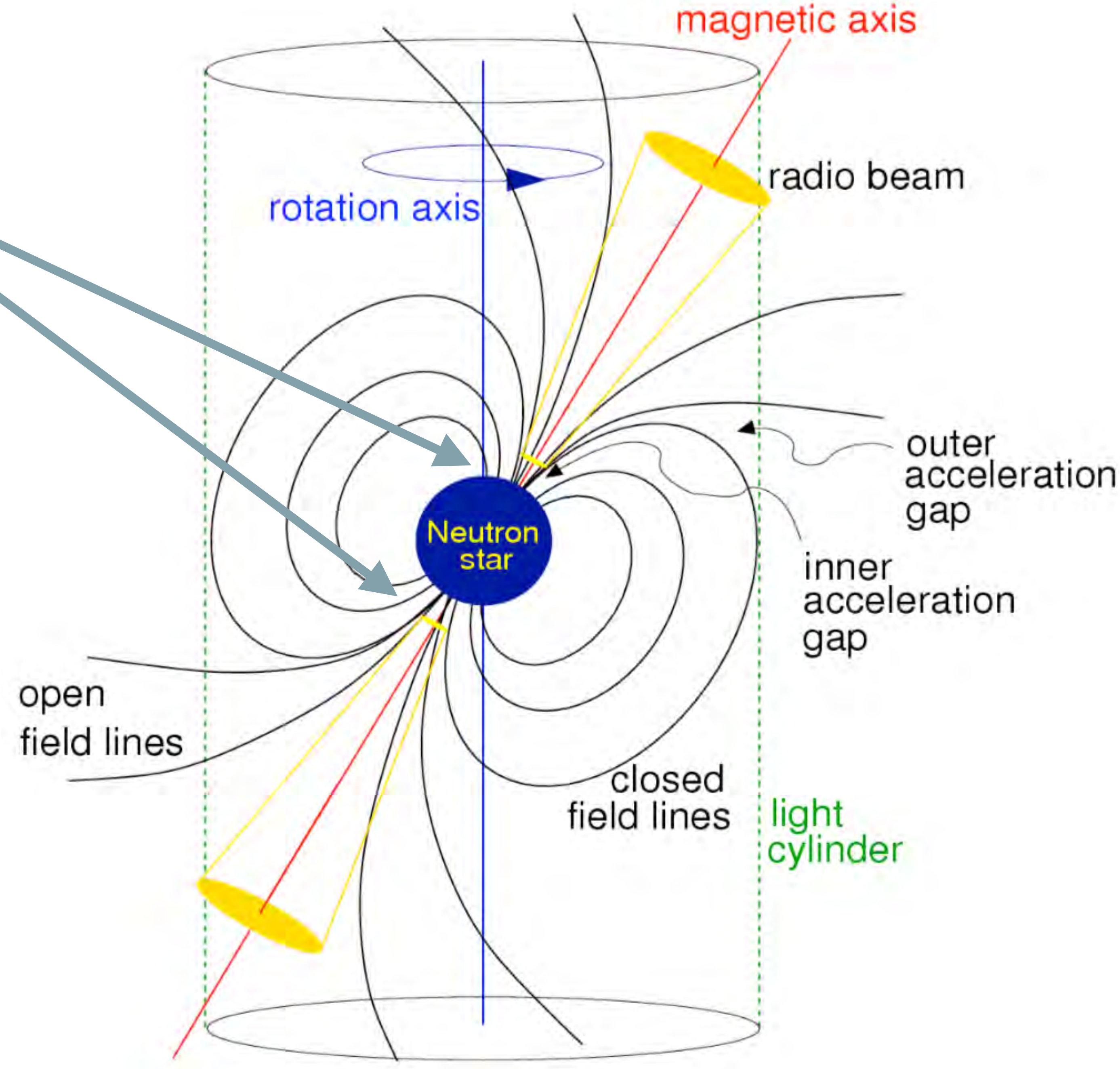
Gravitational Waves and Compact Stars

Why Neutron Stars ?

Ultimate testing place for physics of dense matter

e^+e^- pair creation

$$M = 1.5 \sim 2.0 M_\odot$$
$$R = 10 \sim 15 \text{ km}$$
$$A \sim 10^{57} \text{ nucleons}$$

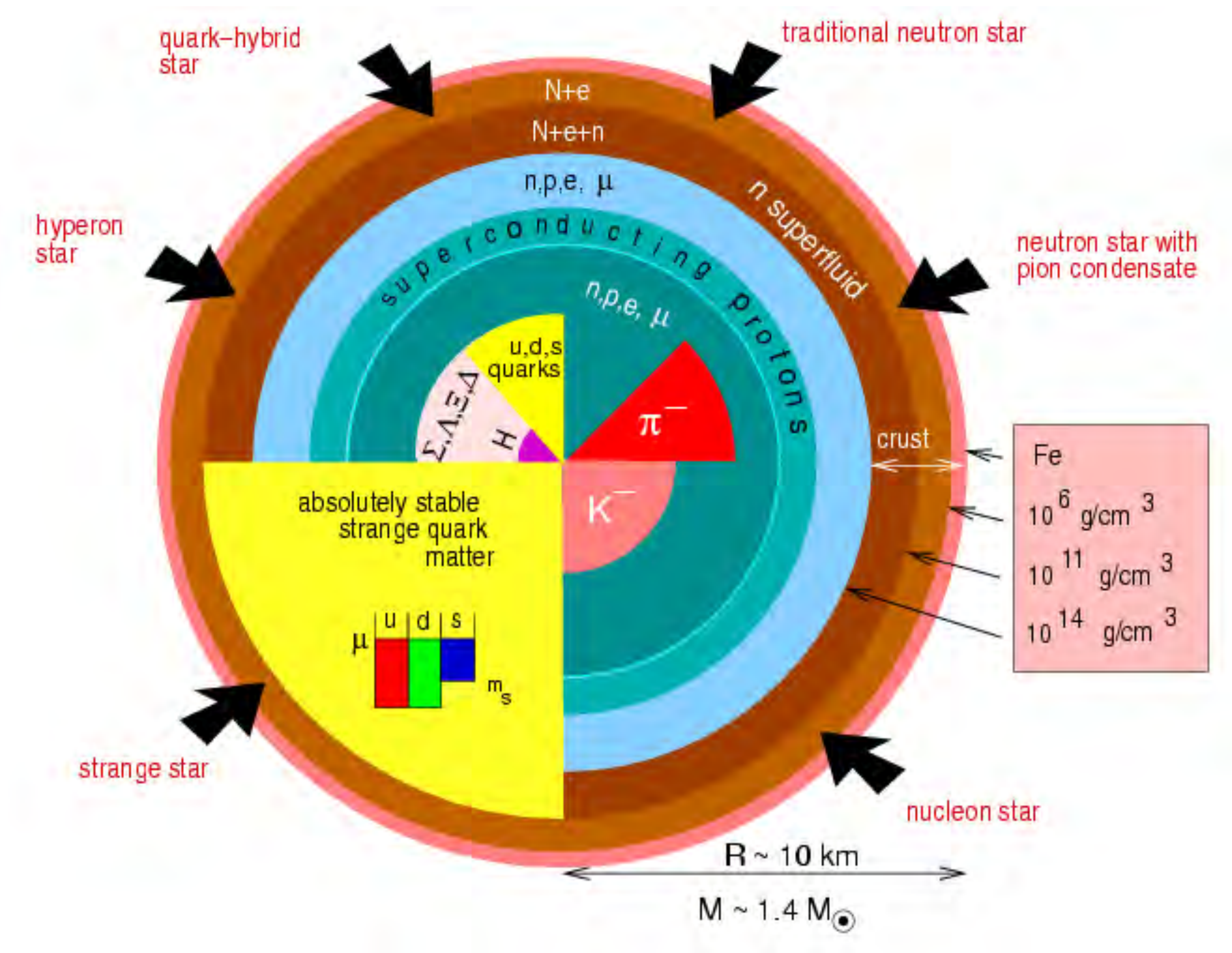


Nuclear matter is not an ideal gas

F. Weber 2005

Condensed Matter:
- *electron degeneracy*
- *EM interaction*

Neutron Star:
- *hadron (p,n) degeneracy*
- *strong interaction*

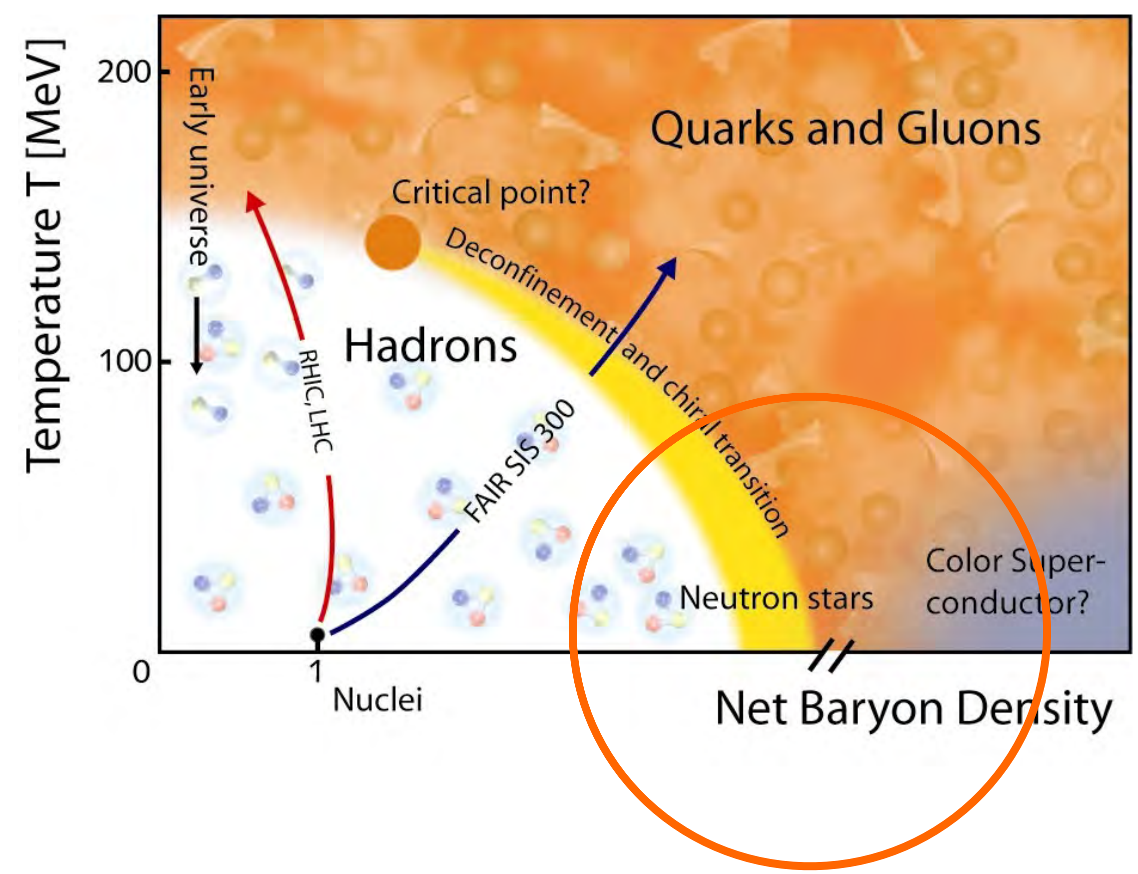


- **still uncertain due to the nature of strong interactions**
- **introduction of 3 body forces**
- **exotic states with strangeness**
- **… …**

Why neutron stars

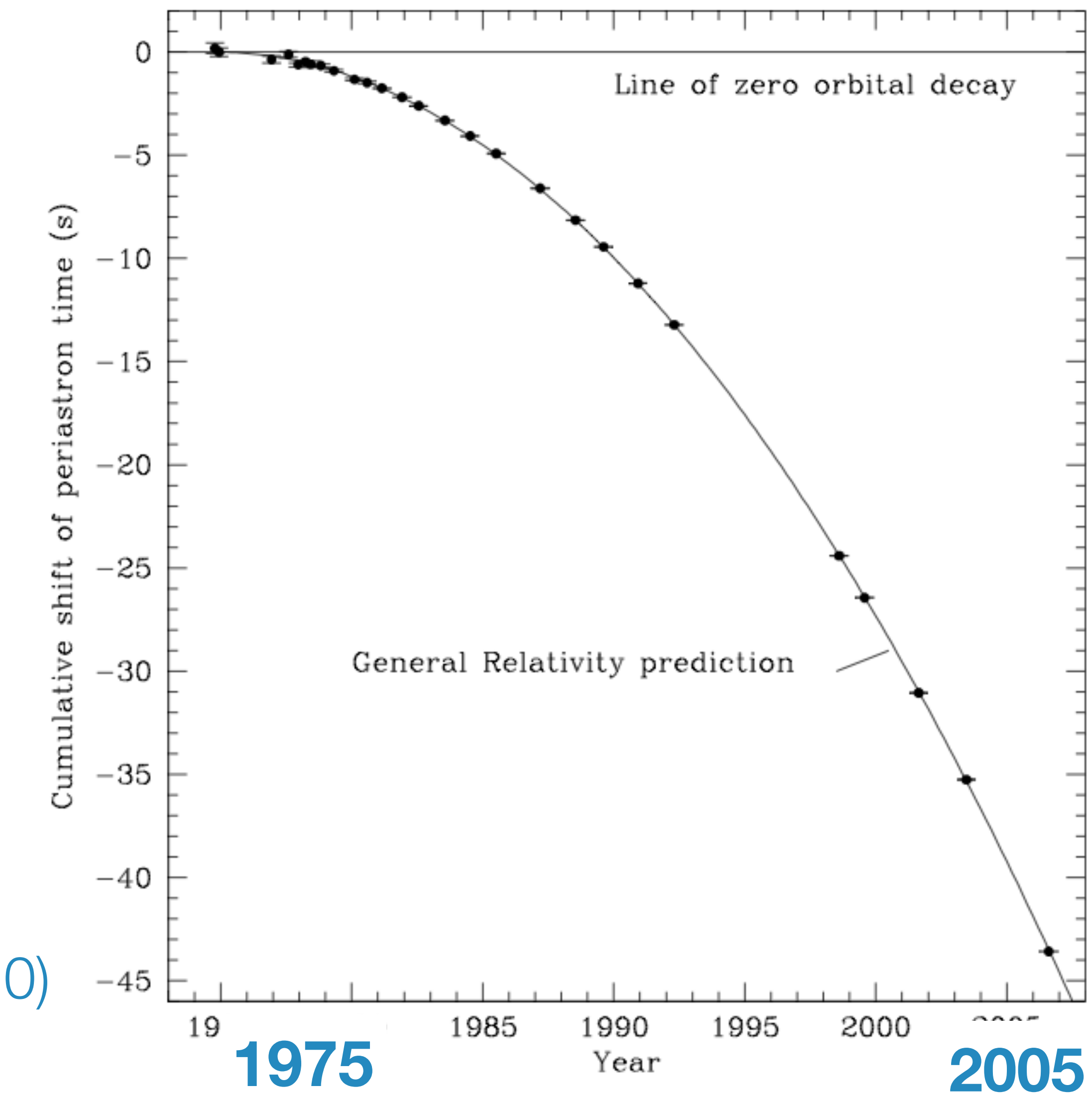
Ultimate testing place for physics of dense matter

- ✓ chiral symmetry restoration
- ✓ color superconductivity
- ✓ color-flavor locking
- ✓ quark-gluon-plasma
- ✓ AdS/QCD
- ✓ symmetry energy
- ✓ tensor forces
- ✓ 3-body forces
- ✓ … …



Gravitational waves from neutron star binaries

- B1913+16 / Hulse & Taylor (1975)
- change in the orbital period due to GW radiation
- 1993 Nobel Prize
- LIGO is based on
  NS binary mergers
- GW expected in **2019**
  d = O(100 Mpc)

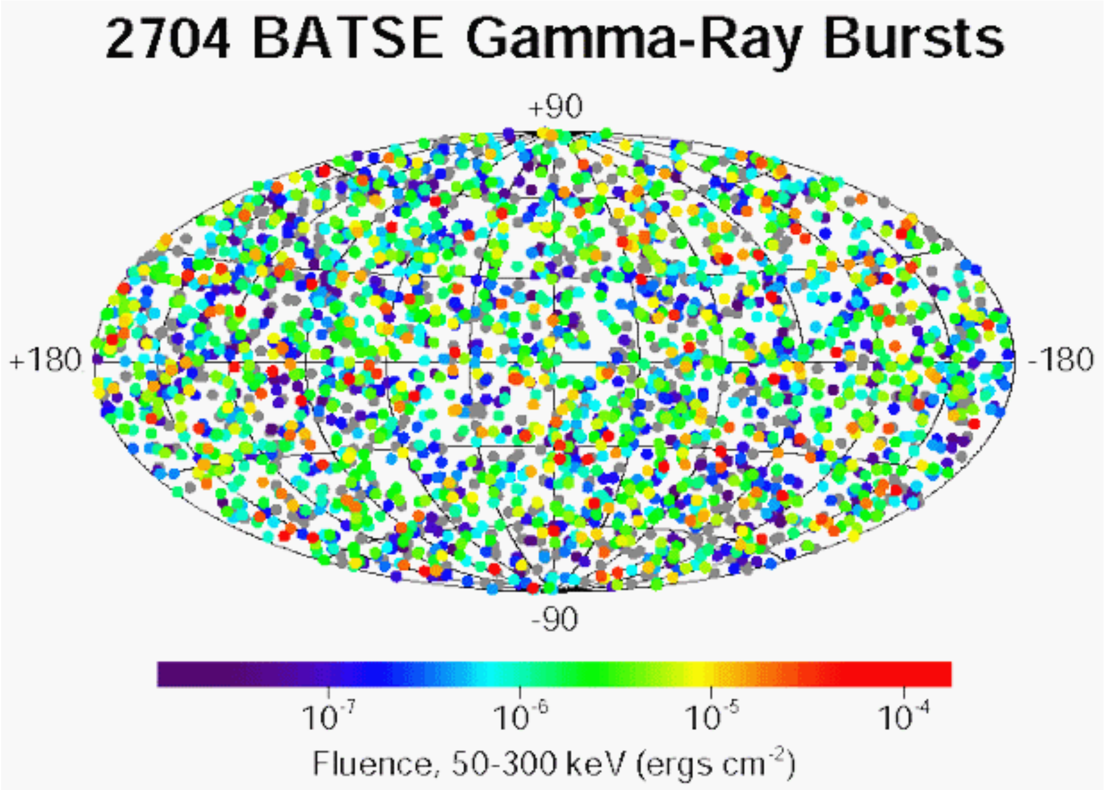


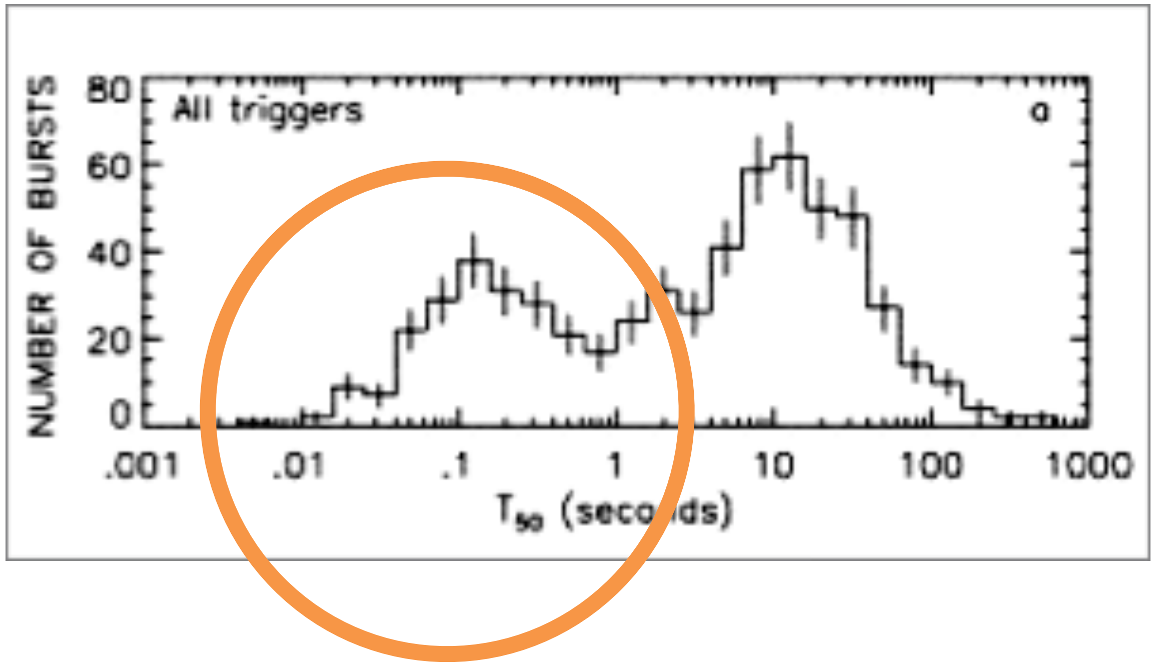
Weisberg, Nice, Taylor, ApJ (2010)

NS (radio pulsar) which will coalesce within Hubble time

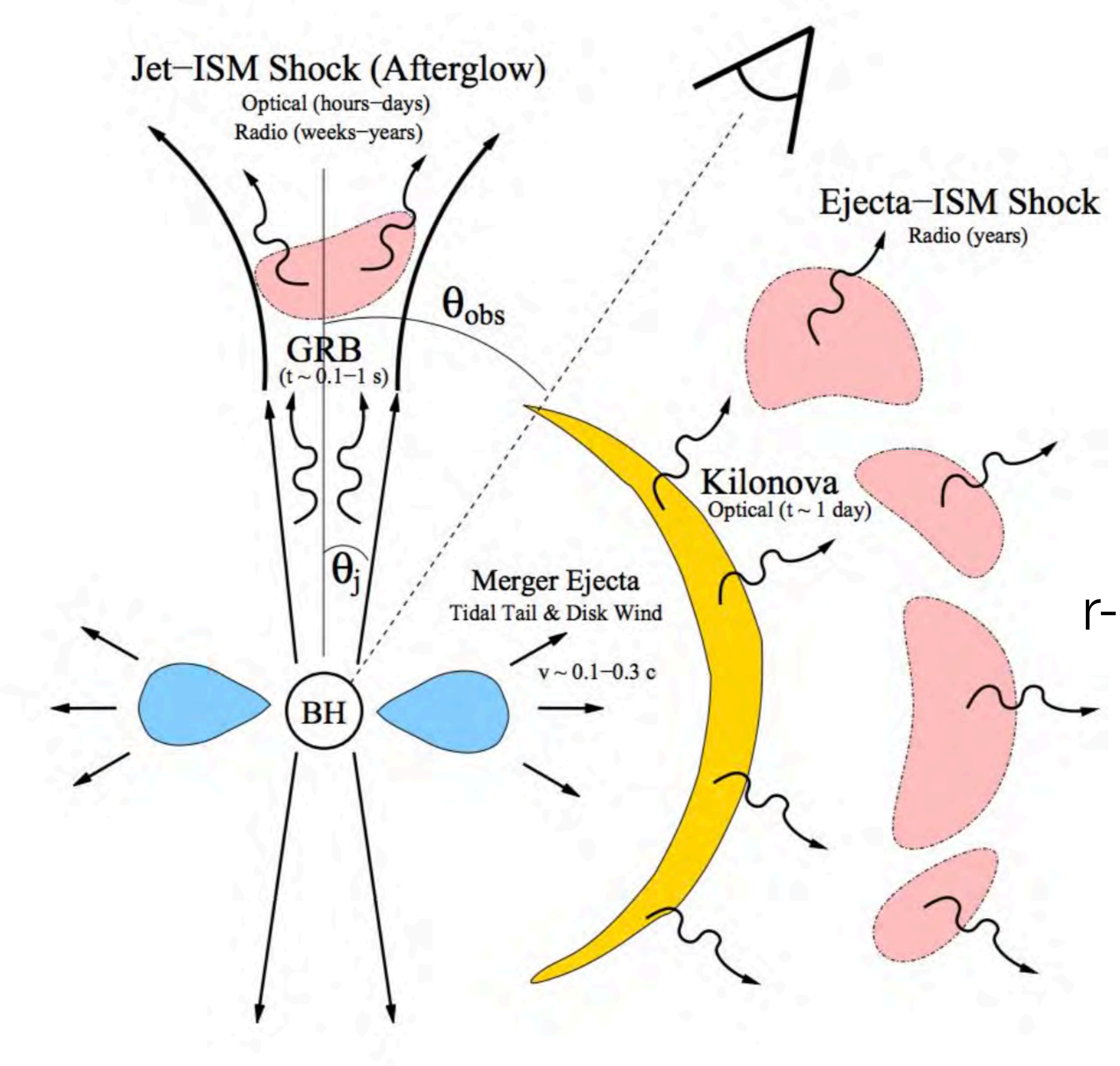
PSR	P (ms)	P_b (hr)	e	Total Mass $M_\odot$	τ_c (Myr)	τ_{GW} (Myr)	
J0737−3039A	22.70	2.45	0.088	2.58	210	87	(2003)
J0737−3039B	2773	2.45	0.088	2.58	50	87	(2004)
B1534+12	37.90	10.10	0.274	2.75	248	2690	(1990)
J1756−2251	28.46	7.67	0.181	2.57	444	1690	(2004)
B1913+16	59.03	7.75	0.617	2.83	108	310	(1975)
B2127+11C	30.53	8.04	0.681	2.71	969	220	(1990)
J1141−6545†	393.90	4.74	0.172	2.30	1.4	590	(2000)

sGRB short-hard gamma-ray bursts from NS mergers





GRB and Kilonova from NS binaries



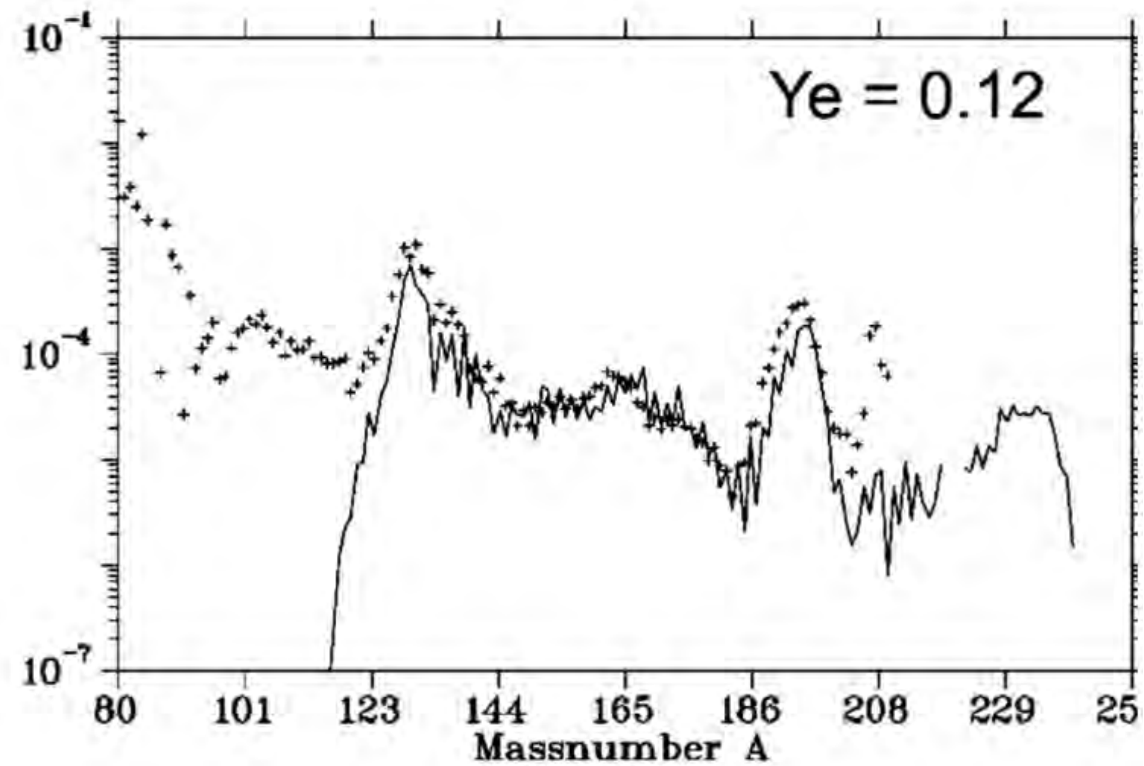
r-process **peak at A~195**

Metzger and Berger 2012

Heavy Elements from NS mergers

Sources of Heavy Elements

S Rosswog 2015

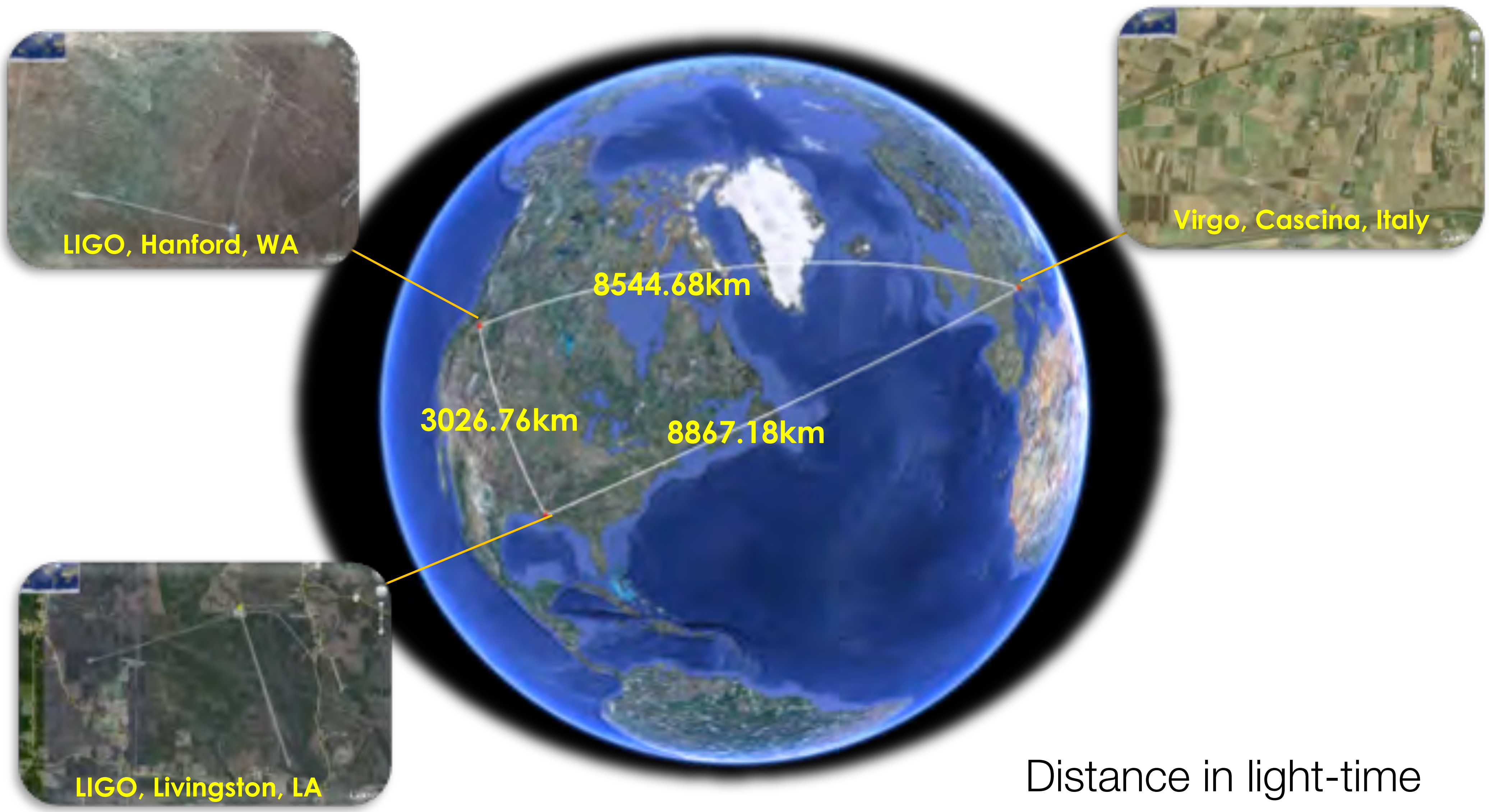


solar pattern vs NS-merger

- **Supernovae:**
  neutrino-driven wind
  r-process **peak at A~130**

- **NS mergers:**
  r-process **peak at A~195**

Network of gravitational wave observatories

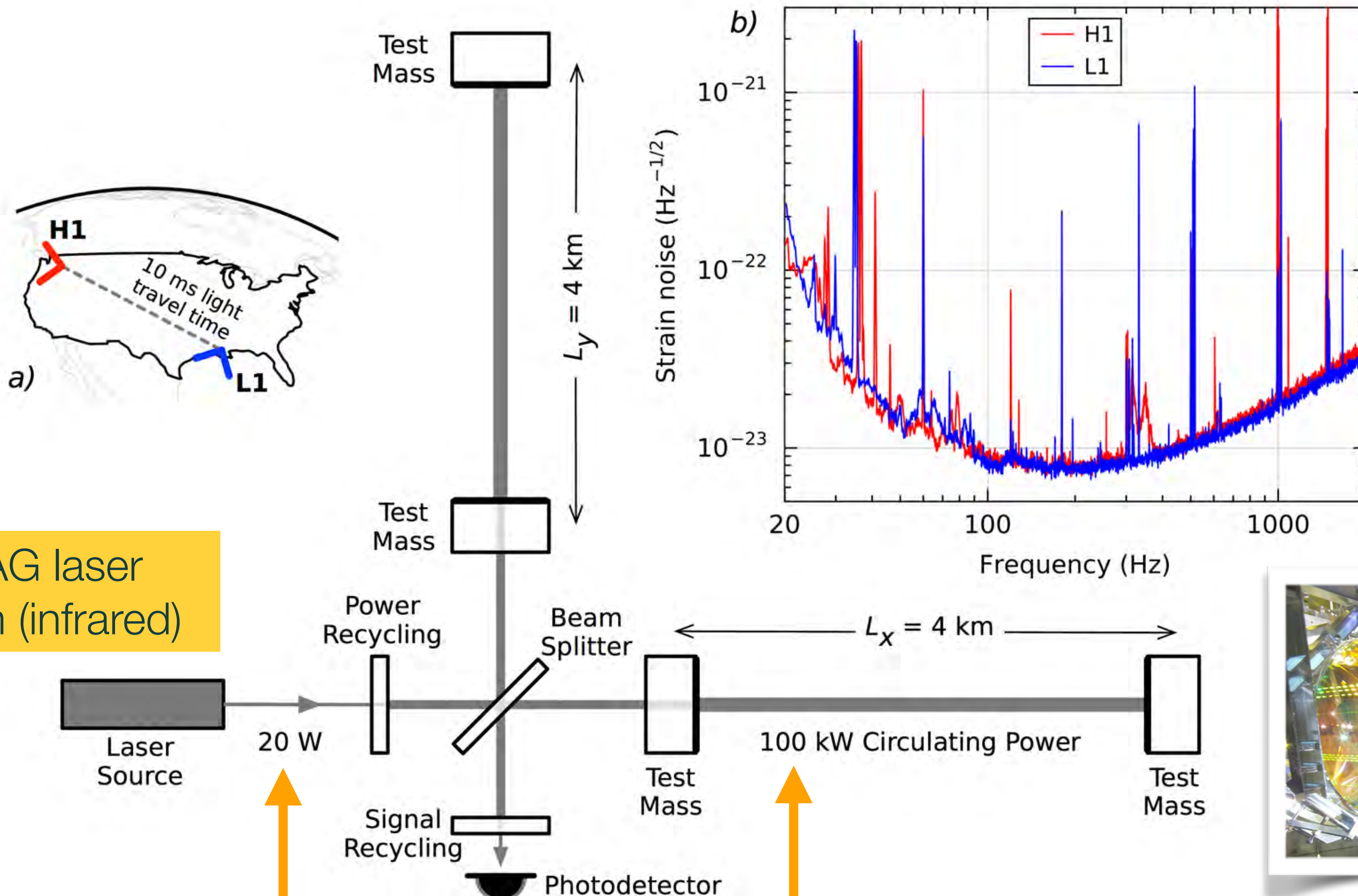


Distance in light-time
HO/LO-Virgo ~ 0.03 sec
HO-LO ~ 0.01 sec

Laser Interferometer Gravitational-wave Observatory

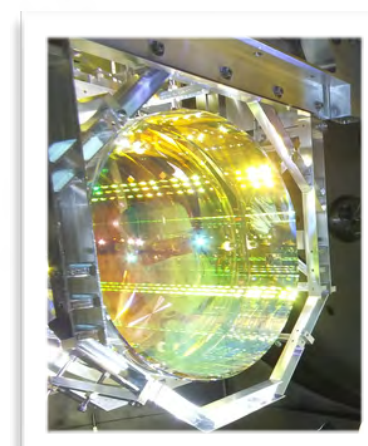
LSC LIGO Scientific Collaboration
VIRGO

Nd:YAG laser
1064nm (infrared)



Goal 200 W

Goal 750 kW



40kg fused Silica (SiO_2)
(absorption < 1ppm)

Sensitivity of Interferometers

한국 중력파 연구 협력단
Open Collaboration for the Gravitational Wave Detection and its Application

If the length resolution is λ_{laser}, detectable strain is

$$h \equiv \frac{\Delta l}{l} = \frac{\lambda_{laser}}{l} = \frac{10^{-6}\mathrm{m}}{10^{3}\mathrm{m}} = 10^{-9}$$

Fabry-Perot cavity (~ 250 bounces) : $\lambda_{\mathrm{GW}} \sim 1000 \text{ km for } 300 \text{ Hz}$

$$h \sim \frac{\Delta l}{l_{eff}} \sim \frac{\lambda_{laser}}{\lambda_{GW}} \sim \frac{10^{-6}\mathrm{m}}{10^{6}\mathrm{m}} = 10^{-12}$$

Due to quantum nature of the photons, the length resolution could be reduced

$$h \sim \frac{1}{\sqrt{N_{photons}}} \frac{\lambda_{laser}}{\lambda_{GW}}$$

Measuring the phase shift by averaging over some period τ

Poisson distribution

$$p(N) = \frac{\bar{N}^N \exp\left(-\bar{N}\right)}{N!}, \quad \Delta N = \sqrt{\bar{N}}$$

N: measured number of photons over period τ

$$E = \hbar \frac{2\pi c}{\lambda_l} N \rightarrow \Delta E = \hbar \frac{2\pi c}{\lambda_l} \sqrt{\bar{N}}$$

Uncertainty in the phase measurement : $\Delta \Phi = 2\pi c \frac{\Delta t}{\lambda_l}$

$$\Delta t \Delta E \geq \frac{\hbar}{2} \rightarrow \Delta t \Delta E = \frac{\Delta \Phi \lambda_l}{2\pi c} \hbar \frac{2\pi c}{\lambda_l} \sqrt{\bar{N}} \geq \frac{\hbar}{2}$$

$$\rightarrow \Delta \Phi \geq \frac{1}{\sqrt{\bar{N}}} \rightarrow N \geq \frac{1}{2(\Delta \Phi)^2}$$

Shot noise

Collect photons for a time of the order of the period of GW wave

$$\tau \sim 1/f_{GW}$$

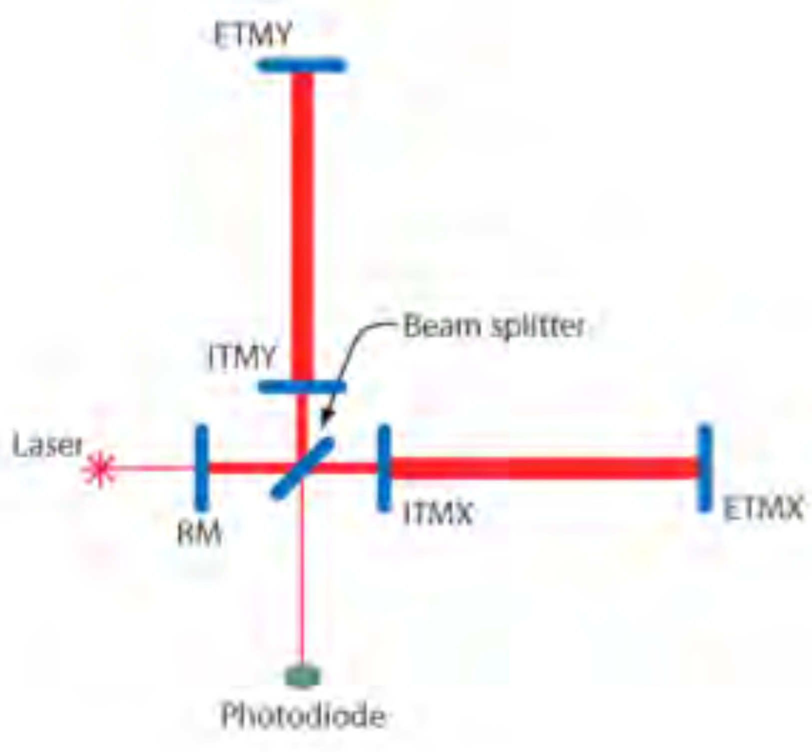
$$N_{photons} = \frac{P_{laser}}{hc/\lambda_{laser}}\tau \sim \frac{P_{laser}}{hc/\lambda_{laser}}\frac{1}{f_{GW}}$$

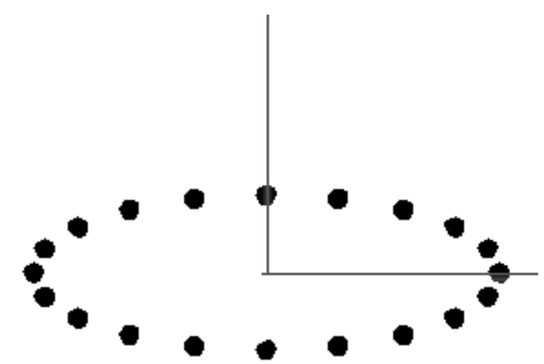
For 1W laser with λ_{laser}=1 μm, f_{GW}=300Hz, $N_{photons}$=10^{16}

$$h \sim \frac{\Delta l}{l_{eff}} \sim \frac{N_{photons}^{-1/2}\lambda_{laser}}{\lambda_{GW}} \sim \frac{10^{-8}\times 10^{-6}\mathrm{m}}{10^{6}\mathrm{m}} = 10^{-20}$$

By adopting high power laser (20W for O1) and Power Recycling, we can reach 'astrophysical sensitivity' of ~10^{-22}.

