Prepared for submission to JHEP

K-hep Lecture note on conformal field theory

Abstract: This is summary for the $4 \times (1 + \frac{1}{4})$ hours lecture on CFT during K-hep workshop in 2018.[1]

[1]Any comments/questions/suggestions are welcome (arima275@snu.ac.kr). You can find update in K-hep Lecture note on CFT

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1 Lecture I, II : General aspects of CFT

1.1 Classical aspects of CFT

1.1.1 Classical Poincare invariant field theory

Ref :
Book "Classical Theory of Gauge fields" By Valery Rubakov

Poincare transformations on D-dimensional Minkowski space-time $\mathbb{R}^{1,D-1}$ $(D>2)$,

$$\begin{aligned} &\mathbb{R}^{1,D-1} = \{x^\mu = (x^0, x^1, \ldots, x^{D-1})\}\ , \quad \text{with metric} \\ &ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -(dx^0)^2 + (dx^1)^2 + \ldots + (dx^{D-1})^2\ . \end{aligned} \tag{1.1}$$

are transformations $x^\mu \to \tilde{x}^\mu(x)$ such that

$$\frac{\partial \tilde{x}^\rho}{\partial x^\mu}\frac{\partial \tilde{x}^\sigma}{\partial x^\nu}\eta_{\rho\sigma} = \eta_{\mu\nu}\ . \tag{1.2}$$

So, it is nothing but isometry on the Minkowski space-time. Infinitesimal poincare transformations, $\tilde{x}^\mu = x^\mu + \epsilon^\mu(x) + o(\epsilon^2)$, satisfy

$$\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) = 0\ . \tag{1.3}$$

The general solutions are

$$\epsilon^\mu(x) = a^\mu + \Lambda^\mu_{\ \nu} x^\nu \ , \quad \text{with } \Lambda_{\mu\nu} = -\Lambda_{\nu\mu} \ . \tag{1.4}$$

Finite transformations are

$$\begin{aligned} &\exp(a^\mu P_\mu) := \exp(a^\mu \partial_\mu) \ : \ x^\mu \to \tilde{x}^\mu = x^\mu + a^\mu \ \ \text{(translation)}, \\ &\exp(\Lambda^{\mu\nu} L_{\mu\nu}) := \exp(\Lambda^\mu_{\ \nu} x^\nu \partial_\mu) \ : \ x^\mu \to \tilde{x}^\mu = \exp(\Lambda)^\mu_{\ \nu} x^\nu \ \ \text{(rotation and Lorentz boost)}. \end{aligned} \tag{1.5}$$

Example : Consider a single real scalar field

$$\phi \ : \ \mathbb{R}^{1,D-1} \to \mathbb{R} \ . \tag{1.6}$$

Its Poincare invariant action is

$$\begin{aligned} &S[\phi] = \int d^D x \mathcal{L}(\phi, \partial\phi) \ , \quad \text{where} \\ &\mathcal{L}(\phi, \partial_\mu \phi) = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g_{m,n}\phi^{2m}(\partial_\mu\phi\partial^\mu\phi)^n \ , \quad (g_{0,1} = 0) \end{aligned} \tag{1.7}$$

Probelem 1.1.1-1: Show that the action is invariant under the following Poincare transformation

$$\begin{aligned} &\phi \to \tilde{\phi} \ , \quad \text{where} \\ &\tilde{\phi}(\tilde{x}(x)) = \phi(x) \ , \quad \tilde{x}^\mu(x) = x^\mu + a^\mu + \exp(\Lambda)^\mu_{\ \nu} x^\nu \ . \end{aligned} \tag{1.8}$$

Noether Theorem Suppose that the Lagrangian is invariant under a infinitesimal transformation, $\phi \to \phi + \delta_a \phi$, up to a total divergence

$$\delta_a \mathcal{L}(\phi, \partial\phi) = \partial_\mu \mathcal{F}^\mu_a \ . \tag{1.9}$$

Then, there is a associated conserved current j^μ_a

$$j^\mu_a := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_a \phi - \mathcal{F}^\mu_a \ . \tag{1.10}$$

Problem 1.1.1-2: Show that the above current are conserved, $\partial_\mu j^\mu_a = 0$, modulo classical equation of motion.

Stress-energy tensor : The conserved current for translation symmetry, $\delta_a : x^\mu \to \tilde{x}^\mu = x^\mu + a^\mu$, can be written as

$$(j_a)_\mu = a^\nu T_{\nu\mu} \ . \tag{1.11}$$

The tensor $T_{\mu\nu}$ is called stress-energy tensor. T_{00} is Hamiltonian and $T_{0i} (i = 1, \ldots, D-1)$ is momentum along i-th direction.

Problem 1.1.1-3 Show that the stress-energy tensor $T_{\mu\nu}$ for the theory in (1.7) with $g_{m,n\geq 1} = 0$ is

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi + \eta_{\mu\nu} \mathcal{L} \tag{1.12}$$

1.1.2 Conformal field theory

Ref:
"Lectures on Confomral Field Theory" arXiv:1511.04074, By Joshua D. Qualls

Conformal transformations are transformations $x^\mu \to \tilde{x}^\mu(x)$ such that

$$\frac{\partial \tilde{x}^\rho}{\partial x^\mu}\frac{\partial \tilde{x}^\sigma}{\partial x^\nu}\eta_{\rho\sigma} = \Omega^2(x)\eta_{\mu\nu}\ , \quad \text{for some } \Omega\ . \tag{1.13}$$

Infinitesimal conformal transformations, $\tilde{x}^\mu = x^\mu + \epsilon^\mu(x) + o(\epsilon^2)$, satisfy

$$\partial_\mu \epsilon_\nu(x) + \partial_\nu \epsilon_\mu(x) = \lambda(x)\eta_{\mu\nu}\ , \quad \text{for some } \lambda\ . \tag{1.14}$$

The general solutions for $D > 2$ are $x^2 := x^\mu x_\mu$

$$\begin{aligned} \epsilon^\mu(x)\partial_\mu &= (a^\mu + b^\nu(2x_\nu x^\mu - x^2\delta^\mu_\nu) + \Lambda^\mu{}_\nu x^\nu + cx^\mu)\partial_\mu\ , \quad \text{with } \Lambda_{\mu\nu} = -\Lambda_{\nu\mu}\ . \\ &:= a^\mu P_\mu + b^\mu K_\mu + \Lambda^{\mu\nu}L_{\mu\nu} + c\mathcal{D}\ . \end{aligned} \tag{1.15}$$

Problem 1.1.2-1: Show that infinitesimal conformal transformation form a $SO(2,D)$ algebra.

Its finite transformations are

$$\begin{aligned} &\exp(a^\mu P_\mu)\ :\ x^\mu \to \tilde{x}^\mu = x^\mu + a^\mu \ \text{ (translation)}\ , \\ &\exp(\Lambda^{\mu\nu}L_{\mu\nu})\ :\ x^\mu \to \tilde{x}^\mu = \exp(\Lambda)^\mu{}_\nu x^\nu \ \text{ (rotation and Lorentz boost)}\ , \\ &\exp(c\mathcal{D})\ :\ x^\mu \to \tilde{x}^\mu = e^c x^\mu \ \text{ (dilatation)}\ , \\ &\exp(b^\mu K_\mu)\ :\ x^\mu \to \tilde{x}^\mu = \frac{x^\mu - x^2 b^\mu}{1 - 2(b\cdot x) + b^2 x^2} \ \text{ (special conformal).} \end{aligned} \tag{1.16}$$

Sometimes, we also consider a discrete inversion symmetry:

$$I\ :\ x^\mu \to \frac{x^\mu}{x^2}\ . \tag{1.17}$$

Using the inversion, the special transformation can be rewritten as

$$\frac{x^\mu}{x^2} = \frac{\tilde{x}^\mu}{x^2} + b\ . \tag{1.18}$$

Example Conformally invariant Lagragian for a single scalar field is

$$\mathcal{L}(\phi, \partial_\mu\phi) = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - g\phi^{\frac{2D}{D-2}}\ . \tag{1.19}$$

Note that the Lagrangian is analytic only when $D = 3, 4$ and 6.

