VOAs from 4d $\mathcal{N}=2$ SCFTs

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1. Big picture / Intro
2. Superconformal algebra $su(2,2|2)$ & shortening conds
3. Chiral algebra from SCFT (VOA)
4. ~~Unitarity bounds~~ SCFT / VOA corresp.
5. Unitary bounds &
6. Higgs branch & Macdonald index
7. To do list.

1. Intro / Summary

- There is NO SOLVABLE (interacting) CFT in $d>2$.
- In $d=2$, conformal algebra $\sim (\text{Virasoro})_c \times (\text{Virasoro})_c$ = ∞-dim'l.
  - $\exists$ thys w/ finite # of Vir. primaries,
  - Crossing symmetry $= $ fixes the OPE coeff. $\Rightarrow$ fully solvable, for certain values of $\boxed{c}$ $\to$ minimal model

- ~~In 4d, d>2, # of primaries are necessarily ∞~~.

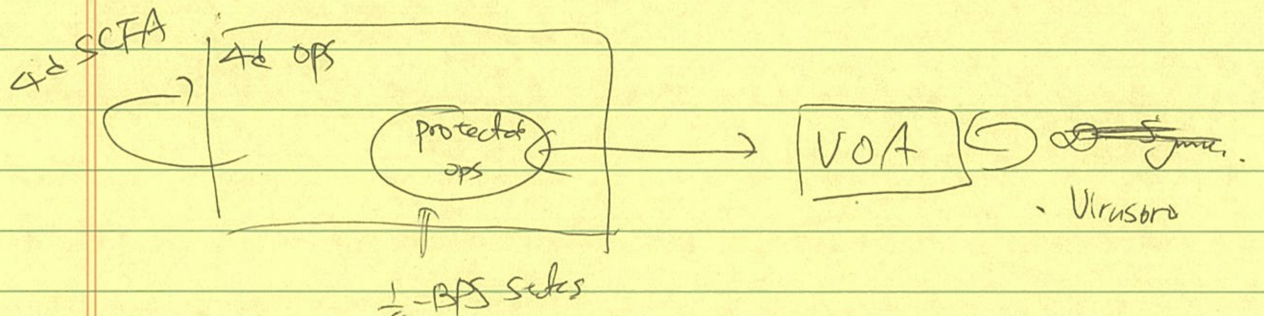
- Can we realize ∞'-dim symmetry in d>2?

$\Rightarrow$ YES! w/ SUSY

$$\begin{cases} d=4 & \mathcal{N}=2 \\ d=6 & \mathcal{N}=(2,0) \end{cases}$$

★ $\{$4d $\mathcal{N}=2$ SCFTs$\}$ $\longrightarrow$ $\{$VOAs$\}$

"chiral algebra" = symmetry alg. acting on 2d CFT.



- $\chi_{2d}(q) = I_{\text{Schur}}(q)$!

$\boxed{\text{Argyres-Douglas thys}}$ $\longleftrightarrow$ $\boxed{\text{BPZ (non-unitary) MMs}}$

- $c_{2d} = -12\, c_{4d}$, $\quad k_{2d} = -\frac{1}{2} k_{4d}$

$G_F \rightsquigarrow g_F$.

✓ $\mathcal{M}_{Higgs} \longleftrightarrow X_V$: associated var.

✓ $\langle \mathcal{O}_1(z_1,\bar z_1) \ldots \mathcal{O}_n(z_n,\bar z_n) \rangle = f(z_1 \ldots z_n)$! holom.

2. $SU(2,2|2)$ & Shortening cond. [D-O '02] [CDI '16] etc. 3

$SU(2,2|2) \supset SO(2,4) \times SU(2)_R \times U(1)_r$.

All states/ops labelled by (Δ, j_1, j_2, R, r)

(*) Recall: Conformal primary: $K_\mu |O_\Delta\rangle = 0$.

[Minwalla '97]

– descendants: $P_{\mu_1} \cdots P_{\mu_r} |O\rangle$

$\leftrightarrow \partial_{\mu_1} \cdots \partial_{\mu_r} O$

$K \uparrow \downarrow P$

	D or E
$\|O_\Delta\rangle$	Δ
$P_\mu \|O_\Delta\rangle$	$\Delta+1$
$P_\mu P_\nu \|O_\Delta\rangle$	$\Delta+2$
$\vdots$	

$(P_\mu^\dagger = K_\mu)$

$[P, K] = D + M$

Unitary bound:

$\Delta \geq 1$ for a scalar.