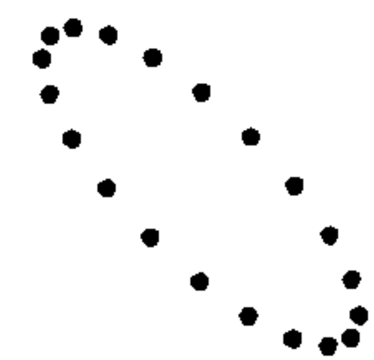
GW propagating in z-direction





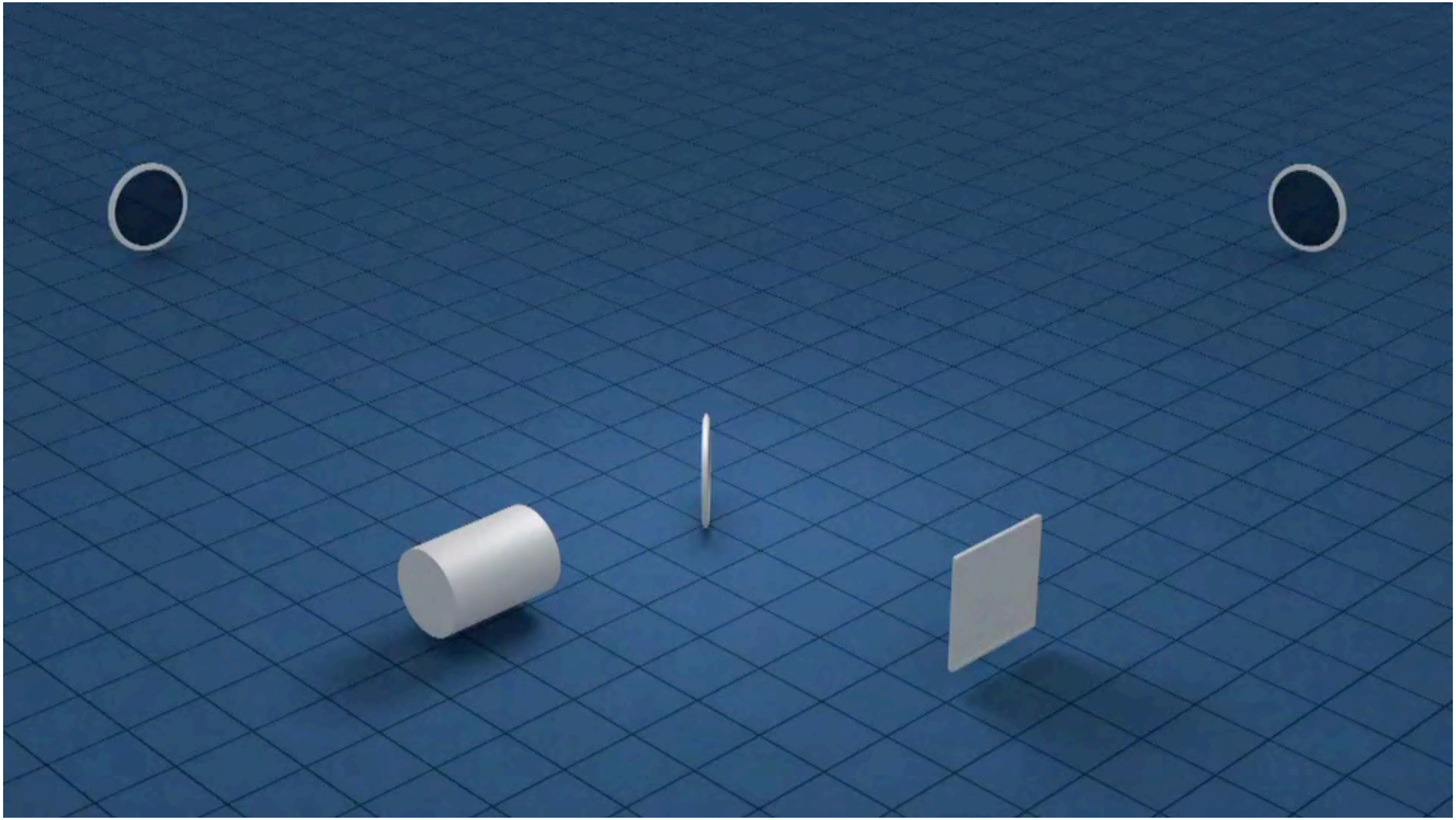
h_+

plus polarisation



$h_\times$

cross polarisation



Principle of GW detection

Image credit: LIGO/T. Pyle

What is interacting with GW, laser light or test mass?

Kip Thorne

- **TT gauge**
  - mirrors & beam splitters remain at rest
  - GW (h) interact with interferometer's light
- **local Lorentz (LL) frame**
  - in the proper reference frame of beam splitter
  - beam splitter is freely falling with GW
  - mirrors are moving
  - tiny correction due to GW & light interaction
- **physical observables should be independent of coord choice**

Detectability of LIGO 2015

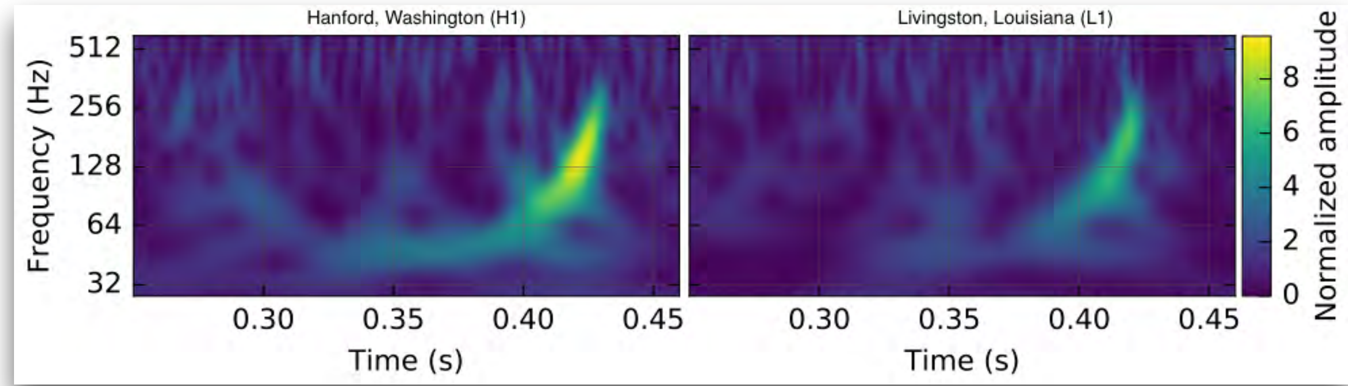
- 1/1000 of proton diameter in a distance of 4km
  (1/10 of hair thickness in a distance of 1 light year)

- Strong LASER power (20 W → 700 W → 100 kW)
  (design goal : 200 W → 750 kW)

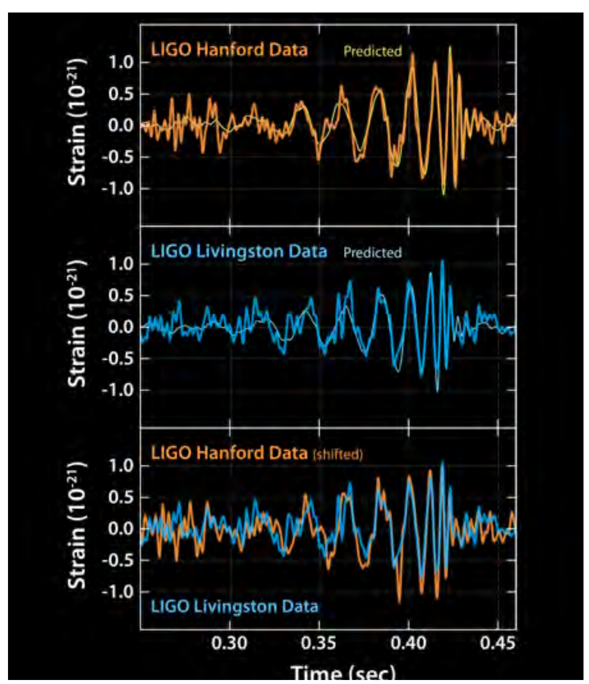
- 280 bounces between mirrors (effective distance :1120 km)

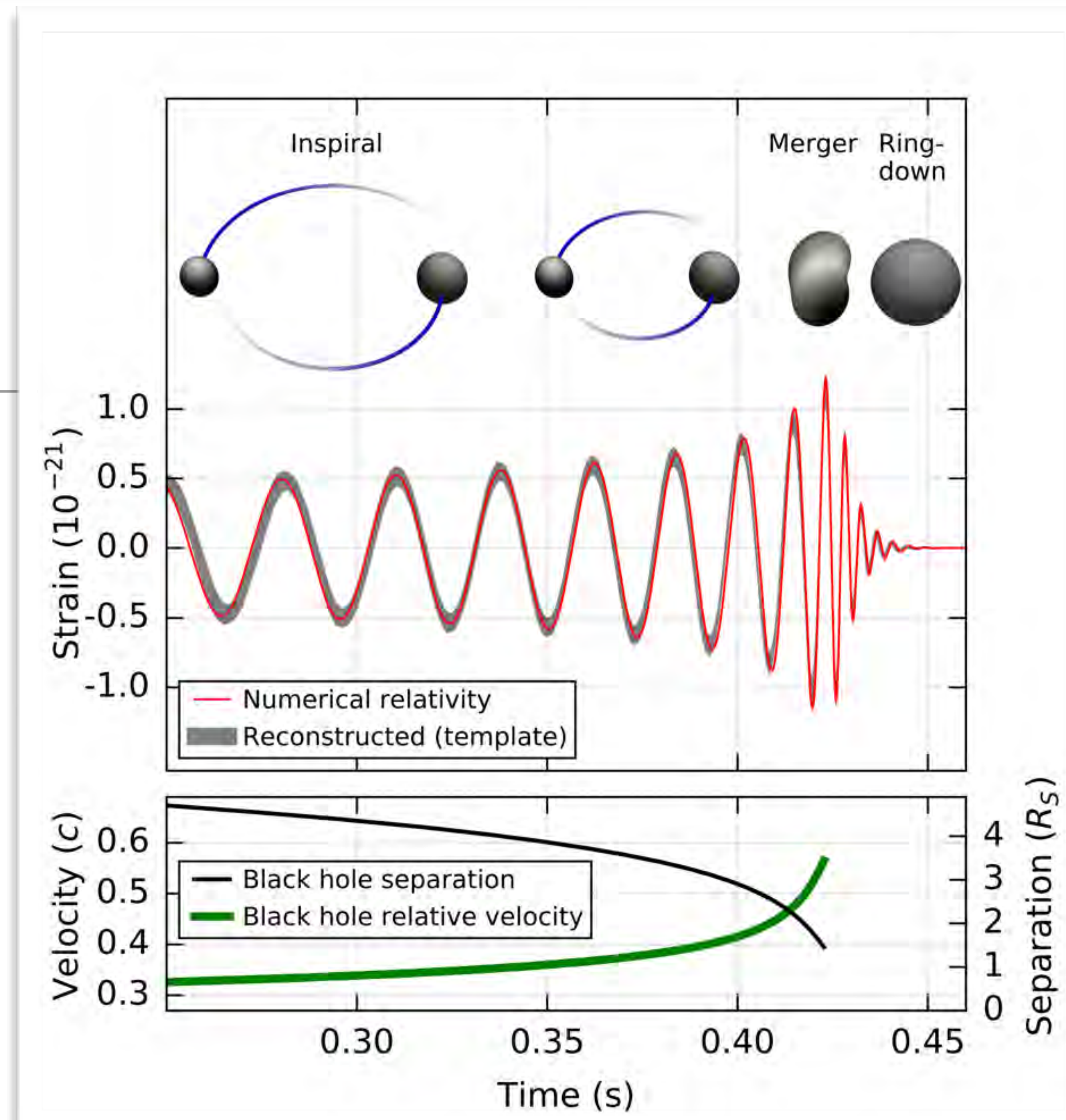
- Detection limit :
  NS binary merger - 10 billion light year
  BH binary merger - 30 billion light year

중력파 검출


Hanford, Washington (H1)
Livingston, Louisiana (L1)
Frequency (Hz)
512
256
128
64
32
0.30 0.35 0.40 0.45
Time (s)
Normalized amplitude
0 2 4 6 8


PHYSICAL
REVIEW
LETTERS
12 FEBRUARY 2016
American Physical Society
APS physics
Volume 116, Number 6


LIGO Hanford Data
Predicted
LIGO Livingston Data
Predicted
LIGO Hanford Data (shifted)
LIGO Livingston Data
Strain (10^{-21})
1.0 0.5 0.0 -0.5 -1.0
0.30 0.35 0.40 0.45
Time (sec)


Inspiral
Merger
Ring-down
Strain (10^{-21})
1.0 0.5 0.0 -0.5 -1.0
Numerical relativity
Reconstructed (template)
Velocity (c)
0.6 0.5 0.4 0.3
Black hole separation
Black hole relative velocity
Separation (R_S)
4 3 2 1 0
0.30 0.35 0.40 0.45
Time (s)

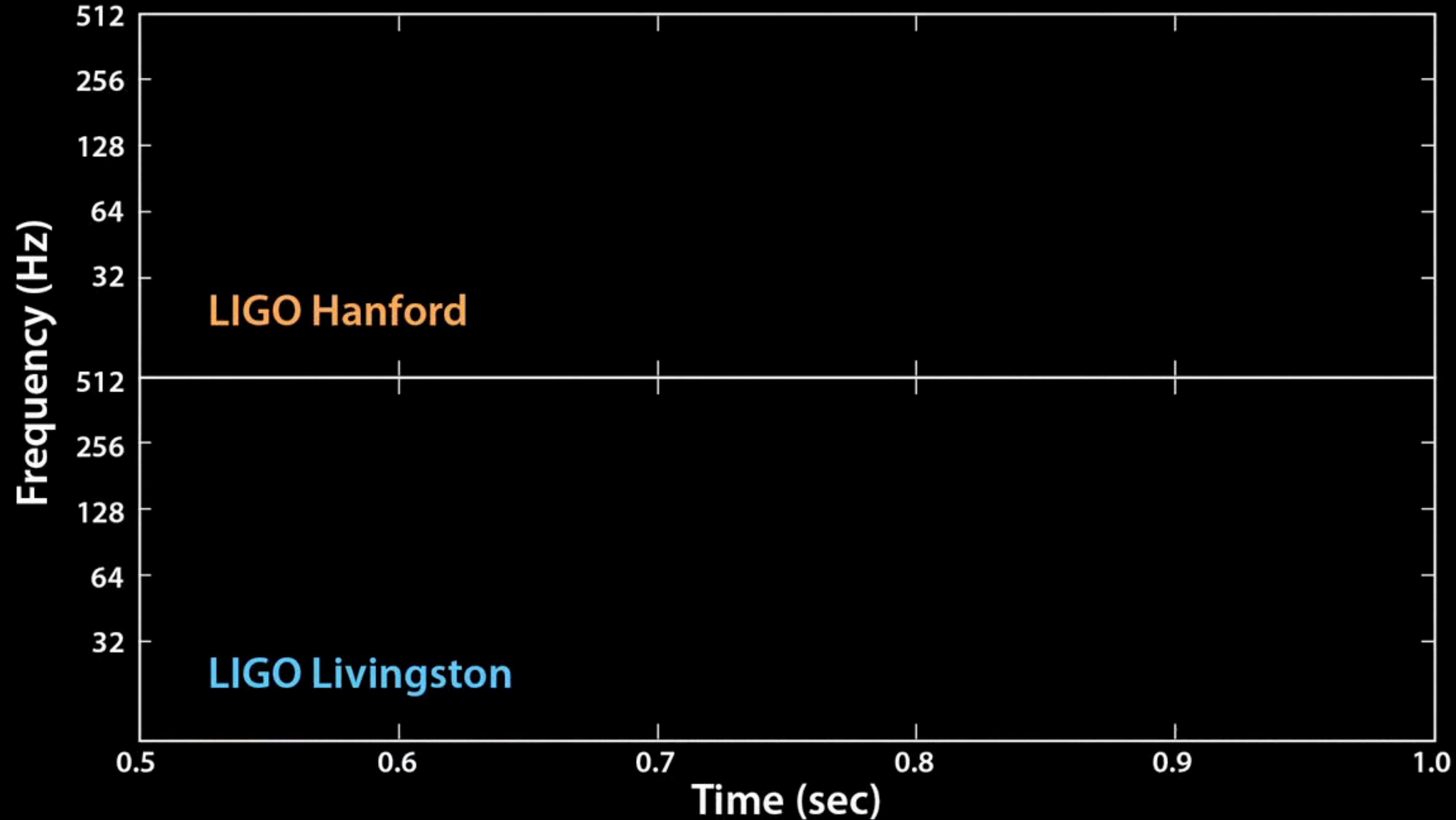
LIGO

1st detection of GW

Sep 14th, 2015

LSC LIGO Scientific Collaboration

Korean TIme Sep 14, 2015, pm 6:51



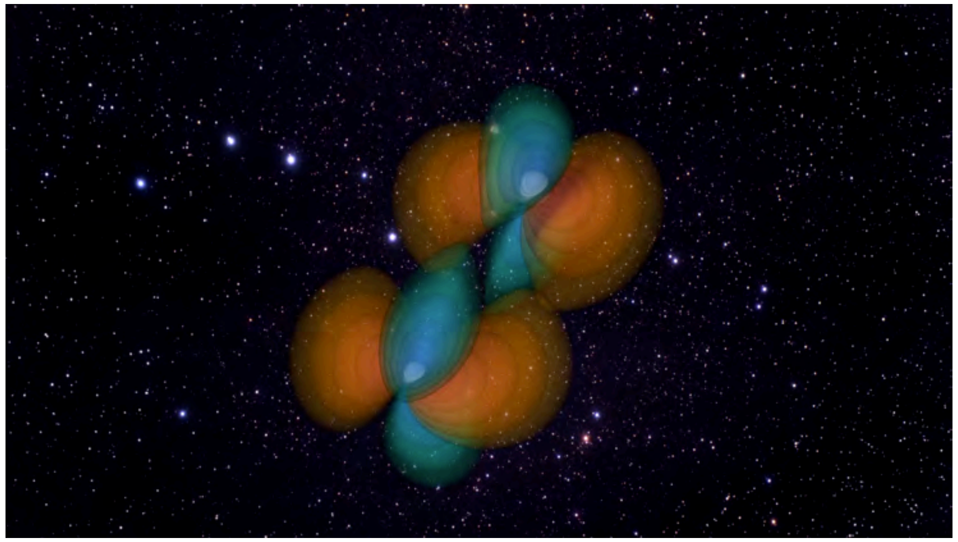
Sound of BH binary merger

Image credit : LIGO

GW150914

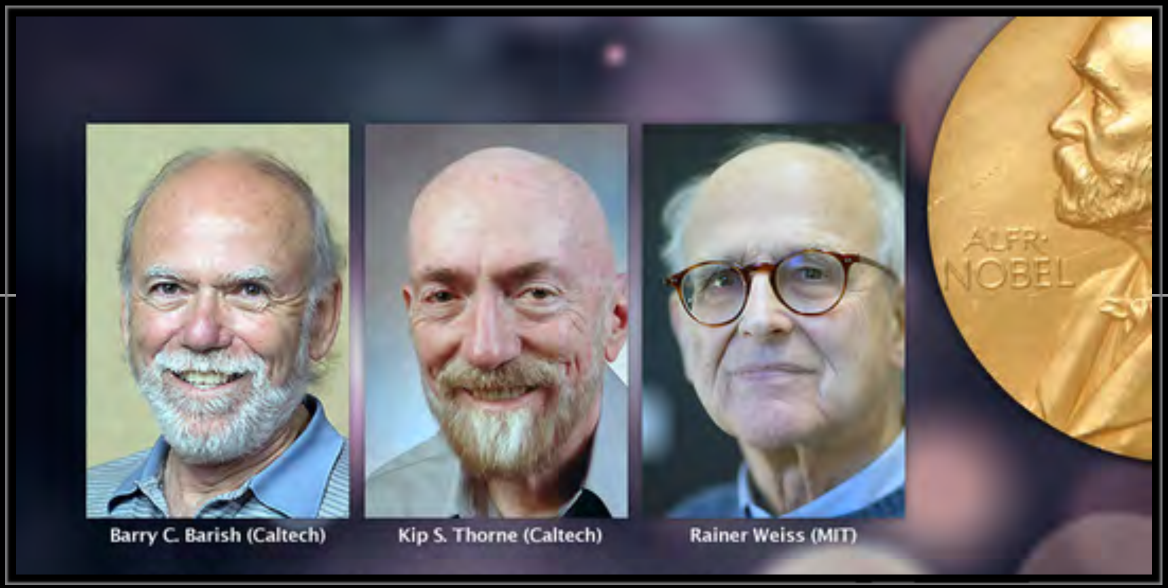
- 36 $M_{\odot}$ BH + 29 $M_{\odot}$ BH (final BH with 62 $M_{\odot}$)
- 3 $M_{\odot}$ in GW energy
  maximum luminosity ~ 50 times of the total light luminosity of the Universe
- distance : 13 billion light year (red shift z=0.09)
- GW frequency : 30-150 Hz
- GW maximum strain : 10^{-21}
  - 4×10^{-16} cm variation in 4km
    (correspond to hair thickness in 1 light year)

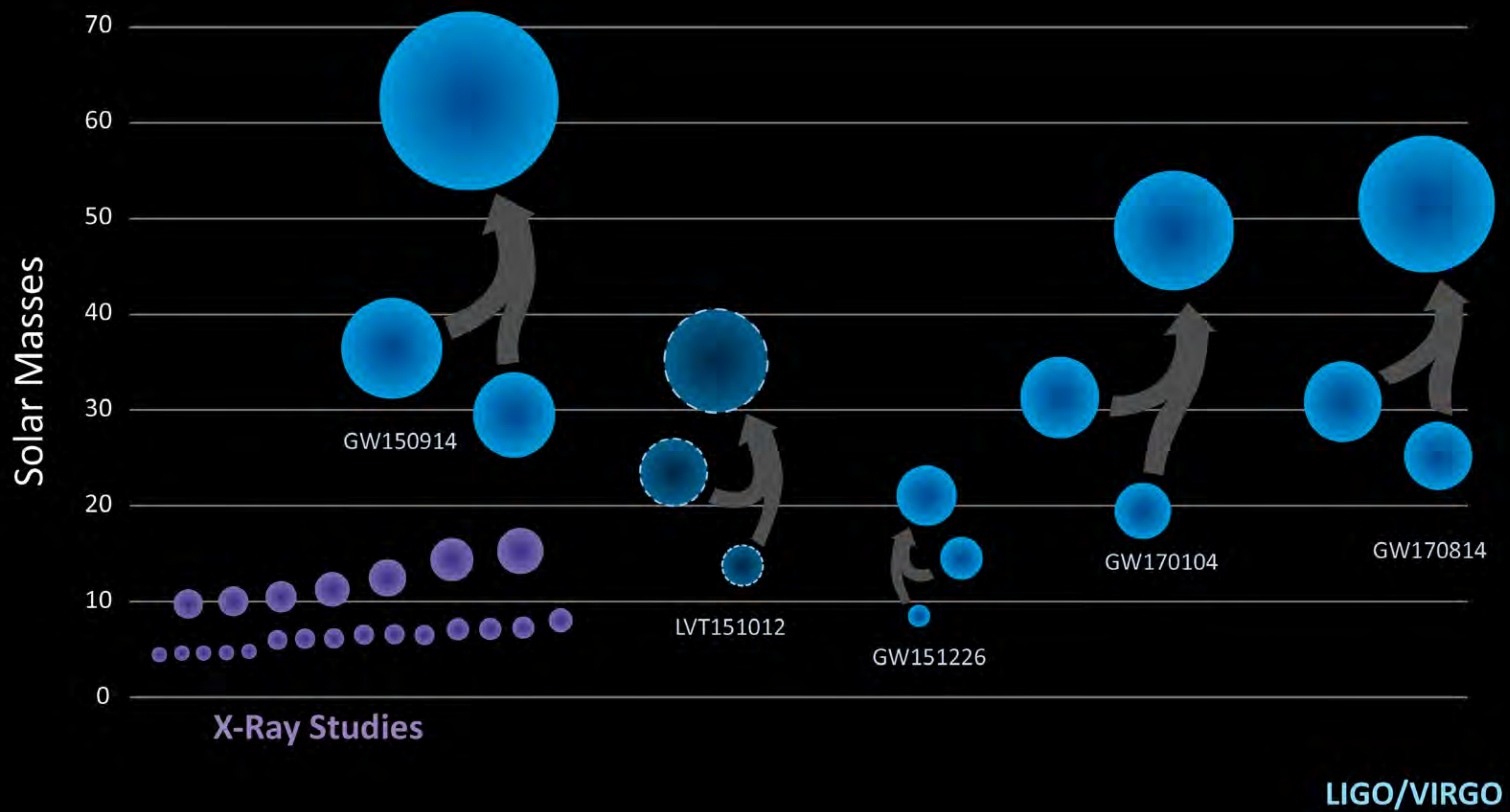
$$m = (1 + z) m^{\text{source}}$$



Georgia Tech animation

2017 Nobel Prize in Physics





Press Release Oct 16, 2017

GW from Binary NS Mergers

*GW 170817 (**d=40 Mpc**)*
GRB 170817A by Fermi-GBM
Kilonova/X-ray/Optical Afterglows

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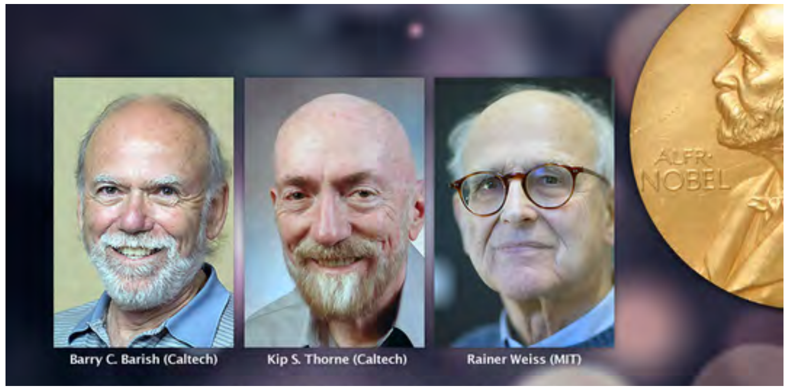
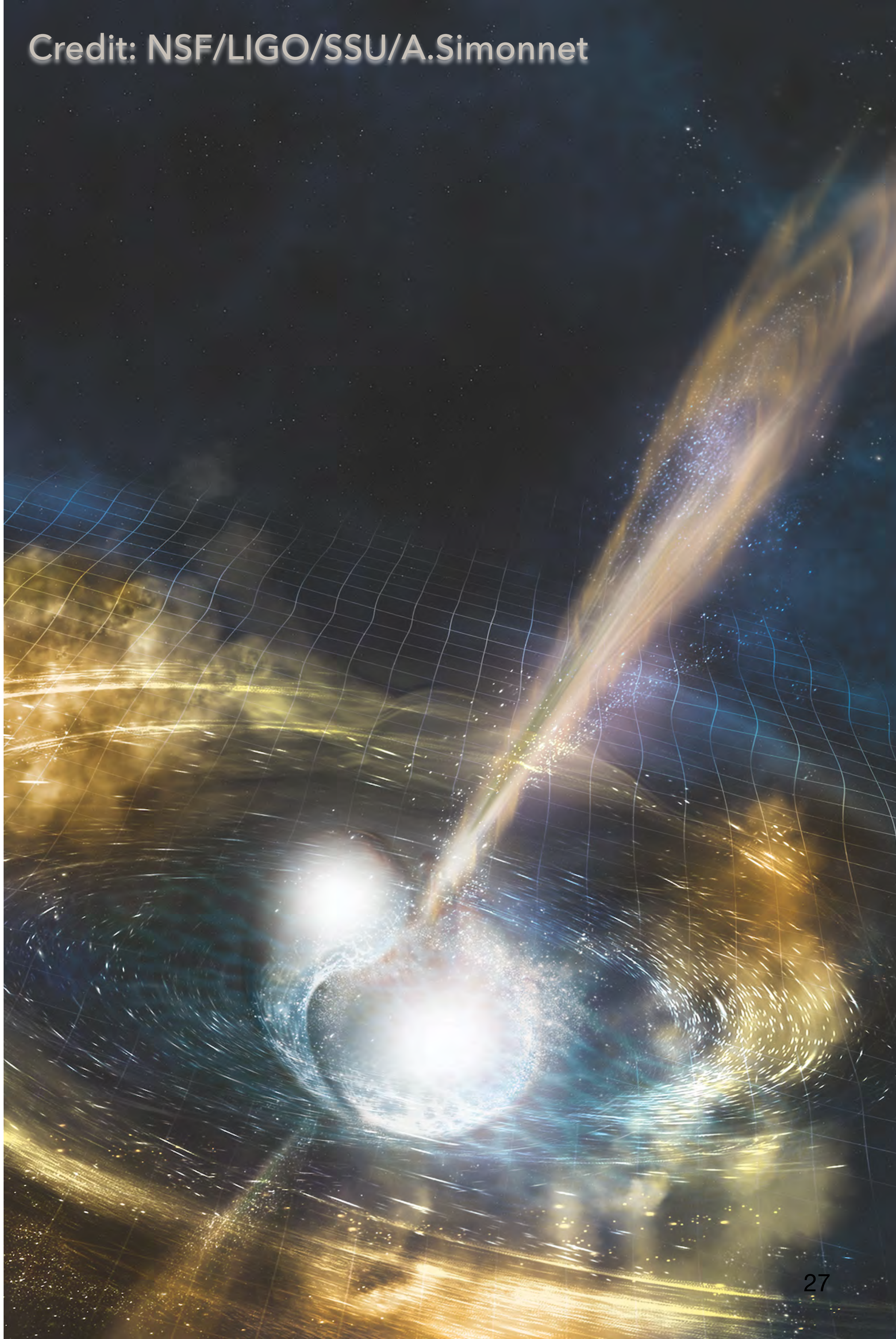
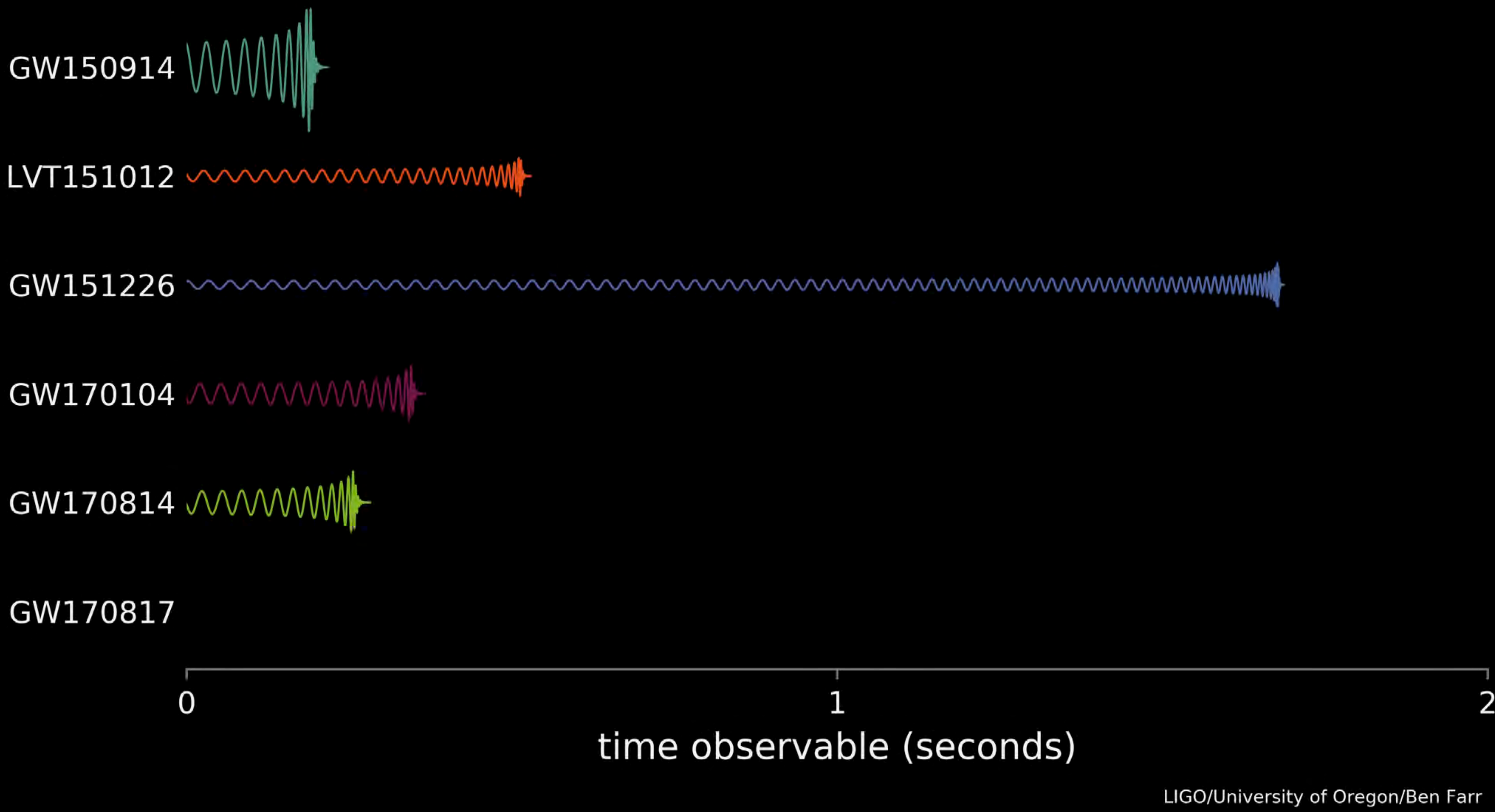
soon after the announcement of
2017 Nobel Prize



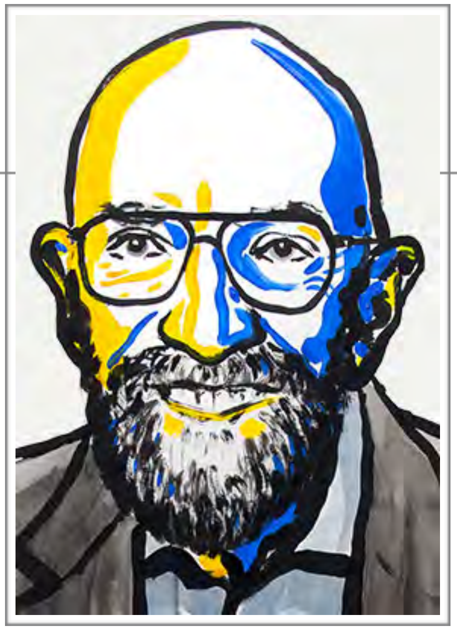




GW170817 | NS binary merger

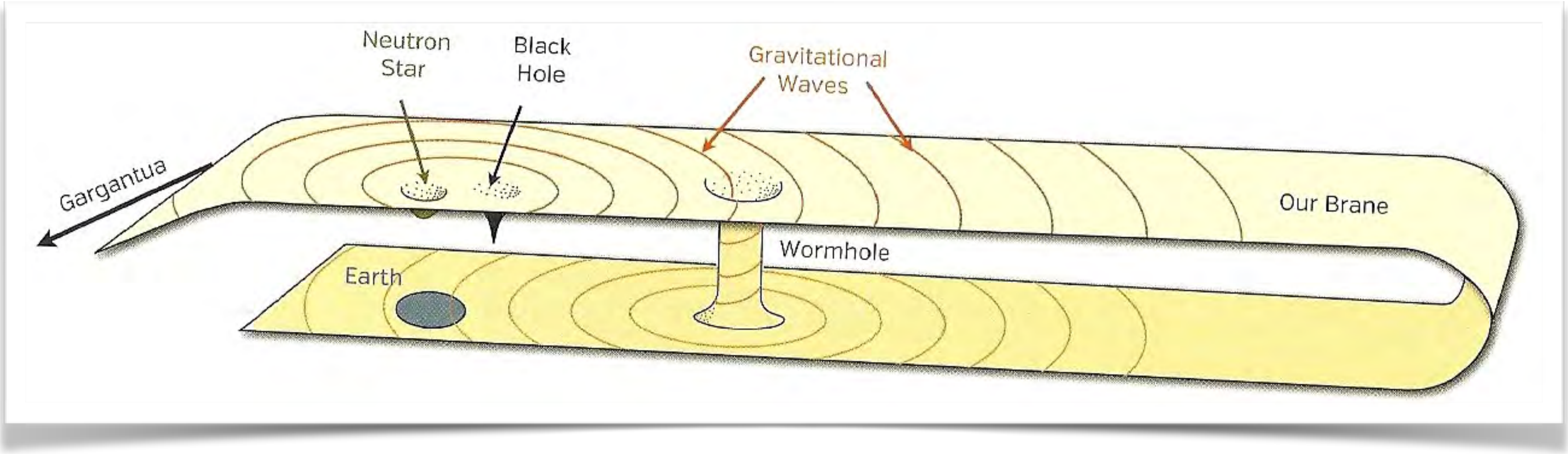
Interstellar original scenario by kip Thorne

2017 Nobel Prize

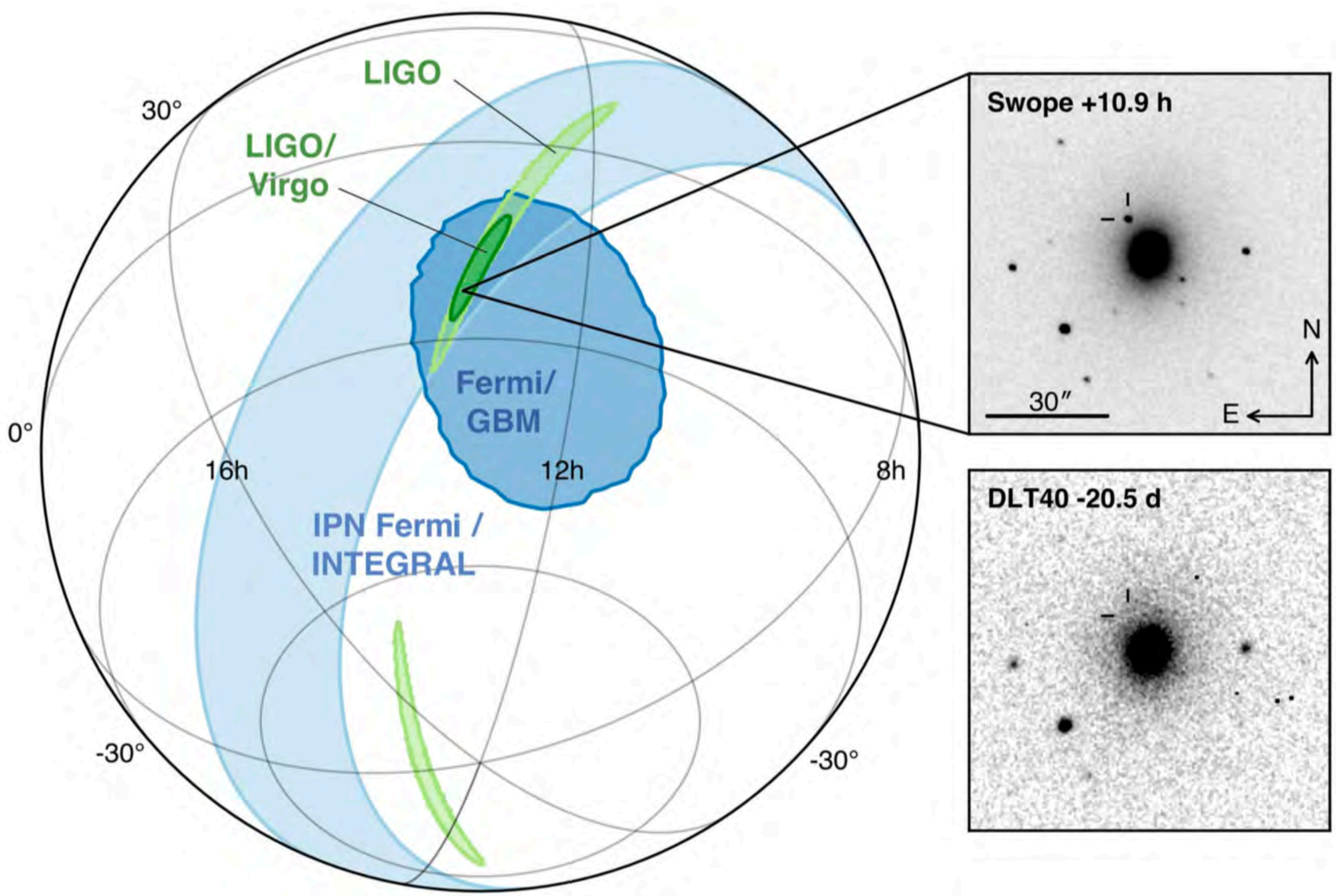


- 2019 LIGO detected GW from Saturn
  - GW from BH/NS binary mergers
  - No BH/NS near Saturn
  - Existence of a Warm Hole near Saturn

- Interstellar starts 40 years after GW detection from Saturn



GW170817 / GRB170817A



TIMELINE

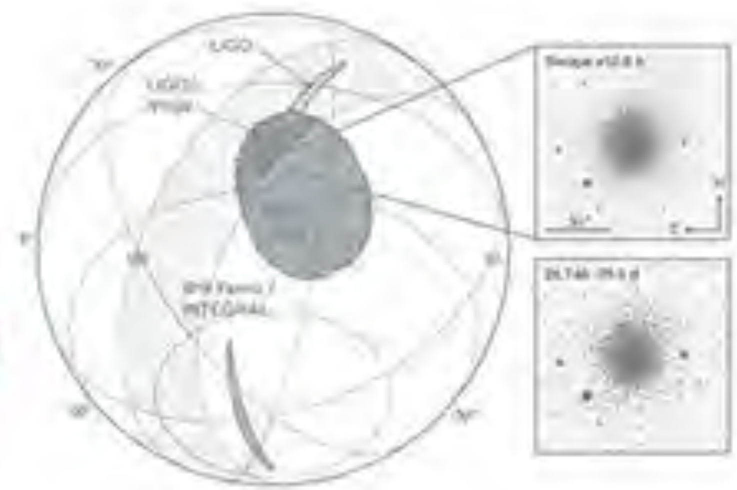
중성자별 충돌에서 발생한 중력파, 감마선, 가시광선, 엑스선 및 전파 관측

2017.08.17.
12:41:04 UTC

라이고 및 비르고
중력파 신호
포착

+2
seconds

페르미 및 인티그랄
감마선 신호
포착



Fermi/Integral gamma-ray

Telescopes in Chile

+11
hours

칠레 천문대
망원경들이
가시광선 신호
포착

http://horizon.kias.re.kr

+21
hours

국내연구진 호주
이상각망원경으로
추적 관측 시작
이후 약 4주간
추적 관측
(KMTNet, BOOTES-5
망원경 등)

Korean Telescopes
Nature 551, 71 (2017)