Probelem 1.1.2-2: Show that the action is invariant under the following conformal transformation

$$\begin{aligned} &\phi \to \tilde{\phi}\ , \quad \text{where} \\ &\tilde{\phi}(\tilde{x}(x)) = |\det \frac{\partial \tilde{x}}{\partial x}|^{-\frac{D-2}{2D}}\phi(x)\ , \quad \tilde{x}^\mu(x) = (\exp[a^\mu P_\mu + b^\mu K_\mu + \Lambda^{\mu\nu}L_{\mu\nu} + c\mathcal{D}]\cdot x)^\mu\ . \end{aligned} \tag{1.20}$$

1.1.3 Conformal field theory on conformally flat Euclidean space-time

Wick rotation to $\mathbb{R}^D$ Let $x_D := ix_0$ and the Eucledian action is

$$S_E = -\int d\tau d^{D-1}x \mathcal{L}(\phi, \partial\phi)\big|_{\partial_0\phi \to i\partial_D\phi} \,. \tag{1.21}$$

After the Wick rotatin, the ϕ is considered to be a function on $\mathbb{R}^D$ with Eucleadian metric

$$ds^2 = \delta_{ab}dx^a dx^b = (dx^1)^2 + \ldots + (dx^D)^2 \,. \tag{1.22}$$

Conformal group on conformally flat space Conformal transformations on general Euclidean metric g_{ab} is a transformation $x \to \tilde{x}(x)$ such that

$$\frac{\partial \tilde{x}^c}{\partial x^a}\frac{\partial \tilde{x}^d}{\partial x^b} g_{cd} = g_{ab} \,. \tag{1.23}$$

For flat Euclidean metric, $g_{ab} = \delta_{ab}$, the conformal transformations form $SO(1, D+1)$ group. More generally, the conformal group for a conformally flat metric

$$g_{ab} = \kappa(x)^2 \delta_{ab} \text{ , for some } \kappa \tag{1.24}$$

is $SO(1, D+1)$.[1]

Problem 1.1.3-1: Show that the usual round metrics on S^D and $\mathbb{R} \times S^{D-1}$ are conformally flat.

Conformal mapping of CFT From a conformally invariant field theory on $\mathbb{R}^D$, there is a canonical way called 'conformal mapping' to construct conformally invariant field theory on a conformally flat Eucleadian space. Let's demonstate the conformal mapping using the conformally invariant scalar theory in (1.19). After Wick-rotation, the Euclidean action becomes

$$S_E = \int d^D x \big(\frac{1}{2}\delta^{ab}\partial_a\phi\partial_b\phi + g\phi^{\frac{2D}{D-2}}\ \big). \tag{1.25}$$

The action can be interpreted as conformally invariant action on a conformally flat space with metric $g_{ab} = \kappa^2\delta_{ab}$ with proper rescaling of Lagrangian density

$$\begin{aligned} S_E &= \int d^D x \sqrt{g}\mathcal{L}_E \,, \quad \text{where} \\ \mathcal{L}_E &= \frac{1}{\sqrt{g}}\big(\delta^{ab}\frac{1}{2}\partial_a\phi\partial_b\phi + g_4\phi^{\frac{2D}{D-2}}\ \big) \,, \quad \sqrt{g} := \sqrt{\det(g)_{ab}} = \kappa^D \,. \end{aligned} \tag{1.26}$$

[1] Unlike conformal group, generally the isometry group of conformally flat metric is different from the isometry group of flat metric

Then, we redefine the field $\phi \to \kappa^{(D-2)/2}\phi$ to have a canonical kinetic term. The resulting Lagrangian is

$$\begin{aligned}
\mathcal{L}_E &= \frac{1}{2}\kappa^{-2}\delta^{ab}\partial_a\phi\partial_b\phi + g_4\phi^{\frac{2D}{D-2}} + \frac{1}{2}\kappa^{-D}\delta^{ab}\partial_a\kappa^{\frac{D-2}{2}}\partial_b\kappa^{\frac{D-2}{2}}\phi^2 + \frac{1}{\sqrt{g}}(\text{total derivative}) \ , \\
&= \frac{1}{2}g^{ab}\partial_a\phi\partial_b\phi + \frac{1}{2}(\frac{D-2}{2})^2(\kappa^{-4}g^{ab}\partial_a\kappa\partial_b\kappa)\phi^2 + g_4\phi^{\frac{2D}{D-2}} + \frac{1}{\sqrt{g}}(\text{total derivative}) \ , \\
&= \frac{1}{2}g^{ab}\partial_a\phi\partial_b\phi + \frac{1}{2}\times\frac{D-2}{4(D-1)}R\phi^2 + g_4\phi^{\frac{2D}{D-2}} + \frac{1}{\sqrt{g}}(\text{total derivative}) \ .
\end{aligned} \tag{1.27}$$

Here we use the fact that

$$R(\text{scalar curvature of metric } g) = (D-1)(D-2)\kappa^{-4}g^{ab}\partial_a\kappa\partial_b\kappa \ . \tag{1.28}$$

Problem 1.1.3-2: Check the above.

Note that after the conformal mapping, there appears a mass-like term, $\frac{1}{2}\times\frac{D-2}{4(D-1)}R\phi^2$, from coupling to the curvature of background metric.

1.2 Quantum aspects of CFT

1.2.1 QFT as RG between CFTs

Ref:
Lecture note on "The renormalization group" By David Tong,
(http://www.damtp.cam.ac.uk/user/tong/sft/three.pdf)

RG in QFT To specify the quantum theory, we need to introduce a cut-off Λ_0

$$\mathcal{Z} = \int [D\phi]_{|p|<\Lambda_0} e^{-S_{E;\Lambda_0}[\phi]} \tag{1.29}$$

The Euclidean action is

$$\begin{aligned}
&S_{E;\Lambda_0}[\phi] = \int d^Dx \mathcal{L}_{E;\Lambda_0}(\phi,\partial\phi) \ , \quad \text{where} \\
&\mathcal{L}_{E;\Lambda_0}(\phi,\partial_\mu\phi) = \frac{1}{2}\partial_a\phi\partial^a\phi + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} g_{\mathcal{O}_{m,n}}\Lambda_0^{D-[\mathcal{O}_{m,n}]}\mathcal{O}_{m,n} \ , \quad (g_{\mathcal{O}_{0,1}}=0) \\
&\mathcal{O}_{m,n} := \phi^{2m}(\partial_a\phi\partial^a\phi)^n \ , \quad [\mathcal{O}_{m,n}] := m(D-2)+nD \ .
\end{aligned} \tag{1.30}$$

Here we introduce dimensionless coupling constants $g_{m,n}$. $[\mathcal{O}]$ denotes the mass-dimension of the operator $\mathcal{O}$.

In measurement of physical quantities, there is characteristic energy-scale Λ (such as momenta of external particles in scattering procedure). For the QFT to be valid at the energy scale Λ, we need to assume that

$$\Lambda < \Lambda_0 \ . \tag{1.31}$$

The physics at the energy scale Λ can be described by an effective action $S_{E,\Lambda}[\phi_\Lambda]$ obtained by integrating fastly oscillating modes $\phi_{(\Lambda,\Lambda_0)}$. Let

$$\begin{aligned} &\phi(x) = \int_{|p|<\Lambda_0} \tilde{\phi}(p) e^{ip\cdot x} = \phi_\Lambda(x) + \phi_{(\Lambda,\Lambda_0)}(x)\ , \quad \text{where} \\ &\phi_\Lambda(x) := \int_{|p|<\Lambda} \tilde{\phi}(p) e^{ip\cdot x}\ , \quad \phi_{(\Lambda,\Lambda_0)} := \int_{\Lambda\leq|p|<\Lambda_0} \tilde{\phi}(p) e^{ip\cdot x}\ . \end{aligned} \tag{1.32}$$

Then,

$$S_{E;\Lambda_0}[\phi] = S_{E;\Lambda_0}[\phi_\Lambda] + S_{E;\Lambda_0}[\phi_{(\Lambda,\Lambda_0)}] + S_{\text{int}}[\phi_\Lambda, \phi_{(\Lambda,\Lambda_0)}]\ . \tag{1.33}$$

Problem 1.2.1-1: Show that $S_{\text{int}}[\phi_\Lambda, \phi_{(\Lambda,\Lambda_0)}] = 0$ for free theory ($g_{\mathcal{O}} = 0$ for all $\mathcal{O}$ except $\mathcal{O}_{1,0}$).

Finally, we define the effecitve action at the scale at $\Lambda < \Lambda_0$ as

$$S_{E,\Lambda}[\phi_\Lambda] := S_{E,\Lambda_0}[\phi_\Lambda] - \log \int [D\phi_{(\Lambda,\Lambda_0)}]_{\Lambda\leq|p|<\Lambda_0} e^{-S_{E,\Lambda_0}[\phi_{(\Lambda,\Lambda_0)}] - S_{\text{int}}[\phi_\Lambda,\phi_{(\Lambda,\Lambda_0)}]}\ . \tag{1.34}$$

By definition,

$$\mathcal{Z} = \int [D\phi]_{|p|<\Lambda_0} e^{-S_{E;\Lambda_0}[\phi]} = \int [D\phi_\Lambda]_{|p|<\Lambda} e^{-S_{E;\Lambda}[\phi_\Lambda]} \tag{1.35}$$

The effective action can be generally written as

$$\begin{aligned} &S_{E;\Lambda}[\phi_\Lambda] = \int d^D x \mathcal{L}_{E;\Lambda}(\phi_\Lambda, \partial\phi_\Lambda)\ , \\ &\mathcal{L}_{E;\Lambda}(\phi_\Lambda, \partial\phi_\Lambda) = \frac{Z_\Lambda(\Lambda)}{2} \partial^\mu\phi_\Lambda\partial_\mu\phi_\Lambda + \sum_{m,n} Z_\Lambda^m \Lambda^{D-[\mathcal{O}_{m,n}]} g_{\mathcal{O}_{n,m}}(\Lambda) \mathcal{O}_{n,m}(x)\ , \\ &\mathcal{O}_{n,m}(x) = \phi_\Lambda^{2n} (\partial_\mu\phi_\Lambda\partial^\mu\phi_\Lambda)^m\ , \quad [\mathcal{O}_{n,m}] = m(D-2) + nD\ . \end{aligned} \tag{1.36}$$