$\Delta \geq \frac{3}{2}$ " spin-$\frac{1}{2}$. $(\frac{1}{2},0)$ or $(0,\frac{1}{2})$

$\Delta \geq 2$ for $(1,0)$ or $(0,1)$

$\Delta \geq 3$ for a vector $(\frac{1}{2},\frac{1}{2})$

$\vdots$

$$\boxed{\Delta \geq f(j_1) + f(j_2)}$$

$f(j) = 0$ for $j = 0$

$f(j) = j+1$ for $j > 0$.

for Scalar $\underline{\underline{\Delta \geq \frac{d-2}{2}}}$

$(j_1 = j_2 = 0)$

$\Rightarrow$

Shortening cond: – $\partial^2 \phi = 0$ $\Rightarrow$ free field.

$\hookrightarrow$ "null state" at level 2.

$\underline{\partial \psi = 0}$ – null at level 1

$\partial_\mu j^\mu = 0$ – conserved current.

* Superconf'l rep.

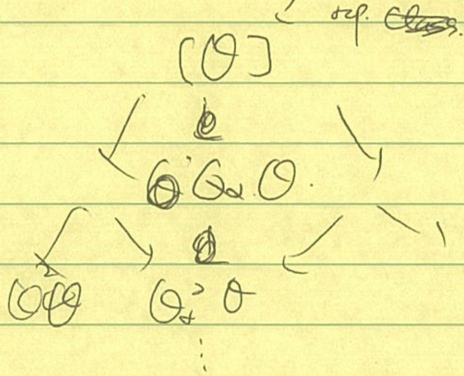
~~S~~ superconf. primary: $S^{\alpha}|O\rangle = 0$

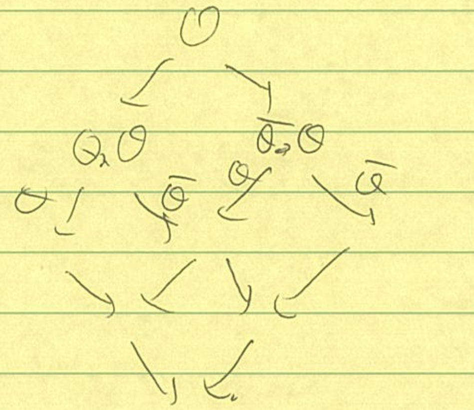
$[D, S^{\alpha}] = -\frac{1}{2} S^{\alpha}$

$[D, Q_{\alpha}] = \frac{1}{2} Q_{\alpha}$.

conf. ~~descendants~~ rep. ~~class~~

$Q^2 \sim P$, $S^2 \sim K$





For $N=2$, 8 Q's.

$Q^{I=1,2}_{\alpha=\pm}$ $\tilde{Q}_{J\dot{\alpha}=\pm}$ ($J=1,2$)

$$\boxed{\{Q^I_{\alpha}, \tilde{Q}_{J\dot{\alpha}}\} = \delta^I_J P_{\alpha\dot{\alpha}}}$$

& 8 S's = superconf'al chg.

$$\boxed{\{\tilde{S}^{J\dot{\alpha}}, S_J^{\alpha}\} = \delta^I_J K^{\dot{\alpha}\alpha}}$$

$$\{Q^I_{\alpha}, S_J^{\beta}\} = \frac{1}{2}\delta^I_J \delta_{\alpha}^{\beta} D + \delta^I_J M_{\alpha}^{\beta} - \delta_{\alpha}^{\beta} R^I_J.$$

$\left(R^1_2 = R^+, \quad R^2_1 = R^-, \quad R^1_1 = \frac{1}{2}r + R, \quad R^2_2 = \frac{1}{2}r - R \right)$

* Q: raising operator.

$\Rightarrow$ In gen'l. for a long mult.

$A^{\Delta}_{R,r(j_1,j_2)}$: ~~2^8 256 states~~

Unitarity $\Rightarrow$ $\begin{cases} \Delta \geq 2 + 2j_1 + 2R + r \\ \Delta \geq 2 + 2j_2 + 2R - r \end{cases}$ $r + j_1 - j^{\dots}$

For $j_1 = 0$, $\Delta = 2R + r$ or $\Delta \geq 2 + 2R + r$.

$j_2 = 0$ $\quad \Delta = 2R - r$ $\quad$ "

Shortening conds for Q's.

DO	CDI			
	L	$[j_1, j_2]_\Delta^{(R, r)}$		$\Delta > 2 + 2j_1 + 2R - r$.
C	A_1	$[j_1, j_2]_\Delta^{(R, r)}$	$j_1 \geq 0$	$\Delta = 2 + 2j_1 + 2R - r$.
	A_2	$[0, j_2]$		$\Delta = 2 + 2R - r$.
B	B_1	$[0, j_2]$		$\Delta = 2R - r$.

also for the $\bar{Q}$'s

$\begin{cases} B^I : \quad Q^I_\alpha |\psi\rangle = 0 \\ C^I = \begin{cases} \varepsilon^{\alpha\beta} Q^I_\alpha |\psi\rangle_\beta = 0 & A^1 \ (j_1 \neq 0) \\ \varepsilon^{\alpha\beta} Q^I_\alpha Q^I_\beta |\psi\rangle = 0 & A^2 \ (j_1 = 0) \end{cases} \end{cases}$