Chandra X-ray

+9
Days

찬드라
우주망원경
X선 신호
포착

VLA radio

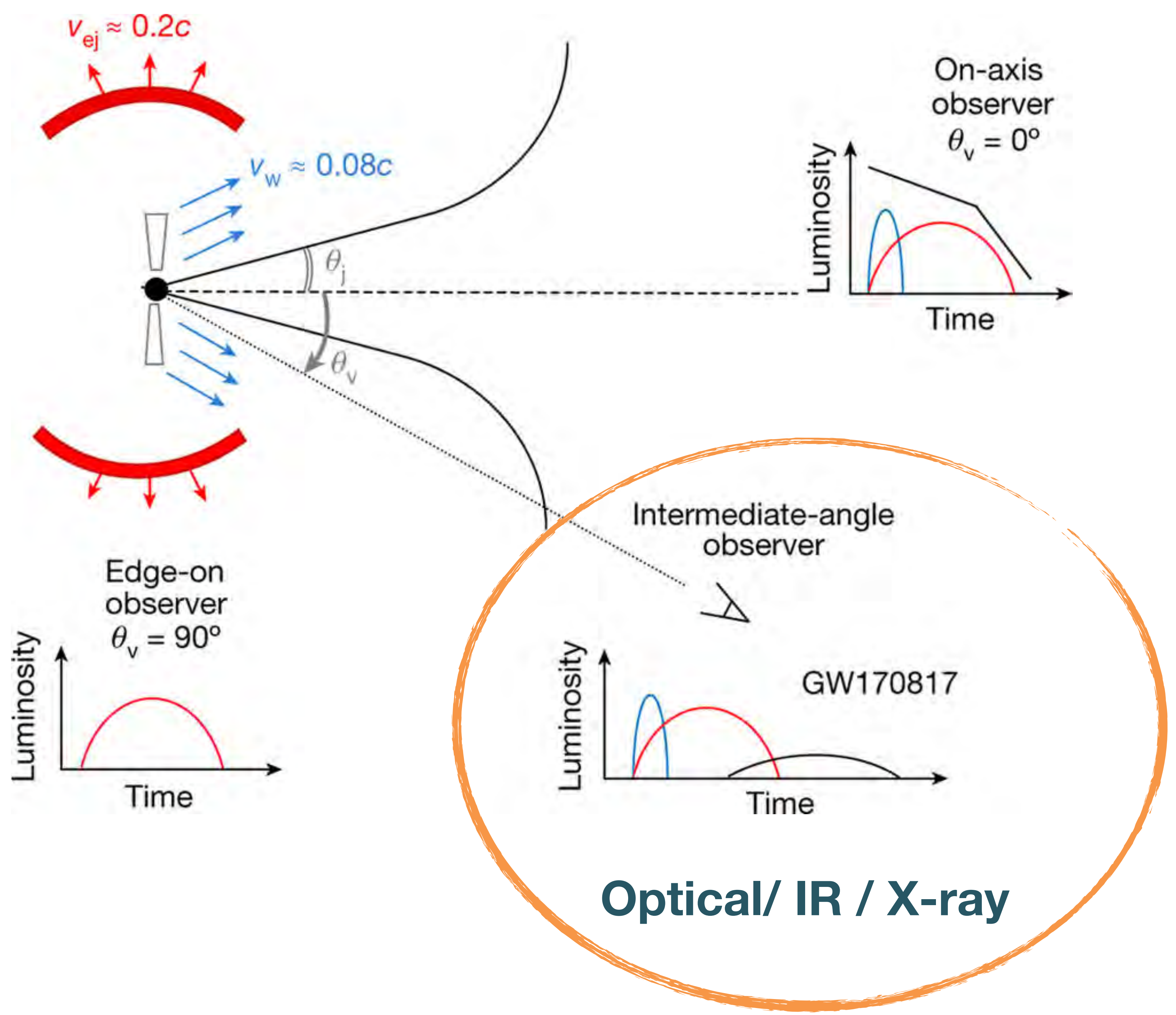
+16
Days

지상 전파 망원경
전파 신호
포착

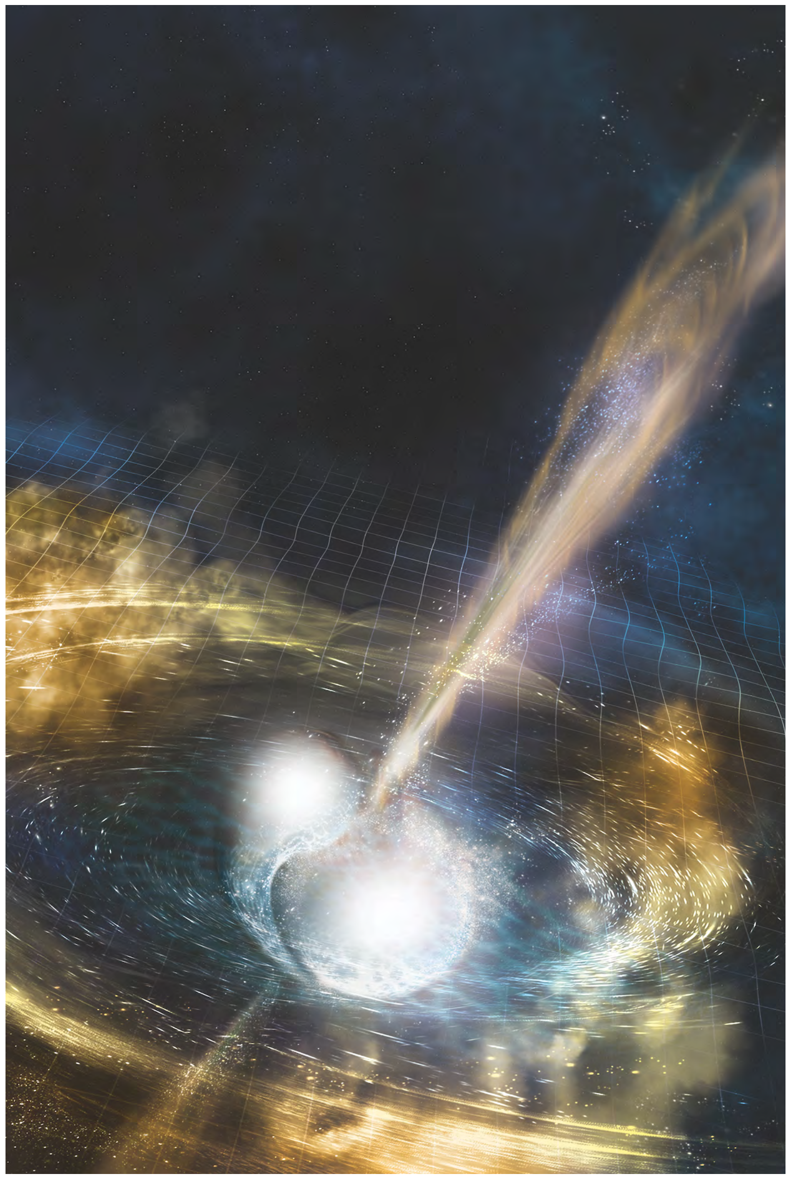
http://horizon.kias.re.kr

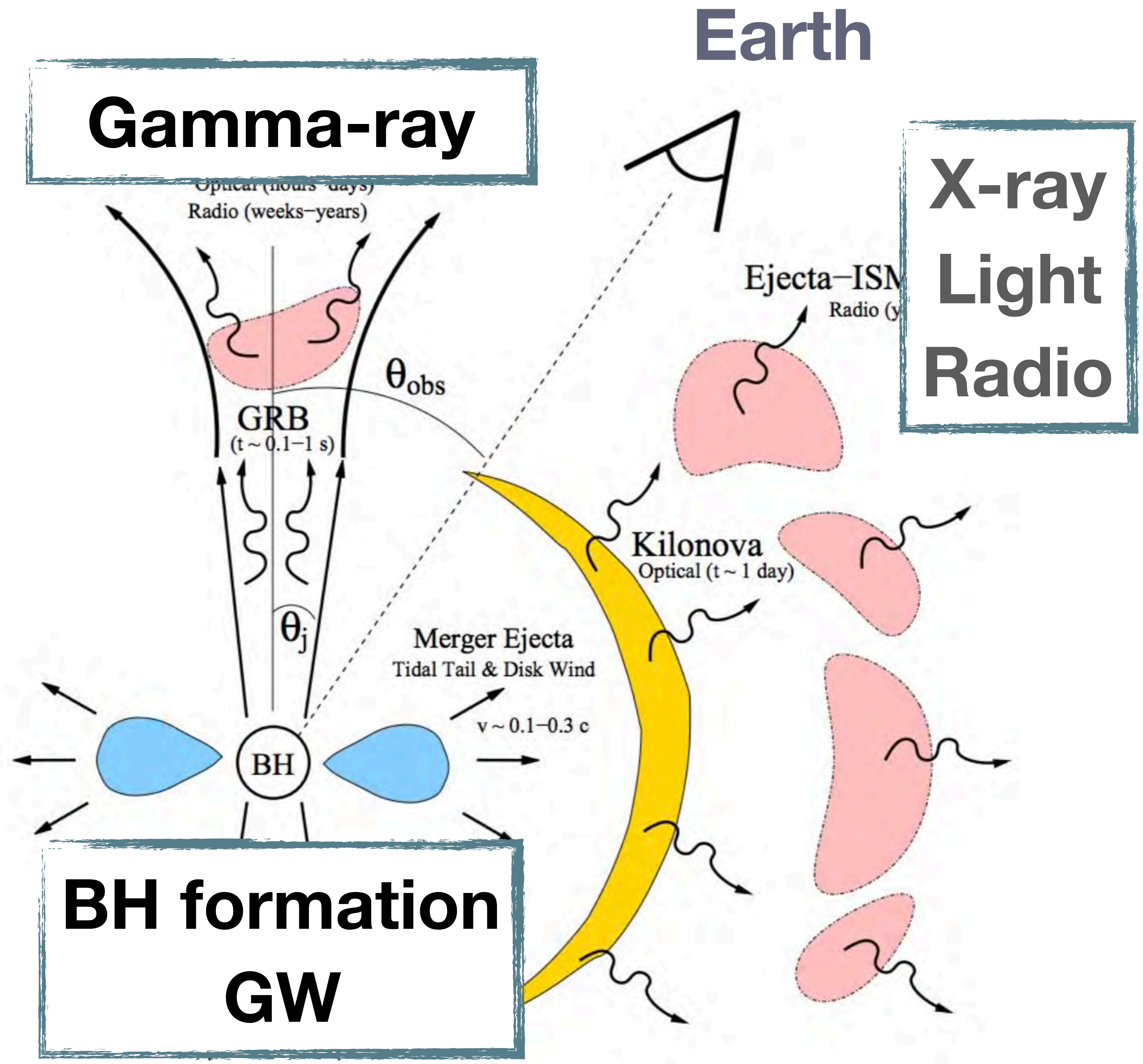
Kilonova / Geometry of GW170817

Nature 551, 71 (2017)



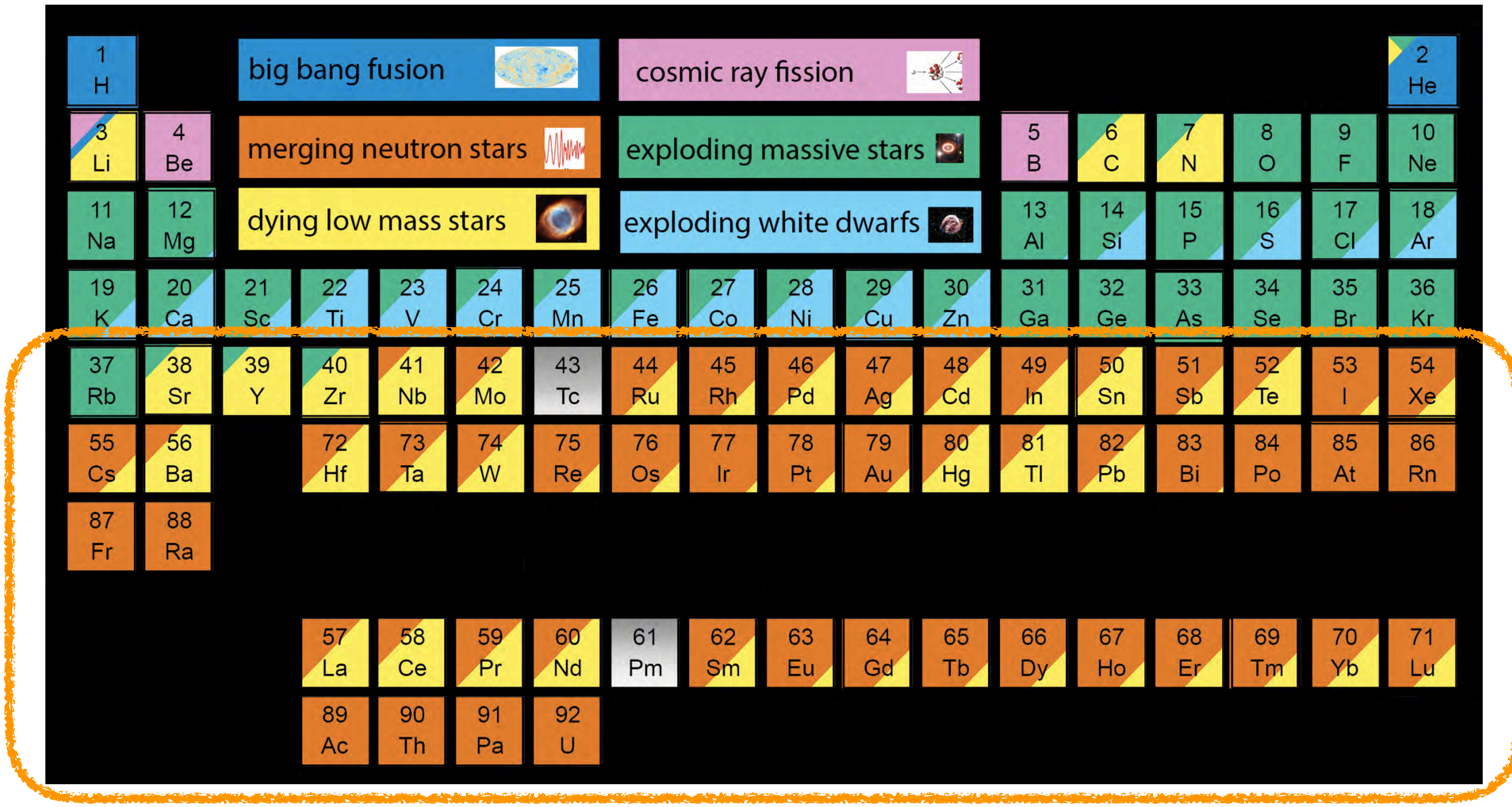
NS binary merger





Origin of Solar System Elements

merging neutron stars



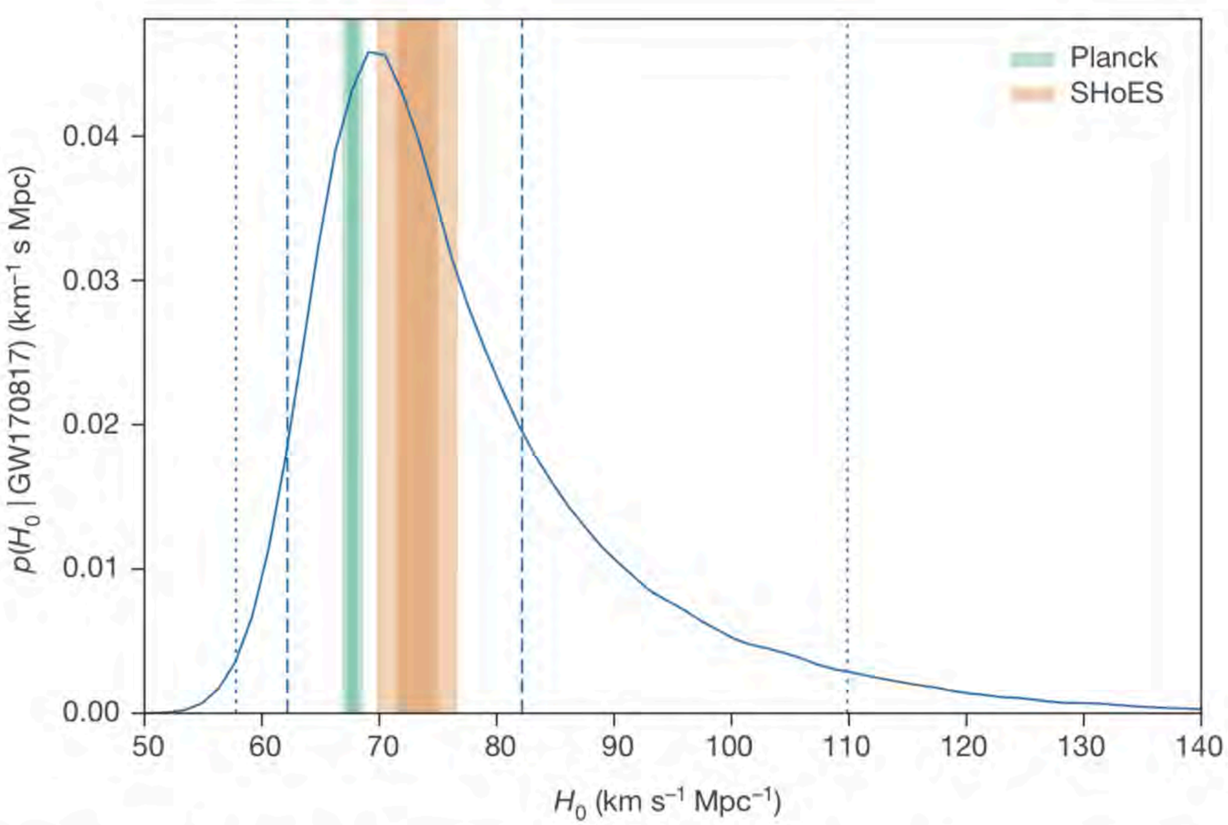
LETTER

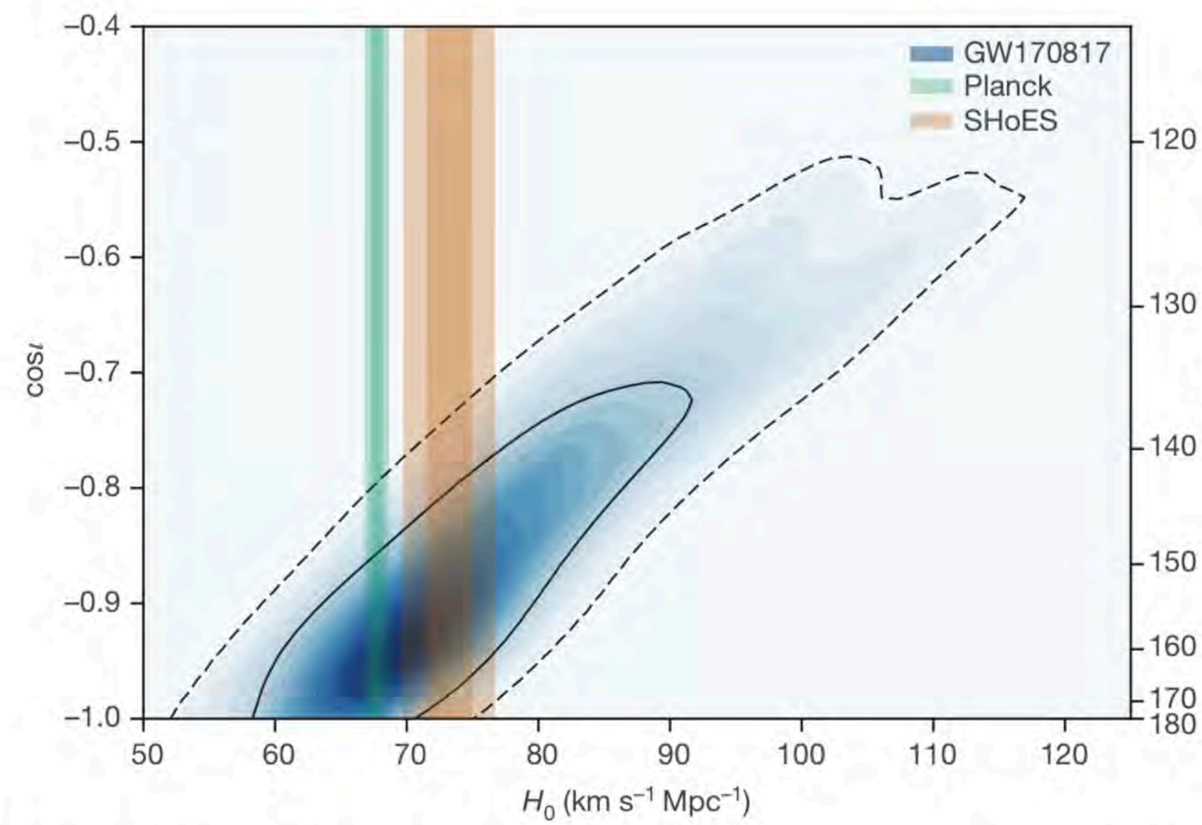
doi:10.1038/nature24471

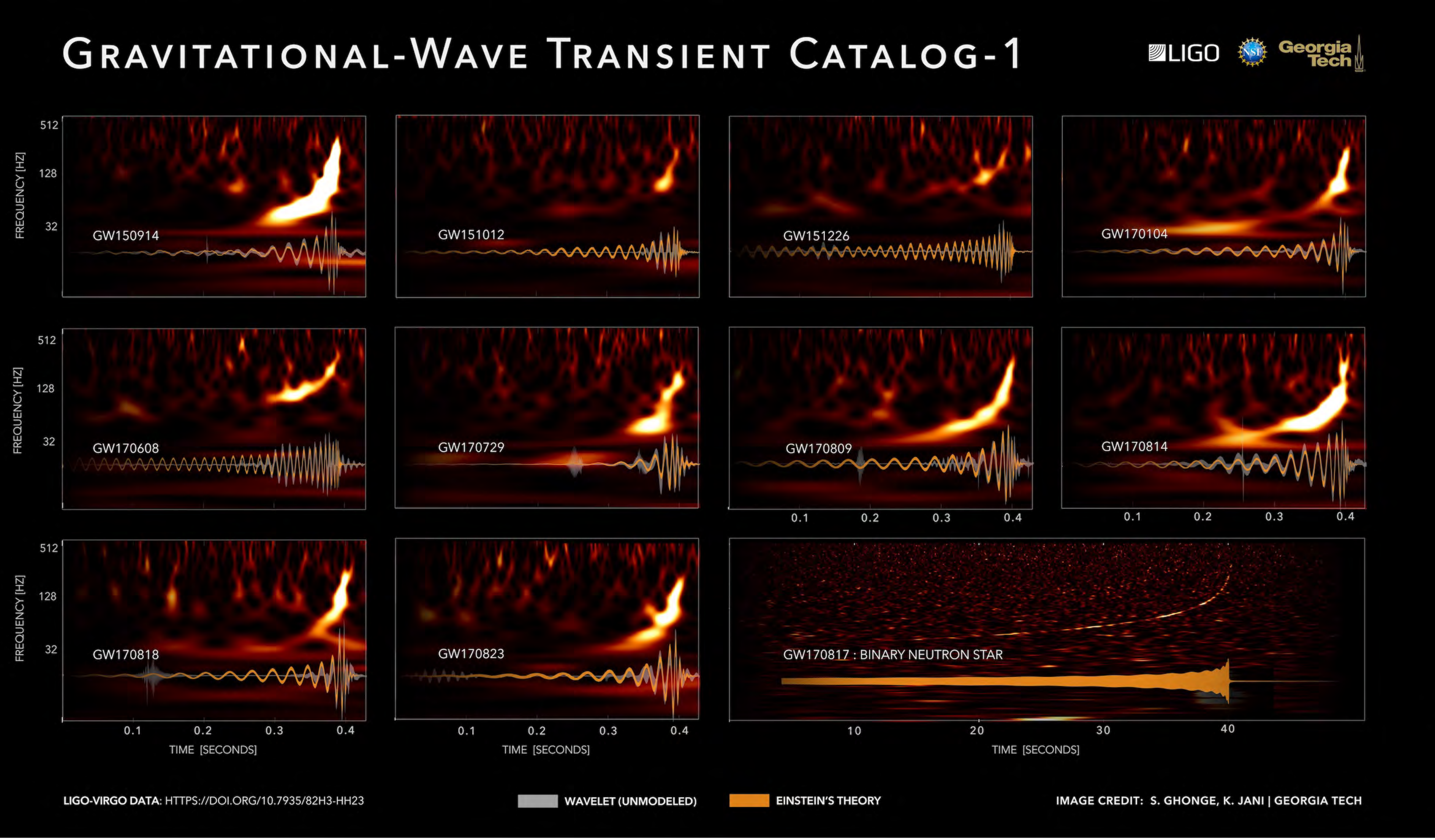
A gravitational-wave standard siren measurement of the Hubble constant

The LIGO Scientific Collaboration and The Virgo Collaboration*, The 1M2H Collaboration*, The Dark Energy Camera GW-EM Collaboration and the DES Collaboration*, The DLT40 Collaboration*, The Las Cumbres Observatory Collaboration*, The VINROUGE Collaboration* & The MASTER Collaboration*

$$H_0 = 70.0^{+12.0}_{-8.0}\ \mathrm{km\ s^{-1}\,Mpc^{-1}}$$

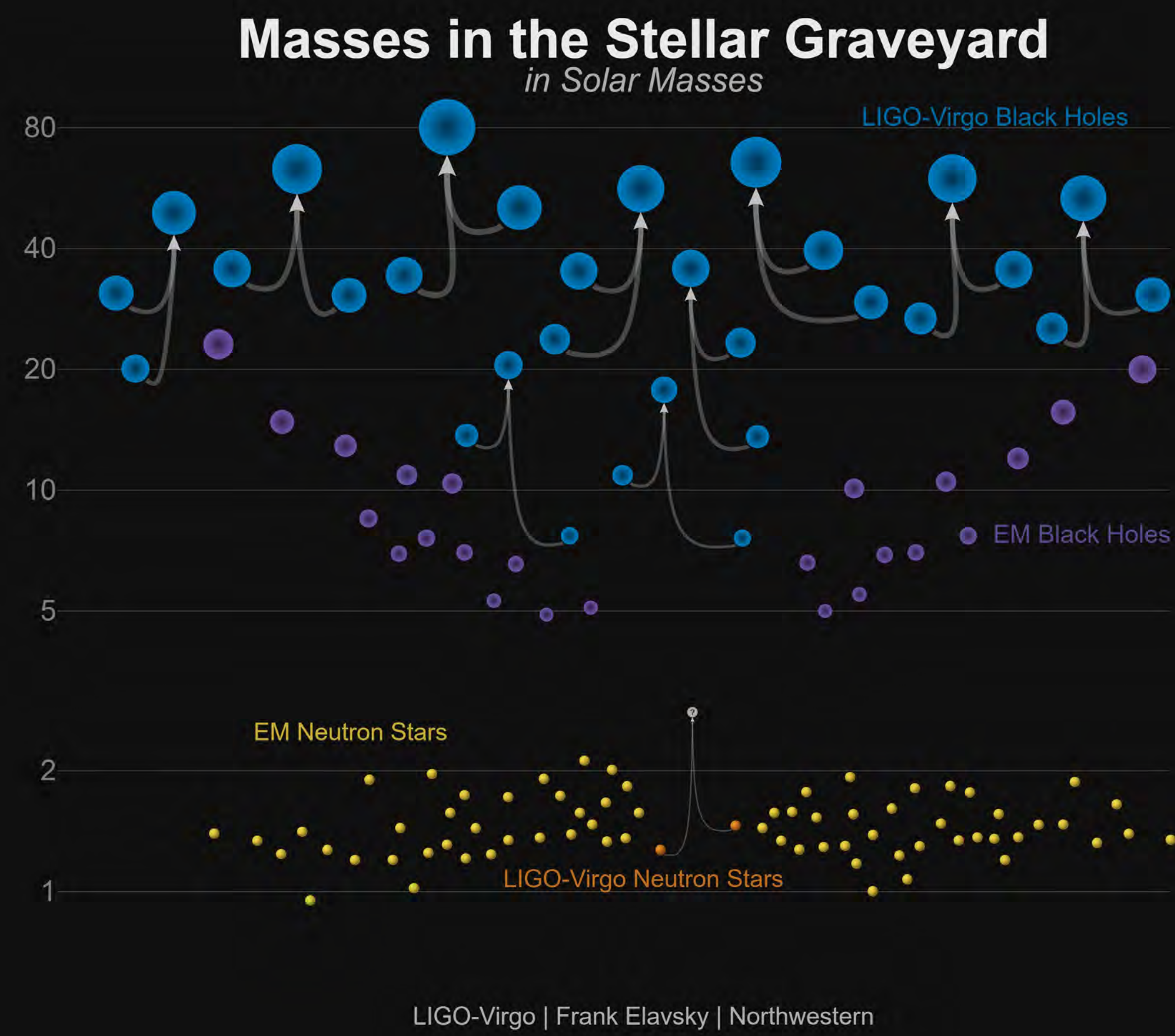






GW Catalog

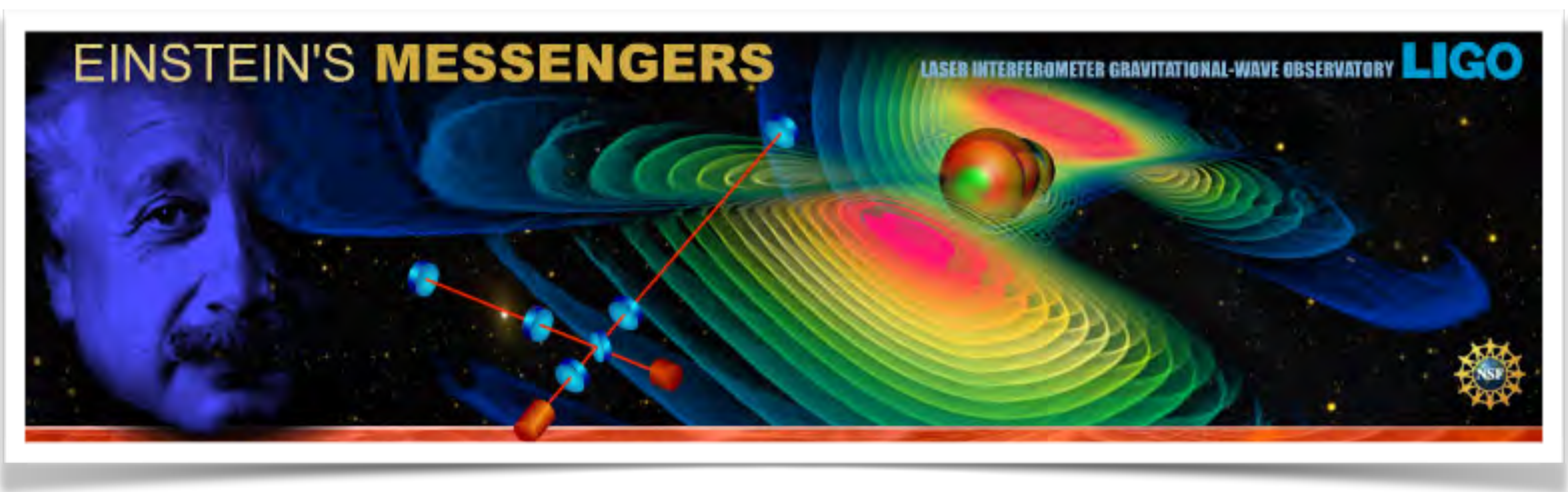
2018.12.05



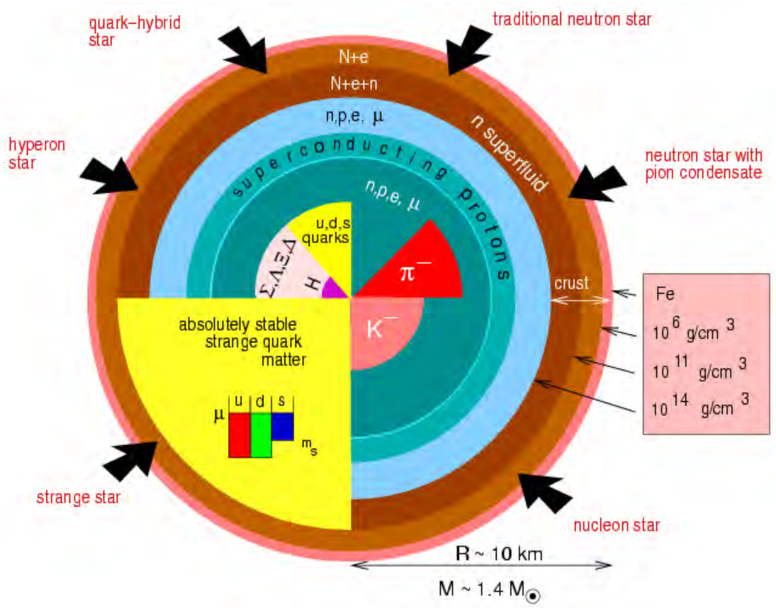
Masses in the Stellar Graveyard
in Solar Masses
LIGO-Virgo Black Holes
EM Black Holes
EM Neutron Stars
LIGO-Virgo Neutron Stars
80
40
20
10
5
2
1
LIGO-Virgo | Frank Elavsky | Northwestern

Gravitational-Wave & Multi-Messenger Astronomy

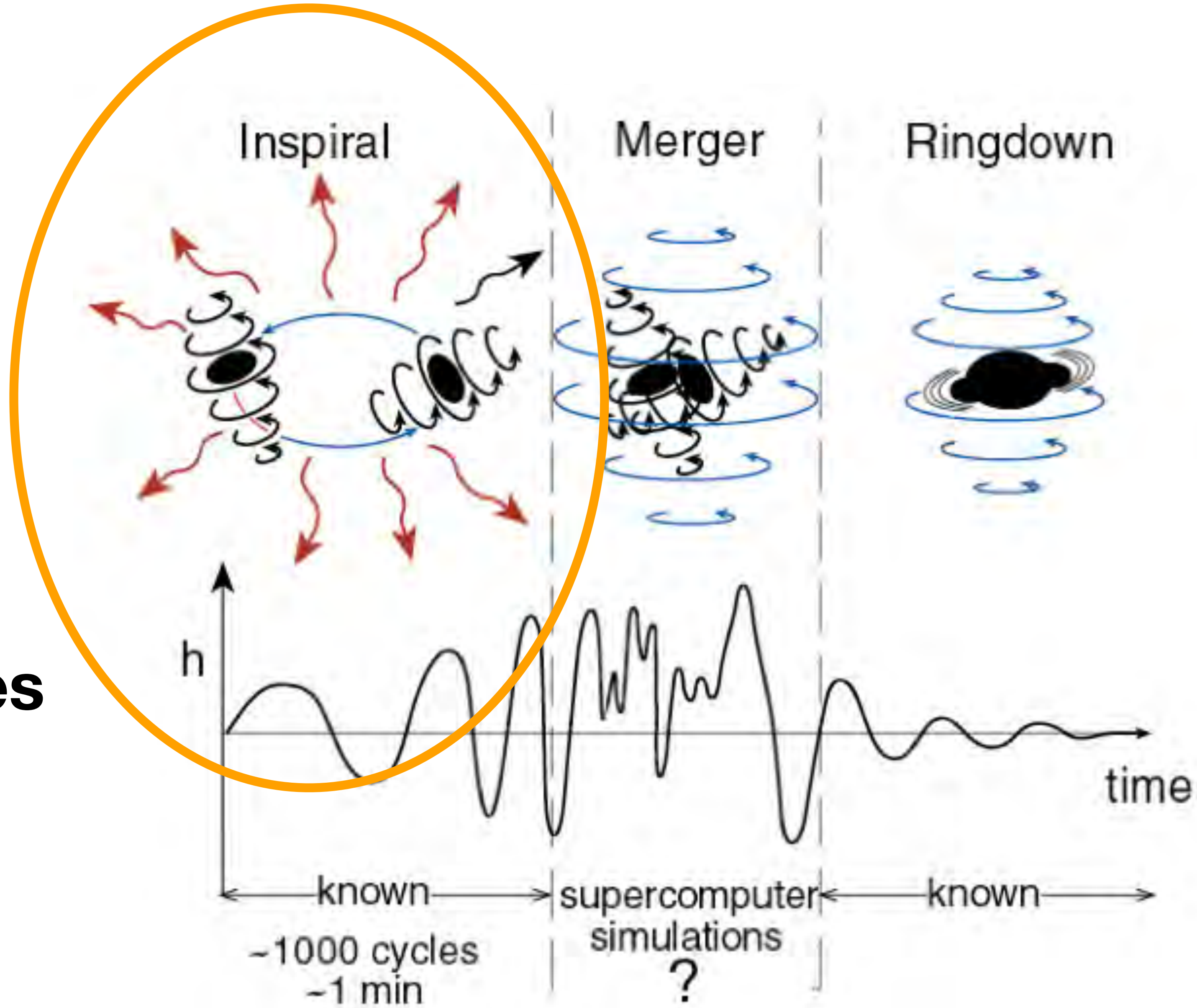
- First direct detection of GW in 2015
- First detection of BHs with masses 30 ~ 60 solar mass
- **GW, Gamma-ray, Optical, X-ray, Radio from NS mergers**
- New era for GW Astronomy & **Multi-Messenger Astronomy**



Response of NS to GW during Inspiral



perturbative approaches



Einstein's field equation with source

$$G_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$$

$$T^{\alpha\beta} = \left(\begin{array}{c|c} \text{energy density} & \text{energy flux} \\ \hline \text{mom. endsity} & \text{stress tensor} \end{array} \right)$$

$$\nabla_\beta T^{\alpha\beta} = 0 \longrightarrow \nabla_\beta G^{\alpha\beta} = 0$$

local conservation of energy-momentum

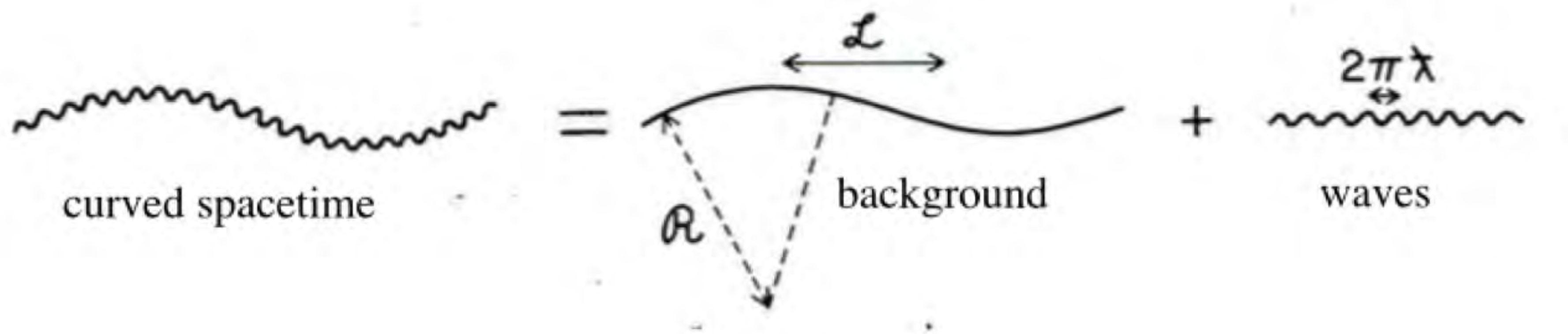
Bianchi identity

Newtonian limit

Principles of Gravitational Waves

$c = 1$ unit

long timescale change



short timescale oscillation

Linearized Vacuum Einstein Equation

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

$$|h_{\mu\nu}| \ll 1$$

$$\eta_{\alpha\beta} = \mathrm{diag}(-1,1,1,1) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\delta R_{\alpha\beta} = \frac{1}{2}\left[-\Box h_{\alpha\beta} + \partial_\alpha V_\beta + \partial_\beta V_\alpha\right] = 0$$

$$\Box \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta = -\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2$$

$$V_\alpha \equiv \partial_\gamma h^\gamma_\alpha - \frac{1}{2}\partial_\alpha h^\gamma_\gamma$$

$$h^\gamma_\alpha = \eta^{\gamma\delta} h_{\delta\alpha}$$

Gauge transformation

$$x'^\alpha = x^\alpha + \xi^\alpha(x)$$

$$h'_{\alpha\beta} = h_{\alpha\beta} - \partial_\alpha\xi_\beta - \partial_\beta\xi_\alpha$$

Lorentz gauge

$$V'_\alpha \equiv \partial_\beta h'^\beta_\alpha - \frac{1}{2}\partial_\alpha h'^\beta_\beta = 0$$

Field equation

$$\Box h'_{\alpha\beta} = 0$$

Linearized Gravitation vs Electromagnetism

	Linearized Gravitation	Electromagnetism
basic potential	$h_{\alpha\beta}(x)$	$\left(\Phi(t,\vec{x}), \vec{A}(t,\vec{x})\right)$
fields	$\delta R_{\alpha\beta\gamma\delta}(x)$	$\vec{E}(t,\vec{x}), \vec{B}(t,\vec{x})$
Gauge transformation	$h_{\alpha\beta} \rightarrow h_{\alpha\beta} - \partial_\alpha \xi_\beta - \partial_\beta \xi_\alpha$	$\vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda$ $\Phi \rightarrow \Phi - \partial\Lambda/\partial t$
Lorentz gauge	$\partial_\beta h^\beta_\alpha - \frac{1}{2}\partial_\alpha h^\beta_\beta = 0$	$\vec{\nabla} \cdot \vec{A} + \partial\Phi/\partial t = 0$
field equation	$\Box h_{\alpha\beta} = 0$	$\Box\vec{A} = 0$ $\Box\Phi = 0$

Transverse-traceless (TT) gauge

impose Lorentz gauge

$$\Box h_{\alpha\beta} = 0$$

wave equation

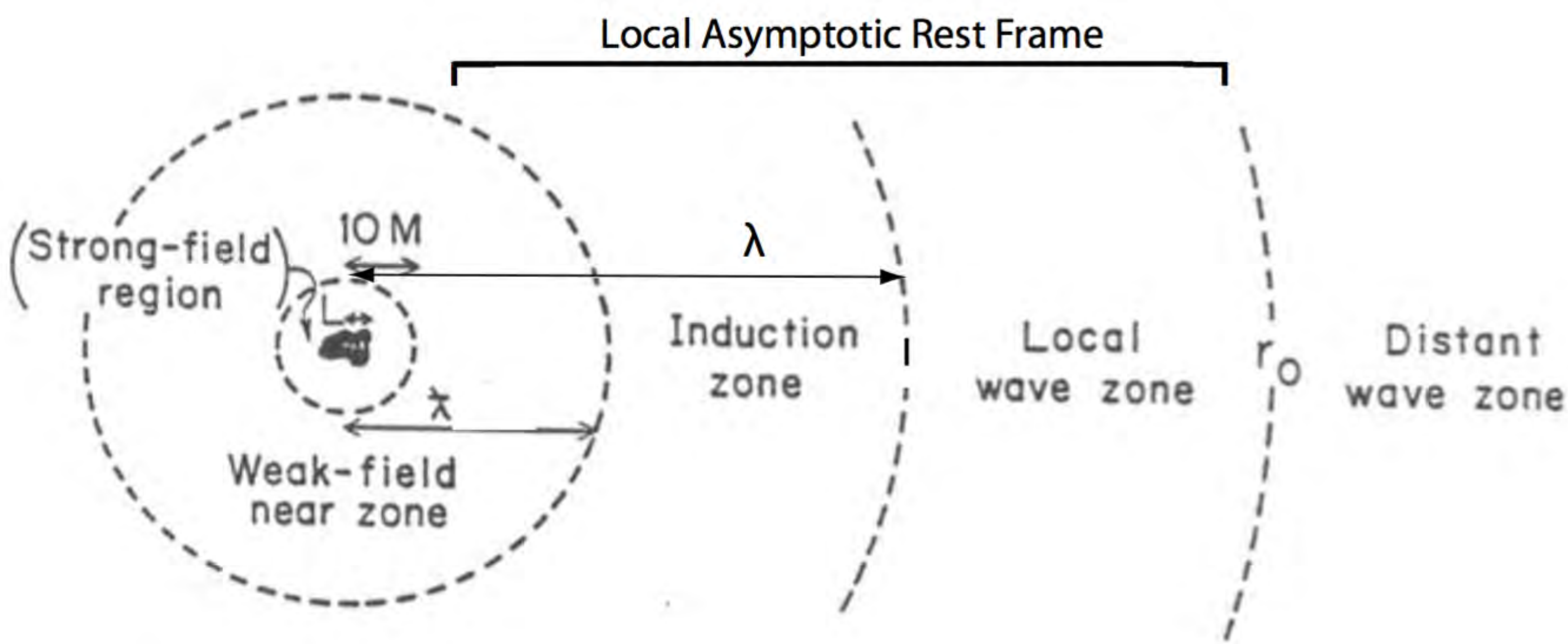
further impose TT gauge

$$h_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \text{a} & \text{b} & 0 \\ 0 & \text{b} & \text{-a} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(\vec{k}\cdot\vec{x}-\omega t)}$$

$$a = \frac{1}{2}\left(h_{xx} - h_{yy}\right)$$

$$b = h_{xy}$$

Regions of space around a source of GW



GW propagating in z-direction **in TT gauge**

plus polarisation **cross polarisation**

$$h_{\mu\nu}^{TT} = h_+(t-z)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + h_\times(t-z)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