Z_Λ is the wavefunction renormalization. Beta-function is defined as

$$\beta_{\mathcal{O}} = \Lambda \frac{\partial g_{\mathcal{O}}(\Lambda)}{\partial\Lambda} = \frac{\partial g_{\mathcal{O}}(\Lambda)}{\partial \log\Lambda}\ . \tag{1.37}$$

The beta-function for the dimensionless couplings takes the form

$$\beta_{\mathcal{O}} = \beta_{\mathcal{O}}^{\text{class}}(g) + \beta_{\mathcal{O}}^{\text{quant}}(g) = ([\mathcal{O}] - D) g_{\mathcal{O}}(\Lambda) + \beta_{\mathcal{O}}^{\text{quant}}(g)\ . \tag{1.38}$$

The first classical term simply comes from dimension analysis of operators $\mathcal{O}_{n,m}$. The second term, on the other-hand, comes from interaction terms S_{int} and can be perturbatively computed using Feynmann diagram. The $\beta_{\mathcal{O}}^{\text{quant}}$ depends on all the coupling constants. According to their mass dimension, we classify local operators $\mathcal{O}_{n,m}$ into 3 categories

$$\mathcal{O} \text{ is called } \begin{cases} \text{relevant}\ ,\ [\mathcal{O}] < D \\ \text{irrelevant}\ ,\ [\mathcal{O}] > D \\ \text{marginal}\ ,\ [\mathcal{O}] = D \end{cases} \tag{1.39}$$

When the coupling constants g are small, the quantum corrections $\beta_{\mathcal{O}}^{\text{quant}}$ are negligible and thus

$$g_{\mathcal{O}} \text{ gets } \begin{cases} \text{stronger}\,, & \text{for relevant } \mathcal{O} \\ \text{weaker}\,, & \text{for irrelevant } \mathcal{O} \\ ??\,, & \text{for marginal } \mathcal{O} \end{cases} \tag{1.40}$$

as we decrease the energy scale Λ (IR limit). The scale transformation (dilatation) $x \to bx$ in the conformal symmetry acts on Λ as

$$\text{Scale transformation}: \ \Lambda \to \frac{1}{b}\Lambda\,. \tag{1.41}$$

So, the true meaning of scale invariance in quantum field theory is

$$\text{Quantum scale invariance}: \ \beta_{\mathcal{O}} = 0 \quad \forall \mathcal{O}\,. \tag{1.42}$$

Note that the classically conformally invariant theory ($g_{\mathcal{O}} = 0$ except for $\mathcal{O} = \mathcal{O}_{m=\frac{D}{D-2},n=0}$) satisfy the $\beta_{\mathcal{O}}^{\text{classical}} = 0$. But the quantum effect β^{quant} generically breaks the scale invariance. Free massless theories ($g_{\mathcal{O}} = 0$ for all $\mathcal{O}$) is a quantum scale invariant theory since there is no quantum effect.

From these quantum analysis, we may conclude that there is no CFT other than free massless theories. The hasty conclusion turns out to be wrong in two ways. First, for quantum field theories with higher enough supersymmety quantum corrections are milder and there are infintely many interacting CFTs with supersymmetry. One famous example is $D = 4$ maximally supersymmetric ($\mathcal{N} = 4$) Yang-Mills theory. Secondly, CFT could arise as an end point of RG running.

CFT at the end of RG On the space of couplings $\{g_{\mathcal{O}}\}$, RG flow (in decreasing Λ) can be thought as a transformation generated by following vector field

$$\text{RG vector field}: \ -\sum_{\mathcal{O}} \beta_{\mathcal{O}} \frac{\partial}{\partial g_{\mathcal{O}}}\,. \tag{1.43}$$

At the fixed point where $\beta_{\mathcal{O}} = 0$ for all $\mathcal{O}$, the theory stop running under the RG and becomes a scale-invariant theory. In QFT, the scale-invariance usually (almost all cases) leads to full conformal-invariance. So, we expect to have an interacting CFT as end of RG running if we tuned the intial coupling constants $\{g_{\mathcal{O}}(\Lambda_{UV})\}$ properly. One most well-studied examle is so-called Wilson-Fisher fixed point in $D = 3$. The starting UV theory is the 3D ϕ^4 theory whose Lagrangian is given by

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - g_2\phi^2 - g_4\phi^4\,. \tag{1.44}$$

The qualitative description of the RG vector field in the space of (g_2, g_4) is given in the figure 1. The qualitative feature of the RG vector field have been supported from various computations, such as perturbative expansion in small g_4, ϵ expansion in $D = 4 - \epsilon$ and $1/N$-expansion regarding the theory as scalar $O(N)$ model with $N = 1$. In the RG analysis, we ignore the infintely many coupling constants corresponding to irrelevant operators since they all finally vanish in the IR limit, $\Lambda \to 0$.

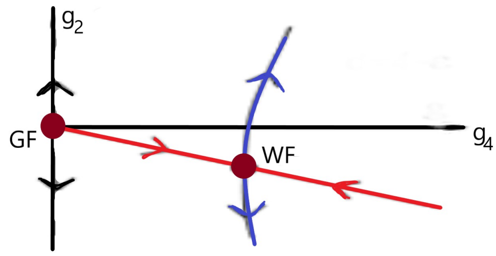

Figure 1. Schematic RG vector field for 3D ϕ^4-theory. There are two fixed point, one is Gaussian fixed point (GF, free theory) at $g_2, g_4 = 0$ and the other is Wilson-Fisher fixed point (WF, interacting).

QFT as RG between CFTs In the above RG analysis, there are two fixed points, Gaussian fixed point and Wilson-Fisher fixed point. For each fixed point, there is an associated CFT. Two CFTs are related by a RG flow (GF→WF) triggered by an deformation $\delta\mathcal{L} = -g_2\phi^2 - g_4\phi^4$ on the UV CFT (GF) with finely tuned coefficients $g_2 < 0$ and $g_4 > 0$.

$$\begin{aligned}&\text{GF} + (\text{deformed by } \delta\mathcal{L} = -g_2\phi^2 - g_4\phi^4 \text{ with fined tuned } g_2/g_4)\\&\xrightarrow{\quad\text{RG}\quad} \text{WF}\ .\end{aligned} \tag{1.45}$$

Other deformations lead to following IR phases

$$\begin{aligned}&\text{GF} + (\text{deformed by irrelevant operators } \delta\mathcal{L} = \sum_i g_i\mathcal{O}_i \text{ with } [\mathcal{O}_i] > 3)\\&\xrightarrow{\quad\text{RG}\quad} \text{GF}\ ,\\&\text{GF} + (\text{deformed by } \delta\mathcal{L} = -g_2\phi^2 - g_4\phi^4 \text{ with } |g_2| >> |g_4| \text{ and } g_2 > 0)\\&\xrightarrow{\quad\text{RG}\quad} \text{Mass gap with unbroken } \mathbb{Z}_2\ ,\\&\text{GF} + (\text{deformed by } \delta\mathcal{L} = -g_2\phi^2 - g_4\phi^4 \text{ with } |g_2| >> |g_4| \text{ and } g_2 < 0)\\&\xrightarrow{\quad\text{RG}\quad} \text{Mass gap with spontaneously broken } \mathbb{Z}_2\ .\end{aligned} \tag{1.46}$$

Physics near the phase transition between two mass gapped theories are described by the WF fixed point. Regarding IR fixed point of the mass gapped theory as a trivial CFT[2], RG always ends at a CFT. Generally we may consider QFT as a study on RG running betweem CFTs. There are in general two ways of triggering RG from a UV CFT.

$$\begin{aligned}&\text{1) Gauging of flavor symmetry of UV CFT}\\&\text{2) By adding relevant (spin 0) primary operators in UV CFT to Lagrangian}\end{aligned} \tag{1.47}$$

The gauging operation is only possible when the UV CFT has an non-trivial flavor symmetry. For odd spacetime dimension ($D = 3, 5$), we can add also CS interactions in addition

[2] CFT with no dynamical local degree of freedom, i.e. no local operator. But it could have non-local operator. We sometimes further distinguished trivally gapped phase (described by a CFT with a single ground state and no non-local oeprators) from topological phase (gapped CFT with degenerate vacua parametrized by VEV of non-local operators, described by a topological quantum field theory).

to the gauge kinetic term in the gauging procedure. In general CFT, a (scalar) primary operator $\mathcal{O}$ with a conformal dimension Δ is defined to be a local operator such that

$$\begin{aligned}
&\exp[b^\mu K_\mu]\cdot \mathcal{O}(x=0) = \mathcal{O}(x=0)\ , \quad \text{for all } b^\mu\ , \\
&\langle O(x)O(0)\rangle = \frac{c}{|x|^{2\Delta}}\ .
\end{aligned} \tag{1.48}$$