Some short multiplets $\quad$ (Log $\quad A^\Delta_{R, r(j_1, j_2)}$)

$B^1 \cap \bar{B}_2$	$\hat{B}_R$: $\Delta = 2R$, $j_1 = j_2 = r = 0$	$\Rightarrow$ "Higgs"
$B^1 \cap \bar{C}_2$	$D_{R(0, j_2)}$: $\Delta = 1 + 2R + j_2$, $r = j_2 + 1$	Hall-Littlewood
$\bar{B}_2 \cap C_1$	$\bar{D}_{R(j_1, 0)}$: $\Delta = 1 + 2R + j_1$, $-r = j_1 + 1$	Higgs
$C^1 \cap \bar{C}_2$	$\hat{C}_{R(j_1, j_2)}$: $\Delta = 2 + 2R + j_1 + j_2$, $r = j_2 - j_1$	Schur
$\bar{B}^1 \cap B^2$	$\mathcal{E}_{r(0, j_2)}$: $\Delta = r$, $R = 0$	$\rightarrow$ Coulomb

* Chiral algebra $\supset$ {Schur ops} $\supsetneq$ {SHL ops} $\supset$ {Higgs}

Coulomb?

* ~~Superconformal index~~ e counts short multiplets up to recombination

e.g.) $A^{\cancel{\Delta = 2+2R+r}}$

$$\Delta = 2 + 2j_1 + 2R + r$$
$$= 2 + 2j_2 + 2R - r$$
$$\Rightarrow r = j_2 - j_1$$

e.g.)

$$A^{2+2R+j_1+j_2}_{R, j_2-j_1, (j_1, j_2)} \qquad \Delta = 2 + j_1 + j_2 + 2R.$$
$$= \hat{C}_{R(j_1,j_2)} \oplus \hat{C}_{R+\frac{1}{2}(j_1-\frac{1}{2}, j_2)} \oplus \hat{C}_{R+\frac{1}{2}(j_1, j_2-\frac{1}{2})} \oplus \hat{C}_{R+1(j_1-\frac{1}{2}, j_2-\frac{1}{2})}$$

e.g.) If some of the quantum numbers above become unphysical.

e.g.) $j_1 = j_2 = 0$

$$A^{2+2R}_{R,0(0,0)} = \hat{C}_{R(0,0)} \oplus \mathcal{D}_{R+1(0,0)} \oplus \bar{\mathcal{D}}_{R+1(0,0)} \oplus \hat{B}_{R+2}$$
$$\mathcal{D}_{R+1(0,0)} = \text{"}\hat{C}_{R+\frac{1}{2}(-\frac{1}{2},0)}\text{"}, \quad \bar{\mathcal{D}}_{R+1(0,0)} = \text{"}\hat{C}_{R+\frac{1}{2}(0,-\frac{1}{2})}\text{"}, \quad \hat{B}_{R+2} = \text{"}\hat{C}_{R+1(-\frac{1}{2},-\frac{1}{2})}\text{"}$$

* $\mathcal{E}, \bar{\mathcal{E}}, \hat{B}_{\frac{1}{2}}, \hat{B}_1, \hat{B}_{\frac{3}{2}}, \mathcal{D}_0, \bar{\mathcal{D}}_0, \mathcal{D}_{\frac{1}{2}}, \bar{\mathcal{D}}_{\frac{1}{2}}$ never recombine

+ Superconformal index ⇒ Counts short mult. up to recomb.

$$\boxed{I_{full}} = \mathrm{Tr}\,(-1)^F\, p^{\frac{1}{2}(E+2j_1-2R-r)}\, q^{\frac{1}{2}(E-2j_1-2R-r)}\, t^{R+r}$$

$$I_{full} \xrightarrow{p\to 0} \boxed{I_M(q,t)}$$

$$I_{full} \xrightarrow{u=\frac{p}{t},\ p,q,t\to 0} \boxed{I_C} \longrightarrow \text{Counts } \mathcal{E},\ D_0$$

$$I_M(q,t) \xrightarrow{t=q} \boxed{I_S(q)} = \mathrm{Tr}(-1)^F q^{E-R}$$

$$I_M(q,t) \xrightarrow{q\to 0} I_{HL}(t) \to \text{Counts } \hat{B}_R,\ D_R$$

Counts $\hat{B}_R$, $\hat{C}_R$, D, $\bar{D}$

⑧

3. Chiral algebra from 4d $\mathcal{N}=2$

Review 2d CFT $\supset$ Vir $\otimes$ Vir

$$\begin{array}{cc} \cup & \cup \\ sl(2) & sl(2) \\ \shortparallel & \\ \{L_{-1}, L_0, L_1\} & \end{array}$$

meromorphicity of local op $\mathcal{O}(z,\bar{z})$ implies chiral sym.