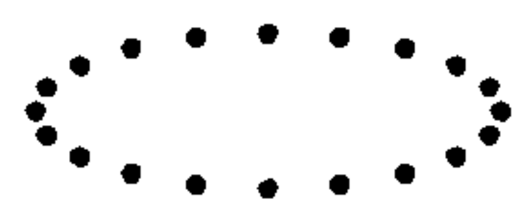
line element

$$ds^2 = -dt^2 + (1+h_+)dx^2 + (1-h_+)dy^2 + dz^2 + 2h_\times dxdy$$

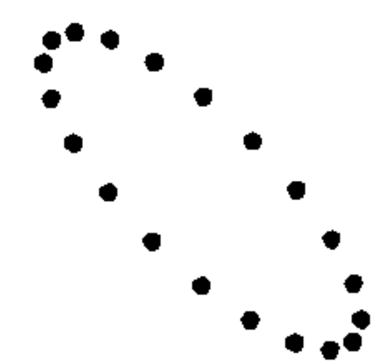
lengths in x- & y-directions

$$L_x = \int_{x_1}^{x_2} \sqrt{1+h_+}dx \approx (1+\frac{1}{2}h_+)L_{x0};$$
$$L_y = \int_{y_1}^{y_2} \sqrt{1-h_+}dy \approx (1-\frac{1}{2}h_+)L_{y0}$$

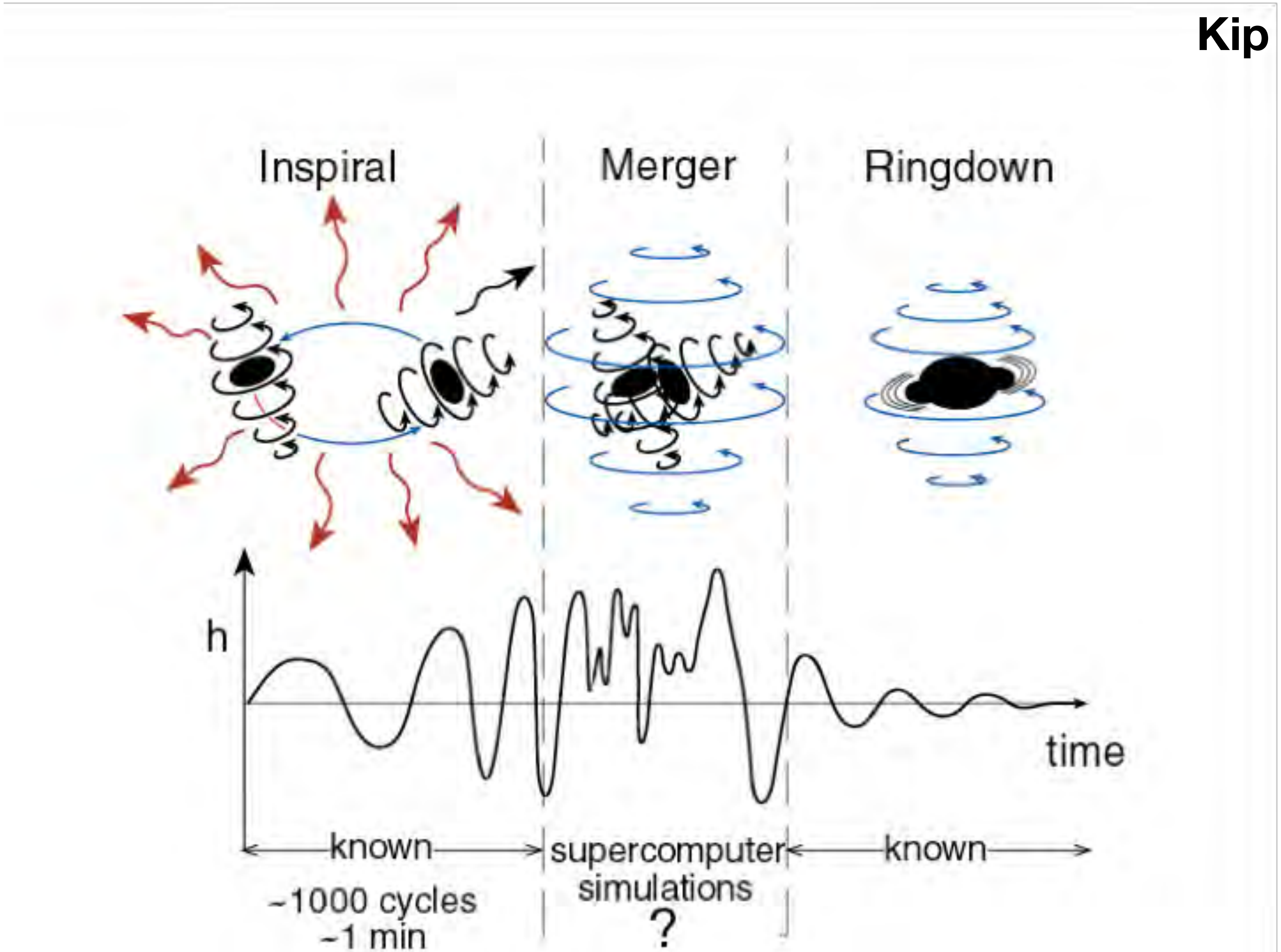
h_+



$h_\times$



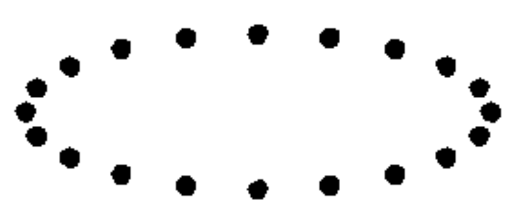
Typical wave forms

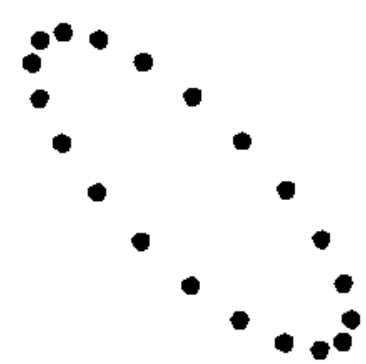


Tidal deformability & Love number

Selected references

- **A.E.H. Love** (1909) - The Yielding of the Earth to Disturbing Forces
- **K.S. Thorne** & A. Campolattaro (1967) - non-radial pulsation of NS
- J.B. Hartle & **K.S. Thorne** (1969) - stability of rotating NS
- … …
- **K.S. Thorne** (1998) - Tidal stabilization of rigid rotating, fully relativistic neutron star
- … …





Tidal deformability & Love number

$$-\frac{(1+g_{tt})}{2} = -\frac{m}{r} - \frac{3Q_{ij}}{2r^3}\left(n^i n^j - \frac{1}{3}\delta^{ij}\right) + \mathcal{O}\left(\frac{1}{r^3}\right) + \frac{\mathcal{E}_{ij}}{2} r^2 n^i n^j + \mathcal{O}(r^3)$$

$\mathcal{E}_{ij}$: external quadrupole tidal field

Q_{ij} : quadrupole moment of NS

$\lambda :\quad$ Tidal deformability

$$Q_{ij} = -\lambda \mathcal{E}_{ij}$$

$$Q_{ij} = \int d^3x \delta\rho(x) \left(x_i x_j - \frac{1}{3} r^2 \delta_{ij}\right)$$

$$n^i = \frac{x^i}{r}$$

dimensionless parameter

$k_2 :\quad l = 2$ Tidal Love number

$$k_2 = \frac{3}{2} G \lambda R^{-5}$$

Hinderer et al. PRD 81 (2010)

Regge-Wheeler gauge

linear $l = 2$ perturbation onto spherically symmetric star

$$
\begin{aligned}
ds^2 = \; & - \; e^{2\Phi(r)} \left[1 + H(r) Y_{20}(\theta,\varphi)\right] dt^2 \\
& + \; e^{2\Lambda(r)} \left[1 - H(r) Y_{20}(\theta,\varphi)\right] dr^2 \\
& + \; r^2 \left[1 - K(r) Y_{20}(\theta,\varphi)\right] \left(d\theta^2 + \sin^2(\theta) d\varphi^2\right)
\end{aligned}
$$

$$K'(r) = H'(r) + 2H(r)\Phi'(r)$$

$$f = \frac{d\epsilon}{dp}$$

$$\left(-\frac{6e^{2\Lambda}}{r^2} - 2(\Phi')^2 + 2\Phi'' + \frac{3}{r}\Lambda' + \frac{7}{r}\Phi' - 2\Phi'\Lambda' + \frac{f}{r}(\Phi' + \Lambda')\right) H + \left(\frac{2}{r} + \Phi' - \Lambda'\right) H' + H'' = 0$$

$$
\begin{aligned}
\frac{dH}{dr} &= \beta \\
\frac{d\beta}{dr} &= 2\left(1 - 2\frac{m_r}{r}\right)^{-1} H \left\{-2\pi\left[5\epsilon + 9p + f(\epsilon + p)\right] \right. \\
& \qquad \left. + \frac{3}{r^2} + 2\left(1 - 2\frac{m_r}{r}\right)^{-1} \left(\frac{m_r}{r^2} + 4\pi r p\right)^2 \right\} \\
& \quad + \frac{2\beta}{r}\left(1 - 2\frac{m_r}{r}\right)^{-1} \left\{-1 + \frac{m_r}{r} + 2\pi r^2(\epsilon - p)\right\}
\end{aligned}
$$

Tidal love number

$$k_2 = \frac{3}{2} G \lambda R^{-5}$$

$k_2 : \quad l = 2$ Tidal Love number

$$\begin{aligned} k_2 \quad = \quad & \frac{8C^5}{5}(1-2C)^2[2+2C(y-1)-y] \\ & \times\Big\{2C[6-3y+3C(5y-8)] \\ & \quad +4C^3[13-11y+C(3y-2)+2C^2(1+y)] \\ & \quad +3(1-2C)^2[2-y+2C(y-1)]\ln(1-2C)\Big\}^{-1} \end{aligned}$$

$$y = \frac{R\beta(R)}{H(R)} \qquad\qquad C = \frac{M}{R}$$

Laser Interferometer

- Consider a simple Michelson interferometer with $l_x \approx l_y \approx l$.

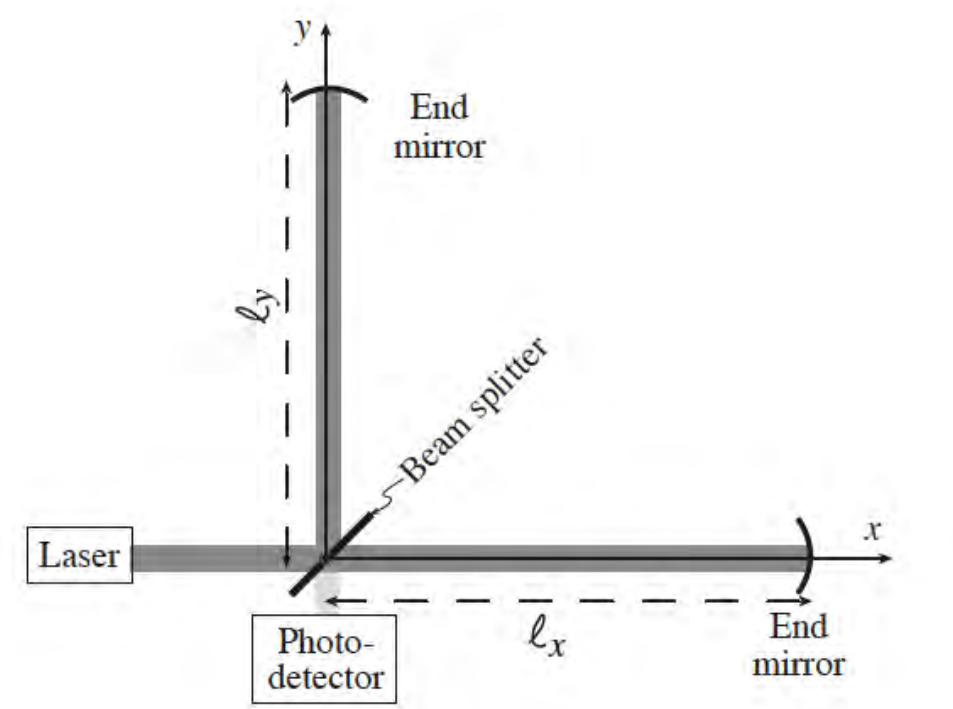
- **The phase difference** of returning lights reflected by x and y ends

$$\Delta\phi = \phi_x - \phi_y \approx 2\omega_0 \left[l_x - l_y + l h(t)\right]$$

- Toward the photo-detector:

$$E_{PD} \propto e^{i\phi_x} - e^{i\phi_y} = e^{i\phi_y}(e^{i\Delta\phi} - 1)$$

- For small $\Delta\phi$,



Time varying part

$$I_{PD} \propto |e^{i\Delta\phi} - 1|^2 \approx |\Delta\phi|^2 \approx 4\omega_0^2(l_x - l_y)^2 + 8\omega_0^2(l_x - l_y)l h(t)$$

PRL 112, 101101 (2014) PHYSICAL REVIEW LETTERS week ending 14 MARCH 2014

Systematic Parameter Errors in Inspiraling Neutron Star Binaries

Marc Favata*

$$\tilde{h}_{\mathrm{T}}(f) = \mathcal{A} f^{-7/6} e^{i\Psi_{\mathrm{T}}(f)}$$

$$\Psi_{\mathrm{T}}(f) = \varphi_c + 2\pi f t_c + \frac{3}{128\eta v^5}(\Delta\Psi^{\mathrm{pp}}_{3.5\mathrm{PN}} + \Delta\Psi^{\mathrm{spin}}_{3\mathrm{PN}} + \Delta\Psi^{\mathrm{ecc.}}_{2\mathrm{PN}} + \Delta\Psi^{\mathrm{tidal}}_{6\mathrm{PN}} + \Delta\Psi^{\mathrm{tm}}_{6\mathrm{PN}})$$

$$v = (\pi f M)^{1/3}$$

$$v/c = (GM\pi f/c^3)^{1/3}$$

6PN

$$\Delta\Psi^{\mathrm{tidal}}_{6\mathrm{PN}} = -\frac{39}{2}\tilde{\Lambda} v^{10} + v^{12}\left(\frac{6595}{364}\delta\tilde{\Lambda} - \frac{3115}{64}\tilde{\Lambda}\right)$$

5PN

$$\tilde{\Lambda} \equiv 32\frac{\tilde{\lambda}}{M^5} = \frac{8}{13}[(1+7\eta-31\eta^2)(\hat{\lambda}_1+\hat{\lambda}_2) - \sqrt{1-4\eta}(1+9\eta-11\eta^2)(\hat{\lambda}_1-\hat{\lambda}_2)].$$

PRL 112, 101101 (2014) PHYSICAL REVIEW LETTERS week ending 14 MARCH 2014

Systematic Parameter Errors in Inspiraling Neutron Star Binaries

Marc Favata*

phase shift vs deformability

$$\left.\frac{d\Psi_{\mathrm{T}}}{dx}\right|_{\mathrm{tidal,5PN}} = -\frac{195}{8}\frac{x^{3/2}}{\eta}\frac{\tilde{\lambda}}{M^5} \propto \frac{\tilde{\lambda}}{M^5}$$

$$x = (\omega M)^{2/3} \Rightarrow \left(\omega\frac{GM}{c^3}\right)^{2/3}$$

$$\eta = m_1 m_2/M^2$$

dimensionless

$$\Lambda = \frac{\lambda}{m^5} \Rightarrow G\lambda\left(\frac{c^2}{Gm}\right)^5 \approx 950.5\left(\frac{m_\odot}{m}\right)^5\left(\frac{\lambda}{10^{36}\ \mathrm{g\ cm^2\ s^2}}\right)$$

$$\Lambda = G\left(\frac{c^2}{Gm}\right)^5 \times \frac{2}{3}\frac{R^5}{G}k_2 = \frac{2}{3}\left(\frac{Rc^2}{Gm}\right)^5 k_2 \approx 9495\left(\frac{R_{10\mathrm{km}}}{m_{M_\odot}}\right)^5 k_2$$

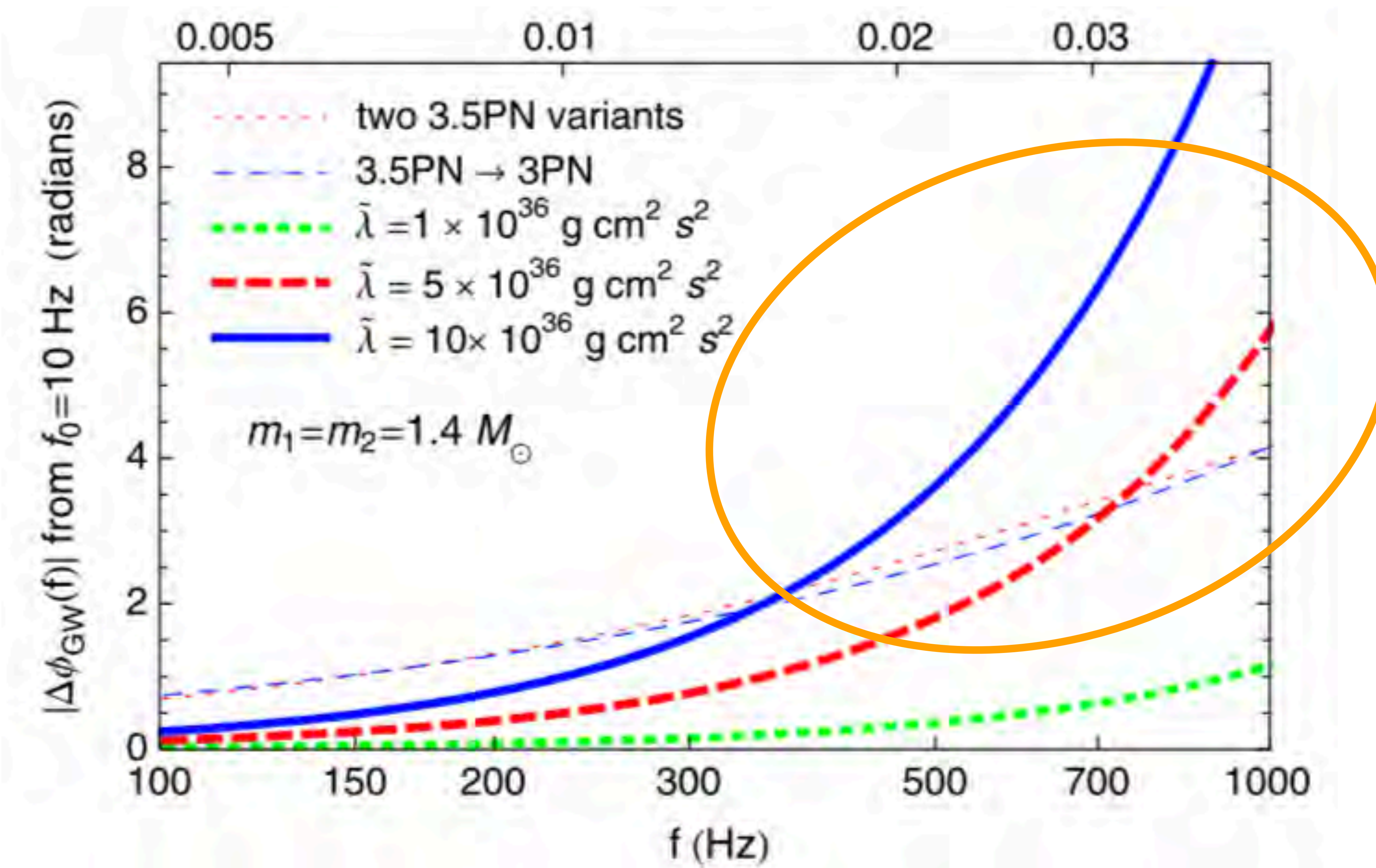
accumulated GW phase

PHYSICAL REVIEW D **81**, 123016 (2010)

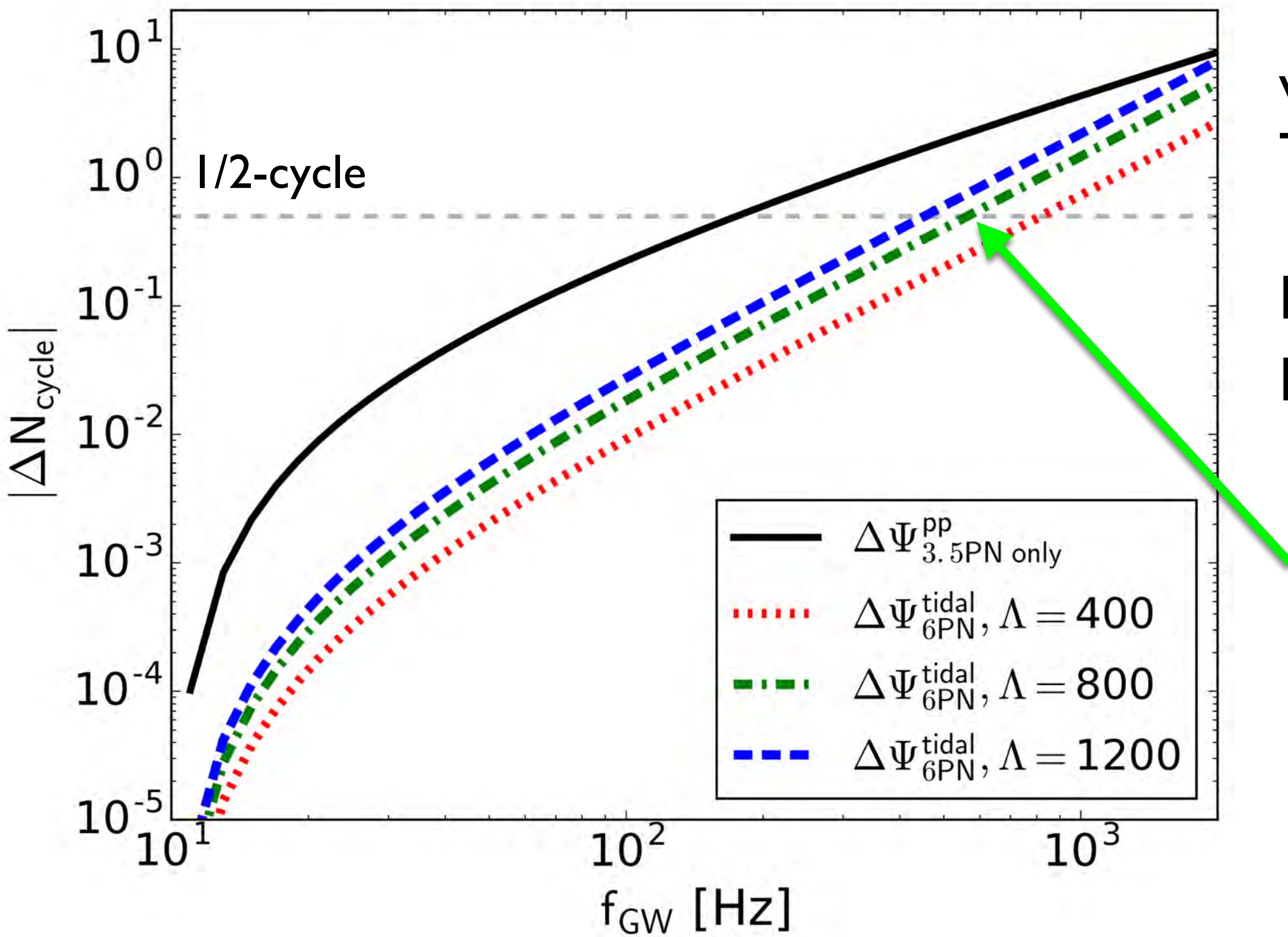
Tidal deformability of neutron stars with realistic equations of state and their gravitational wave signatures in binary inspiral

Tanja Hinderer,[1] Benjamin D. Lackey,[2] Ryan N. Lang,[3,4] and Jocelyn S. Read[5]

$$|\Delta\phi_{\rm GW}(f)| = \left|\Psi(f)_{\rm pp(3.5PN)} - \Psi(f)_{\rm pp(3.5PN)+tidal(5PN)}\right|$$



accumulated GW phase



waveform model:
TaylorF2(SPA)

Mch=1.188M$_\odot$

M1=M2=1.365M$_\odot$

~600 Hz

Measurement error vs. source distance

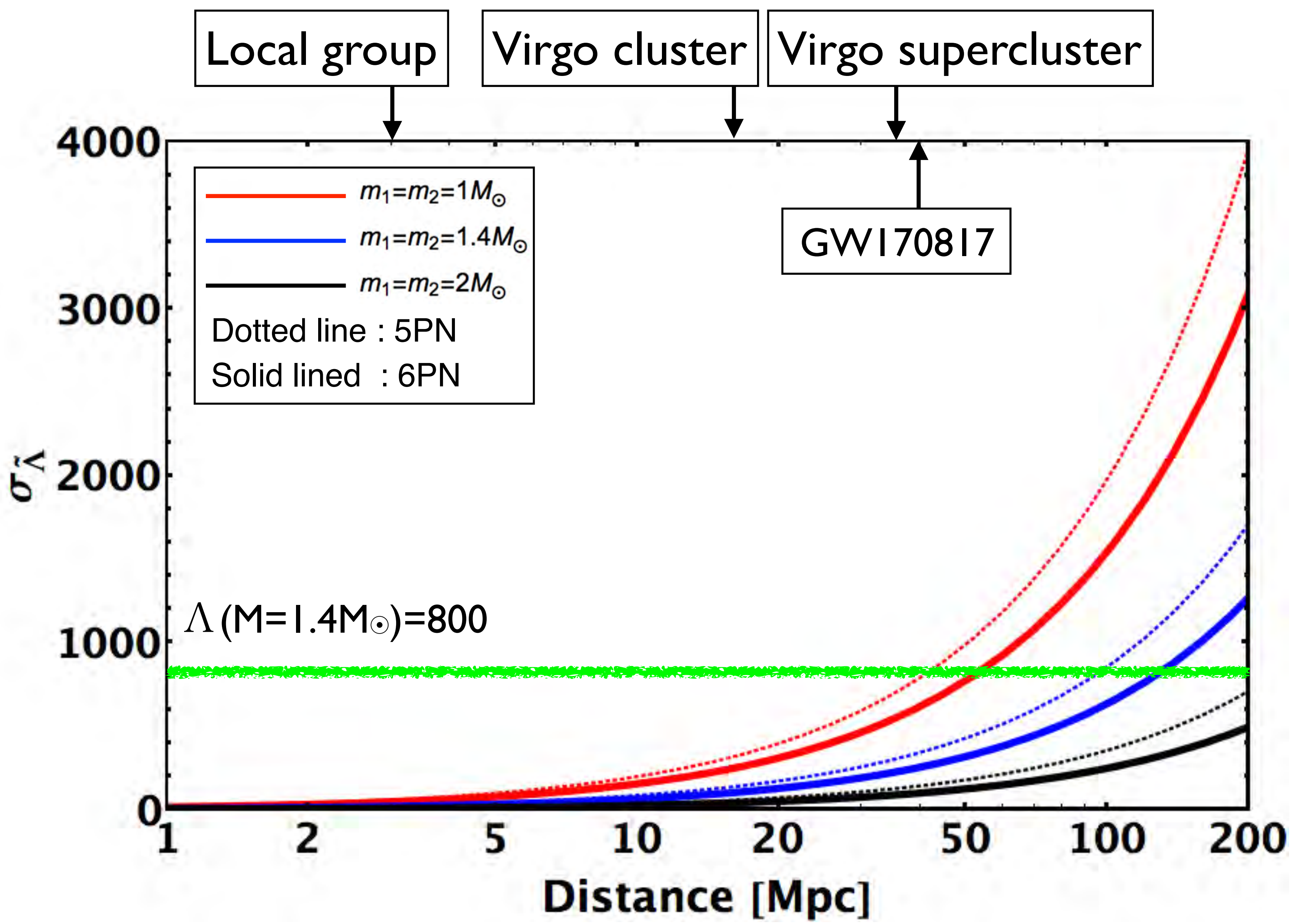


Fig. 1: Tidal deformability measurement error vs distance to the source. distances to galaxy clusters and GW20170817 distance are marked.

Y.B. Choi, H. S. Cho, C.-H. Lee

Press Release Oct 16, 2017

GW from Binary NS Mergers

*GW 170817 (**d=40 Mpc**)*
GRB 170817A by Fermi-GBM
Kilonova/X-ray/Optical Afterglows

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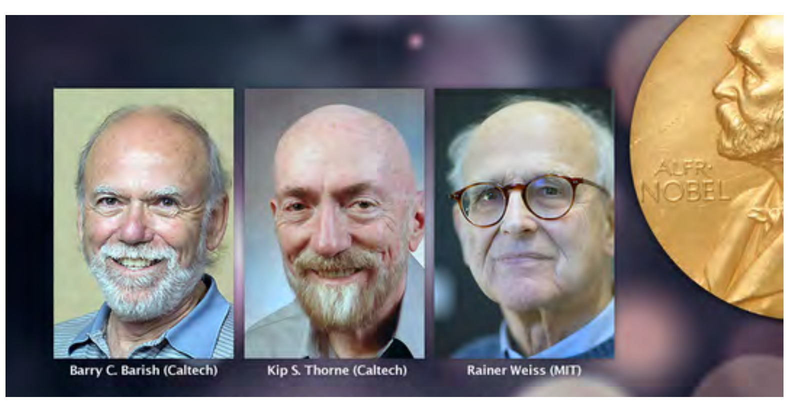
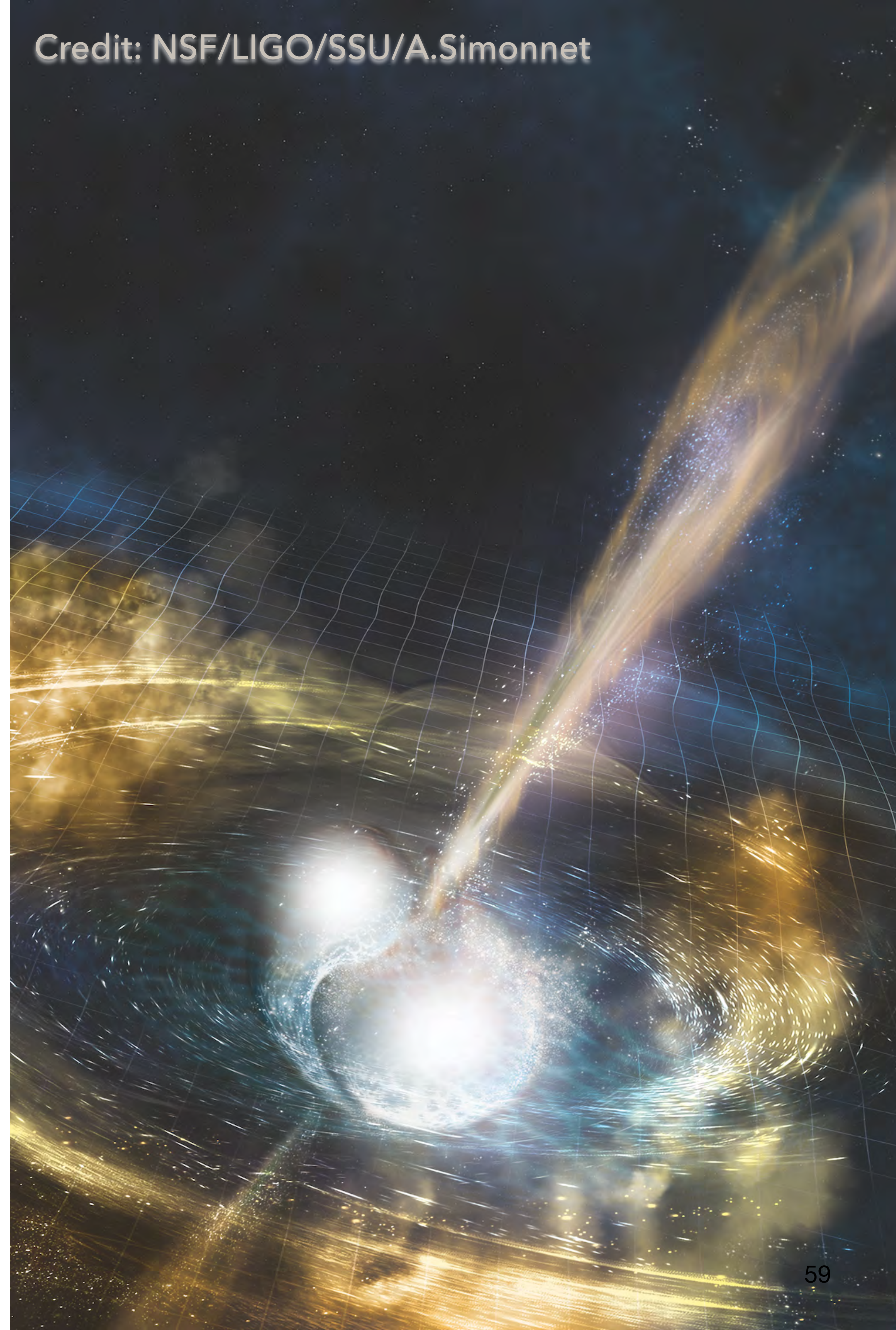
soon after the announcement of
2017 Nobel Prize





PRL **119,** 161101 (2017)

Selected for a Viewpoint in *Physics*

PHYSICAL REVIEW LETTERS

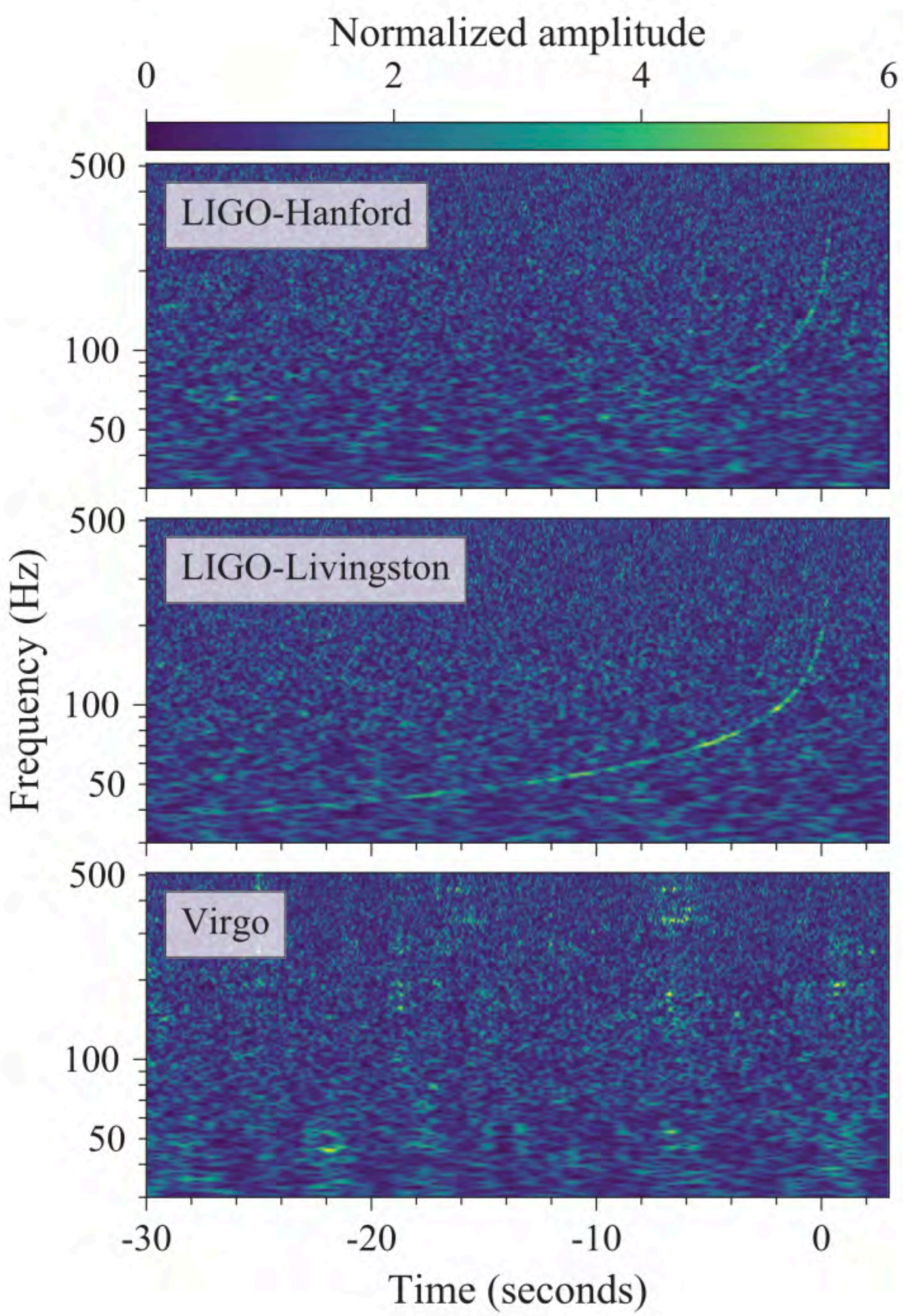
week ending
20 OCTOBER 2017

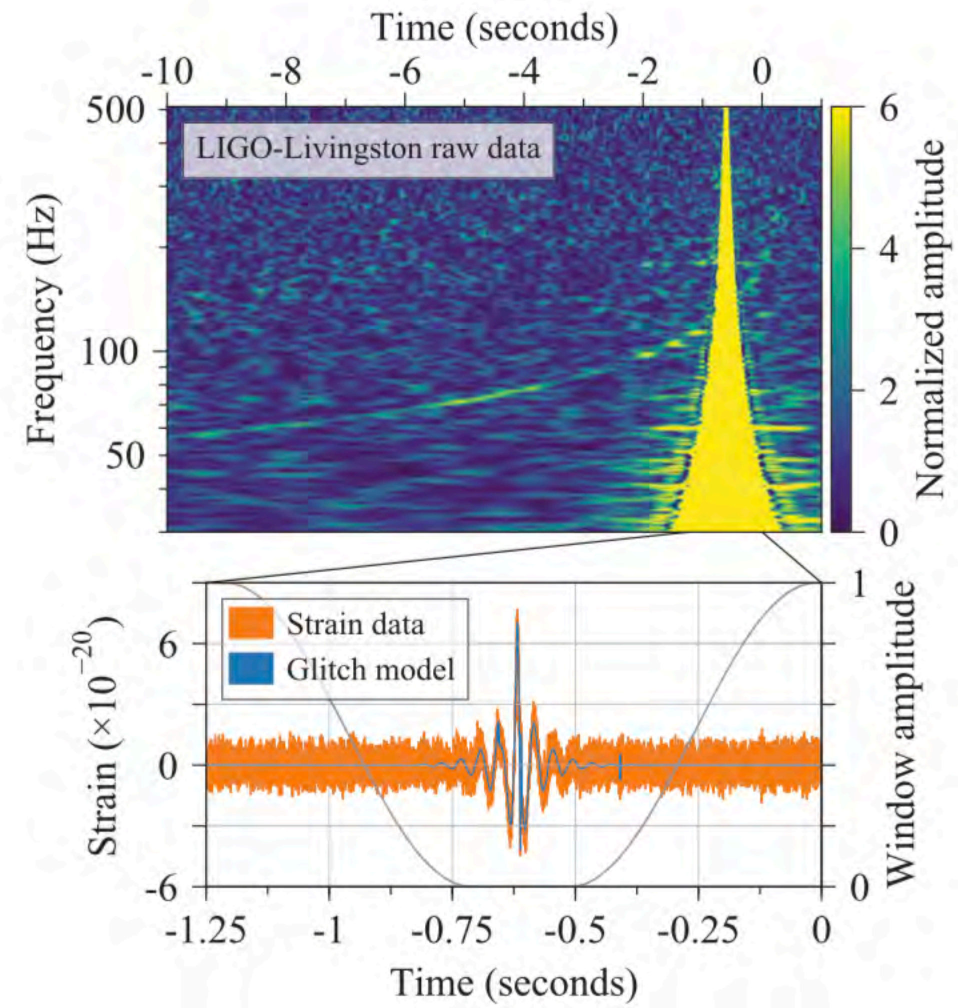
GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral

B. P. Abbott *et al.**

(LIGO Scientific Collaboration and Virgo Collaboration)

(Received 26 September 2017; revised manuscript received 2 October 2017; published 16 October 2017)





PRL 119, 161101 (2017) | Selected for a Viewpoint in Physics
PHYSICAL REVIEW LETTERS | week ending 20 OCTOBER 2017

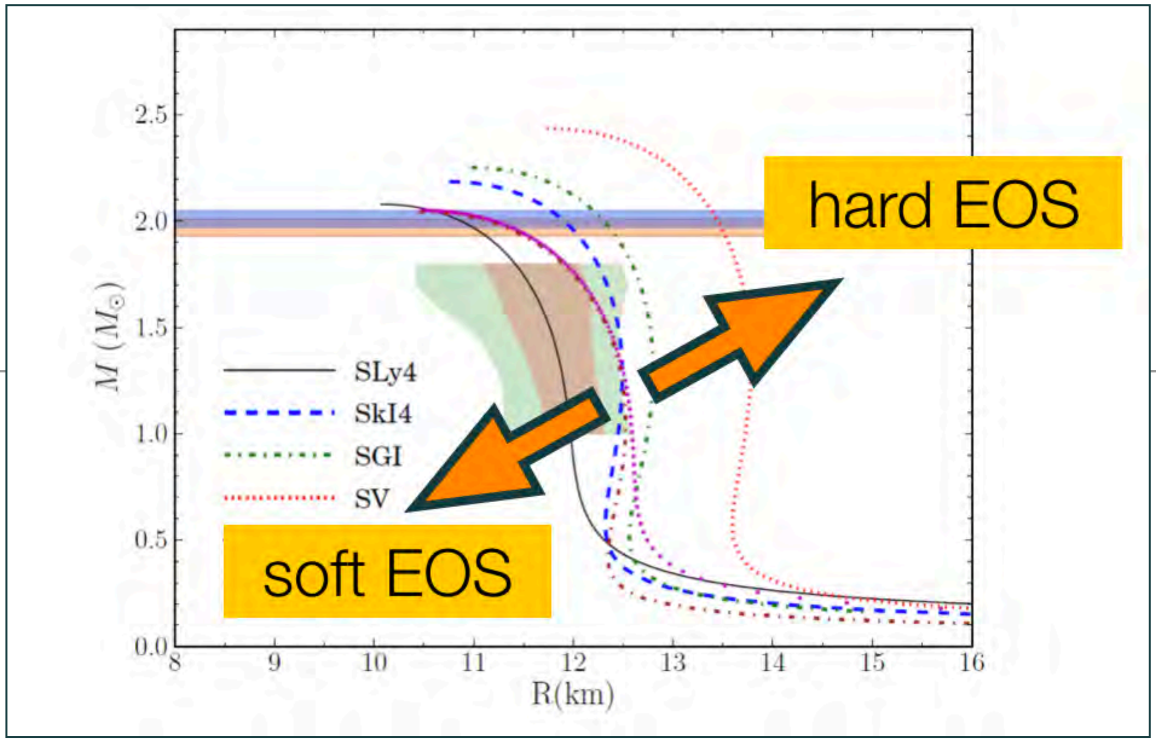
GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral

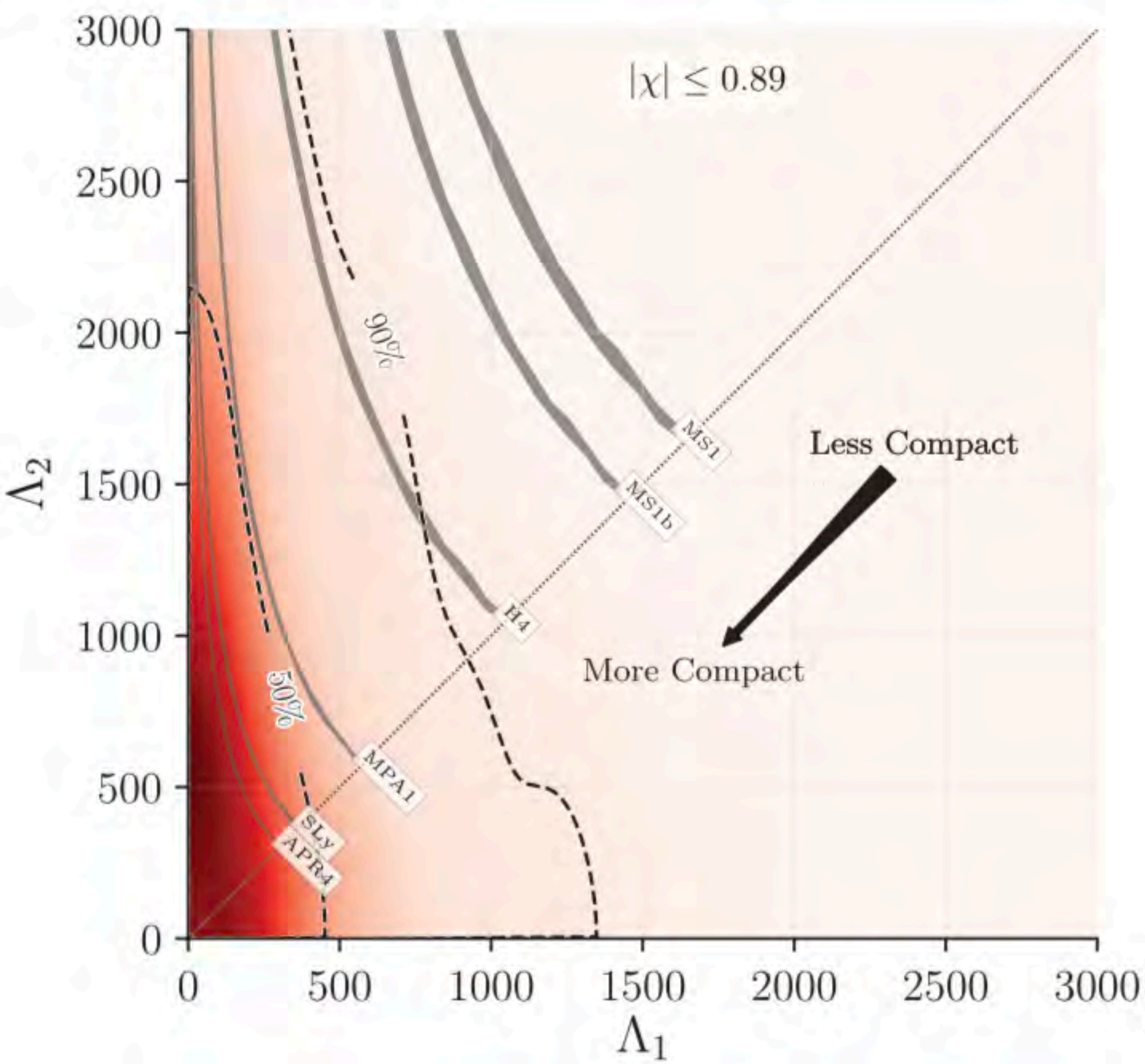
TABLE I. Source properties for GW170817: we give ranges encompassing the 90% credible intervals for different assumptions of the waveform model to bound systematic uncertainty. The mass values are quoted in the frame of the source, accounting for uncertainty in the source redshift.