The conformal dimension Δ is a generalization of mass dimension $[\mathcal{O}]$ in free field theory and we define

$$\mathcal{O} \text{ is } \begin{cases} \text{relevant}\ ,\ \Delta(\mathcal{O}) < D \\ \text{irrelevant}\ ,\ \Delta(\mathcal{O}) > D \\ \text{marginal}\ ,\ \Delta(\mathcal{O}) = D \end{cases} \tag{1.49}$$

In conventional QFT textbook, the starting UV CFT is chosen as a free massless theories (free scalars or free fermions). But, we may choose an interacting CFT, such as WF, as starting UV CFT. These QFTs, studying RG between an interacting CFT and another CFT, are usually called non-Lagrangian QFTs. At WF point, there are two relevant operators, $\mathcal{O}_\phi$, $\mathcal{O}_{\phi^2}$. The $\mathcal{O}_\phi$ is odd under the $\mathbb{Z}_2$ symmetry in the WF theory and orginated from an local operator ϕ in the UV GF theory. Its conformal dimension is given by

$$\begin{aligned}
&\Delta(\mathcal{O}_\phi) = \frac{1}{2} + \gamma_\phi\ , \text{ where} \\
&\gamma_\phi = -\frac{1}{2}\Lambda \frac{\partial \log Z_\Lambda}{\partial \Lambda}| \quad @\ \text{WF}\ .
\end{aligned} \tag{1.50}$$

Here γ_ϕ is called anomalous dimension. The currently known most precise way of computing the anomalous dimension is so-called conformal bootstrap method (reviwed by Junchen's Lectures). The numerical value is

$$\Delta(\mathcal{O}_\phi) = 0.5182(3) \tag{1.51}$$

The $\mathbb{Z}_2$-even operator $\mathcal{O}_{\phi^2}$ comes from a mixing of operators ϕ^2 and ϕ^4 in the UV GF theory under the RG. From the mixing, we have two scalar primary operators $\mathcal{O}_{\phi^2}$ (with smaller Δ) and $\mathcal{O}_{\phi^4}$ (with bigger Δ) in the WF theory whose conformal dimensions are

$$\Delta(\mathcal{O}_{\phi^2}) = 1.413(1)\ , \quad \Delta(\mathcal{O}_{\phi^4}) = 3.84(4)\ , \tag{1.52}$$

Note that $\mathcal{O}_{\phi^2}$ is relevant while $\mathcal{O}_{\phi^4}$ is irrelevant. The deformation on WF CFT triggered by the two operators corresponds to blue-line (for $\mathcal{O}_{\phi^2}$) and red-line (for $\mathcal{O}_{\phi^4}$) in the figure 1. From the figure, we see that

$$\begin{aligned}
&\text{WF} + (\text{deformed by } \delta\mathcal{L} = \mathcal{O}_{\phi^2}) \\
&\xrightarrow{\text{RG}} \text{Mass gap}\ . \\
&\text{WF} + (\text{deformed by } \delta\mathcal{L} = \mathcal{O}_{\phi^4}) \\
&\xrightarrow{\text{RG}} \text{WF}\ .
\end{aligned} \tag{1.53}$$

1.2.2 Radial quantization

Note that $\mathbb{R} \times S^{D-1}$ is conformally flat since

$$\begin{aligned} ds^2(\mathbb{R} \times S^{D-1}) &= d\tau^2 + ds^2(S^{D-1}) = \frac{dr^2}{r^2} + ds^2(S^{D-1}) \\ &= \frac{1}{r^2}(dr^2 + r^2 ds^2(S^{D-1})) = \frac{1}{r^2} ds^2(\mathbb{R}^D) \ . \end{aligned} \tag{1.54}$$

and thus we can uniquely put the theory on the manifold by requiring invariance under the full conformal symmetry, $SO(1, D+1)$. Under the conformal mapping, the radial direction r on $\mathbb{R}^D$ is related to a time coordinate τ on $\mathbb{R} \times S^{D-1}$ by a relation $r = e^\tau$.

$$\mathcal{D} = r\frac{\partial}{\partial r} = \frac{\partial}{\partial \tau} \ . \tag{1.55}$$

Considering the τ-direction as time direction, we can construct Hilbert space associated to the constant time slice, $\tau = \tau_0$ (fixed). The quantization is called radial quantization since we use the S^{D-1} at fixed radius $r_0 = e^{\tau_0}$ as constant time-slice in the quantization. For CFT in arbitary dimension D, there is an isomorphism between

$$\begin{aligned} &(\text{Space of local operators } \{\mathcal{O}\} \text{ on } \mathbb{R}^D) \\ &\simeq (\text{Radially quantized Hilbert-space } \mathcal{H}(S^{D-1}) = \{|\mathcal{O}\rangle\} \text{ on } \mathbb{R} \times S^{D-1}) \ . \end{aligned} \tag{1.56}$$

Under the isomorphism,

$$(\text{Conformal dimension } \Delta \text{ of } \mathcal{O}) = (\text{Energy } E \text{ of the state } |\mathcal{O}\rangle) \ . \tag{1.57}$$

Example : free massless scalar theory The Lagrangian for free massless scalar theory on general conformally flat space-time is given in (1.27). Using the fact that the scalar curvature $R(S^{D-1})$ of unit round $(D-1)$-dimensional sphere is $(D-1)(D-2)$, the Lagrangian becomes

$$S_E[\phi] = \int d\tau d\Omega_{D-1} \left(\frac{1}{2}(\partial_\tau \phi)^2 - \frac{1}{2}\phi \nabla^2_{\Omega_{D-1}} \phi + \frac{1}{2}(\frac{D-2}{2})^2 \phi^2 \right) \ . \tag{1.58}$$

Here $d\Omega_{D-1}$ is the measure $\sqrt{\det(g_{ij})} \prod_{i=1}^{D-1} dx^i$ where $x^{i=1,\ldots,D-1}$ are coordinates on S^{D-1} and g_{ij} is the metric on S^{D-1}. In the above expression, we use

$$\begin{aligned} &\sqrt{\det(g)} g^{ij} \partial_i \phi \partial_j \phi = -\phi \partial_i (\sqrt{\det(g)} g^{ij} \partial_j \phi) + (\text{total divergence}) \\ &= -\phi \nabla^2_{\Omega_{D-1}} \phi \ . \end{aligned} \tag{1.59}$$

g is the metric tensor on S^{D-1} and $\nabla^2_{\Omega_{D-1}}$ is the Laplacian operator acting on scalar on S^{D-1}. Let us expand the scalar field in terms of harmonics on S^{D-1},

$$\begin{aligned} &\phi(\tau, \Omega_{D-1}) = \sum_{\ell,m} \phi_{\ell,m}(\tau) Y_{\ell,m}(\Omega_{D-1}) \ , \\ &\nabla^2_{\Omega_{D-1}} Y_{\ell,m} = -\ell(\ell + D - 2) Y_{\ell,m} \ . \end{aligned} \tag{1.60}$$

More explicitly, the spherical harmonics can be represented as

$$
\begin{aligned}
&Y_{\ell,m}(\Omega_{D-1}) = m^{a_1\dots a_\ell} y_{a_1}\dots y_{a_\ell} \quad (S^{D-1} \ : \ \sum_{a=1}^{D} y_a^2 = 1) \ \text{ where} \\
&m^{a_1\dots a_\ell} \ \text{ is (properly normalized) symmetric and traceless tensor} \ .
\end{aligned}
\tag{1.61}
$$

The system can be treated as 1D QM with infinitely many decoupled harmonic oscillators (HOs)

$$
S_E[\phi] = \sum_{\ell,m} \int d\tau \left(\frac{1}{2}(\partial_\tau \phi_{\ell,m})^2 + \frac{1}{2}\omega_{\ell,m}^2 \phi_{\ell,m}^2 \right) , \quad \text{with } \omega_{\ell,m}^2 = (\ell + \frac{D-2}{2})^2 \ . \tag{1.62}
$$

Quantizaing the infintely many HOs, the Hamiltonian is given by

$$
\begin{aligned}
\hat{H} &= \sum_{\ell,m} \omega_{\ell,m} (a^\dagger_{\ell,m} a_{\ell,m} + \frac{1}{2}) \ , \\
&= \sum_{\ell,m} \omega_{\ell,m} a^\dagger_{\ell,m} a_{\ell,m} + \epsilon_0 \ .
\end{aligned}
\tag{1.63}
$$

Here $a^\dagger_{\ell,m}$ and $a_{\ell,m}$ are the usual creating and annhilating operators respectively for each harmonic modes. General states in the radially quantized Hilbert spaces $\mathcal{H}(S^{D-1})$ are of the form

$$
\begin{aligned}
&\prod_{\ell,m} (a^\dagger_{\ell,m})^{\mathcal{N}_{\ell,m}} |0\rangle \quad (\mathcal{N}_{\ell,m} \geq 0) \ , \ \text{whose energy is} \\
&E = \sum_{\ell,m} \mathcal{N}_{\ell,m} (\ell + \frac{D-2}{2}) \ , \ \text{(we choose } \epsilon_0 = 0) \ .
\end{aligned}
\tag{1.64}
$$