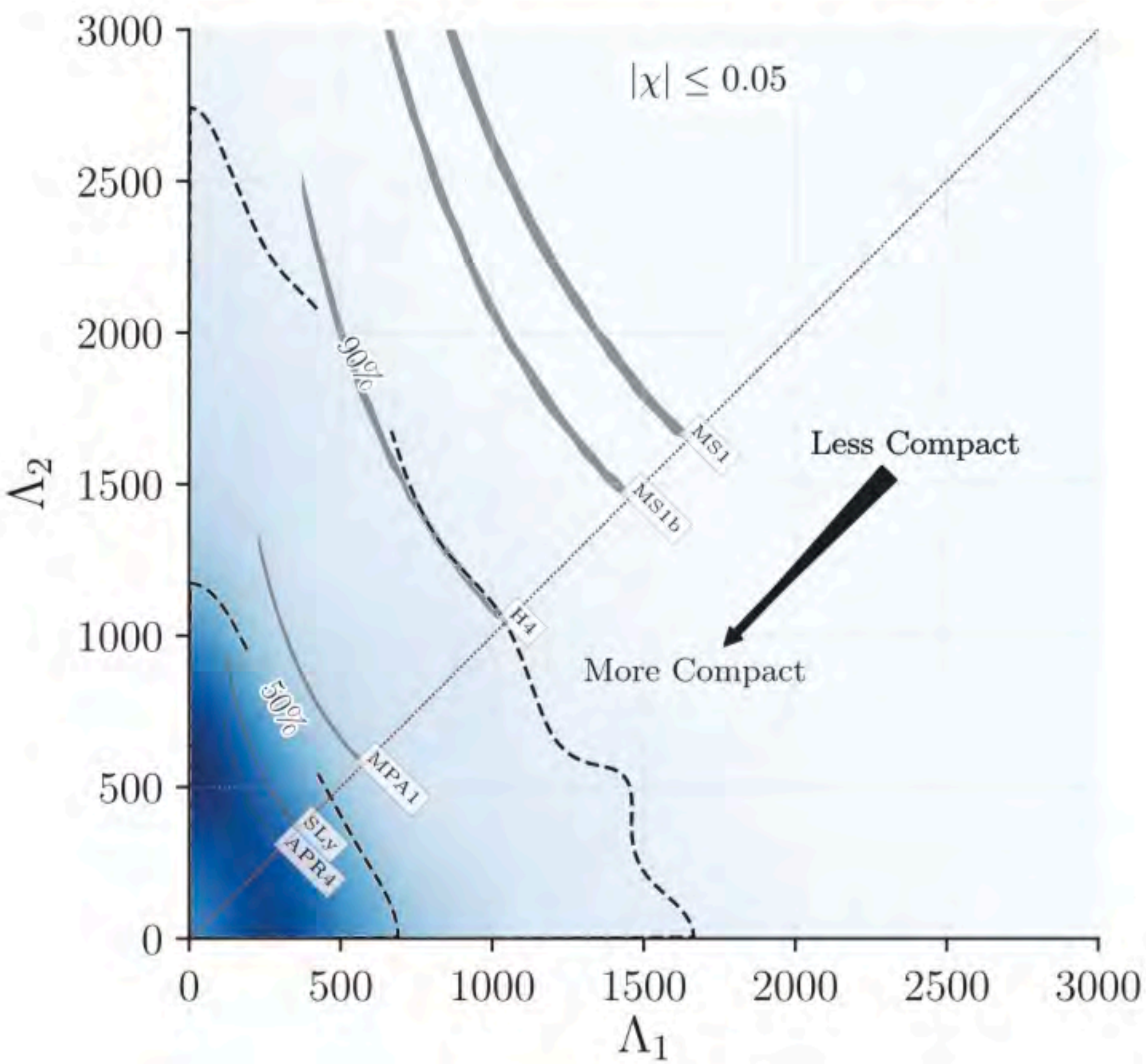
	Low-spin priors ($\|\chi\| \leq 0.05$)	High-spin priors ($\|\chi\| \leq 0.89$)
Primary mass m_1	1.36–1.60 $M_\odot$	1.36–2.26 $M_\odot$
Secondary mass m_2	1.17–1.36 $M_\odot$	0.86–1.36 $M_\odot$
Chirp mass $\mathcal{M}$	$1.188^{+0.004}_{-0.002} M_\odot$	$1.188^{+0.004}_{-0.002} M_\odot$
Mass ratio m_2/m_1	0.7–1.0	0.4–1.0
Total mass $m_{\rm tot}$	$2.74^{+0.04}_{-0.01} M_\odot$	$2.82^{+0.47}_{-0.09} M_\odot$
Radiated energy $E_{\rm rad}$	$> 0.025 M_\odot c^2$	$> 0.025 M_\odot c^2$
Luminosity distance $D_{\rm L}$	40^{+8}_{-14} Mpc	40^{+8}_{-14} Mpc
Viewing angle Θ	$\leq 55°$	$\leq 56°$
Using NGC 4993 location	$\leq 28°$	$\leq 28°$
Combined dimensionless tidal deformability $\tilde{\Lambda}$	≤ 800	≤ 700
Dimensionless tidal deformability $\Lambda(1.4 M_\odot)$	≤ 800	≤ 1400

GW170817
Information of Neutron Star Structure has been revealed by Gravitational Waves

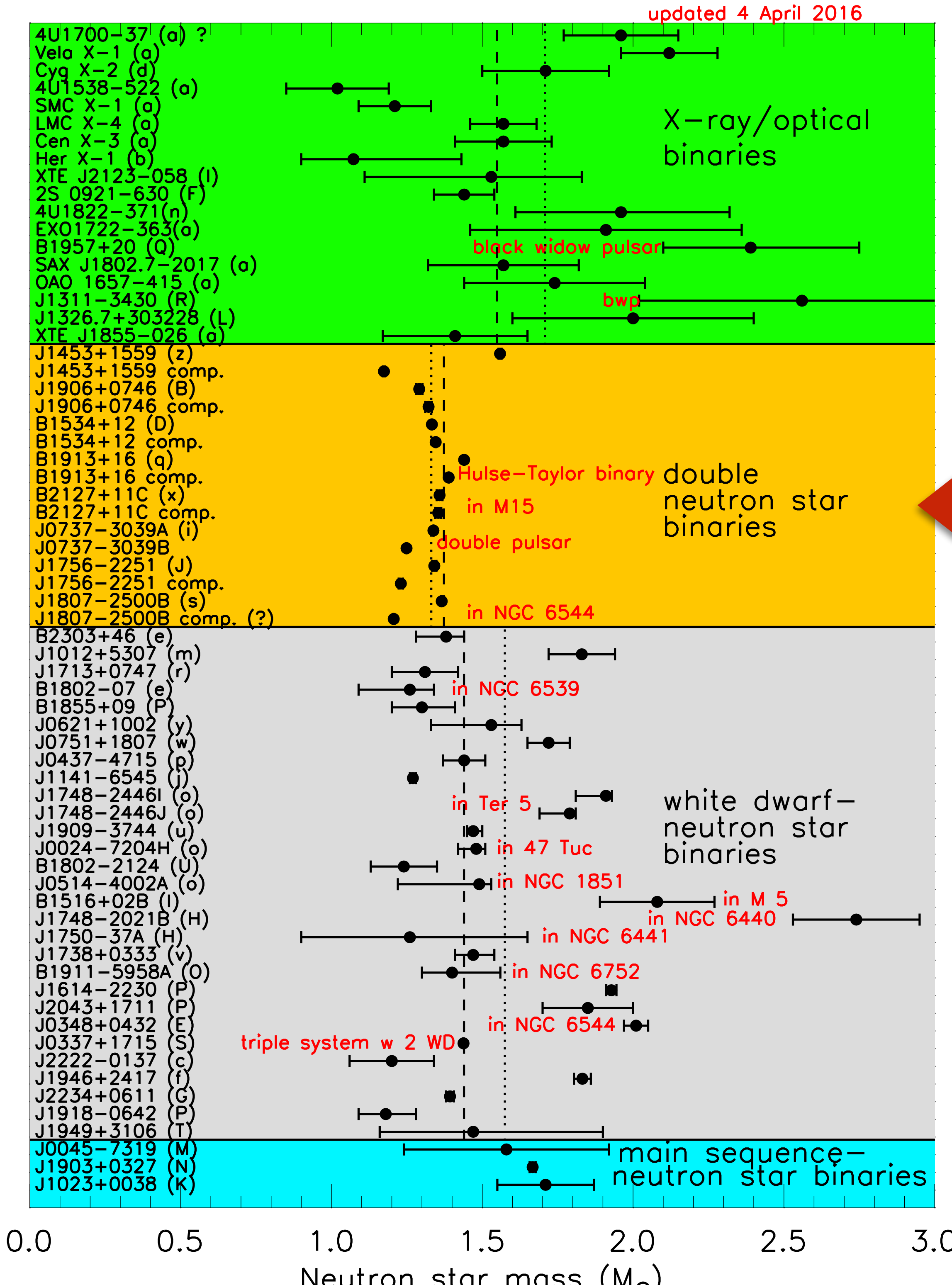
perfer lower Λ (soft EOS)







Neutron Star of Known Mass



GW170817: BNS

M1: 1.36~1.60 $M_\odot$ (1.36~2.26)

M2: 1.17~1.36 $M_\odot$ (0.86~1.36)

J. Lattimer, Annu.Rev.Nucl.Part.Sci.62,485(2012) and https://stellarcollapse.org by C. Ott

Slide from YMKim

Prospects of the Observing Runs

"Prospects for Observing and Localizing Gravitational-Wave Transients with Advanced LIGO, Advanced Virgo and KAGRA", arXiv:1304.0670v4, LIGO-P1200087-v45, Living Rev. Relativity, 21, 3 (2018)

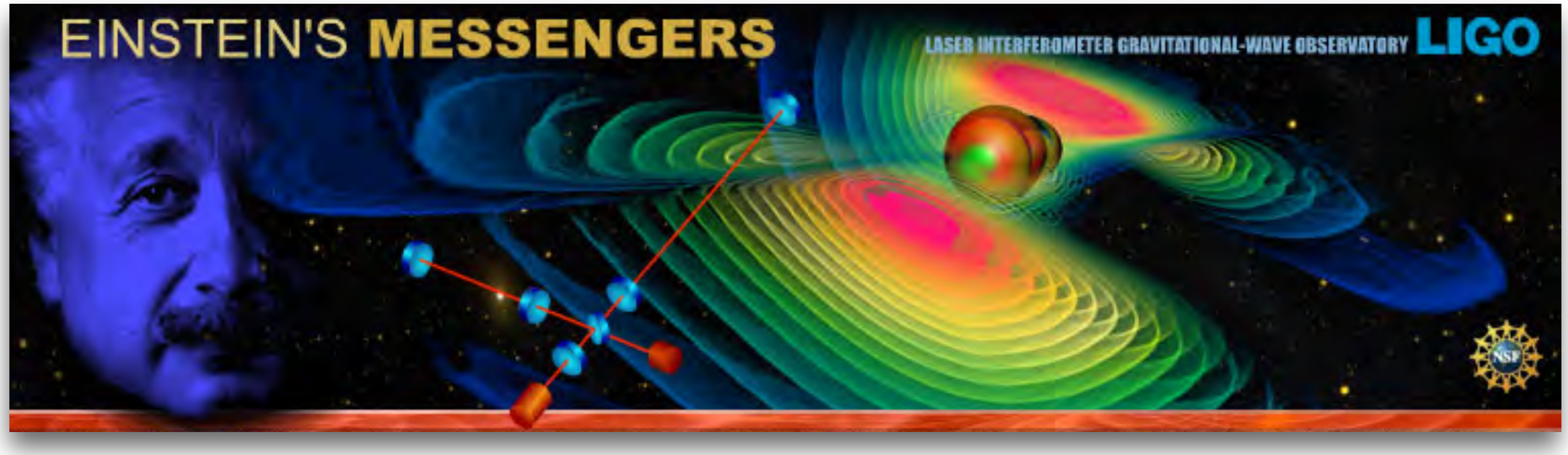
Epoch			2015 – 2016	2016 – 2017	2018 – 2019	2020+	2024+
Planned run duration			4 months	9 months	12 months	(per year)	(per year)
Expected burst range/Mpc		LIGO	40 – 60	60 – 75	75 – 90	105	105
		Virgo	—	20 – 40	40 – 50	40 – 70	80
		KAGRA	—	—	—	—	100
Expected BNS range/Mpc		LIGO	40 – 80	80 – 120	120 – 170	190	190
		Virgo	—	20 – 65	65 – 85	65 – 115	125
		KAGRA	—	—	—	—	140
Achieved BNS range/Mpc		LIGO	60 – 80	60 – 100	—	—	—
		Virgo	—	25 – 30	—	—	—
		KAGRA					
Estimated BNS detections			0.05 – 1	0.2 – 4.5	1 – 50	4 – 80	11 – 180
Actual BNS detections			0	1	—	—	—
90% CR	% within	5 deg^2	< 1	1 – 5	1 – 4	3 – 7	23 – 30
		20 deg^2	< 1	7 – 14	12 – 21	14 – 22	65 – 73
		median/deg^2	460 – 530	230 – 320	120 – 180	110 – 180	9 – 12
Searched area	% within	5 deg^2	4 – 6	15 – 21	20 – 26	23 – 29	62 – 67
		20 deg^2	14 – 17	33 – 41	42 – 50	44 – 52	87 – 90

We expect to observe more BNS and/or NS-BH

Slide by YMKim

LIGO

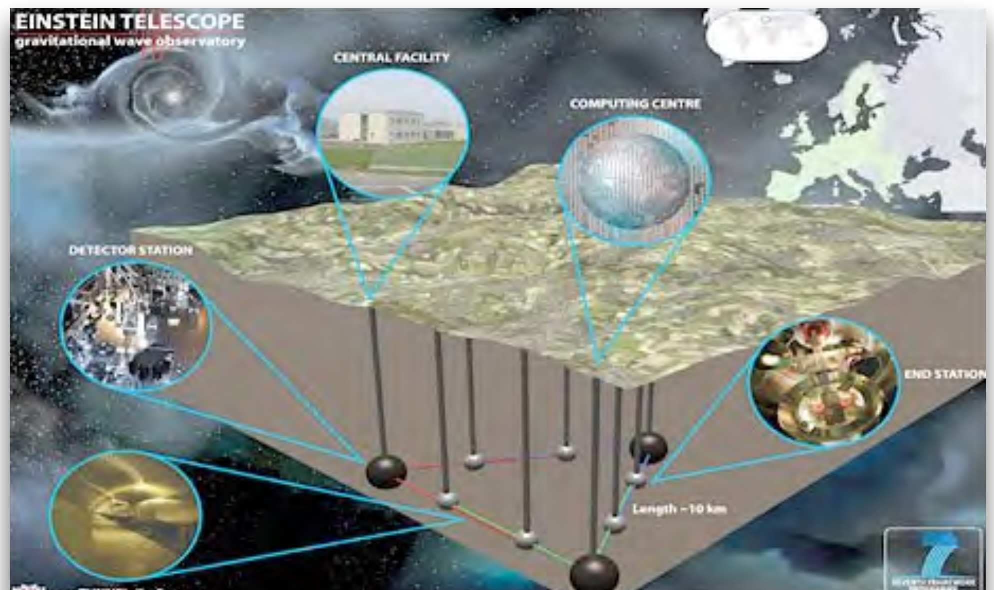
- First detection of gravitational-waves
- First detection of black hole binary
- First observation of heavy black holes
- New possibilities: grativational-wave astronomy
- Neutron stars, black holes, supernovae, gamma-ray bursts, …
- … …



Future Gravitational-Wave Observatories

Einstein Telescope

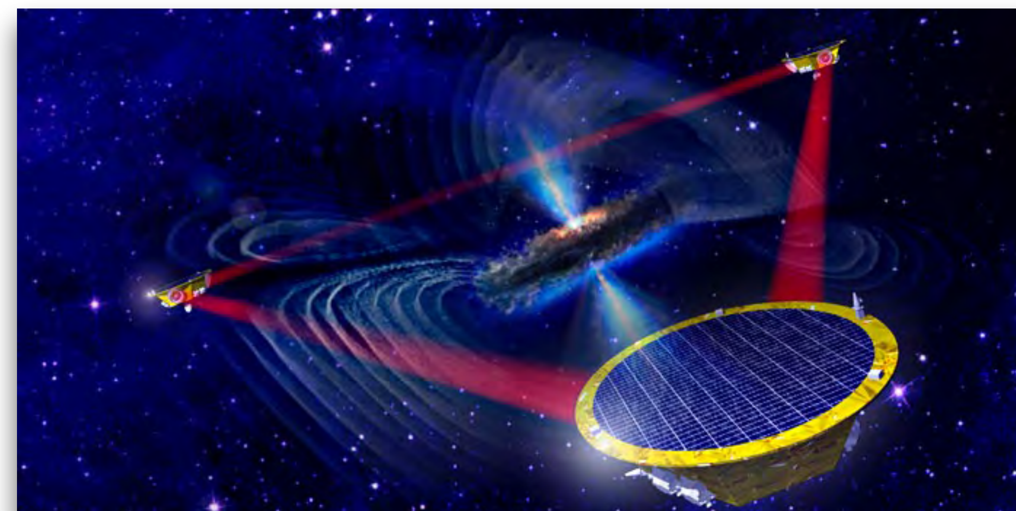
ESA / 2030? (designing stage)



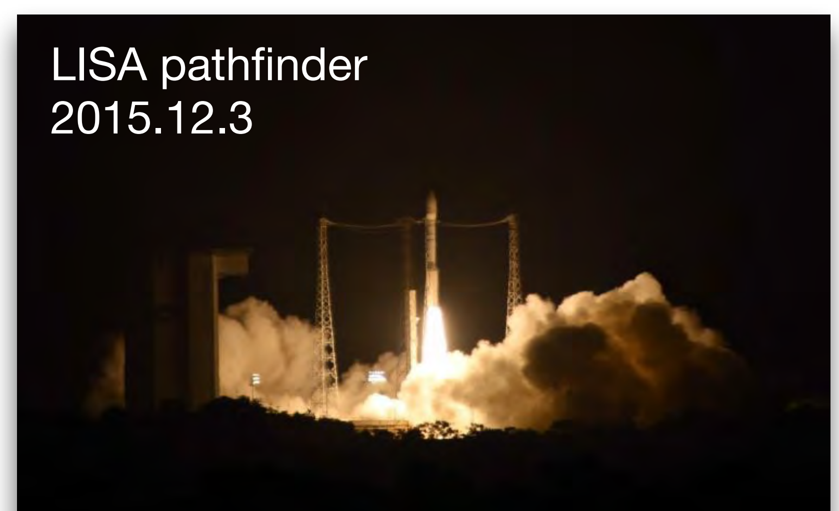
10 km

eLISA

2034



10^6 km

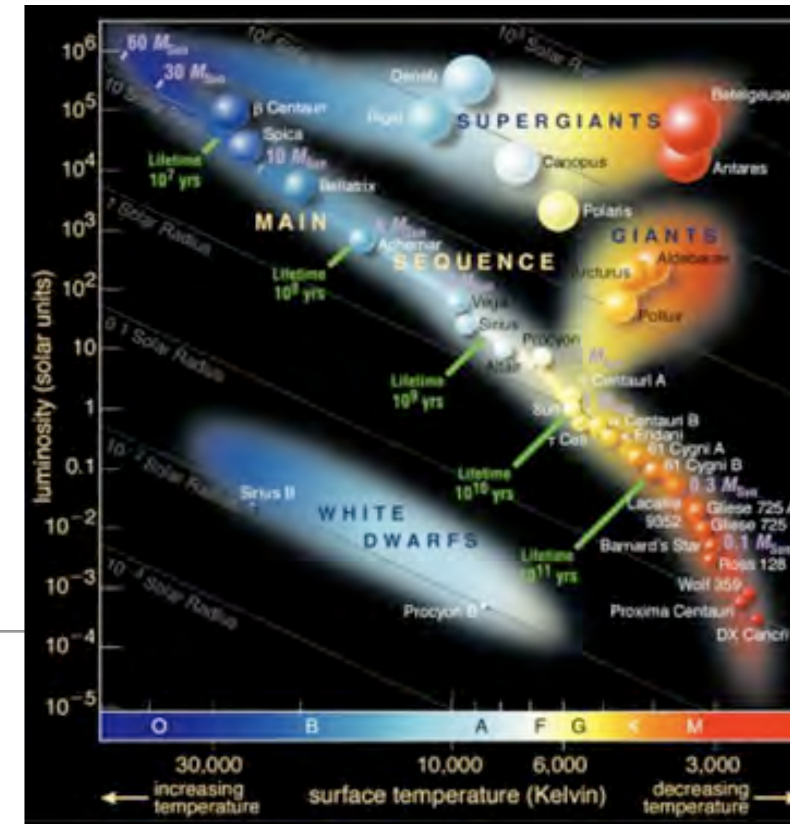


LISA pathfinder
2015.12.3

Part 2

Neutron Star Equation of State

Physics of White Dwarfs



S. Chandrasekhar (1967)
Stellar Structure

S.L. Shapiro & S.A. Teukolsky (1983)
Black Holes, White Dwarfs, and Neutron Stars : The Physics of Compact Object

D. Maoz (2006)
Astrophysics in a Nutshell

Thermodynamics

without nuclear burning

zero temperature limit

Hydrostatic equilibrium	$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$
Mass continuity	$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$
Radiative energy transport	$$\frac{dT(r)}{dr} = -\frac{3L(r)\kappa(r)\rho(r)}{4\pi r^2 4acT(r)^3}$$
Energy conservation	$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\epsilon(r)$$

$$P = P(\rho, T, \text{composition})$$

$$\kappa = \kappa(\rho, T, \text{composition})$$

$$\epsilon = \epsilon(\rho, T, \text{composition})$$

Pressure

$$n = N/V$$

number density

- Radiation pressure of photon & neutrino

$$P_{\text{rad}} = \frac{1}{3} a T^4$$

- Thermal (kinetic) pressure of ideal gas

$$P_{\text{kin}} = nkT \quad (PV = NkT)$$

- Quantum degeneracy pressure of ideal gas

$$P_{\text{deg}} = ? \qquad \text{(Pauli exclusion)}$$

- Pressure from strong interactions

$$P_{\text{nuclear matter}} = ?$$

Virial theorem

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$\int_0^R \frac{dP}{dr} 4\pi r^3 dr = \int_0^R \left(-\frac{GM}{r^2}\rho\right) 4\pi r^3 dr$$

$$\int_0^R \frac{dP}{dr} 4\pi r^3 dr = \cancel{P \cdot 4\pi r^2 \Big|_0^R} - \int_0^R 3P \cdot 4\pi r^2 dr$$

$$\int_0^R \left(-\frac{GM}{r}\right) 4\pi r^2 \rho dr$$

$$E_{\text{therm}} = \int_0^R \left(\frac{3}{2}nkT\right) 4\pi r^2 dr = \frac{1}{2}\int_0^R 3P \cdot 4\pi r^2 dr$$

$$-2E_{\text{therm}} = E_G$$

$$E_{\text{therm}} = -\frac{1}{2}E_G = \frac{1}{2}|E_G|$$

$$E_{\text{tot}} = E_G + E_{\text{therm}} = \frac{1}{2}E_G = -\frac{1}{2}|E_G| < 0$$

Virial theorem

$$E_{\rm therm} = -\frac{1}{2}E_G = \frac{1}{2}|E_G|$$

$$E_{\rm tot} = E_G + E_{\rm therm} = \frac{1}{2}E_G = -\frac{1}{2}|E_G| < 0$$

- Star forms by slow gravitational contraction
- Total energy becomes more negative
  - T & density increase as star loses energy

i) start nuclear burning (thermal pressure dominates)
ii) reach quantum state (degeneracy pressure dominates)
- contraction stops and cools down

n_e, n_+ number density

Ideal Degenerate Electron Gas

$$\Delta x \Delta p_x > h \qquad d^3p dV > h^3$$

$$\rho_{\rm quantum} \approx \frac{m_p}{(\lambda_e/2)^3} = \frac{8m_p(3m_e kT)^{3/2}}{h^3}$$

$$\rho_{\rm quantum}(\text{sun, center}) \approx 640 \text{ g/cm}^3$$

$$\rho_{\rm quantum}(T = 10^8 \text{ K}) \approx 11,000 \text{ g/cm}^3$$

$$n_e = \int_0^{p_f} \frac{8\pi}{h^3} p^2 dp = \frac{8\pi}{3h^3} p_f^3 = Z n_+ = Z \frac{\rho}{A m_p}$$

$$\rho_{\rm WD} \sim 10^6 \text{ g/cm}^3$$

$$T_{\rm WD} \sim 10^7 \text{ K}$$

Quantum degeneracy pressure dominates : *thermal pressure negligible*

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$p \propto \rho^\gamma \propto \rho^{(n+1)/n}$$

$$P_{\rm deg} \approx 3 \times 10^{22} \text{dyne/cm}^2$$

$$P_{\rm th} \approx 2 \times 10^{20} \text{dyne/cm}^2$$

n : polytropic index

Polytropic Structure

adiabatic process (thermal pressure)

$$pV^\gamma = \text{constant}$$

$$\gamma = \frac{c_p}{c_V}$$ ratio of specific heats

$$E = \frac{3}{2}\frac{N}{V}kT$$

$$P_{\text{kin}} = \frac{N}{V}kT$$

ideal Fermi gas $p \propto \rho^\gamma \propto \rho^{(n+1)/n}$

n : polytropic index

Polytropic Structure

n : polytropic index

ideal Fermi gas $\quad p \propto \rho^{\gamma} \propto \rho^{(n+1)/n}$

nonrelativistic

$$\gamma = \frac{5}{3} \qquad n = \frac{3}{2} \qquad P_e = \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{20 m_e m_p^{5/3}} \left(\frac{Z}{A}\right)^{5/3} \rho^{5/3}$$

ultrarelativistic

$$\gamma = \frac{4}{3} \qquad n = 3 \qquad P_e = \left(\frac{3}{8\pi}\right)^{1/3} \frac{hc}{4 m_p^{4/3}} \left(\frac{Z}{A}\right)^{4/3} \rho^{4/3}$$

Lane-Emden Equation

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

$$P = K\rho^\gamma$$

$$\rho = \rho_c \theta^n$$

$$r = a\xi \qquad a = \sqrt{\frac{(n+1)K_n \rho_c^{(1-n)/n}}{4\pi G}}$$

$$\frac{1}{\xi}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} = -\theta^n$$

Boundary conditions

$$\rho = \rho_c \theta^n$$

center $\theta(0) = 1, \quad \theta'(0) = 0$

surface $\theta(\xi_1) = 0 \;\rightarrow\; P = \rho = 0$

$$M = 4\pi \left[\frac{(n+1)K_n}{4\pi G} \right]^{3/2} \rho_c^{(3-n)/2n} \left| \left(\xi^2 \frac{d\theta}{d\xi} \right)_{\xi=\xi_1} \right|$$

$$r = a\xi \qquad a = \sqrt{\frac{(n+1)K_n \rho_c^{(1-n)/n}}{4\pi G}}$$

$$\rho = \rho_c \theta^n$$

center $\theta(0) = 1, \quad \theta'(0) = 0$

surface $\theta(\xi_1) = 0$ → $P = \rho = 0$

$$M = 4\pi \left[\frac{(n+1)K_n}{4\pi G} \right]^{3/2} \rho_c^{(3-n)/2n} \left| \left(\xi^2 \frac{d\theta}{d\xi} \right)_{\xi=\xi_1} \right|$$

$$R = \sqrt{\frac{(n+1)K_n \rho_c^{(1-n)/n}}{4\pi G}}\, \xi_1$$

Constants of Lane-Emden equation

$$r = a\xi$$

n	ξ_1	$-\xi_1^2 \left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$	$\rho_c/\bar{\rho}$
0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
2.5	5.35528	2.18720	23.40646
3.0	6.89685	2.01824	54.1825
3.25	8.01894	1.94980	88.153
3.5	9.53581	1.89056	152.884
4.0	14.97155	1.79723	622.408
4.5	31.83646	1.73780	6189.47
4.9	169.47	1.7355	934800
5.0	∞	1.73205	∞

Polytropic Structure

n : polytropic index

ideal gas $$p \propto \rho^{\gamma} \propto \rho^{(n+1)/n}$$

nonrelativistic

$$\gamma = \frac{5}{3} \quad n = \frac{3}{2} \qquad P_e = \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{20 m_e m_p^{5/3}} \left(\frac{Z}{A}\right)^{5/3} \rho^{5/3}$$

ultrarelativistic

$$\gamma = \frac{4}{3} \quad n = 3 \qquad P_e = \left(\frac{3}{8\pi}\right)^{1/3} \frac{hc}{4 m_p^{4/3}} \left(\frac{Z}{A}\right)^{4/3} \rho^{4/3}$$

Chandrasekhar Mass

$\mu_e = A/Z \approx 2 \text{ for He, C, O,..}$

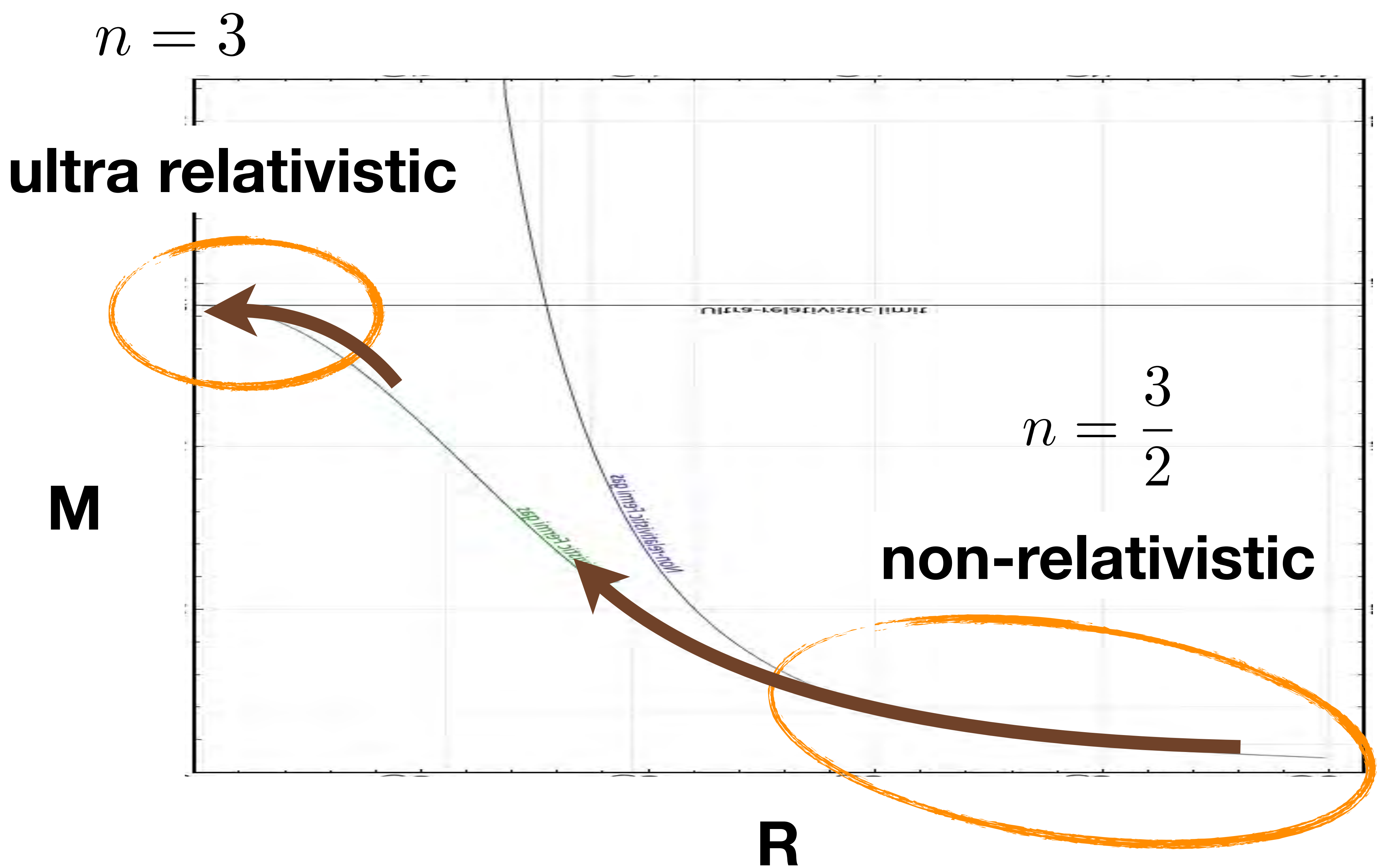
Ultra-relativistic $\gamma = \frac{4}{3} \quad n = 3$

$$R = 3.347 \times 10^4 \left(\frac{\rho_c}{10^6 \text{ g cm}^{-3}} \right)^{-1/3} \left(\frac{2}{\mu_e} \right)^{2/3} \text{ km}$$

$$M = 1.457 \left(\frac{2}{\mu_e} \right)^2 M_\odot$$

independent of central density

White Dwarfs



Physics of Neutron Stars

S.L. Shapiro & S.A. Teukolsky (1983)

Black Holes, White Dwarfs, and Neutron Stars: The Physics of Compact Object

J. M. Lattimer (2016)

Neutron Stars are Gold Mines (G.E.Brown Memorial, World Scientific, 2017)

… …

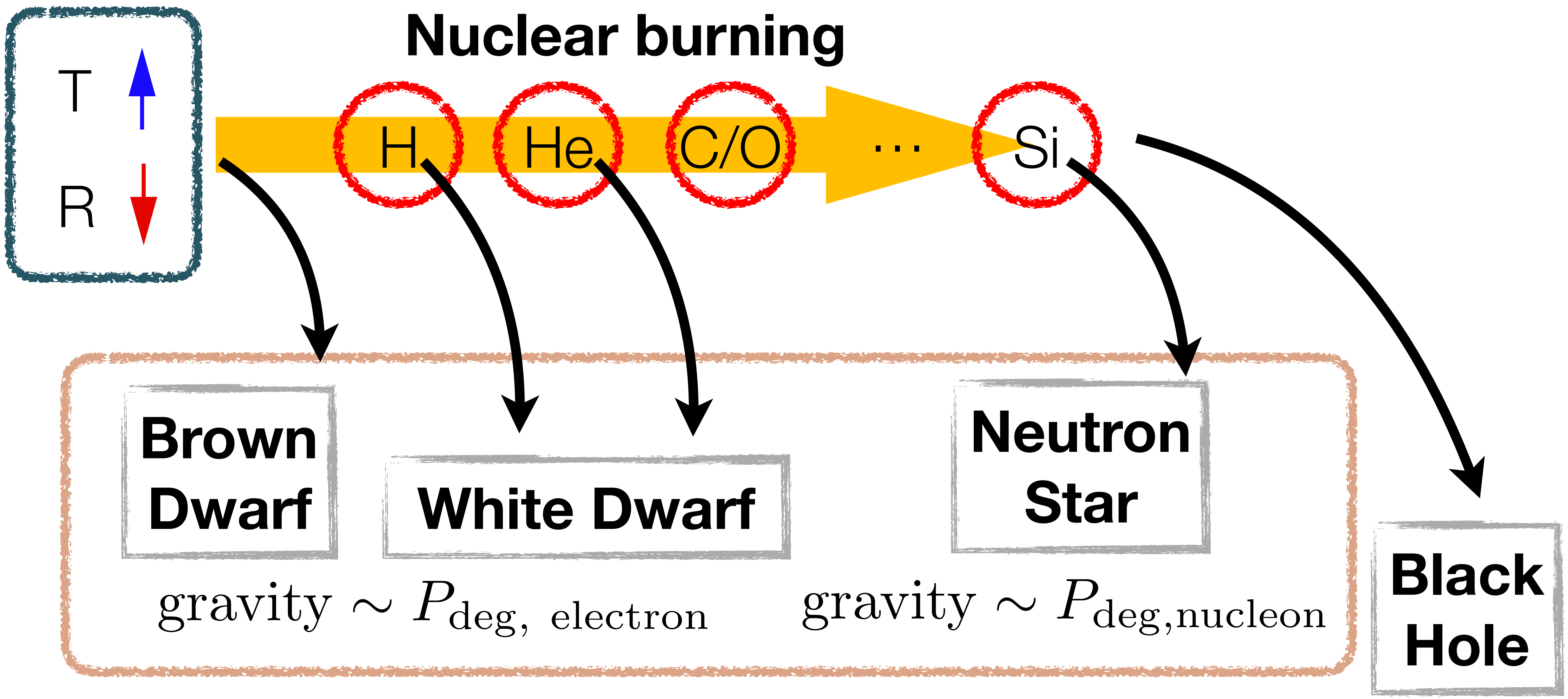
Q) What about neutron star?