Here the vacuum $|0\rangle$ is chosen such that

$$
a_{\ell,m}|0\rangle = 0 \ , \quad \text{for all } (\ell, m) \ . \tag{1.65}
$$

The state corresponds to following local operator on $\mathbb{R}^D$

$$
\begin{aligned}
&\prod_{\ell,m} (\partial^\ell_m \phi)^{\mathcal{N}_{\ell,m}} \ , \ \text{whose scaling dimension is} \\
&\Delta = \sum_{\ell,m} \mathcal{N}_{\ell,m} (\ell + \frac{D-2}{2}) \ .
\end{aligned}
\tag{1.66}
$$

Here

$$
\begin{aligned}
&\partial^\ell_m \phi = m^{a_1\dots a_\ell} \partial_{a_1\dots a_\ell} \phi \ , \ \text{where } a_i = 1, \dots, D \text{ and} \\
&m^{a_1\dots a_\ell} \ \text{ are symmetric and traceless tensor}
\end{aligned}
\tag{1.67}
$$

Problem 1.2.2-1: Why do we need to impose the traceless condition?

So we confirm the isomorphism (1.56) for a free massless real scalar theory in general space-time dimension D.

The above discussion can be easily generalized to a free massless complex scalar Φ case, $\Phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, by doubling the oscillatory modes

$$(a, a^\dagger) \to \{(a_1, a_1^\dagger), (a_2, a_2^\dagger)\} \, . \tag{1.68}$$

In the case, the thoery has $u(1)$ flavor symmetry whose charge $\mathcal{F}$ is

$$\mathcal{F}(\Phi) = 1 \, , \quad \mathcal{F}(\overline{\Phi}) = -1 \, . \tag{1.69}$$

So, it is better to introduce

$$\begin{aligned}
&a_\Phi = \frac{1}{\sqrt{2}}(a_1 - ia_2) \, , \quad a_{\overline{\Phi}} = \frac{1}{\sqrt{2}}(a_1 + ia_2) \, , \\
&a^\dagger_\Phi = \frac{1}{\sqrt{2}}(a^\dagger_1 + ia^\dagger_2) \, , \quad a^\dagger_{\overline{\Phi}} = \frac{1}{\sqrt{2}}(a^\dagger_1 - ia^\dagger_2) \, , \\
&\text{These complexified quanta have following } u(1)\text{-charges} \\
&\mathcal{F}(a^\dagger_\Phi, a_{\overline{\Phi}}) = 1 \, , \quad \mathcal{F}(a^\dagger_{\overline{\Phi}}, a_\Phi) = 1
\end{aligned} \tag{1.70}$$

Basis of the radially quantized Hilbert-space $\mathcal{H}(S^{D-1})$ for the free complex scalar field theory is

$$\begin{aligned}
&\sum_{\ell,m} (a^\dagger_{\Phi,\ell,m})^{\mathcal{N}_{\ell,m}} (a^\dagger_{\overline{\Phi},\ell,m})^{\mathcal{M}_{\ell,m}} |0\rangle \, , \\
&E = \sum_{\ell,m} (\mathcal{N}_{\ell,m} + \mathcal{M}_{\ell,m})(\ell + \frac{D-2}{2}) \, , \quad \mathcal{F} = \sum_{\ell,m} (\mathcal{N}_{\ell,m} - \mathcal{M}_{\ell,m}) \, .
\end{aligned} \tag{1.71}$$

Problem 1.2.2-2: Confirm that the partiton function

$$Z_{\text{free } \Phi}(q, u) := \text{Tr}_{\mathcal{H}(S^{D-1})} q^E u^{\mathcal{F}} \, , \tag{1.72}$$

is given by

$$Z_{\text{free } \Phi}(q, u) = \prod_{\ell,m} \frac{1}{(1 - q^{\ell + \frac{D-2}{2}} u)(1 - q^{\ell + \frac{D-2}{2}} u^{-1})} \, . \tag{1.73}$$

States in gauged HO For later use (will be used in the study of monopole operators in 3d Chern-Simons matter theories), let construct Hilbert-space for gauged HO system which is described by following 1D Euclidian action

$$S_E[A, \{\Phi_\alpha\}] = \int d\tau \sum_\alpha \left(|(\partial_\tau + A_\tau)\Phi_\alpha|^2 + \omega_\alpha^2 |\Phi_\alpha|^2 \right) + q_{\text{ext}} \int A \, . \tag{1.74}$$

Before the gauging, the Hilbert-space for two-dimensional HO is spanned by ($\epsilon_0 = \sum_\alpha \omega_\alpha$)

$$\prod_\alpha (a^\dagger_{\Phi_\alpha})^{\mathcal{N}_\alpha} (a^\dagger_{\bar{\Phi}_\alpha})^{\mathcal{M}_\alpha}|0\rangle \text{ with } E = \sum_\alpha \omega_\alpha(\mathcal{N}_\alpha + \mathcal{M}_\alpha) + \epsilon_0 \text{ and } \mathcal{F} = \sum_\alpha (\mathcal{N}_\alpha - \mathcal{M}_\alpha) \ . \tag{1.75}$$

and the ptn $\text{Tr} q^E u^{\mathcal{F}}$ is

$$Z^{\text{before gauging}} = q^{\epsilon_0} \prod_\alpha \frac{1}{(1-q^{\omega_\alpha}u)(1-q^{\omega_\alpha}u^{-1})} \ . \tag{1.76}$$

After gauging (and introducing external charge term $q_{\text{ext}} \int A$), we need to impose the following condition for gauge invariance

$$\sum_\alpha (\mathcal{N}_\alpha - \mathcal{M}_\alpha) + q_{\text{ext}} = 0 \ \ \text{(guage-invariance)} \ . \tag{1.77}$$

The term $q_{\text{ext}} \int A$ introduce an external charged particle of charge q_{ext} and the vacuum $|0\rangle$ has charge q_{ext}. The gauge charge shold be cancelled by exciting oscillatory modes. So the Hilbert-space of the gauged HO is spanned by

$$\prod_\alpha (a^\dagger_{\Phi_\alpha})^{\mathcal{N}_\alpha} (a^\dagger_{\bar{\Phi}_\alpha})^{\mathcal{M}_\alpha}|0\rangle \text{ with } \sum_\alpha (\mathcal{N}_\alpha - \mathcal{M}_\alpha) + q_{\text{ext}} = 0 \ . \tag{1.78}$$

So the ptn after gauging is

$$\begin{aligned} Z^{\text{after gauging}}(q) &= \oint_{|u|=1} \frac{du}{2\pi i u} u^{q_{\text{ext}}} Z^{\text{before gauging}}(q,u) \ , \\ &= q^{\epsilon_0} \oint_{|u|=1} \frac{du}{2\pi i u} u^{q_{\text{ext}}} \prod_\alpha \frac{1}{(1-q^{\omega_\alpha}u)(1-q^{\omega_\alpha}u^{-1})} \ . \end{aligned} \tag{1.79}$$

Problem 1.2.2-3: Derive the ptn from following Eucledian path-integral

$$\begin{aligned} &Z^{\text{after gauging}}(q) = \int \frac{[DA]\prod_\alpha [D\Phi_\alpha]}{\text{(gauge)}} e^{-S_E[A,\{\Phi_\alpha\}]} \ , \\ &\text{Periodic b.c. : } A_\tau(\tau+\beta) = A_\tau \ , \ \Phi_\alpha(\tau+\beta) = \Phi_\alpha(\tau) \end{aligned} \tag{1.80}$$

Here q is related to the radius ($\frac{\beta}{2\pi}$) of the thermal circle as follows

$$q = e^{-\beta} \tag{1.81}$$

Step I : Show that

$$\begin{aligned} &\int \prod_\alpha [D\Phi_\alpha] e^{-S^0_E[\{\Phi_\alpha\}]} = q^{\epsilon_0} \prod_\alpha \frac{1}{(1-q^{\omega_\alpha})^2} \ , \\ &\text{where } S^0_E[\{\Phi_\alpha\}] = \int d\tau \sum_\alpha |\partial_\tau \Phi_\alpha|^2 + \omega^2 |\Phi_\alpha|^2 \ , \\ &\text{with periodic b.c : } \Phi(\tau) = \Phi(\tau+\beta) \ . \end{aligned} \tag{1.82}$$

Step II : Show that ($|u|=1$)

$$
\begin{aligned}
&\int \prod_\alpha [D\Phi_\alpha] e^{-S_E^0[\{\Phi_\alpha\}]} = q^{\epsilon_0} \prod_\alpha \frac{1}{(1-q^{\omega_\alpha} u)(1-q^{\omega_\alpha} u^{-1})} \\
&\text{where } S_E^0[\{\Phi_\alpha\}] = \int d\tau \sum_\alpha |\partial_\tau \Phi_\alpha|^2 + \omega_\alpha^2 |\Phi_\alpha|^2 \ , \\
&\text{with twisted b.c : } \Phi(\tau) = u\Phi(\tau+\beta) \ .
\end{aligned} \tag{1.83}
$$