Nucleon degeneracy pressure

Evolution of star

Viral theorem

as star radiates



as star radiates R ~ fixed
T ↓

$$\text{gravity} \gg P_{\text{deg,nucleon}}$$

$$\text{gravity} \sim P_{\text{quantum}}?$$

What's inside black hole ?

quantum gravity

$$\text{gravity} \sim P_{\text{quantum}} ?$$

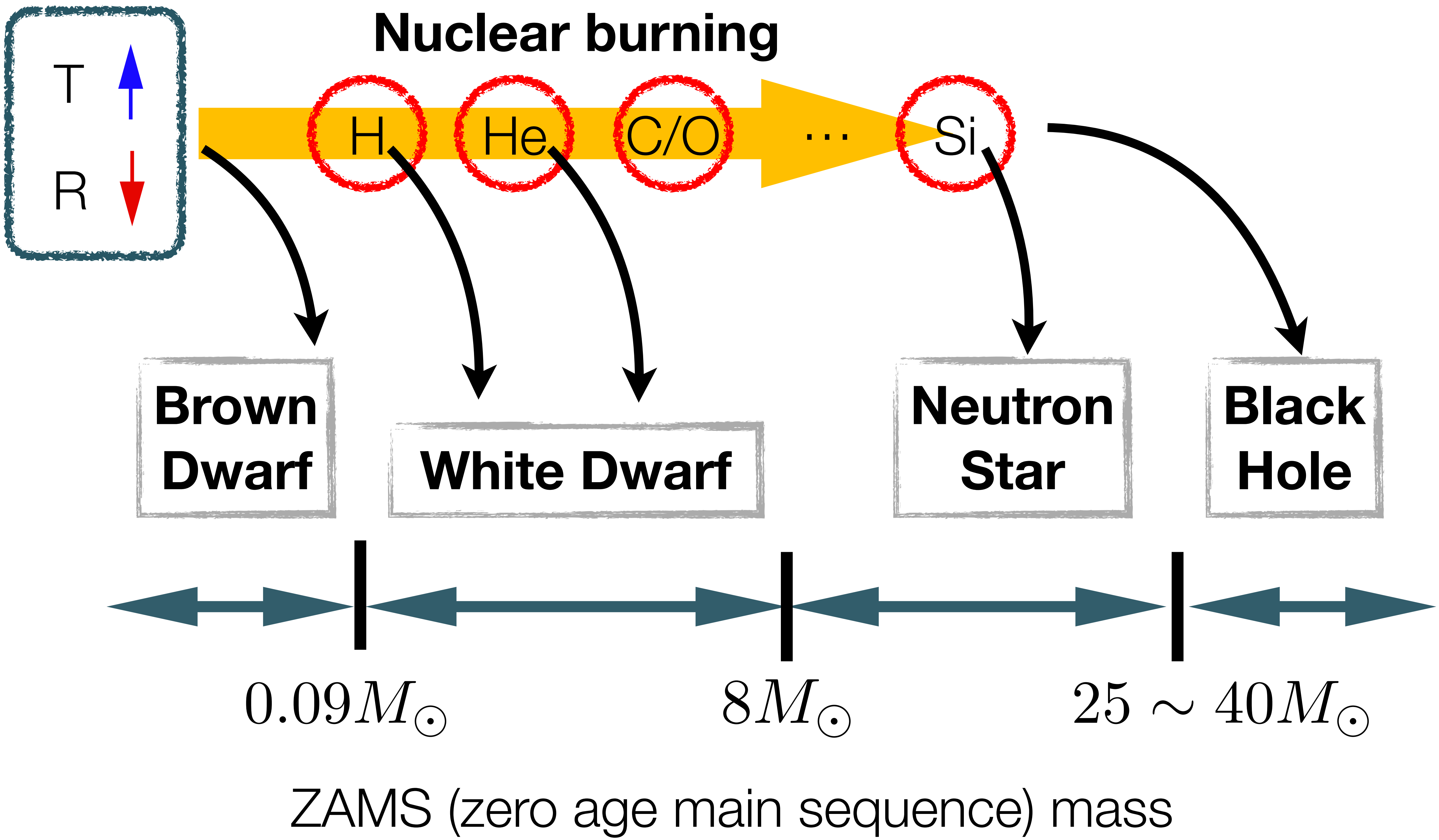
$$\frac{GMm}{l} \sim \frac{G}{l}\frac{E}{c^2}m \sim \frac{G}{lc^2}\frac{\hbar c}{l}m = \frac{G\hbar}{l^2 c}m \sim mc^2$$

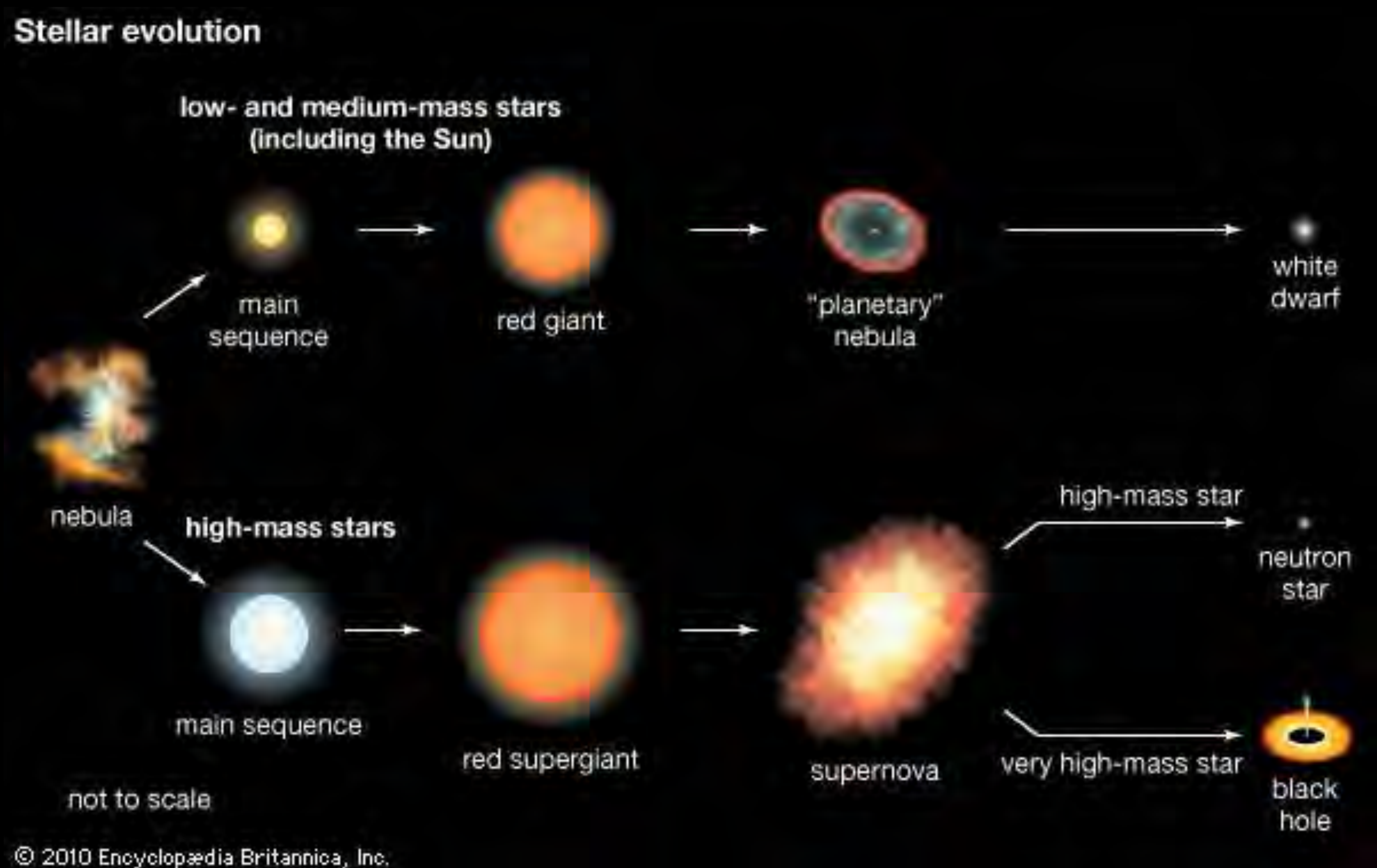
$$M \sim \frac{E}{c^2} \qquad E \sim \frac{\hbar c}{l} \qquad l \sim \sqrt{\frac{G\hbar}{c^3}}$$

$$l_{\text{Planck}} \sim 10^{-33}\ \text{cm}$$

$$t_{\text{Planck}} = \frac{l_{\text{Planck}}}{c} \sim 10^{-43}\ \text{s}$$

Initial mass of progenitor star




Stellar evolution
low- and medium-mass stars
(including the Sun)
main
sequence
red giant
"planetary"
nebula
white
dwarf
nebula
high-mass stars
high-mass star
neutron
star
main sequence
red supergiant
supernova
very high-mass star
black
hole
not to scale
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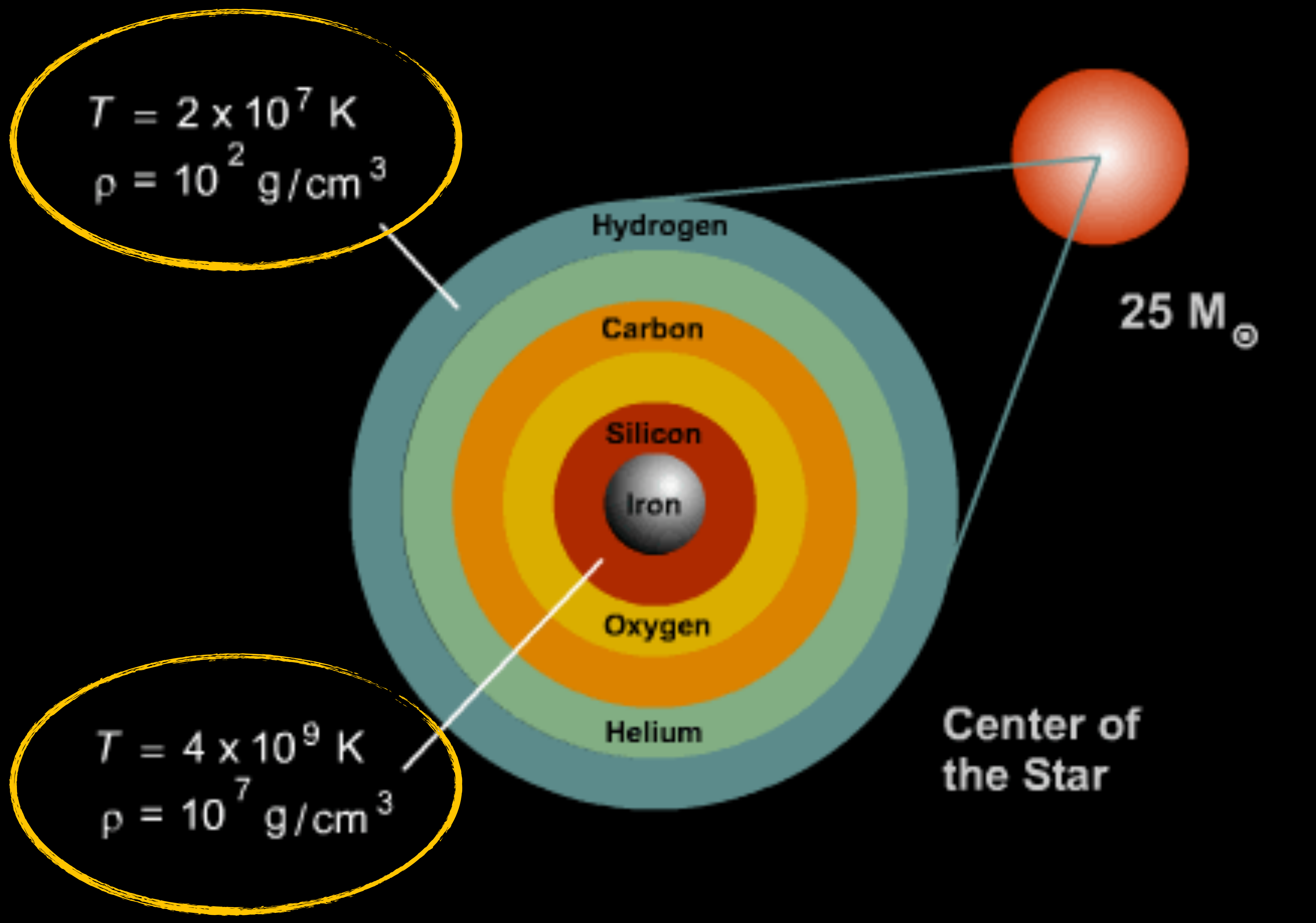


T = 2 x 10^7 K
ρ = 10^2 g/cm^3
Hydrogen
Carbon
Silicon
Iron
Oxygen
Helium
T = 4 x 10^9 K
ρ = 10^7 g/cm^3
25 M☉
Center of the Star

Neutron Star Formation from Fe Core Collapse

When Fe core reaches Chandrasekhar Limit

Nuclear Photodisintegration

$$\gamma + {}^{56}\mathrm{Fe} \rightarrow 13\ {}^{4}\mathrm{He} + 4\ \mathrm{n}\ (124\ \mathrm{MeV\ is\ consumed})$$

$$\gamma + {}^{4}\mathrm{He} \rightarrow 2\ \mathrm{p} + 2\ \mathrm{n}\ (28.3\ \mathrm{MeV\ is\ consumed})$$

$$E_{\mathrm{consumed}}(10^{57}\mathrm{p}) \sim 1.4 \times 10^{52}\ \mathrm{erg} \sim 10^{11}\mathrm{year\ with}\ L_{\odot}$$

(n : neutron)

Neutralisation

$$\mathrm{e}^{-} + \mathrm{p} \rightarrow \mathrm{n} + \nu_{\mathrm{e}}$$

$$\mathrm{e}^{-} + {}^{56}\mathrm{Fe} \rightarrow {}^{56}\mathrm{Mn} + \nu_{\mathrm{e}}$$

$$\mathrm{e}^{-} + {}^{\mathrm{Mn}}\mathrm{Fe} \rightarrow {}^{56}\mathrm{Cr} + \nu_{\mathrm{e}}$$

core collapse → NS + Supernova

$$R_{\mathrm{Fe}} \sim 1500\mathrm{km}$$

$$R_{\mathrm{NS}} \sim \mathcal{O}(10\mathrm{km})$$

$$\tau_{\mathrm{fall}} \sim \mathrm{a\ few\ seconds}$$

Properties of Neutron Star

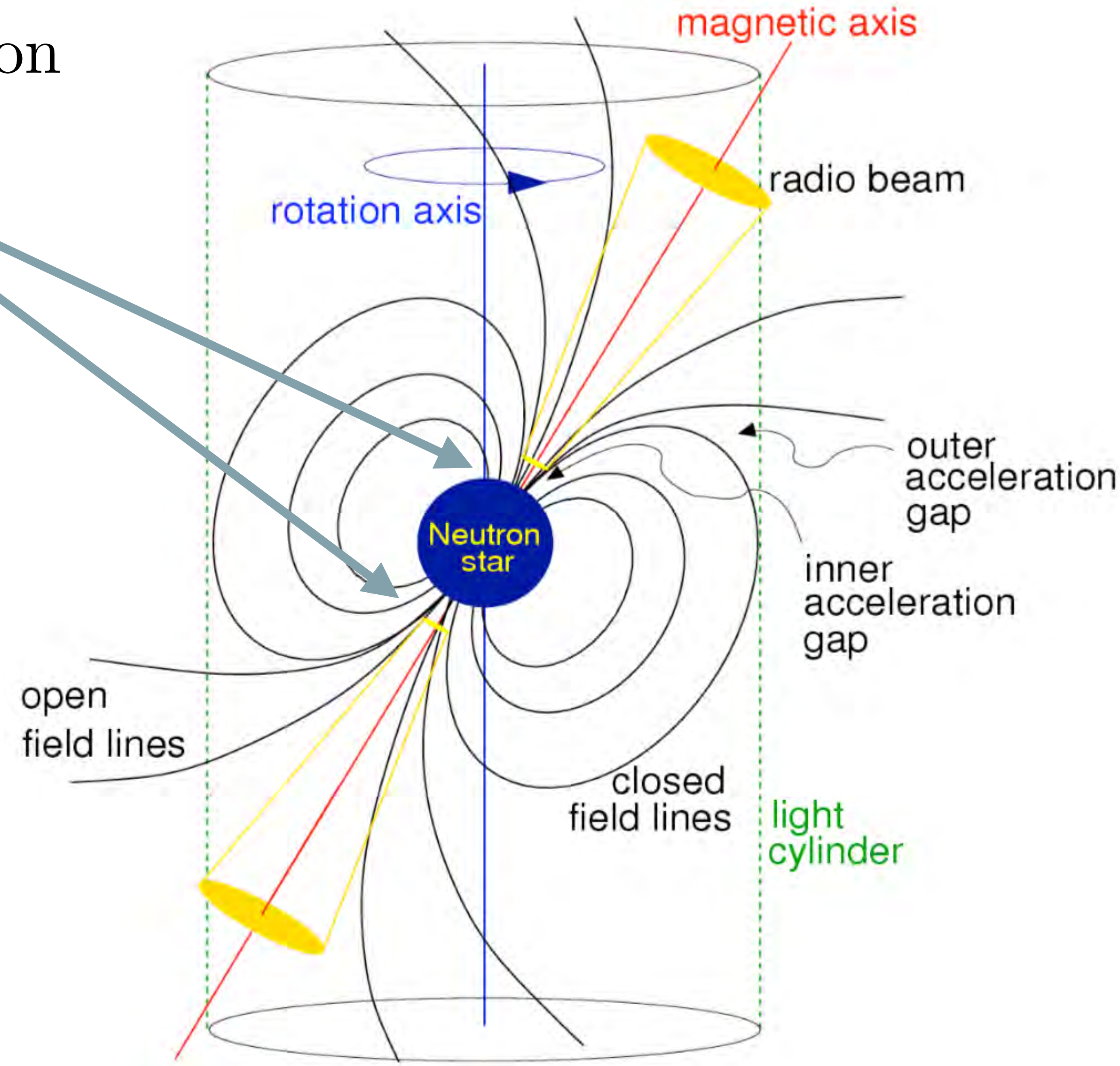
e^+e^- pair creation

Pulsar

$$M = 1.5 \sim 2.0 M_\odot$$

$$R = 10 \sim 15 \text{ km}$$

$$A \sim 10^{57} \text{ nucleons}$$



Educated Guesses

white dwarfs

$$\rho_{\rm WD} \sim 10^6 \ {\rm g/cm^3}$$

$$T_{\rm WD} \sim 10^7 \ {\rm K}$$

$$R = 3.347 \times 10^4 \left(\frac{\rho_c}{10^6 \ {\rm g \ cm^{-3}}} \right)^{-1/3} \left(\frac{2}{\mu_e} \right)^{2/3} \ {\rm km}$$

$$M = 1.457 \left(\frac{2}{\mu_e} \right)^2 M_\odot$$

Quantum degeneracy pressure of protons/neutrons

neutron stars

$$\mu_e = A/Z \rightarrow \mu_n = 1$$

$$R_{\rm NS} \approx 2.3 \times 10^9 \ {\rm cm} \left(\frac{m_e}{m_n} \right) \left(\frac{1}{\mu_n} \right)^{5/3} \left(\frac{M}{M_\odot} \right)^{-1/3} \approx 14 \ {\rm km} \left(\frac{M}{1.4 M_\odot} \right)^{-1/3}$$

$$M_{\rm NS,Chan} \approx 0.2 \left(\frac{2}{\mu_n} \right)^2 \left(\frac{hc}{G m_p^2} \right)^{3/2} m_p \approx 4 \times 1.4 M_\odot = 5.6 M_\odot$$

$$\rho \sim 10^{14} {\rm g/cm^3}$$

General Relativity

hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \qquad \frac{dM}{dr} = 4\pi r^2 \rho$$

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}\left(1+\frac{P}{\rho c^2}\right)\left(1+\frac{4\pi P r^3}{Mc^2}\right)\left(1-\frac{2GM}{rc^2}\right)^{-1}$$

$$\frac{dM}{dr} = 4\pi r^2\left(\frac{\epsilon}{c^2}\right)$$

include all energy sources

physics of dense nuclear matter (strong interaction)

Without general relativity

n : polytropic index

ideal gas $p \propto \rho^{\gamma} \propto \rho^{(n+1)/n}$

nonrelativistic

$$\gamma = \frac{5}{3} \quad n = \frac{3}{2} \qquad P_e = \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{20 m_e m_p^{5/3}} \left(\frac{Z}{A}\right)^{5/3} \rho^{5/3}$$

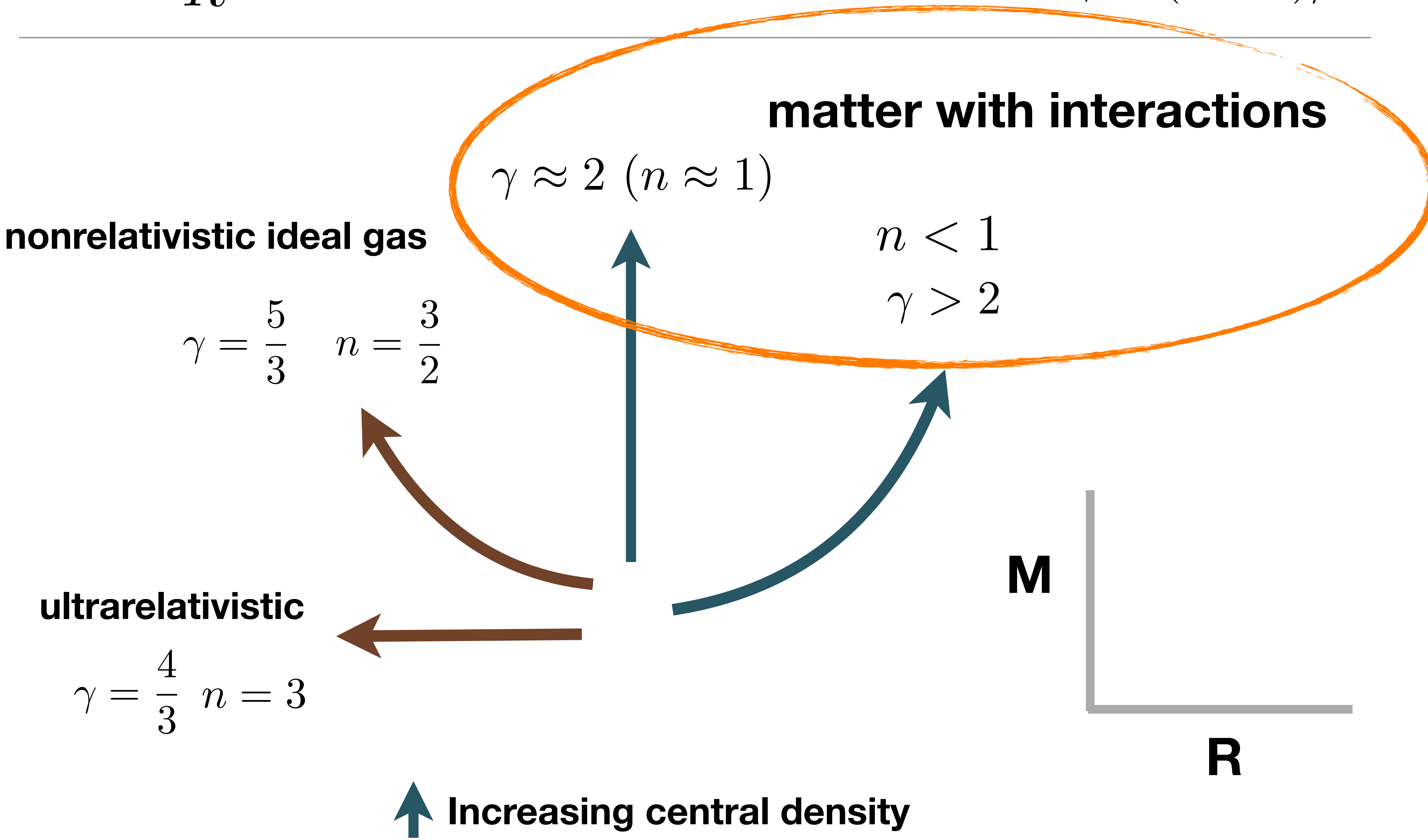
ultrarelativistic

$$\gamma = \frac{4}{3} \quad n = 3 \qquad P_e = \left(\frac{3}{8\pi}\right)^{1/3} \frac{hc}{4 m_p^{4/3}} \left(\frac{Z}{A}\right)^{4/3} \rho^{4/3}$$

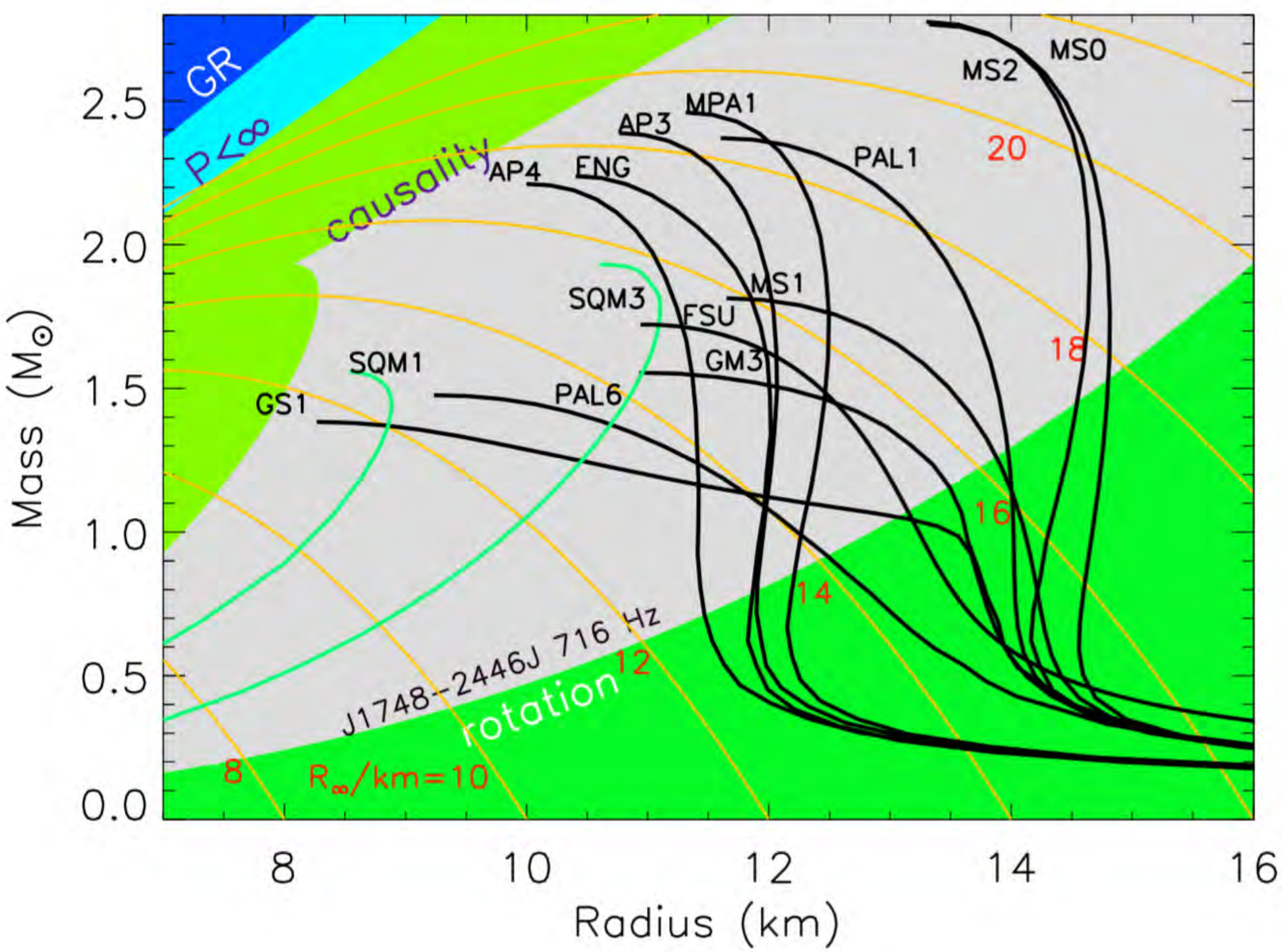
$$M \propto R^{(\gamma-2)/(3\gamma-4)}$$

$$n = 1/(\gamma - 1)$$

$$\gamma = (n+1)/n$$

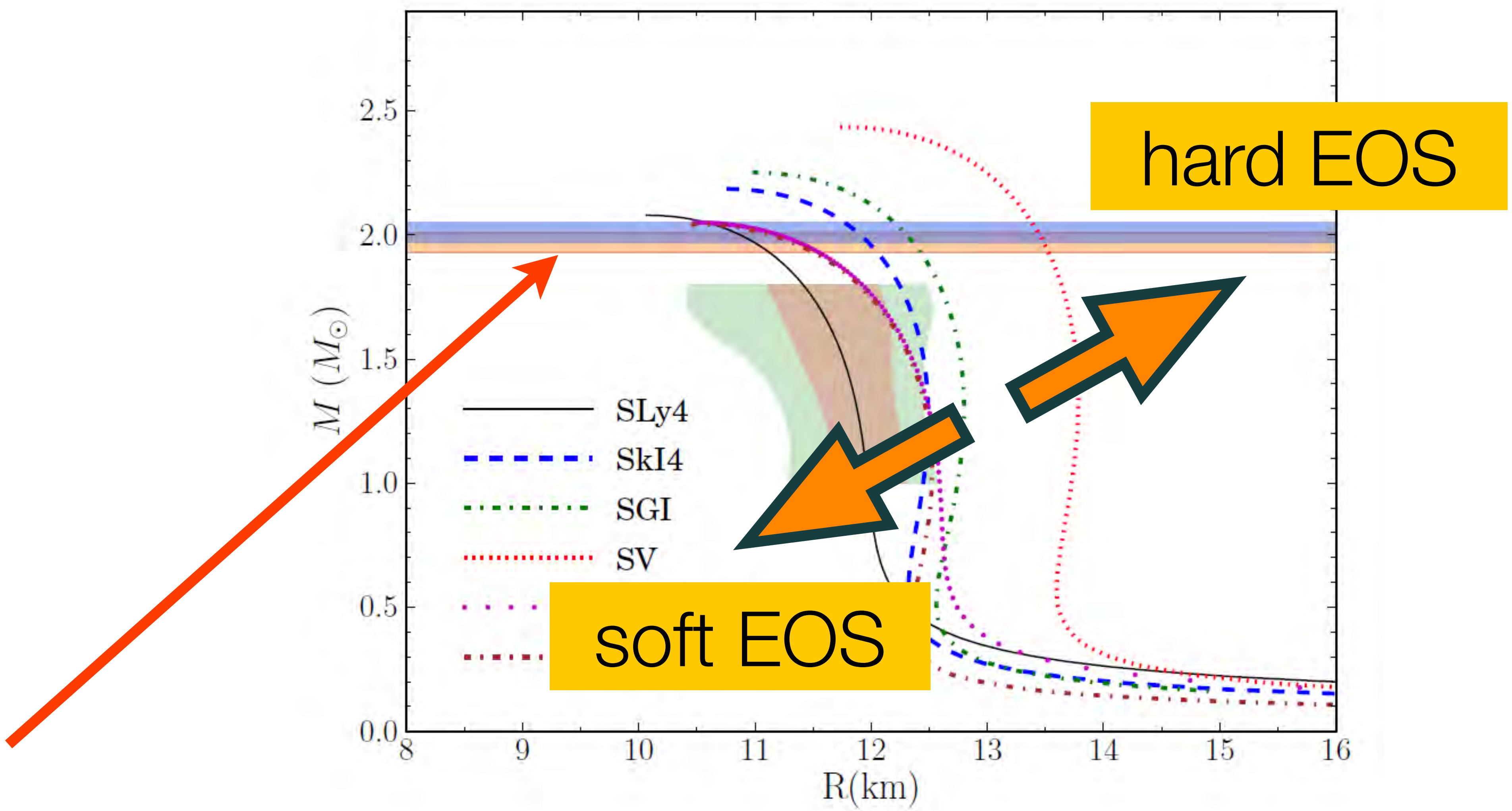


Neutron Star Equation of States



with general relativity & strong interactions !

Mass & radius of neutron star



Neutron Star-White Dwarf Binaries

1.97 solar mass NS : Nature 467 (2010) 1081

2.01 solar mass NS : Science 340 (2013) 6131

Nuclear matter is not an ideal gas

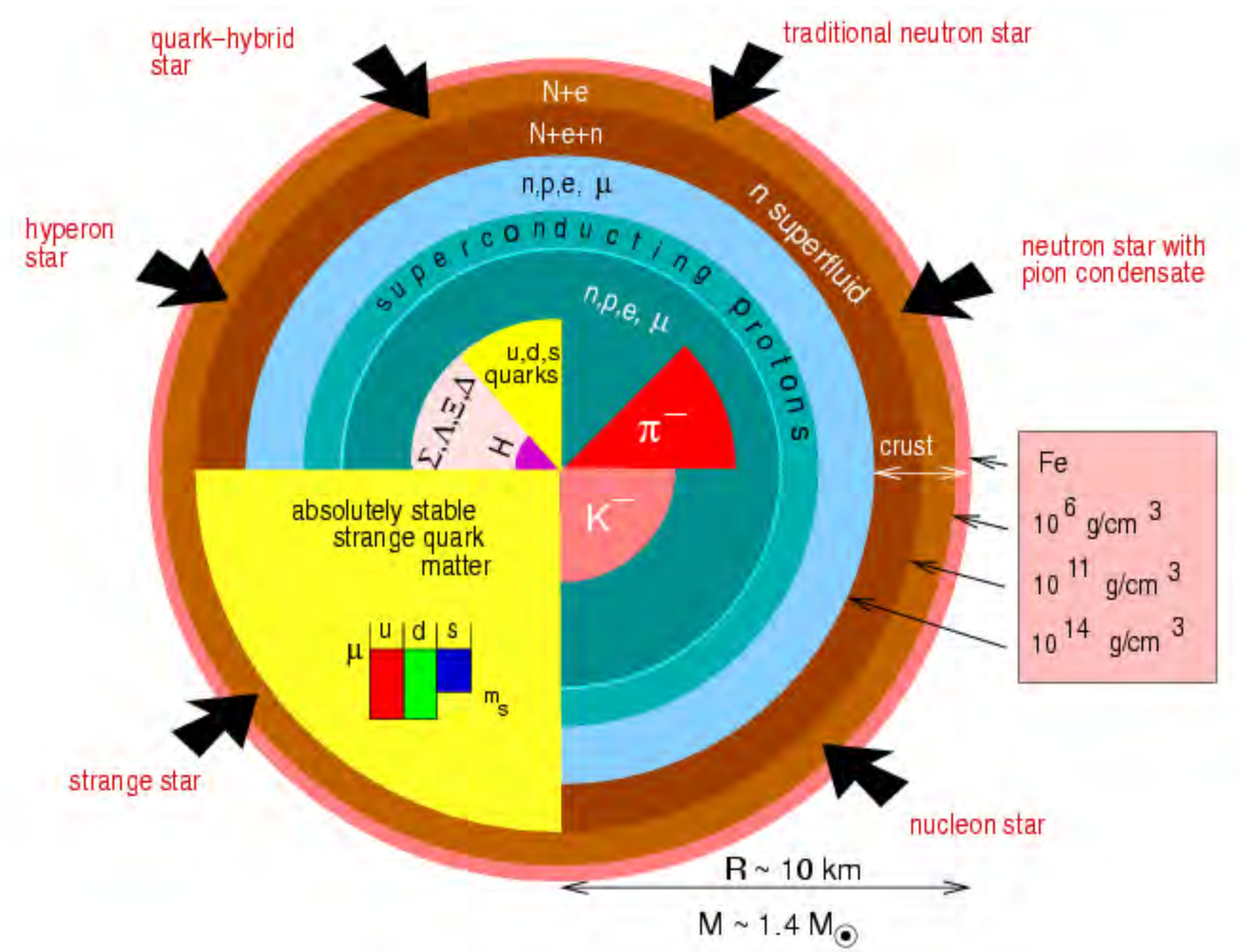
nonrelativistic $\gamma = \frac{5}{3}$

ultrarelativistic $\gamma = \frac{4}{3}$

NS eos includes

$$\gamma \approx 2 \ (n \approx 1)$$

F. Weber 2005



- **still uncertain due to the nature of strong interactions**
- **introduction of 3 body forces**
- **exotic states with strangeness**
- **… …**

Quark stars

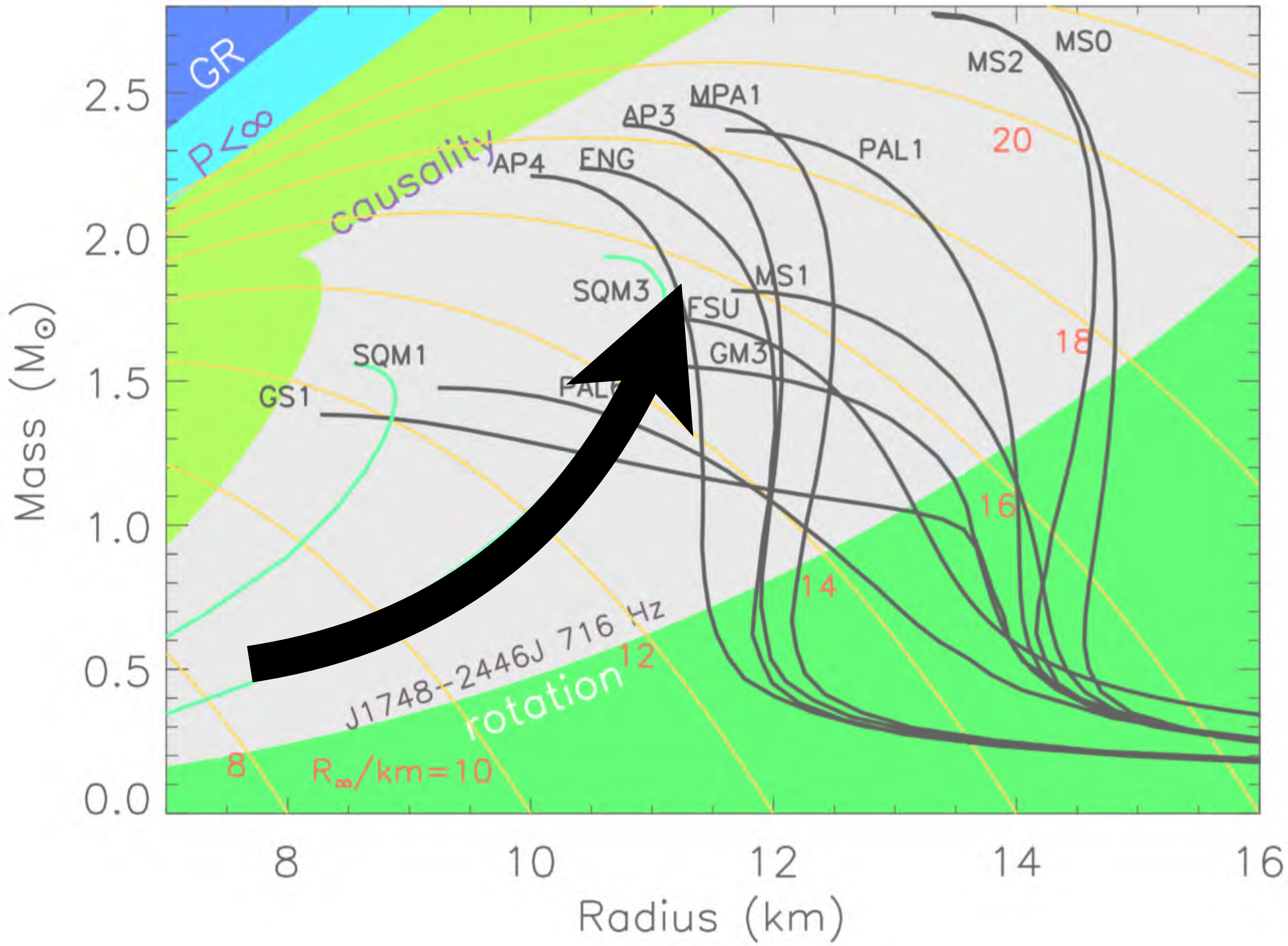
$$p \propto \rho^{\gamma} \propto \rho^{(n+1)/n}$$

strong interaction dominates

$$(\mathrm{u}, \mathrm{d}, \mathrm{s}) \text{ quarks} \Rightarrow \text{charge neutrality without electron}$$

$$n < 1$$

$$\gamma > 2$$



Nuclear Equation of States

$$x = \frac{\rho_p}{\rho_p + \rho_n}$$

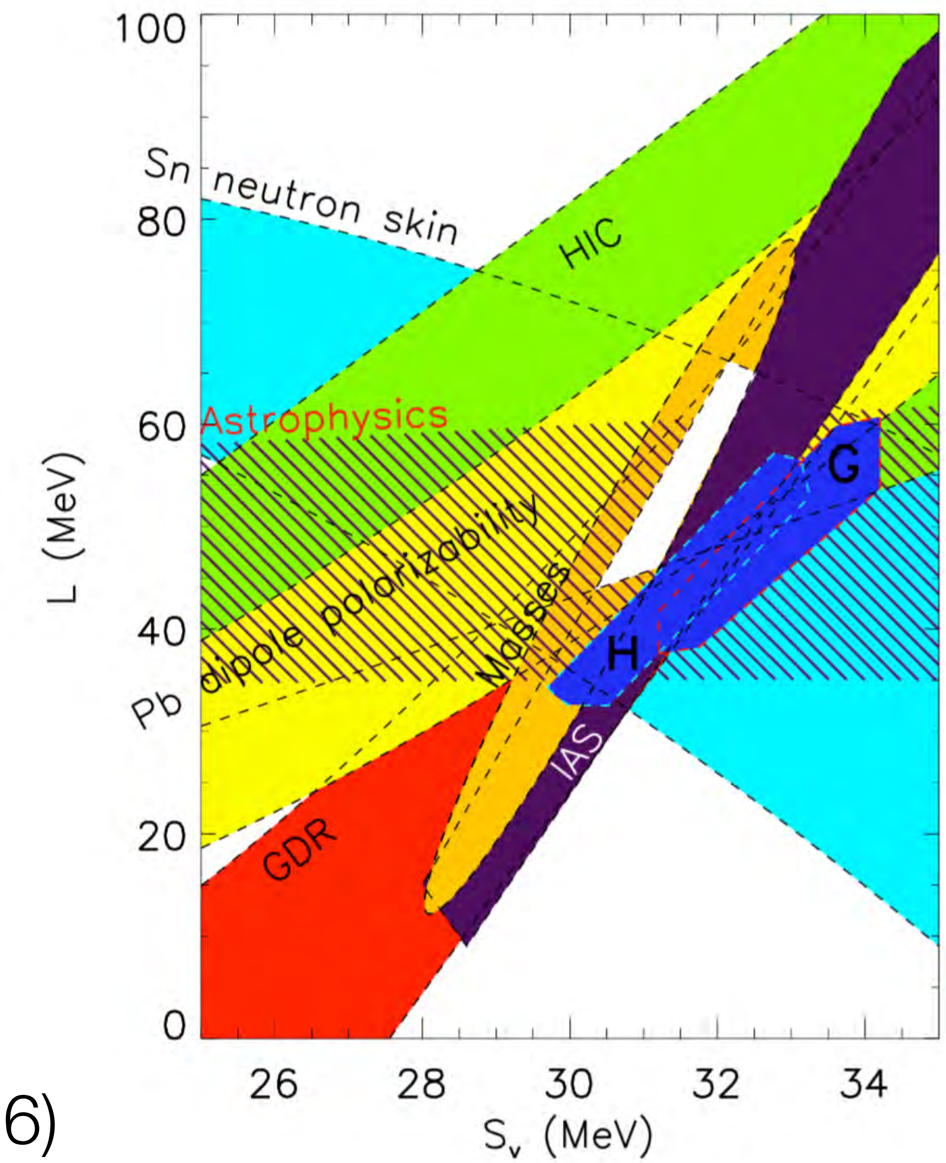
$$E(\rho, x) = -B + \frac{K_0}{18}\left(\frac{\rho}{\rho_0} - 1\right)^2 + \frac{K_0'}{162}\left(\frac{\rho}{\rho_0} - 1\right)^3 + E_{\rm sym}(\rho)(1-2x)^2 + \cdots$$

Incompressibility $K_0 \simeq 230\ \mathrm{MeV}$

Skewness $K_0' \sim -2000\ \mathrm{MeV}$

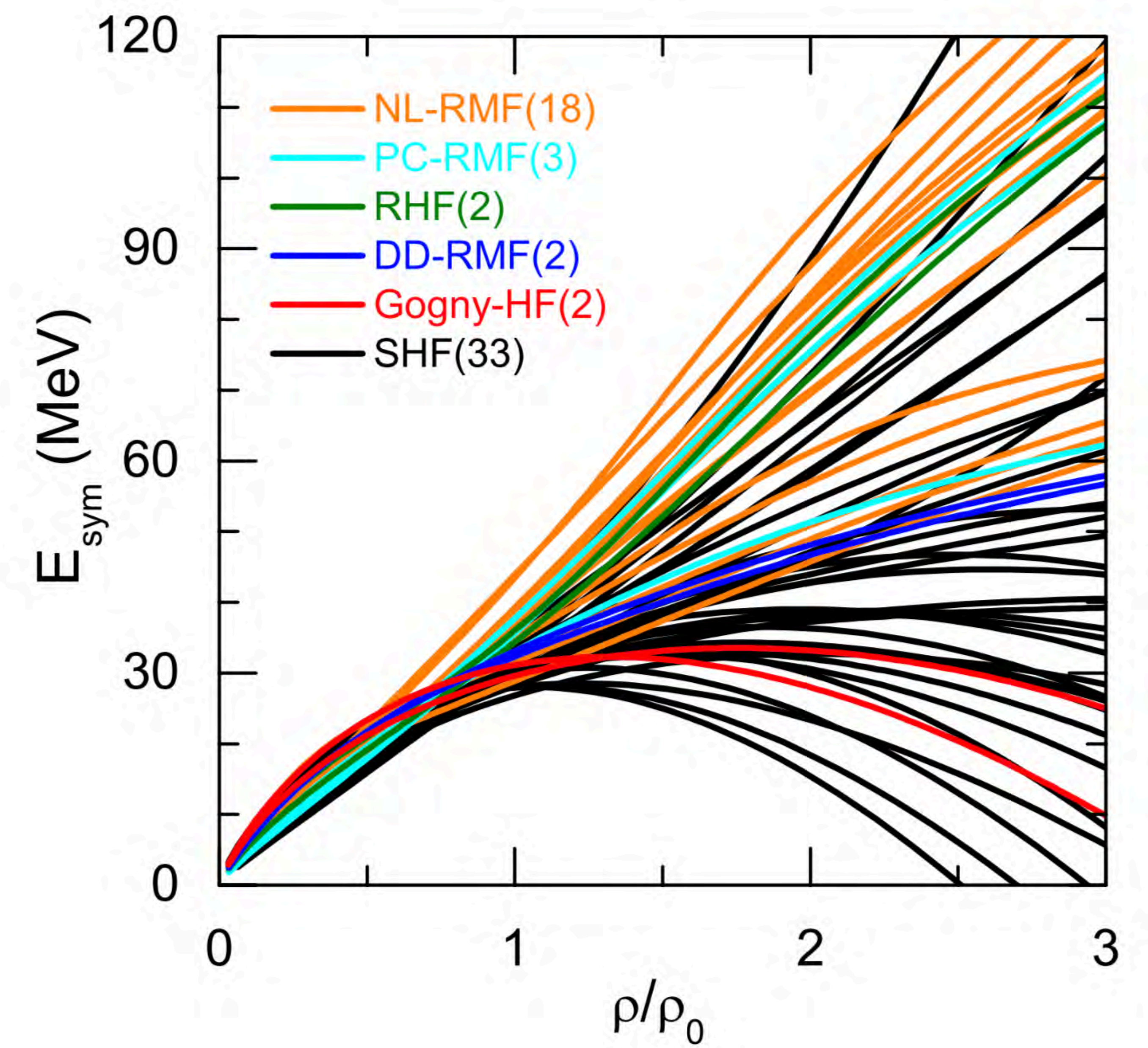
$$S_0 \equiv E_{\rm sym}(\rho_0)$$ **Symmetry Energy**

$$L \equiv 3\rho \left.\frac{\partial E_{\rm sym}}{\partial \rho}\right|_{\rho_0}$$



J.M. Latter, Y. Lim (2016)

Symmetry energy / Theoretical uncertainties

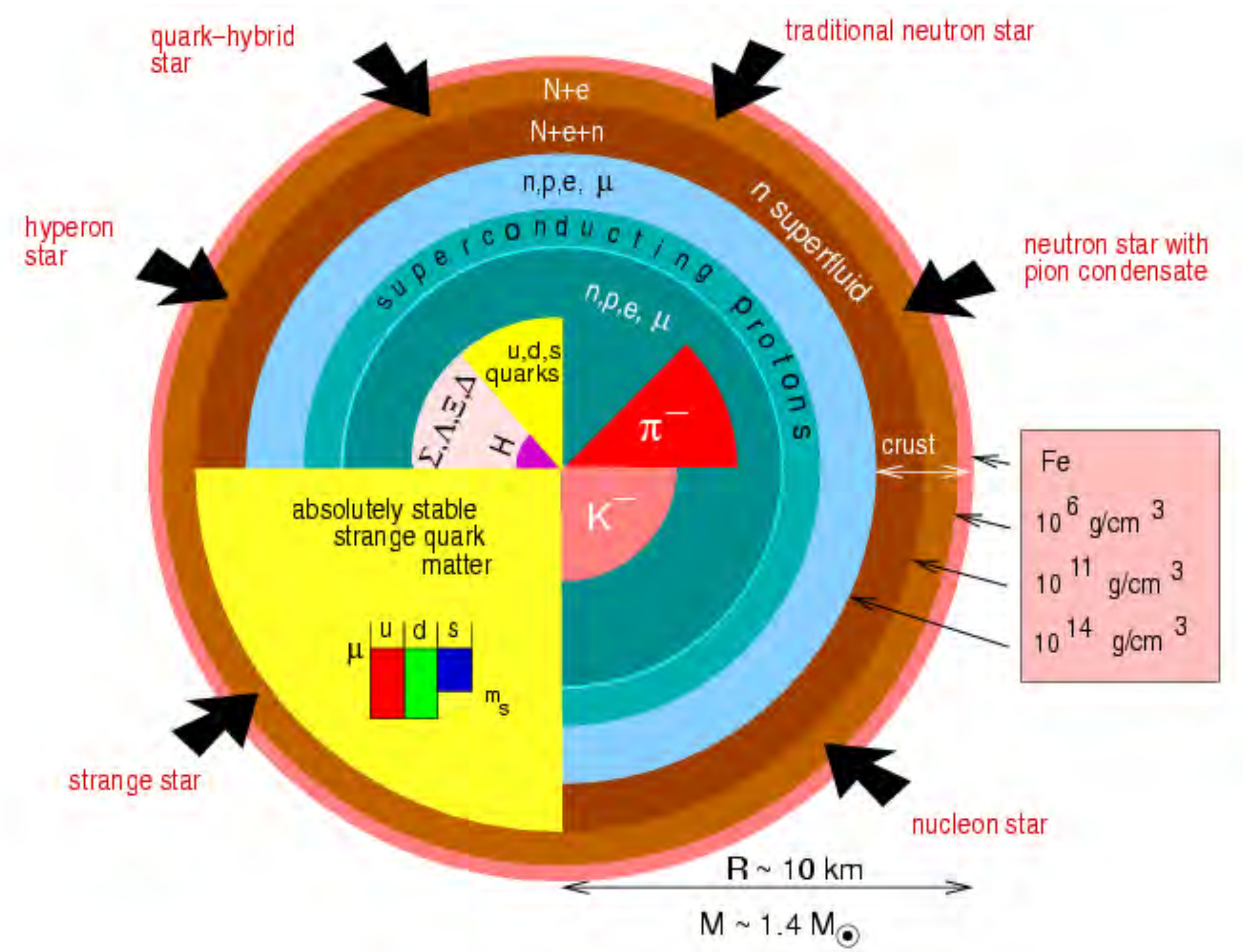


Courtesy of Lie-Wen Chen

waiting for RAON@IBS

piecewise polytropic EoS

$$P(\rho) = K_i \rho^{\Gamma_i}$$



Spectral expansion of adiabatic index [Lindblom et al.]

$$\Gamma(P) = \frac{d(\ln P)}{d(\ln \rho)}$$

$$\Gamma(p) = \exp\left[\sum_k \gamma_k \Phi_k(p)\right]$$

$$\Gamma(p) = \frac{\epsilon + p}{p}\frac{dp}{d\epsilon}$$

$$\epsilon(p) = \frac{\epsilon_0}{\mu(p)} + \frac{1}{\mu(p)}\int_{p_0}^{p}\frac{\mu(p')}{\Gamma(p')}dp'$$

$$\mu(p) = \exp\left[-\int_{p_0}^{p}\frac{dp'}{p'\,\Gamma(p')}\right]$$

Prob 3-a)

$$\Gamma(P) = \frac{d(\ln P)}{d(\ln \rho)}$$

$$d\left(\frac{\varepsilon}{\rho}\right) = -p\, d\left(\frac{1}{\rho}\right)$$

$$\frac{d\varepsilon}{\rho} - \varepsilon \frac{d\rho}{\rho^2} = p \frac{d\rho}{\rho^2}$$

$$d\varepsilon = (\varepsilon + p) \frac{d\rho}{\rho} = (\varepsilon + p) \frac{1}{\Gamma} \frac{dP}{P}$$

$$\Gamma = \left(\frac{p+\varepsilon}{p}\right) \frac{dP}{d\varepsilon}$$

Prob 3-b)

$$\frac{d\epsilon}{dp} = \frac{\varepsilon + p}{p\Gamma}$$

$$d\epsilon = \frac{\varepsilon + p}{p\Gamma} dp = (\varepsilon + p)\frac{dp}{p\Gamma}$$

$$\mu = \exp\left[-\int_{p_0}^{p} \frac{dp'}{p'\Gamma(p')}\right]$$

$$\frac{d\mu}{dp} = -\frac{1}{p\Gamma(p)}\mu$$

$$\frac{d\mu}{\mu} = -\frac{dp}{p\Gamma(p)}$$

$$d\epsilon = -(\varepsilon + p)\frac{d\mu}{\mu} \Rightarrow \mu\, d\epsilon = -(\varepsilon + p)\, d\mu$$

$$\Rightarrow \mu\, d\varepsilon + \varepsilon\, d\mu = -p\, d\mu$$

$$d(\varepsilon\mu) = d\varepsilon\,\mu + \varepsilon\, d\mu = -p\, d\mu$$

$$= \frac{\mu}{\Gamma} dp$$

$$\therefore \varepsilon\mu - e_0 = \int \frac{\mu}{\Gamma} dp$$

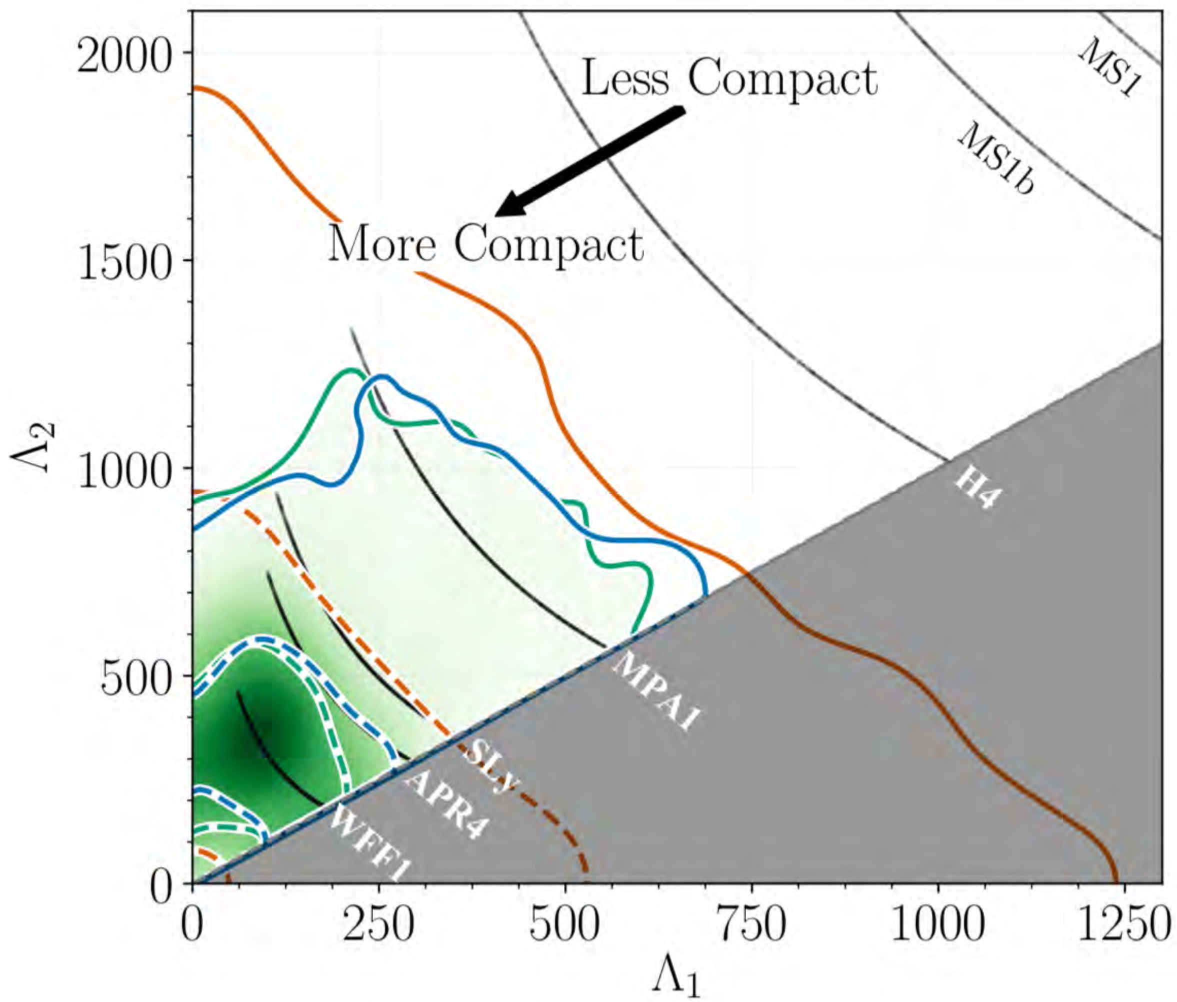
$$\epsilon\mu = \varepsilon_0 + \int \frac{\mu}{\Gamma} dp.$$

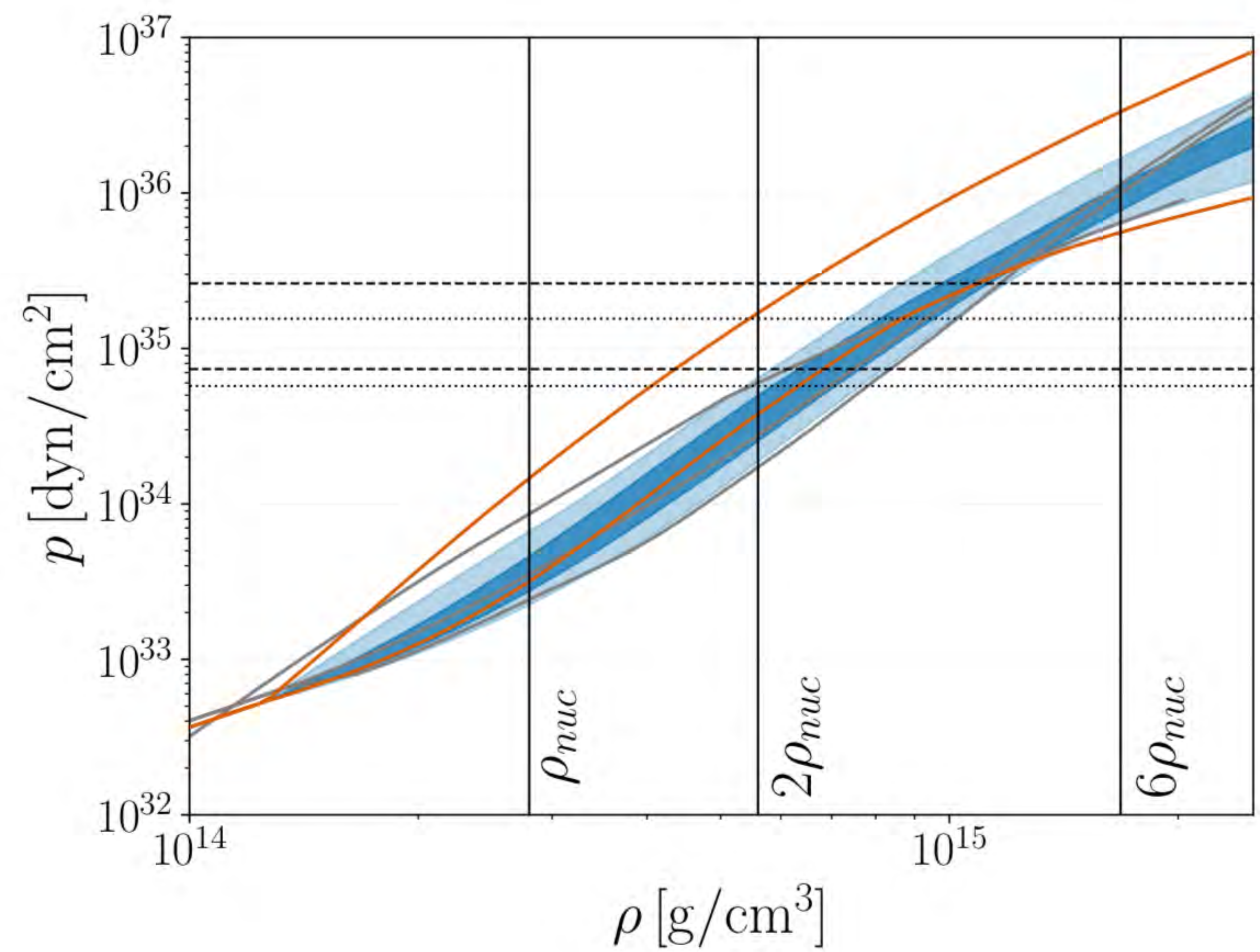
New constraint by GW observation 1

$\Lambda(1.4M_{\odot})=190^{+390}_{-120}$

$P(2\ \rho_{nuc}) = 3.5^{+2.7}_{-1.7} \times 10^{34}$ dyne/cm^2

$P(6\ \rho_{nuc}) = 9.0^{+7.9}_{-2.6} \times 10^{35}$ dyne/cm^2





Abbott et al. (LSC and Virgo), arxiv:1805.11581 (PRL accepted)

$\rho_{nuc} = 2.8 \times 10^{14}$ g/cm^3

Universal (Eos-insensitive) relations

Yagi & Yunes, PR 681, 1 (2017)

I-Love-Q relation, …

- Moment of inertia (I)
- Tital Love number (Love)
- Quardupole moment (Q)

Applications

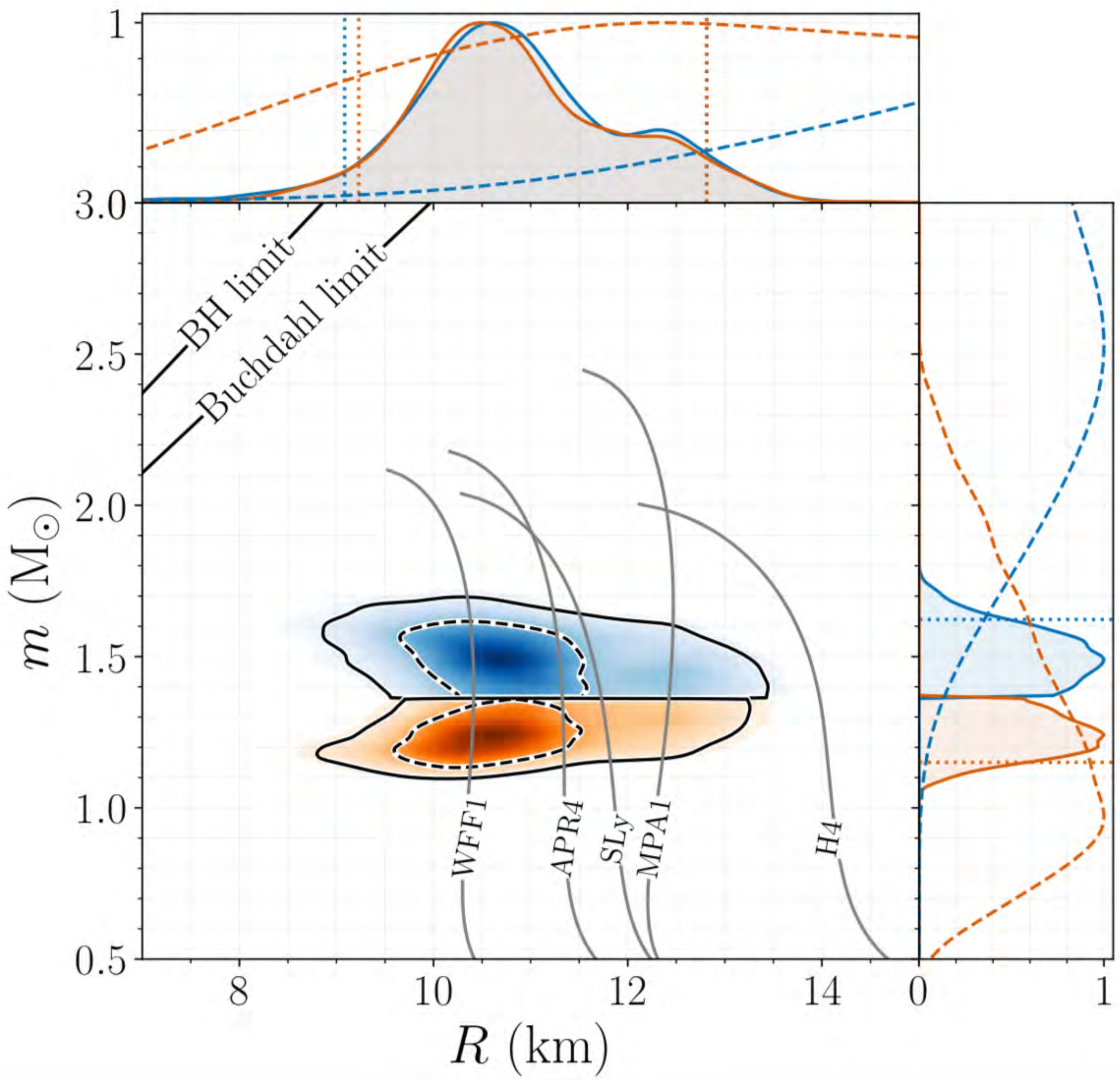
- X-ray observations
- Gravitational-wave measurements
- Gravitational & astrophysical test of GR

New constraint by GW observation 2

EoS insensitive relations (Yagi&Yunes,PR2017)

$R_1 = 10.8\ ^{+2.0}_{-1.7}$ km

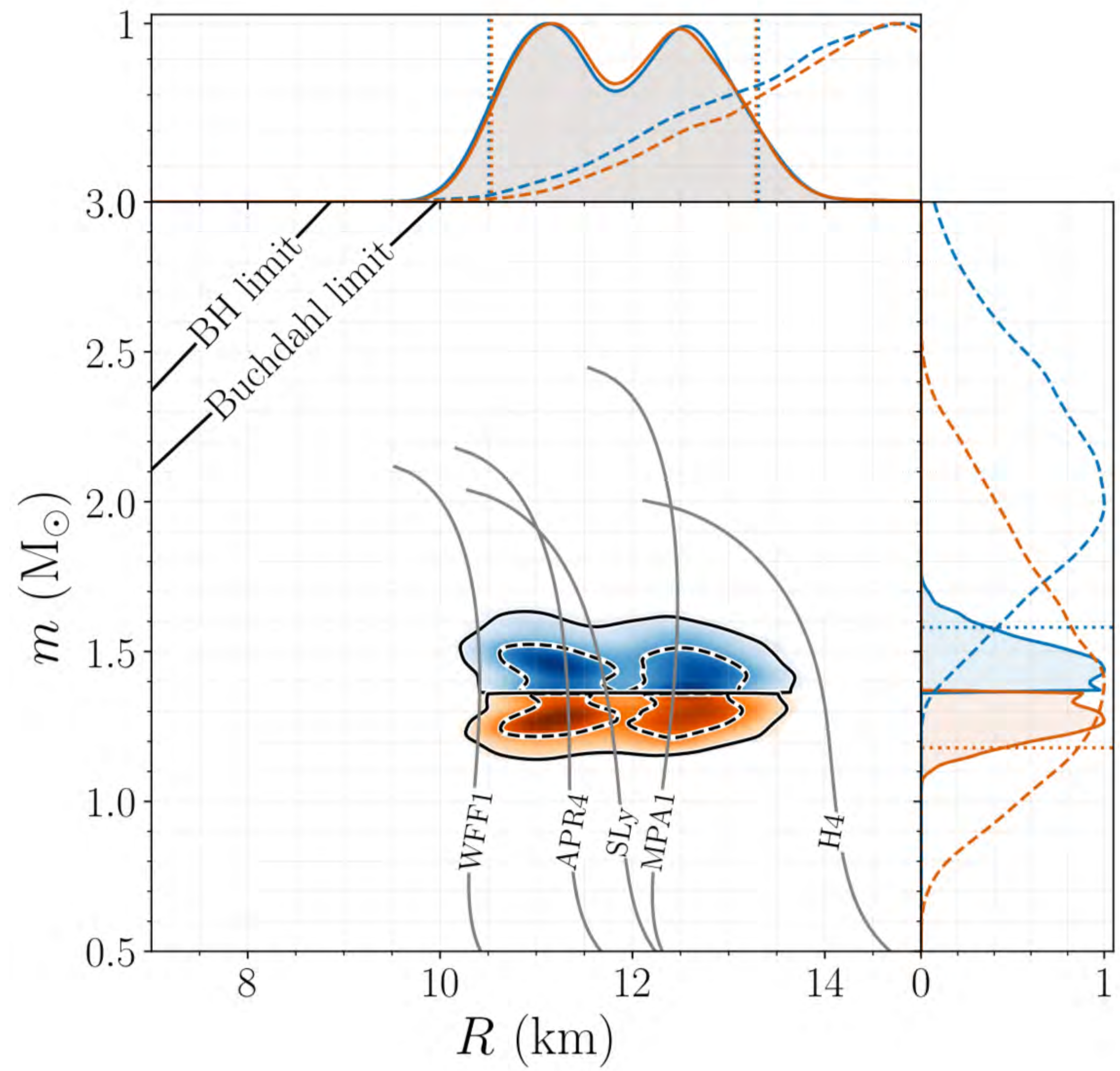
$R_2 = 10.7\ ^{+2.1}_{-1.5}$ km



Parametrized EoS: M_{max} >= 1.97 $M_\odot$

$R_1 = 11.9\ ^{+1.4}_{-1.4}$ km

$R_2 = 11.9\ ^{+1.4}_{-1.4}$ km



Abbott et al. (LSC and Virgo), arxiv:1805.11581 (PRL accepted)

Neutron Star Observations before GW

- Radio pulsars
- Cooling
- Low-mass X-ray binaries

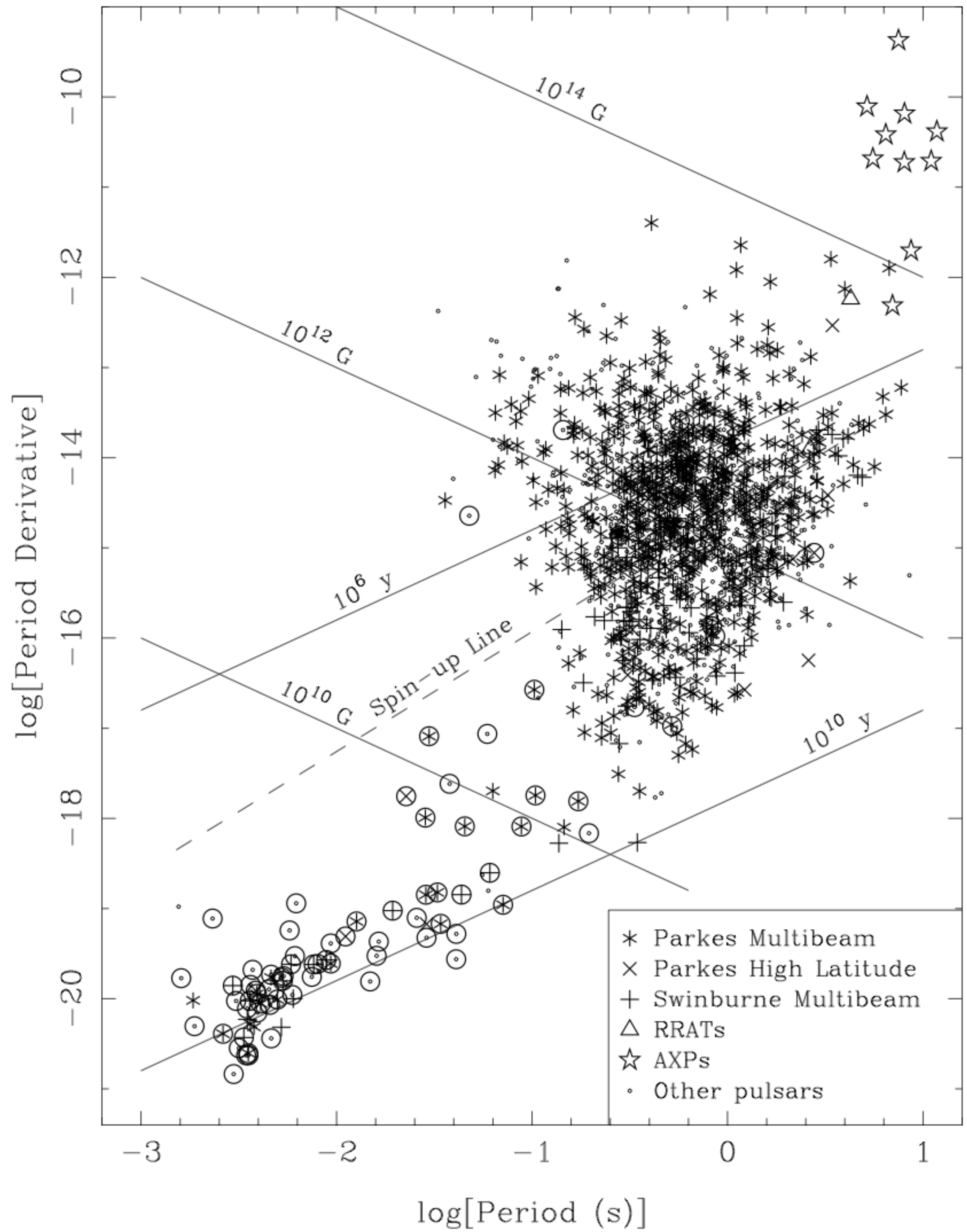
Millisecond Pulsars

Dipole Radiation

$$\dot{E}_{\mathrm{rot}} = I\Omega\dot{\Omega}$$

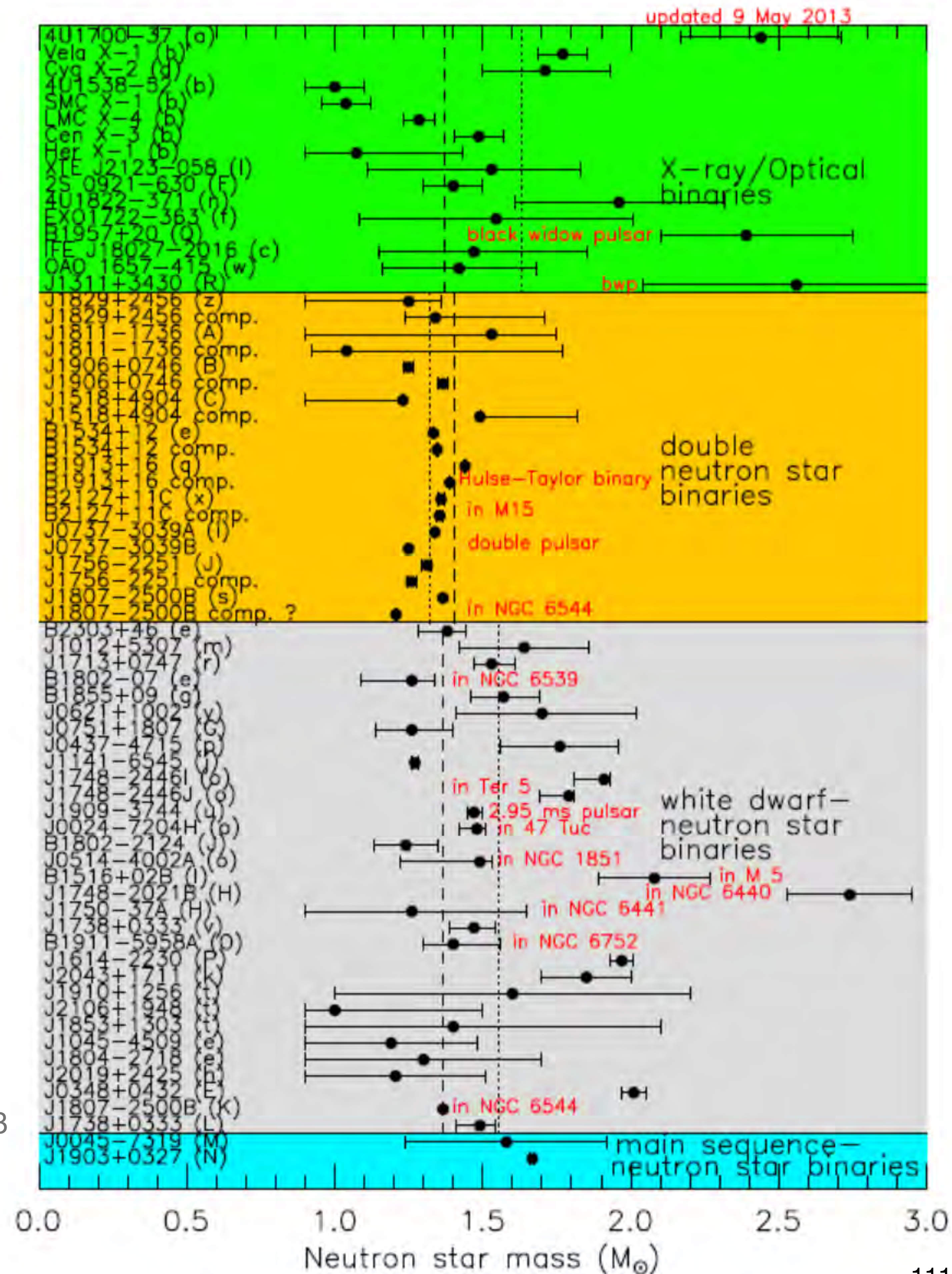
$$\dot{E}_{\mathrm{dipole}} = -\frac{B_\perp^2 R^6 \Omega^4}{6c^3}$$

$$\dot{\Omega} = -\frac{B_\perp^2 R^6 \Omega^3}{6Ic^3}$$



Masses

- High-mass neutron stars in X-ray binaries & white dwarf-NS binaries (2010 & 2013)
- Less than 1.5 solar mass in double NS binaries
- Maximum NS mass is still uncertain

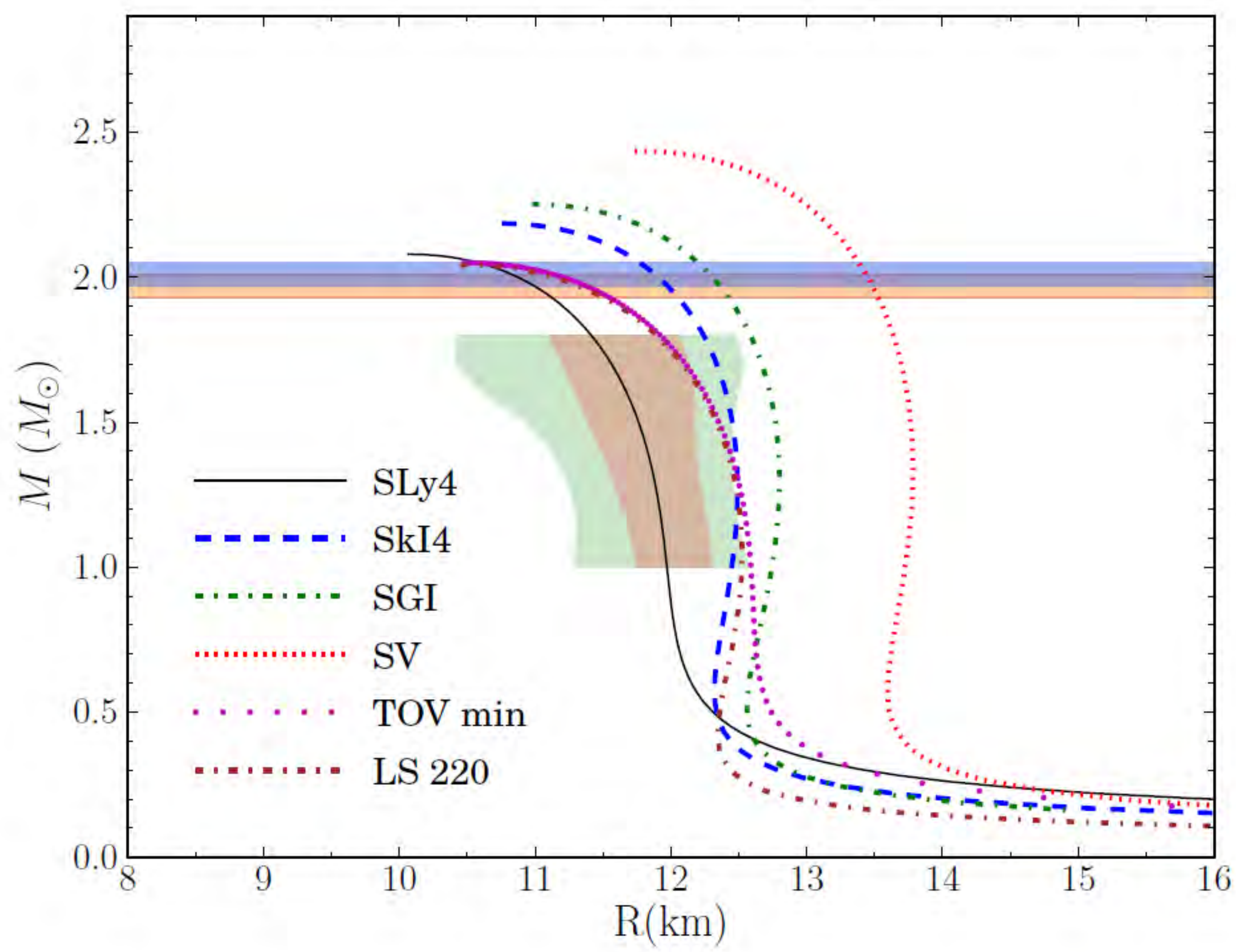


Prakash 2013

Maximum Mass of Neutron Stars

Neutron Star-White Dwarf Binaries

[Nature 467 (2010) 1081; Science 340 (2013) 6131]



$2.01 M_\odot$

$1.97 M_\odot$

Moment of Inertia / Glitches

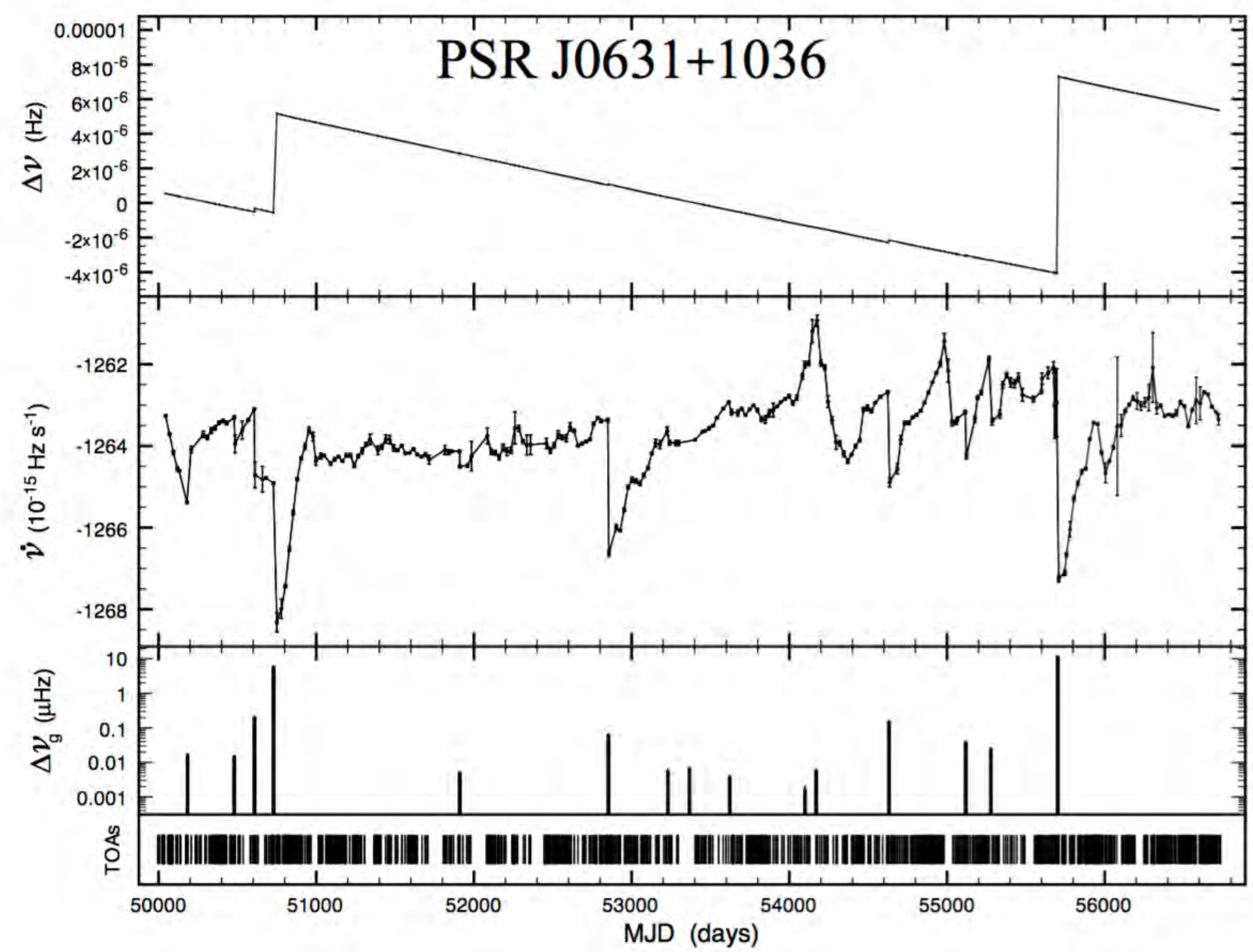
$$\dot{E}_{\rm rot} = I\Omega\dot{\Omega}$$

$$\dot{E}_{\rm dipole} = -\frac{B_\perp^2 R^6 \Omega^4}{6c^3}$$

$$\dot{\Omega} = -\frac{B_\perp^2 R^6 \Omega^3}{6Ic^3}$$

$$\dot{\Omega} \propto -\Omega^n$$

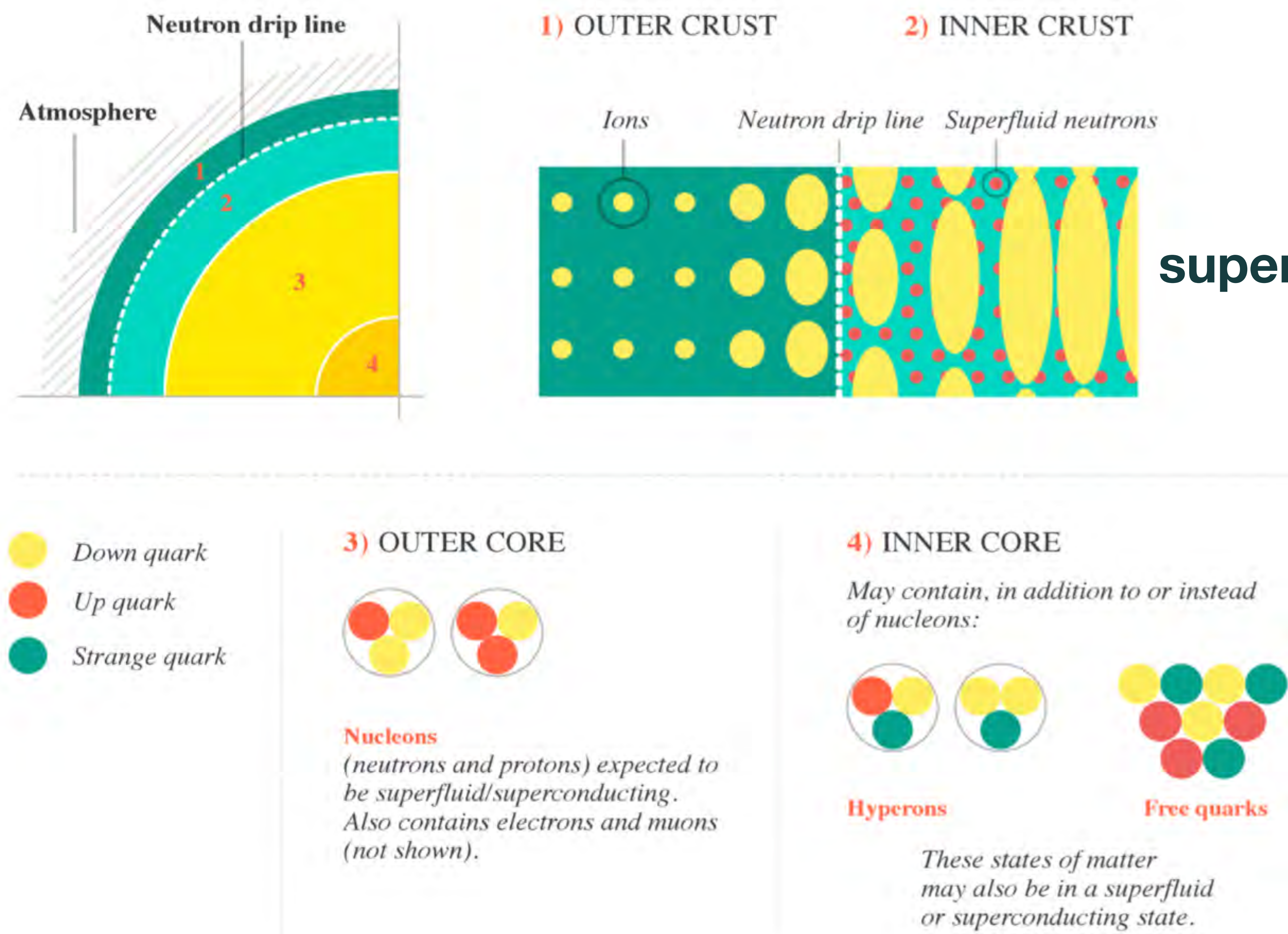
$$\tau_{\rm pulsar} = \frac{\Omega}{(1-n)\dot{\Omega}}$$