Step III : Show that (hint : redefine $\Phi \to e^{-\frac{\log u}{\beta}\tau}\Phi$ and related to the computation in Step II.)

$$
\begin{aligned}
&\int \prod_\alpha [D\Phi_\alpha] e^{-S_E^0[\{\Phi_\alpha\}, A=\frac{\log u}{\beta} d\tau]} = q^{\epsilon_0} \prod_\alpha \frac{1}{(1-q^{\omega_\alpha} u)(1-q^{\omega_\alpha} u^{-1})} \ , \\
&\text{where } S_E^0[\{\Phi_\alpha\}, A] = \int d\tau \sum_\alpha |(\partial_\tau + A_\tau)\Phi_\alpha|^2 + \omega^2 |\Phi_\alpha|^2 \ \text{ and} \\
&\text{with periodic b.c : } \Phi(\tau) = \Phi(\tau+\beta) \ .
\end{aligned} \tag{1.84}
$$

Step IV : Show that

$$
\begin{aligned}
&\int \frac{[DA]\prod_\alpha [D\Phi_\alpha]}{(\text{gauge})} e^{-S_E^0[\{\Phi_\alpha\},A] + q_{\text{ext}} \int A} \quad (\text{with periodic b.c : } A(\tau) = A(\tau+\beta)) \\
&= \oint_{|u|=1} \frac{u}{2\pi i u} u^{q_{\text{ext}}} \int \prod_\alpha [D\Phi_\alpha] e^{-S_E^0[\{\Phi_\alpha\}, A=\frac{\log u}{\beta} d\tau]}
\end{aligned} \tag{1.85}
$$

(hint : Using gauge transformation we can always make $A_\tau = \frac{\log u}{\beta}$ for $|u| = 1$. Then, residual gauge transformation on thermal circle S^1_β (with radius $\frac{\beta}{2\pi}$) make $\log u$ as $2\pi i$-periodic variable. See (2.8))

2 Lecture III, IV : 3d CFT from 3d gauge theory

2.1 Pure Chern-Simons interaction

Ref:
"Remarks on the canonical quantization of the Chern-Simons-Witten theory" by S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg

The general action (with Poincare invariance and gauge invariance) for pure (without matter) $u(1)$ gauge theory on Euclidean 3 dimensional manifold M is

$$
\begin{aligned}
&S_{\text{Maxwell-CS action}}[A] = \frac{1}{g^2} S^{\text{maxw}}[A] - ikCS[A] \ , \quad \text{where} \\
&S^{\text{maxw}}[A] := \int_M dA \wedge dA = \int_M \sqrt{g} F_{ab} F^{ab} \ , \\
&CS[A] := \frac{1}{4\pi} \int_M AdA = \frac{1}{4\pi} \int_M \epsilon^{abc} A_a \partial_b A_c
\end{aligned} \tag{2.1}
$$

The action is invariant under the following gauge transformation

$$\begin{aligned} &A \to A + d\Lambda \ , \text{ where} \\ &e^{\Lambda} \ : \ M \to u(1) \ . \end{aligned} \tag{2.2}$$

Problem 2.1-1: In the Maxwell-CS theory, show that the gauge field has mass proportional to $g^2 k$.

The Maxwell term is irrelevent $[F_{\mu\nu}F^{\mu\nu}] = 4$, i.e. $g^2 \to \infty$ as $\Lambda \to 0$, and the term drop out in the IR and we only have

$$S_{CS}[A] = -\frac{ik}{4\pi}\int_M AdA \ , \tag{2.3}$$

in the IR. When the gauge field is not coupled to any matter field, the theory is called pure CS theory and does not depend on metric of space-time. These metric-independent theories are called topological quantum field theory (TQFT). All physical observables of TQFT are topological invariants of the 3-manifold M. Every TQFTs have vanishing stress-energy tensor

$$T_{\mu\nu} = 0 \ . \tag{2.4}$$

The pure Chern-Simons theory is a trivial local CFT. The only local operator in the theory is identity operator. Note that $F_{\mu\nu}$ is gauge-invariant but e.o.m requires that

$$\frac{\delta S_{CS}[A]}{\delta A} = 0 \ \Rightarrow \ F_{\mu\nu} = 0 \ . \tag{2.5}$$

Hilbert-space on a torus $\mathbb{T}^2$ We put the $u(1)_k$ CS theory on $M = \mathbb{R}_t \times \mathbb{T}^2$

$$\mathbb{R}_t \times \mathbb{T}^2 = \{(t, \theta_1, \theta_2) \ : \ \theta_1 \sim \theta_1 + 1 \ , \ \theta_2 \sim \theta_2 + 1\} \ . \tag{2.6}$$

General solutions to the e.o.m, $F_{\mu\nu} = 0$, on M modulo gauge tansformation are

$$A = i\alpha_1 d\theta_1 + i\alpha_2 d\theta_2 \ , \tag{2.7}$$

where the variable α_i are periodic variables due to the following large gauge transformation

$$\text{Large gauge transformation } e^{\Lambda} = e^{2\pi i n_1 \theta_1 + 2\pi i n_2 \theta_2} \ : \ (\alpha_1, \alpha_2) \sim (\alpha_1 + 2\pi n_1, \alpha_2 + 2\pi n_2) \ . \tag{2.8}$$

Here n_1, n_2 are chosen to be integers for the gauge tranformation e^{Λ} is well-defined on $\mathbb{T}^2$. For zero-modes on $\mathbb{T}^2$ given in (2.7), the CS theory becomes a simple 1d QM described by following action

$$\pm\frac{k}{2\pi}\int dt\dot{\alpha}_1\alpha_2 \tag{2.9}$$

The sign depends on the choice of orientation of the $\mathbb{R}_t \times \mathbb{T}^2$. We choose an orientation such that the sign becomes $+1$. From the action, we see that α_1 and α_2 are canonically conjugate to each other.

$$P_{\alpha_1} = \frac{\delta S_{CS}}{\delta \dot{\alpha_1}} = \frac{k}{2\pi}\alpha_2 \ . \tag{2.10}$$

Quantizing the QM system, we get following quantum commutation relation

$$[\hat{X}, \hat{P}] = \frac{2\pi i}{k} \ . \tag{2.11}$$

We choose $X = \alpha_1$ and $P = \alpha_2$. To construct the Hilbert-space, we introduce a position basis $|X\rangle$ on which the quantum operators act as

$$\hat{X}|X\rangle = X|X\rangle \ , \quad e^{i\epsilon \hat{P}}|X\rangle = |X + \frac{2\pi\epsilon}{k}\rangle \ . \tag{2.12}$$

Since the (X, P) are 2π-periodic variables, we need to impose following conditions on the basis

$$\begin{aligned} &(e^{2\pi \frac{\partial}{\partial X}} - 1)|X\rangle = |X + 2\pi\rangle - |X\rangle = 0 \ , \\ &(e^{2\pi \frac{\partial}{\partial P}} - 1)|X\rangle = (e^{-ik\hat{X}} - 1)|X\rangle = (e^{-ikX} - 1)|X\rangle = 0 \ . \end{aligned} \tag{2.13}$$

By imposing the two conditions, we see that the resulting Hilbert-space is finite-dimensional

$$\begin{aligned} &\mathcal{H}(\mathbb{T}^2) = \text{span}\{|X = \frac{2\pi n}{k}\rangle \ : \ n = 0, \ldots, |k| - 1\} \ , \\ &\text{dim}\mathcal{H}(\mathbb{T}^2) = |k| \ . \end{aligned} \tag{2.14}$$

As a topological theory, all states in the Hilbert-space have zero-energy and thus these are vacua of the theory. For the consistency of the quantization, we need impose that

$$k \in \mathbb{Z} \ . \tag{2.15}$$

Unlike the usual quantum field theory, these vacua are characterized by the VEV of non-local operators called Wilson loops. The Wilson for abelian gauge theory is defined by

$$W_C[A] = \exp \oint_C A \tag{2.16}$$

The operator is supported on a closed curve C.

Problem 2.1-2: Show that the Wilson loop is invariant under gauge transformations.

On the vacua, the Wilson loop for $C = \{(t, \theta_1, \theta_2) = (0, s, 0)\}_{s=0}^{1}$ takes following VEV

$$\langle X = \frac{2\pi n}{k}|W_C|X = \frac{2\pi n}{k}\rangle = \exp\left(i\oint_C \frac{2\pi n}{k} d\theta_1\right) = \exp\left(i\frac{2\pi n}{k}\right) \ . \tag{2.17}$$

The analys can be extended to the case $M = \mathbb{R}_t \times \Sigma_g$ with a 2d Riemann of genus g and the result is

$$\text{dim}\mathcal{H}(\Sigma_g) = |k|^g \ . \tag{2.18}$$

2.2 Topological $u(1)$ symmetry

For $u(1)$ gauge theory (either with CS interaction or not) coupled to matter fields in 3d space-time, there is gauge-invariant conserved current, $J^\mu_{\rm top}$, made of field strength of the gauge field

$$J^\mu_{\rm top} := \frac{1}{2\pi}\epsilon^{\mu\nu\rho}F_{\nu\rho}\ , \quad \partial_\mu J^\mu_{\rm top} = 0\ . \tag{2.19}$$

For pure Chern-Simons theory case, the F vanishes by e.o.m and there is no conserved current. From the Noether theorem, the existence of the conserved current is equivalent to the existence of a $u(1)$ flavor symmetry whose conserved charged is the $j^\mu_{\rm top}$. We call the $u(1)$ symmetry associated to the dyanmical $u(1)$ gauge field is called 'topological $u(1)$ symmetry' which is usually denoted as $u(1)_{\rm top}$.

2.3 Witten's $SL(2,\mathbb{Z})$ action

Ref:
het-th/0307041 "SL(2,Z) Action on Three-Dimensional Conformal Field Theories With Abelian Symmetry" by E. Witten

Let $\mathcal{T}$ be a 3d CFT with a $u(1)$ flavor symmetry. We can deform the theory by gauging the $u(1)$ flavor symmetry with CS level k. Let denote the resulting 3d CFT be

$$(ST^k)\cdot\mathcal{T} := (\text{a 3d CFT obtained by gauging } u(1) \text{ with CS level } k)\ . \tag{2.20}$$

Under the gauging, the number of $u(1)$ symmetry is preserved since gauging $u(1)$ flavor symmetry introduce a topological $u(1)$ symmetry whose charge is given by field strength of the $u(1)$ gauge field. So, we can act the gauging operation successively and consider following theory

$$\ldots\cdot(ST^{k_3})\cdot(ST^{k_2})\cdot(ST^{k_1})\cdot\mathcal{T} \tag{2.21}$$

Witten proved that

$$(ST)\cdot(ST)\cdot(ST)\cdot\mathcal{T} = \mathcal{T}\ , \quad S\cdot S\cdot\mathcal{T} = \mathcal{C}\cdot\mathcal{T}\ , \quad S\cdot S\cdot S\cdot S\cdot\mathcal{T} = \mathcal{T}\ . \tag{2.22}$$

Here $\mathcal{C}$ is a charge conjugation operation, $\mathcal{C}^2 = 1$, and S is considered to be $ST^{k=0}$. So, the field-theoretic action form a $SL(2,\mathbb{Z})$ group.

Problem 2.3-1: Check the $SL(2,\mathbb{Z})$ by reading the Witten's paper (which is quite readable).

2.4 Monopole operator

Ref :
ArXiv:1710.00654, "Monopole Operators in U(1) Chern-Simons-Matter Theories" by S. Chester, L. lliesiu, M. Mezei and S. Pufu.

For $u(1)$ CS matter theory, there is an $u(1)$ topological symmetry associated to the dynamical $u(1)$ gauge field. One natural question one may ask is what local operators are charged under the $u(1)_{\rm top}$? Operators charged under the topological symmetry are called monopole operator. The local operator can not be written in a simple way using elementary fields in the theory. The (conceptually and computationally) best way to see these monopole operators is using the radial quantization introduced in (1.2.2). We will study the CS matter theory on $\mathbb{R} \times S^2$ with a deformation by a Maxwell-term parameterized by λ. The $\lambda = 0$ corresponds to the case, we are interested while we can solve (find spectrum of states in) the system in the $\lambda \to 0$ limit. After solving the $\lambda \to \infty$ limit first, then we will interpolate the spectrum of states to $\lambda = 0$. Under the continous interpolation, discrete quantum numbers (such as Lorentz spin (j, m) or charge Q under $u(1)$ topological symmetry) are invariant and we can say exactly on these quantum number.[3]

Monopole operators in $u(1)$ CS theory coupled to a complex scalar The Euclidean action on $\mathbb{R}^3$ for the CS matter theory is

$$S_E[\Phi, A] = \int_{\mathbb{R}^3} d^3x \sqrt{g} \Big(|(\partial + A)\Phi|^2 \Big) + \frac{k}{4\pi i} \int_{\mathbb{R}^3} AdA \ . \tag{2.23}$$

The theory is at least classically conformal. After radial quantization, the theory is mapped to on $\mathbb{R}_\tau \times S^2$,

$$S_E[\Phi, A] = \int_{\mathbb{R}\times S^2} d^3x \sqrt{g} \Big(|(\partial + A)\Phi|^2 + \frac{1}{4}|\Phi|^2 \Big) + \frac{k}{4\pi i} \int_{\mathbb{R}\times S^2} AdA \tag{2.24}$$

The ptn for the theory can be computed using path-integral

$$\begin{aligned} Z(q) &:= \text{Tr}_{\mathcal{H}(S^2)} q^\Delta \\ &= \int \frac{[DA][D\Phi]}{(\text{gauge})} e^{-S_E[\Phi,A]} \ \text{(with periodic b.c } \tau \sim \tau + \beta) \text{ where } q = e^{-\beta} \ . \end{aligned} \tag{2.25}$$

Now we put the theory on $S^1_\beta \times S^2$ by imposing periodic boundary condition. The path-integral is hard to perform for fixed k since there is no tunable parameter. To have a control over the path-integral, we introduce an additional parameter, say λ, as follows

$$\begin{aligned} Z(q, \lambda) &:= \text{Tr}_{\mathcal{H}_\lambda(S^2)} q^\Delta \\ &= \int \frac{[DA][D\Phi]}{(\text{gauge})} \Big|_{\tau \sim \tau+\beta} e^{-S_E[\Phi,A] - \lambda S_E^{\rm maxw}[A]} \ , \text{ where} \\ S_E^{\rm maxw}[A] &= \int F \wedge *F \ . \end{aligned} \tag{2.26}$$

The maxwell-term is gauge-invariant and poincare-invariant (isometry on $\mathbb{R}_\tau \times S^2$) but not conformally invariant. When $\lambda \to 0$, the deformed theory on $S^1_\beta \times S^2$ becomes the original radially quantized theory and recover the classical conformal invariance. The spectrum $\{\Delta, (j, m), Q\}$ of the Hilbert-space $\mathcal{H}_\lambda(S^2)$ depends on the parameter λ where Δ is energy

[3] Some states in $\lambda \to \infty$ could disapper when $\lambda \to 0$ if the energy of the state diverges in the limit. For simplicity, let us ignore the subtle possiblity.

and (j,m) is a Lorentz spin on S^2 and Q is the charge for the $u(1)_{\rm top}$ symmetry.[4] We will first perform the path-integral, or equivalently construct the Hilbert-space $\mathcal{H}_\lambda(S^2)$, in an asymptotic limit where $\lambda \to \infty$. Then, we will interpolate the spectrum to $\lambda \to 0$. In the interpolation, discrete quantum numbers ((j,m) and Q) are expected to be intact under the continuous deformation and we can read off the spectrum of these quantum numbers in the original theory ($\lambda = 0$).

To perform the path-integral in the asymptotic limit $\lambda \to \infty$, we first expand gauge field as follows

$$A = A_*^{(N)}(u) + \frac{1}{\sqrt{\lambda}}\delta A_N \ , \tag{2.27}$$

where the $A_*^{(N)}(u)$ is the classical configuration (modulo gauge transformation) extremizing the Maxwell action

$$\frac{\delta S^{\rm maxw}[A]}{\delta A}\bigg|_{A=A_*^{(N)}(u)} = 0 \ . \tag{2.28}$$

The general solutions for the classical e.o.m for the Maxwell theory on $S^1_\beta \times S^2$ are

$$A_*^{(N)}(u) \ : \ A_\tau = \frac{1}{\beta}\log u \ , \quad F_{\theta\phi} d\theta \wedge d\phi = \frac{N}{2}\sin\theta d\theta\wedge d\phi \ . \tag{2.29}$$

The charge Q of $u(1)_{\rm top}$ is (see (2.19))

$$Q = \int_{\tau=\tau_0} d\Omega_2 J^\tau_{\rm top} = \frac{1}{2\pi}\int_{S^2} F = N \ . \tag{2.30}$$

The classical solutions are parametrized by

$$\begin{aligned} &N \in \mathbb{Z} \ , \ \text{(Dirac quantization)} \\ &\log u \sim \log u + 2\pi i \ , \ |u| = 1 \ , \ \text{(Large gauge transformation, (2.8))} \end{aligned} \tag{2.31}$$

Problem 2.4-1: Suppose that the scalar field has $u(1)$ guage charge $q_{\rm gauge}$. Then, how should the Dirac quantization condition be modified?