D. Antonopoulou (U. Amsterdam, 2015)

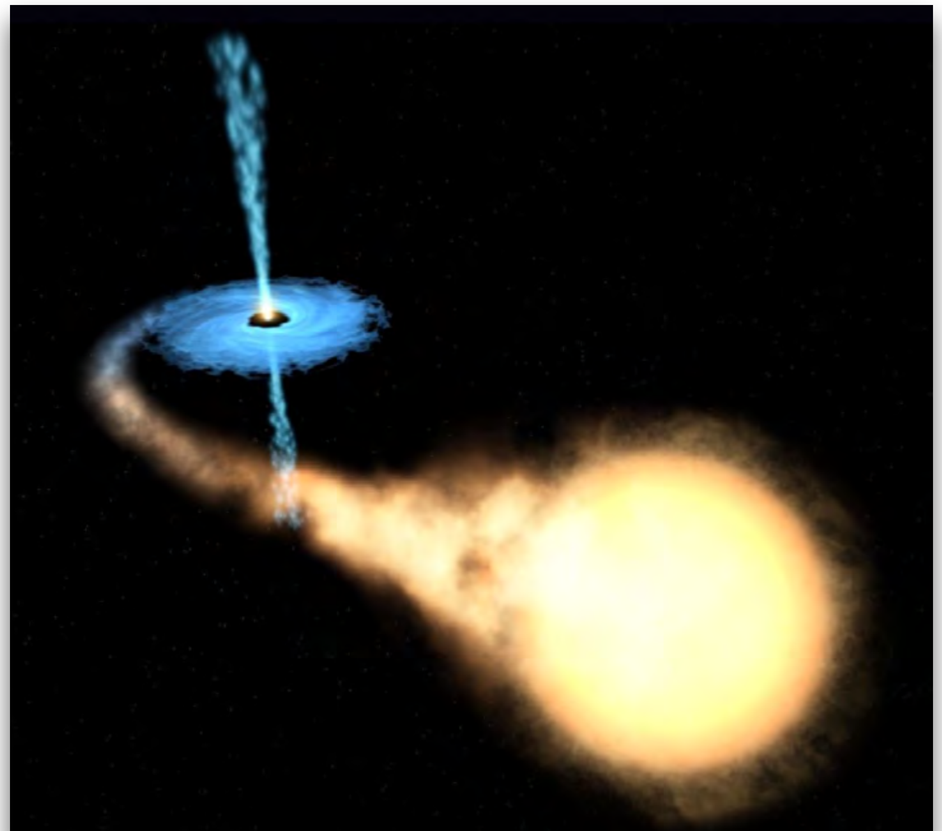
Superfluid Neutrons

D. Antonopoulou (U. Amsterdam, 2015)



superfluid

Low-Mass X-ray Binaries (LMXB)



Accreting Object: NS or BH

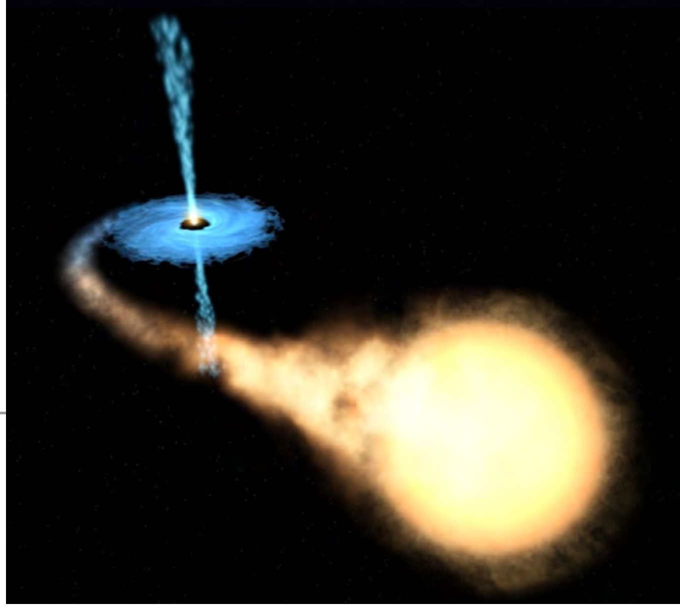
Companion: Low-Mass Main Sequence

Age: Old (> 10^9 year)

Accretion timescale: 10^7 - 10^9 year

X-ray energy: Soft (< 10 keV)

low-mass & high-mass X-ray binaries



Properties	LMXBs	HMXBs
Accreting object	Low B-field NS or BH	High B-field NS or BH
Companion	Low-mass main sequence ($L_{opt}/L_x \ll 0.1$)	High-mass (O or B type) main sequence ($L_{opt}/L_x > 1$)
Stellar population	Old($> 10^9$ yr)	Young($< 10^7$ yr)
Mechanism	Roche-lobe overflow	Stellar wind
Accretion timescale	$10^7 - 10^9$ yr	10^5 yr
Variability	X-ray bursts, Transient behavior	Regular X-ray pulsation
X-ray spectra	Soft (≤ 10 keV)	Hard (≥ 15 keV)

Table 1: Summary of LMXBs and HMXBs (Rosswog et al. 2011)

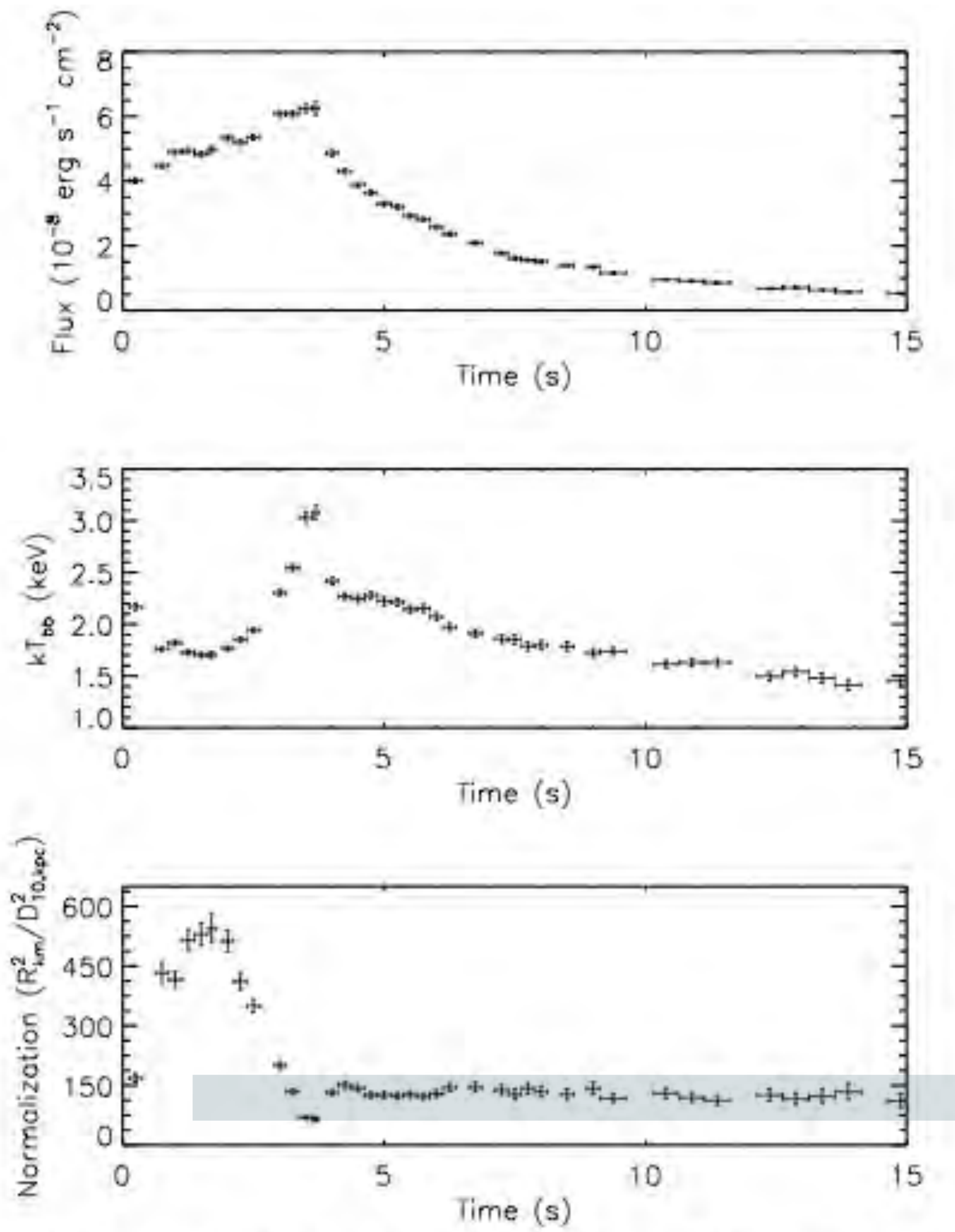
M & R from LMXB

with Myungkuk Kim, Young-Min Kim, Kyujin Kwak

flux

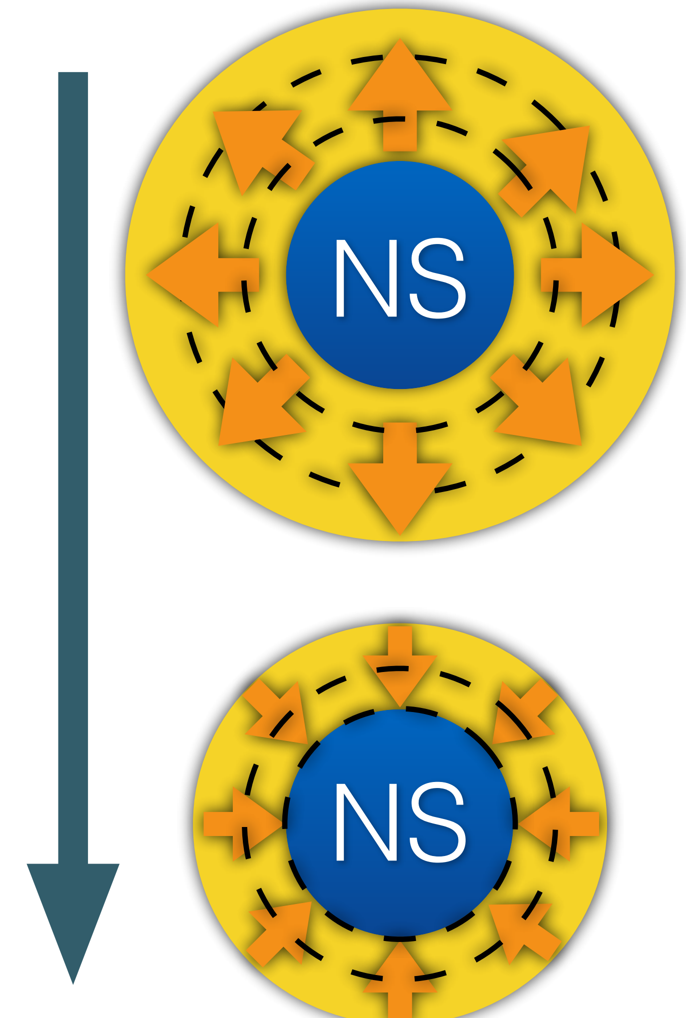
temperature

radius



Ozel et al. 2009

expansion



touchdown

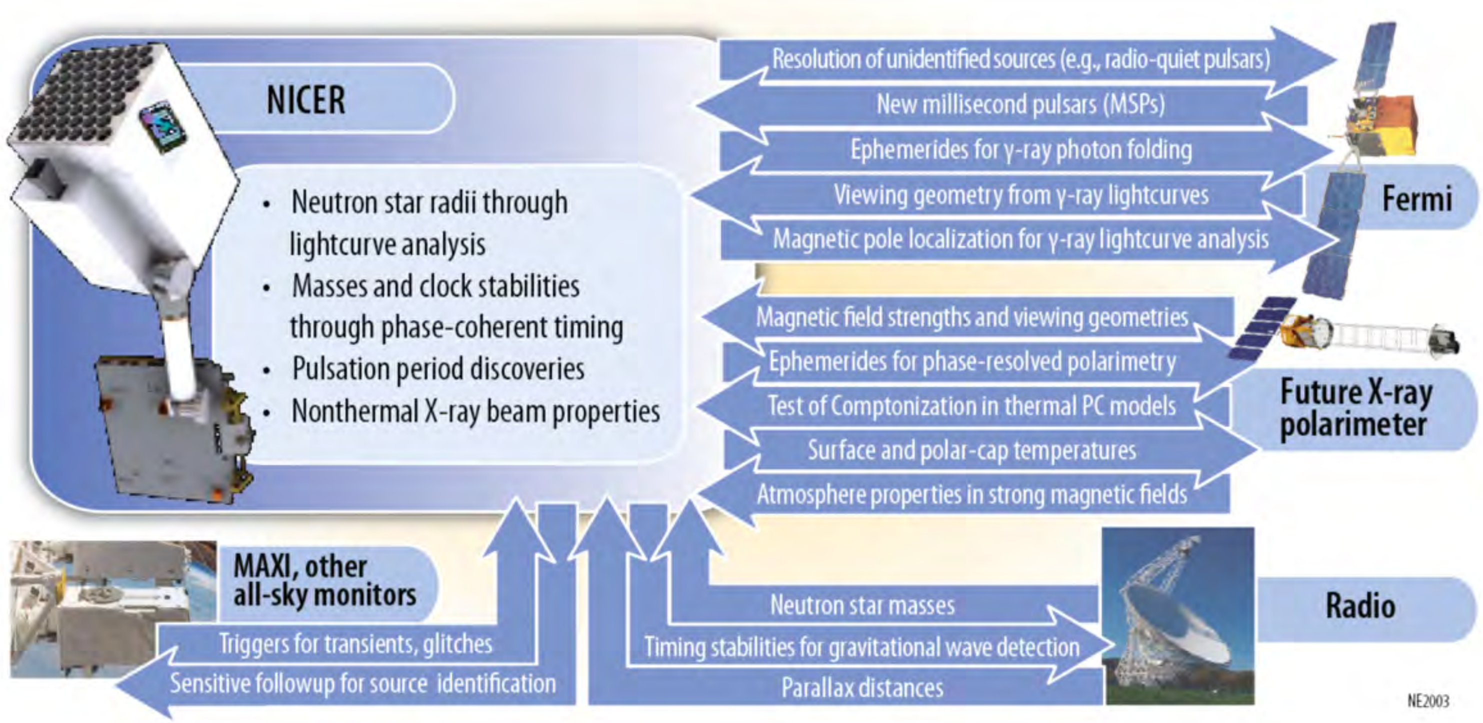
Low-Mass X-ray Binaries (LMXB)

Steiner, Lattimer, Brown, ApJ, 2010

Object	M ($M_\odot$)	R (km)	M ($M_\odot$)	R (km)
	$r_{ph} = R$		$r_{ph} \gg R$	
4U 1608–522	$1.52^{+0.22}_{-0.18}$	$11.04^{+0.53}_{-1.50}$	$1.64^{+0.34}_{-0.41}$	$11.82^{+0.42}_{-0.89}$
EXO 1745–248	$1.55^{+0.12}_{-0.36}$	$10.91^{+0.86}_{-0.65}$	$1.34^{+0.450}_{-0.28}$	$11.82^{+0.47}_{-0.72}$
4U 1820–30	$1.57^{+0.13}_{-0.15}$	$10.91^{+0.39}_{-0.92}$	$1.57^{+0.37}_{-0.31}$	$11.82^{+0.42}_{-0.82}$
M13	$1.48^{+0.21}_{-0.64}$	$11.04^{+1.00}_{-1.28}$	$0.901^{+0.28}_{-0.12}$	$12.21^{+0.18}_{-0.62}$
ω Cen	$1.43^{+0.26}_{-0.61}$	$11.18^{+1.14}_{-1.27}$	$0.994^{+0.51}_{-0.21}$	$12.09^{+0.27}_{-0.66}$
X7	$0.832^{+1.19}_{-0.051}$	$13.25^{+1.37}_{-3.50}$	$1.98^{+0.10}_{-0.36}$	$11.3^{+0.95}_{-1.03}$

NICER Neutron star Interior Composition ExploreR

- **launch**: June 2017, SpaceX
- **platform**: ISS ELC (ExPRESS Logistics Carrier)
- **instrument**: X-ray (0.2-12 keV)
- **objective**
  - ***structure***: neutron star radii to 5%, cooling timescales
  - ***dynamics***: stability of pulsars as clocks, properties of outbursts, oscillations, and precession
  - ***energetics***: intrinsic radiation patterns, spectra, and luminosities



Principles of NS Equation of State

Workout Session

[Problem 1] The ideal Fermi gas equation of states of white dwarfs or neutron stars, in which quantum degeneracy pressure dominates, can be represented by polytropic form;

$$P_{\rm deg} = K_\Gamma \rho^\Gamma = K_n \rho^{(n+1)/n}$$

where $K_\Gamma(K_n)$ and Γ are constants, ρ is density and n is called the polytropic index.

$$\Gamma(P) = \frac{d(\ln P)}{d(\ln \rho)}$$

a) For the comparison, consider an adiabatic expansion of an ideal monoatomic gas for which thermal (kinetic) pressure, $P_{\rm kin} = (\rho/m)kT$, dominates and degeneracy pressure is negligible. Show that

$$TV^{\gamma-1} = \text{constant}, \qquad P_{\rm kin}V^\gamma = \text{constant}, \qquad P_{\rm kin} \propto \rho^\gamma$$

where $\gamma = c_P/c_V = 5/3$ is the ratio of specific heats (c_P: specific heat at constant pressure, c_V: specific heat at constant volume).

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1) 부피가 일정할 때의 몰 비열 → 등적 몰비열 (C_V)

$$Q = nC_V\Delta T$$

$\Delta E_{\rm int} = Q - W$ ← **(열역학 제1법칙)**

$= nC_V\Delta T - W = nC_V\Delta T - p\Delta V = nC_V\Delta T$ ← **(ΔV = 0 : 등적)**

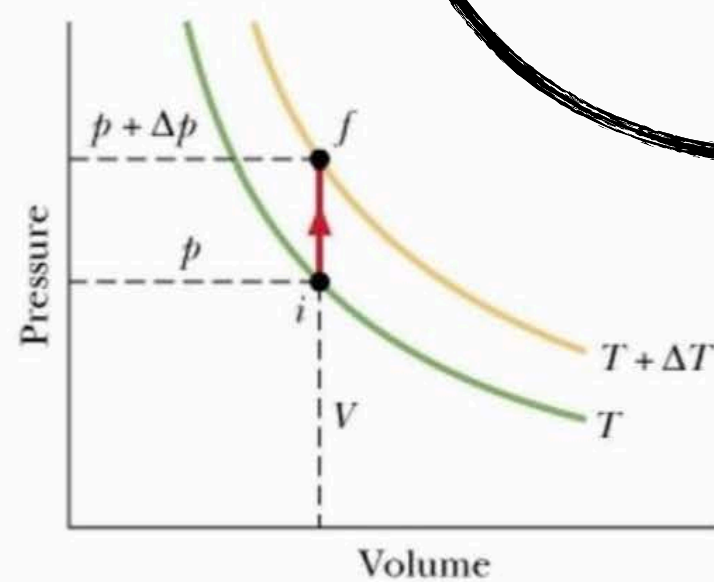
$\therefore\ \Delta E_{\rm int} = nC_V\Delta T$

즉, $\Delta E_{\rm int} = Q$

$$C_V = \frac{Q}{n\Delta T} = \frac{\Delta E_{\rm int}}{n\Delta T} = \frac{1}{n}\left(\frac{\frac{3}{2}nR\Delta T}{\Delta T}\right)$$

$$\Rightarrow\ C_V = \frac{3}{2}R = 12.5\,[J/mol\cdot K]$$

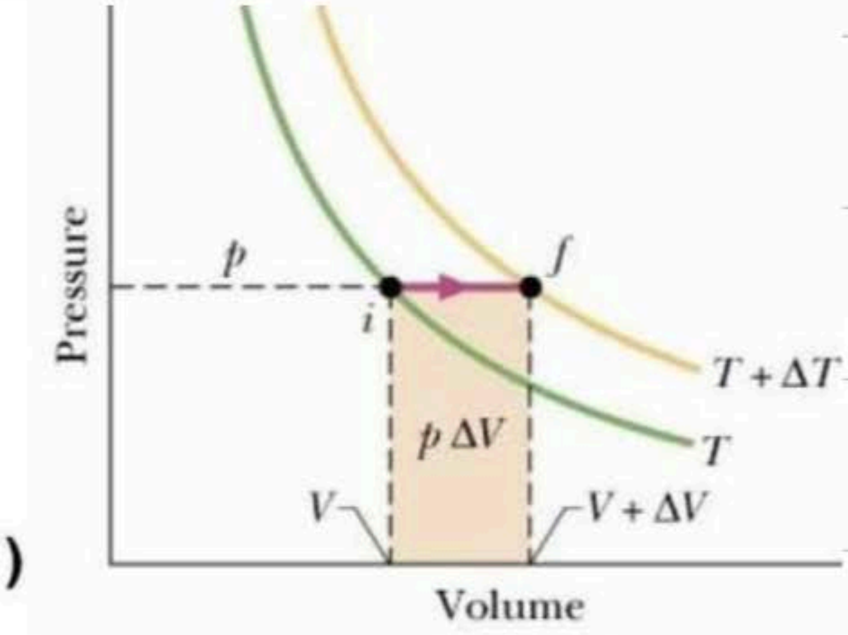
→ 단, 단원자로 된 기체의 경우임.



$$E = \frac{3}{2}nkT$$

$$P_{\rm kin} = nkT$$

2) 압력이 일정할 때의 몰 비열 → 등압 몰비열 (C_p)



$$\Delta Q = nC_p\Delta T$$

$$\Delta E_{int} = Q - W$$ ← **(열역학 제1법칙)**

$$= nC_p\Delta T - W = nC_p\Delta T - p\Delta V$$

$$= nC_p\Delta T - nR\Delta T$$ ← **($pV = nRT$ 이므로, $p\Delta V = nR\Delta T$)**

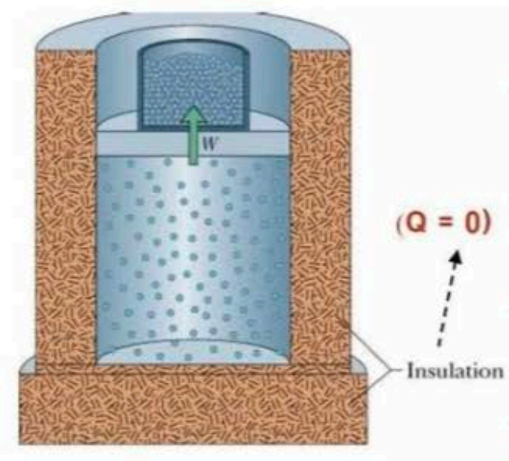
$$\frac{\Delta E_{int}}{n\Delta T} = C_p - R \longrightarrow \left\{\frac{\Delta E_{int}}{n\Delta T} = C_V \text{ 이므로}\right\}$$

⇨ $$C_p = C_V + R$$

1. 단열과정 (Adiabatic process) : Q = 0

단열적으로 부피가 미소 dV 만큼 변하였을 때, 내부에너지는

$$dE_{int} = Q - dW = -pdV$$



그런데,

$$dE_{int} = nC_V dT \text{ (과정에무관하므로)}$$

$$\therefore\ nC_V dT + pdV = 0$$

$$ndT = -\left(\frac{p}{C_V}\right)dV$$

$$pdV + Vdp = nRdT \ \leftarrow \{pV = nRT\}$$

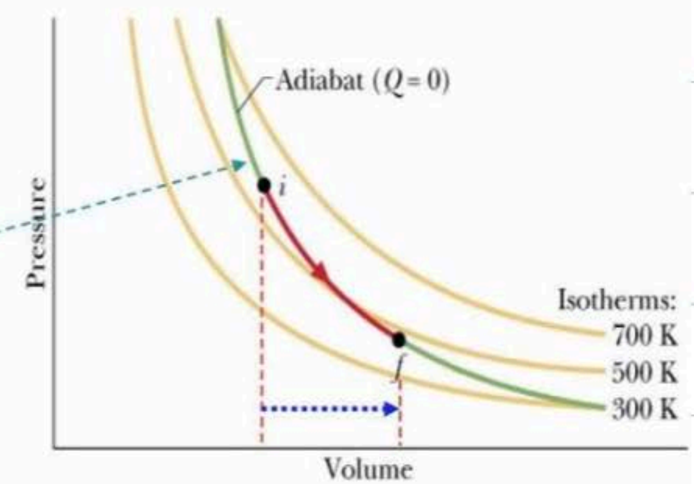
$$ndT = \frac{pdV + Vdp}{C_p - C_V} \ \leftarrow \{R = C_p - C_V\}$$

$$\frac{dp}{p} + \left(\frac{C_p}{C_V}\right)\frac{dV}{V} = 0$$

$$\frac{dp}{p} + \gamma\frac{dV}{V} = 0 \quad \leftarrow \left(\gamma \equiv \frac{C_p}{C_V}\right)$$ **: 몰 비열 간의 비**

양변을 적분

$$\ln p + \gamma \ln V = A \text{(일정한상수)}$$



$$pV^{\gamma} = e^{A} = \text{상수}$$

$$pV^{\gamma} = \text{상수 (단열과정)}$$

$$\Rightarrow \ p_i V_i^{\gamma} = p_f V_f^{\gamma}$$

$$pV^{\gamma} = \left(\frac{nRT}{V}\right)V^{\gamma} \ \Rightarrow \ TV^{\gamma-1} = \text{상수} \ \Rightarrow \ T_i V_i^{\gamma-1} = T_f V_f^{\gamma-1}$$

b) For ideal Fermi gas, in the zero temperature limit, show that the number density of gas (n_g) and Fermi momentum (p_F) are related by

$$n_g = g \times \frac{2\pi}{3h^3} p_F^3$$

where g is the degeneracy.

Consider simple case ($T=0$)

isotropic distribution

$d^3p \to 4\pi p^2 dp$

Fermi momentum

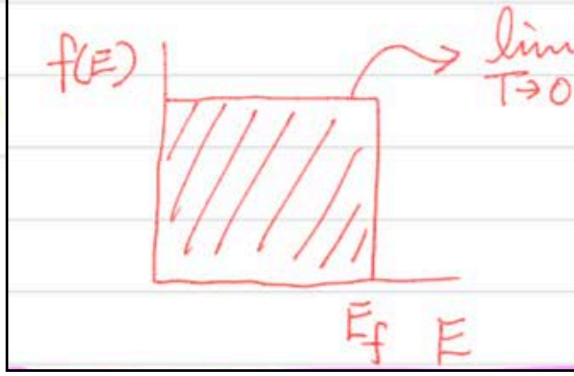
$$dN(p)dp = 2\times \frac{d^3p\,dV}{h^3} = \begin{cases} 2\times \frac{4\pi p^2 dp\, dV}{h^3} & \text{if } |\vec{p}| \le p_F \\ 0 & \text{if } |\vec{p}| > p_F \end{cases}$$

$$n_e(p)dp = \frac{8\pi p^2 dp}{h^3}$$

$$n_e = \int_0^{p_F} \frac{8\pi}{h^3} p^2 dp = \frac{8\pi}{3h^3} p_F^3$$

Normal gas $P(T=0)=0$.

Quantum gas $P(T=0)\neq 0$ from $\Delta x^3 \Delta p^3 > h^3$



All states with $E \le E_F$ are occupied (degenerate)

c) In the non-relativistic limit, show that

$$P_{\rm deg} = g \times \frac{4\pi}{3h^3} \times \frac{1}{5m} \times p_F^5$$

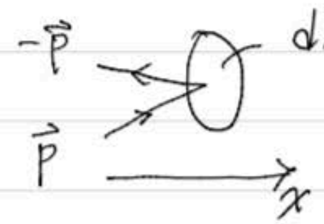
and $\Gamma = 5/3$ and $n = 3/2$. Why is Γ the same as γ obtained in a) despite the difference in their physical origin?

d) In the full relativistic limit, show that

$$P_{\rm deg} = g \times \frac{4\pi}{3h^3} \times \frac{c}{4} \times p_F^4$$

and $\Gamma = 4/3$ and $n = 3$.

9.2 Degeneracy pressure



$$\Delta p_x = 2p_x$$

$$v_x = \frac{dx}{dt}$$

Change of momentum

\# of particles

$$\frac{dF_x}{dA} = \frac{2p_x}{dA\,dt} = \frac{2p_x}{dA\,(dx/v_x)} = \frac{2p_x v_x}{dV}$$

↑ The force per unit area due to each collision.

$$P = \int_0^\infty \frac{dN(p)}{2} \frac{2p_x v_x}{dV}\, dp$$

↑ only half of electrons are moving $+x$ at any given time.

$$p_x v_x = \frac{1}{3} p v$$

$$\begin{cases} v = |\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{3 v_x^2} \\ p = \sqrt{3 p_x^2} \\ vp = 3\, p_x v_x \end{cases}$$

$$P_e = \frac{1}{3} \int_0^\infty n_e(p)\, p\, v\, dp$$

$$P_e = \frac{8\pi}{3h^3} \int_0^{p_F} v\, p^3\, dp$$

$$P_e = \frac{8\pi}{3h^3}\int_0^{p_F} v\, p^3 dp$$

$$p = \gamma m v \quad , \quad E = \gamma m c^2$$

$$v = \frac{p}{m\gamma} = \frac{pc^2}{E} = \frac{pc^2}{\sqrt{p^2c^2 + m^2c^4}}$$

$$P_e = \frac{8\pi}{3h^3}\int_0^{p_F} \frac{p^4c^2}{\sqrt{p^2c^2 + m_e^2c^4}}\, dp$$

$$n_e = \frac{X\rho}{m_H} + \frac{(1-X)\rho}{2m_H} = \frac{\rho}{2m_H}(1+X)$$

$$n_e = \frac{\rho}{\mu_e m_H} \quad ; \quad \mu_e = \frac{2}{1+X}$$ (mean molecular weight of electron)

from

$$n_e = \frac{8\pi}{3h^3} p_F^3$$

$$p_F = \left(\frac{3h^3\rho}{8\pi \mu_e m_H}\right)^{1/3}$$

c) In the non-relativistic limit, show that

$$P_{\rm deg} = g \times \frac{4\pi}{3h^3} \times \frac{1}{5m} \times p_F^5$$

and $\Gamma = 5/3$ and $n = 3/2$. Why is Γ the same as γ obtained in a) despite the difference in their physical origin?

$$P_e = \frac{8\pi}{3h^3} \int_0^{P_F} \frac{p^4 c^2}{\sqrt{p^2c^2 + m_e^2 c^4}} \, dp$$

i) non-relativistic $\quad \sqrt{p^2c^2 + m_e^2c^4} \approx m_e c^2$

$$P = \frac{8\pi}{15\, h^3 m_e} P_F^5$$

$$P = K_1 \rho^{5/3}$$

$$K_1 = \frac{3^{2/3}}{20\,\pi^{2/3}} \frac{h^2}{m_e m_H^{5/3} \mu_e^{5/3}} = \frac{1.00 \times 10^7}{\mu_e^{5/3}}$$

d) In the full relativistic limit, show that

$$P_{\text{deg}} = g \times \frac{4\pi}{3h^3} \times \frac{c}{4} \times p_F^4$$

and $\Gamma = 4/3$ and $n = 3$.

$$P_e = \frac{8\pi}{3h^3} \int_0^{P_F} \frac{p^4 c^2}{\sqrt{p^2c^2 + m_e^2 c^4}} \, dp$$

ii) fully relativistic $\sqrt{p^2c^2 + m_e^2c^4} \approx pc$

$$P = \frac{2\pi c}{3h^3} P_F^4$$

$$P = K_2 \rho^{4/3}$$

$$K_2 = \frac{3^{1/3}}{8\pi^{1/3}} \frac{hc}{m_H^{4/3} \mu_e^{4/3}} = \frac{1.24 \times 10^{10}}{\mu_e^{4/3}}$$

e) For typical white dwarfs ($\rho \sim 10^6$ g cm^{-3}) and neutron stars ($\rho \sim 10^{14}$ g cm^{-3}), compare the magnitudes of degeneracy pressure (P_{deg}), kinetic pressure of the ideal gas (P_{kin}) and radiation pressure at $kT = 1$ MeV, 1 keV and 1 eV.

$$P_{\text{total}} = P_{\text{deg}} + \frac{\rho}{m}kT + \frac{1}{3}aT^4$$

where $k = 1.4 \times 10^{-16}$ erg K^{-1} $= 8.6 \times 10^{-5}$ eV K^{-1}.

$$\left(20^\circ\text{C} = 293K \approx \frac{1}{40}\text{eV}\right)$$

Without nuclear reaction, star will contract to the quantum limit.

de Broglie wavelength

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}} \approx \frac{h}{\sqrt{3mkT}}$$

$$E \sim \frac{3}{2}kT$$

* electron will reach quantum domain first

Check.

If we assume inter-particle separation $\sim \frac{\lambda}{2}$

→ quantum domain

$$\rho_{\text{quantum}} \approx \frac{m_p}{(\lambda/2)^3} = \frac{8m_p(3m_e kT)^{3/2}}{h^3}$$

With $T = 15 \times 10^6$ K

$$\rho_{\text{quantum}} \approx 640 \text{ g/cm}^3$$

$$\rho_{\text{center, sun}} \sim 150 \text{ g/cm}^3$$

↑ classical

[Problem 2] At zero temperature limit, when the compact star is in a hydrostatic equilibrium with spherical symmetry, compact star equation of state can be obtained by solving TOV (Tolman-Oppenheimer-Volkoff) equation

$$\frac{dP(r)}{dr} = -\frac{GM(r)\epsilon(r)}{c^2 r^2}\left(1+\frac{P(r)}{\epsilon(r)}\right)\left(1+\frac{4\pi r^3 P(r)}{M(r)c^2}\right)\left(1-\frac{2GM(r)}{c^2 r}\right)^{-1}$$

where $P(r)$ is the pressure, $\epsilon(r)$ is the energy density and $M(r)$ is the enclosed gravitational mass $M_G(r)$ for a given radius r. The gravitational and baryon masses of the star are defined by

$$M_G(r) = \int_0^R 4\pi r^2 \frac{\epsilon(r)}{c^2}$$

$$M_A(r) = m_A \int_0^R dr 4\pi r^2 n(r)\left(1-\frac{2GM(r)}{c^2 r}\right)^{-1/2}$$

where m_A is baryon mass and $n(r)$ is the baryon number density.

a) In the Newtonian limit ($P \ll \epsilon$ and $GM/c^2 \ll r$), with polytropic EOS, show that TOV equation can be reduced to Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\xi^2\frac{d\theta}{d\xi} = -\theta^n$$

with $\rho = \rho_c\theta^n$, $r = a\xi$, and

$$a = \sqrt{\frac{(n+1)K_n\rho_c^{(1-n)/n}}{4\pi G}},$$

where ρ_c is the central density of a star.

$$\frac{dM_r}{dr} = 4\pi r^2 \rho$$

$$\frac{dP}{dr} = -\frac{GM_r}{r^2}\rho$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = \frac{1}{r^2}\frac{d}{dr}\left(-GM_r\right) = -4\pi G\rho$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

b) The Lane-Emden equation can be solved with boundary conditions at the center;

$$\theta(0) = 1, \quad \theta'(0) = 0.$$

For $n < 5$ (or $\Gamma > 6/5$), the solution decreases monotonically and have a zero at a finite value $\xi = \xi_1$: $\theta(\xi_1) = 0$ (see Table 1 for numerical values). This point corresponds to the surface of the star, where $P = \rho = 0$. Show that the mass of the star is given as

$$M = 4\pi \left[\frac{(n+1)K_n}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/2n} \left| \left(\xi^2 \frac{d\theta}{d\xi}\right)_{\xi=\xi_1} \right|.$$

Table 1: The constants of the Lane-Emden functions.

n	ξ_1	$-\xi_1^2 \left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$	$\rho_c/\bar{\rho}$
0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
2.5	5.35528	2.18720	23.40646
3.0	6.89685	2.01824	54.1825
3.25	8.01894	1.94980	88.153
3.5	9.53581	1.89056	152.884
4.0	14.97155	1.79723	622.408
4.5	31.83646	1.73780	6189.47
4.9	169.47	1.7355	934800
5.0	∞	1.73205	∞

c) In the full relativistic and Newtonian limit, show that the mass become independent of radius. This implies that there exist maximum mass (Chandrasekhar mass) for the compact stars (for which quantum degeneracy pressure dominates). What is the value of the radius-independent mass?

$$M = 4\pi \left[\frac{(n+1)K_n}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/2n} \left| \left(\xi^2 \frac{d\theta}{d\xi}\right)_{\xi=\xi_1} \right|$$

n	ξ_1	$-\xi_1^2 \left(\frac{d\theta_n}{d\xi}\right)_{\xi = \xi_1}$	$\rho_c/\bar{\rho}$
3.0..........	6.89685	2.01824	54.1825

$$M_{Ch} = \frac{\sqrt{6}}{32\pi} \left(\frac{\hbar c}{G}\right)^{3/2} \left(\frac{2}{\mu_e}\right)^2 \frac{\xi_1^2 |\theta'(\xi_1)|}{m_H^2}$$

$$\xi_1^2 |\theta'(\xi_1)| = 2.018$$

$$M_{Ch} = 1.46 \left(\frac{2}{\mu_e}\right)^2 M_\odot$$

d) In the Newtonian limit, show that the radius becomes independent of mass when $n = 1$. Note that $n = 1$ is possible only when the system is far from an ideal Fermi gas; i.e., interactions are non-negligible. What is the value of the mass-independent radius?

---

with $\rho = \rho_c \theta^n$, $r = a\xi$, and

$$a = \sqrt{\frac{(n+1) K_n \rho_c^{(1-n)/n}}{4\pi G}};$$

Table 1: The constants of the Lane-Emden functions.

n	ξ_1	$-\xi_1^2 \left(\frac{d\theta_n}{d\xi}\right)_{\xi=\xi_1}$	$\rho_c/\bar{\rho}$
0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987

But, we don't know K_1 unless we know the interactions !

[Problem 3] Any realistic neutron star equation of state cannot be represented by a single polytropic index and piecewise polytropic EOS is used in many literatures; $P(\rho) = K_i \rho^{\Gamma_i}$. In the piecewise polytropic EOS, different values of polytropic index are used for different density region. As a result, at the boundaries, discontinuities occur. Hence, instead of piecewise polytropics, in the recent analysis of tidal deformability of neutron stars [arXiv:1805.11581], spectral expansion of adiabatic index is used [arXiv:1009.0738, arXiv:1207.3744, arXiv:1807.02538];

$$\Gamma(P) = \exp\left[\sum_k \gamma_k \Phi_k(P)\right]$$

with $\Phi_k = [\ln(P/P_0)]^k$. Note that the adiabatic index is defined as

$$\Gamma(P) = \frac{d(\ln P)}{d(\ln \rho)}$$

and $P \propto \rho^\Gamma$ when Γ is constant.

a) From the first law of thermodynamics, show that

$$\Gamma(P) = \frac{\epsilon + P}{P}\frac{dP}{d\epsilon}.$$

$$\Gamma(P) = \frac{d(\ln P)}{d(\ln \rho)}$$

* Number of particles is fixed.

$$d\left(\frac{\varepsilon}{\rho}\right) = -p\, d\left(\frac{1}{\rho}\right)$$

$$\frac{d\varepsilon}{\rho} - \varepsilon\frac{d\rho}{\rho^2} = p\frac{d\rho}{\rho^2}$$

$$d\varepsilon = (\varepsilon + p)\frac{d\rho}{\rho} = (\varepsilon + p)\frac{1}{\Gamma}\frac{dP}{P}$$

$$\Gamma = \left(\frac{p+\varepsilon}{p}\right)\frac{dP}{d\varepsilon}$$

b) Show that $\epsilon(P)$ can be obtained by

$$\epsilon(P) = \frac{1}{\mu(P)}\left(\epsilon_0 + \int_{P_0}^{P} \frac{\mu(p')}{\Gamma(p')} dp'\right)$$

$$\mu(p) = \exp\left[-\int_{P_0}^{p} \frac{dp'}{p'\Gamma(p')}\right]$$

where $\epsilon_0 = \epsilon(P_0)$ is the constant integration needed to fix the solution.

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b) $$\frac{d\epsilon}{dP} = \frac{\varepsilon + P}{P\Gamma}$$

$$d\epsilon = \frac{\varepsilon + P}{P\Gamma} dP = (\varepsilon + p)\frac{dp}{p\Gamma}$$

$$\mu = \exp\left[-\int_{P_0}^{p} \frac{dp'}{p'\Gamma(p')}\right]$$

$$\frac{d\mu}{dP} = -\frac{1}{P\Gamma(p)}\mu$$

$$\frac{d\mu}{\mu} = -\frac{dP}{P\Gamma(P)}$$

$$d\epsilon = -(\varepsilon + p)\frac{d\mu}{\mu} \Rightarrow \mu d\epsilon = -(\varepsilon + p) d\mu$$

$$\Rightarrow \mu d\varepsilon + \varepsilon d\mu = -p d\mu$$

$$d(\varepsilon\mu) = d\varepsilon\mu + \varepsilon d\mu = -p d\mu$$

$$= \frac{\mu}{\Gamma} dp$$

$$\therefore \varepsilon\mu - e_0 = \int \frac{\mu}{\Gamma} dp$$

$$\epsilon\mu = \varepsilon_0 + \int \frac{\mu}{\Gamma} dP.$$