Then, the path-integral simplify as follows in the limit $\lambda \to \infty$

$$\begin{aligned} &Z(q = e^{-\beta}, \lambda \to \infty) \\ &= \sum_{\text{classical solutions}} \oint \frac{\prod_N [D\delta A^{(N)}][D\Phi]}{\text{(gauge)}} e^{-S_E[\Phi, A = A_*^{(N)}(u) + \frac{1}{\sqrt{\lambda}}\delta A_N] - \lambda S_E^{\rm maxw}[A = A_*^{(N)}(u) + \frac{1}{\sqrt{\lambda}}\delta A_N]} \\ &= \sum_N \oint_{|u|=1} \frac{du}{2\pi i u} e^{-\lambda S_E^{\rm maxw}[A_*^{(N)}(u)] + ikCS[A_*^{(N)}(u)]} \int \frac{\prod_N [D\delta A^{(N)}][D\Phi]}{\text{(residual gauge)}} e^{-S_0[\{\delta A^{(N)}\},\Phi] + o(\frac{1}{\sqrt{\lambda}})} \\ &\text{where } S_0[\{\delta A^{(N)}\},\Phi] := |(\partial + A_*^{(N)}(u))\Phi|^2 + \frac{1}{4}|\Phi|^2 + |\partial(\delta A^{(N)})|^2 \ . \end{aligned} \tag{2.32}$$

[4]Note that the deformation does not break any symmetry associated to these quantum numbers. Otherwise, we can consider these quantum numbers in non-zero λ

The actions for the classical solutions are

$$e^{-\lambda S_E^{\rm maxw}[A_*^{(N)}(u)]+ikCS[A_*^{(N)}(u)]} = e^{-2\pi^2 N^2 \lambda} u^{kN} \ . \tag{2.33}$$

So, the path-integral becomes

$$\begin{aligned} &Z(q=e^{-\beta},\lambda\to\infty) \\ &= \sum_N e^{-4\pi^2 N^2\lambda} \oint_{|u|=1} \frac{du}{2\pi i u} u^{kN} \int [D\Phi] e^{-S_0[\Phi]+o(\frac{1}{\sqrt{\lambda}})} \\ &\text{where } S_0[\Phi] := \int d\tau d\Omega_2 \big(|(\partial_\tau + \frac{1}{\beta}\log u)\Phi|^2 - \Phi^* (\nabla^2_{N,\Omega_2} - \frac{1}{4})\Phi \big) \ . \end{aligned} \tag{2.34}$$

Here we ignore an overall factor (independent on u and N) comming from integrtaion of the $\delta A^{(N)}$. ∇^2_{N,Ω_2} is the Laplacian on S^2 in monopole background with monopole charge N ($F = \frac{N}{2}\sin\theta d\theta d\phi$). By expanding the Φ by monopole harmonics on S^2,

$$\begin{aligned} &\Phi(\tau,\Omega_2) = \sum_{\ell=\frac{|n|}{2},m} \Phi_{\ell,m}(\tau) Y_{N,\ell,m}(\Omega_2) \ , \\ &\nabla^2_{n,\Omega_2} Y_{N,\ell,m} = -\ell(\ell+1) Y_{N,\ell,m} \ , \quad \ell = \frac{|N|}{2}, \frac{|N|}{2}+1, \dots, m = -\ell, -\ell+1, \dots, \ell \ . \end{aligned} \tag{2.35}$$

the ptn can be computed as

$$\begin{aligned} &Z(q=e^{-\beta},\lambda\to\infty) = \sum_N e^{-2\pi^2 N^2\lambda} \oint_{|u|=1} \frac{du}{2\pi i u} u^{kN} \int \prod_{\ell,m} [D\Phi_{N,\ell,m}] e^{-\sum_{\ell,m} S_0[\{\Phi_{N,\ell,m}\}]} \ , \\ &S_0[\{\Phi_{N,\ell,m}\}] := \int d\tau |(\partial_\tau + \frac{1}{\beta}\log u)\Phi_{N,\ell,m}|^2 + \omega^2_{N,\ell,m} |\Phi_{N,\ell,m}|^2 \ . \end{aligned} \tag{2.36}$$

Note that for fixed N, the system is equivalent to the gauged QM system studied in Problem 1-3.4 with $\alpha \to (N,\ell,m)$ and $q_{\rm ext} = kN$. From (1.78), our Hilbert-space $\mathcal{H}_{\lambda\to\infty}(S^2)$ is given by

$$\begin{aligned} &\mathcal{H}_{\lambda\to\infty}(S^2) = \oplus_{N=-\infty}^{\infty} \mathcal{H}_N(S^2) \ , \quad \text{where} \\ &\mathcal{H}_N(S^2) = \text{span}\Big\{ \prod_{\ell,m} (a^\dagger_{\Phi_{N,\ell,m}})^{\mathcal{N}_{N,\ell,m}} (a^\dagger_{\overline{\Phi}_{N,\ell,m}})^{\mathcal{M}_{n,\ell,m}} |N\rangle \ : \ \mathcal{F} = \sum_{\ell,m} (\mathcal{N}_{N,\ell,m} - \mathcal{M}_{N,\ell,m}) + kN = 0 \Big\} \ . \end{aligned} \tag{2.37}$$

$|N\rangle$ is the vacuum in the topological sector with monopole charge N,

$$a_{\Phi_{N,\ell,m}} |N\rangle = a_{\overline{\Phi}_{N,\ell,m}} |N\rangle = 0 \ , \quad \text{for all } \ell = \frac{|N|}{2} + \mathbb{Z}_{\geq 0} \text{ and } m \in \mathbb{Z} \ (|m| \leq \ell) \ . \tag{2.38}$$

The spectrum of monopole operators are summarized in the Table below.

	$u(1)$ gauge charge $\mathcal{F}$	Topologica $u(1)$ charge Q	Lorentz-spin (j,m)
$a^\dagger_{\Phi_{N,\ell\geq \frac{\lvert N\rvert}{2},m}}$	$+1$	0	(ℓ,m)
$a^\dagger_{\overline{\Phi}_{N,\ell\geq \frac{\lvert N\rvert}{2},m}}$	-1	0	(ℓ,m)
$\lvert N\rangle$	kN	N	$(0,0)$

Table 1. $\mathcal{H}_\lambda(S^2)$. We computed the spectrum in $\lambda\to\infty$ but it is expected to work even for $\lambda=0$ case. $|N\rangle$ (bare monopole state) corresponds to the semi-classical monopole configuration in (2.29). The configuration is symmetric on S^2 and the state does not carry Lorentz spin. Due to a CS action, the bare monopole state has non-zero gauge charge $\mathcal{F}=kN$. Only gauge-invariant operators have physical meaning. To obtain gauge-invariant operator we need to excite the oscillatory modes $(a^\dagger_\Phi, a^\dagger_{\overline{\Phi}})$ around the monpole background. The modes always carry non-zero Lorentz spin when $N\neq 0$.

2.5 3d IR dualities

Ref :
arXiv:1606.01989, "A Duality Web in 2+1 Dimensions and Condensed Matter Physics" by S. Seiberg, T. Senthil, C. Wang and E. Witten

IR duality : Two different UV theories that flow to the same IR fixed point.

Example : The following gauge theory

$$\begin{aligned}\mathcal{T}^{UV}\ :\ & u(1)\text{ gauge field coupled to a complex scalar (of gauge charge }+1) \\ & \text{with CS level }+1\text{ and quartic potential }|\Phi|^4\ .\end{aligned} \tag{2.39}$$

is claimed to flow to a free massless complex Dirac fermion theory in IR

$$\mathcal{T}^{IR}\ :\ S=\int d^3x\overline{\Psi}(i\gamma^\mu\partial_\mu)\Psi\ . \tag{2.40}$$

As an zeroth order check of duality, we can see that both theories have same flavor symmetry $u(1)$. For the UV theory, the flavor symmetry is realized as topological $u(1)$ symmetry while it appears as conventional flavor symmetry (rotating elementary field) in the IR theory. As an next, check we can compare some lowest gauge-invariant operators. The non-identity lowest (with smallest Δ) operators in the free-fermion theory are

$$\Psi\text{ and }\overline{\Psi}\text{ with }(\Delta,j,Q)=(1,1/2,1)\text{ and }(1,1/2,-1). \tag{2.41}$$

Here Q is the charge under the $u(1)$ flavor symmetry. On the other-hand, the lowest operators in the UV CS matter-theory is

$$\begin{aligned}& a_{\overline{\Phi}_{N=1,\ell=\frac12}}|N=1\rangle\text{ with }(\Delta,j,Q)=(?,1/2,1)\ ,\quad\text{and}\\ & a_{\Phi_{N=-1,\ell=\frac12}}|N=-1\rangle\text{ with }(\Delta,j,Q)=(?,1/2,-1)\end{aligned} \tag{2.42}$$

Since the conformal dimensioni Δ is a continuous quantum number and it is very difficult to compute it from the UV description. Except the undetermined the conformal diemsnion, the lowest operator spectrum nicely matches in both sides of duality. The duality predicts that the gauge-invariant operator should have conformal dimension 1.

Problem 2.5-1: Check other exmples of IR dualities proposed in arXiv:1606.01989