$$\partial_{\bar{z}} \mathcal{O}(z,\bar{z}) = 0 \rightsquigarrow \mathcal{O}(z,\bar{z}) = \mathcal{O}(z)$$

$$\mathcal{O}(z) = \sum_{n\in\mathbb{Z}} \frac{\mathcal{O}_n}{z^{h+n}} \Rightarrow \mathcal{O}_n = \oint \frac{dz}{2\pi i} z^{h+n-1} \mathcal{O}(z).$$

OPE of mero-op $\Rightarrow$ contains mero. ops only

eg) Stress tensor $T(z)$

$$\partial^\mu T_{\mu\nu} = T^\mu{}_\mu = 0$$

$$\Rightarrow \partial_{\bar{z}} T_{zz}(z,\bar{z}) = 0 \quad \Rightarrow \quad T_{zz}(z,\bar{z}) \equiv T(z).$$

$$\left\{ \begin{array}{l} T(z)T(w) \sim \dfrac{c/2}{(z-w)^4} + \dfrac{2T(w)}{(z-w)^2} + \dfrac{\partial T(w)}{z-w} \\[2ex] T(z) = \displaystyle\sum_{n\in\mathbb{Z}} \frac{L_n}{z^{n+2}} \Rightarrow [L_n, L_m] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n} \end{array} \right.$$

eg) Currents $J^A(z)$

$$\left\{ \begin{array}{l} J^A(z) J^B(w) \sim \dfrac{k\delta^{AB}}{(z-w)^2} + \displaystyle\sum_C \frac{i f^{ABC} J^C(w)}{z-w} \\[2ex] J^A(z) = \displaystyle\sum_n \frac{J^A_n}{z^{n+1}}, \quad [J^A_n, J^B_m] = i f^{ABC} J^C_{n+m} + mk\delta^{AB}\delta_{m+n,0} \end{array} \right.$$

The algebra of all meromorphic ops

$\Rightarrow$ chiral algebra of a 2d CFT!

(VOA) vertex operator algebra,

"meromorphic CFT"

meromorphy is the key!

* We hope to find $sl(2) \times \overline{sl(2)} \subset SU(2,2|2)$?

but make it "holomorphic".

How? $\to$ need to remove $\overline{sl(2)}$ - dependence somehow

$\to$ pass to cohomology!

If $\exists Q$ s.t. $\{Q, \cdot\} = \overline{sl(2)}$?

$H^*_Q = \{Q\text{-closed}\}/\{Q\text{-exact}\}$?

$\overline{sl(2)}$ - dependence will be removed in H^*_Q.

$\to$ ~~Unfortunately,~~

(f) $\boxed{\text{"chiral ring"}}$ $\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = P_{\alpha\dot\alpha}$

$\Rightarrow$ ~~$[P_{\alpha\dot\alpha}, \mathcal{O}(x)] = \{Q_\alpha, \mathcal{O}'(x)\}$~~

$\{Q_\alpha, \mathcal{O}(x)] = 0 \quad \alpha = 1.$

$\Rightarrow$ ~~such operators ex~~ derivative of $\mathcal{O}(x)$ as exac

Q-cohomology $\to$ $[\mathcal{O}_i(x)] := \mathcal{O}_i$

$\langle \mathcal{O}_1(x_1) \cdots \mathcal{O}_n(x_n) \rangle = \langle [\mathcal{O}_1(x_1)] \cdots [\mathcal{O}_n(x_n)] \rangle$

$= \langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \rangle.$

10

Holomorphy from Cohomology. $H^*_{\mathbb{Q}}$.

$$sl(2) \times \widehat{sl(2)} \subset sl(4|2)$$

$D_{2h-2+r/2} = D_{3h-2}$, D_4, D_7, $(A_2, A_3) \subset I_{3,\omega}^{(\ldots)}$, $I_{2,6}$, (A_2, A_4)

Wanted:

- $\mathbb{Q}^2 = 0$
- $sl(2)$, $\widehat{sl(2)}$ acts as hol/anti-hol gen. on $\mathbb{C} \subset \mathbb{R}^4$.
- $[\mathbb{Q}, sl(2)] = 0$
- $\widehat{sl(2)} = \{\mathbb{Q}, *\}$.

Then $H^*_{\mathbb{Q}} = \{\mathbb{Q}\text{-closed}\} / \{\mathbb{Q}\text{-exact}\}$ will be holomorphic.

$sl(2) \times sl(2|2) \subset sl(4|2)$

$$\begin{cases} x_1 = x_2 = 0 \\ z = x_3 + i x_4 \\ \bar{z} = x_3 - i x_4 \end{cases}$$

$L_{-1} = P_{+\dot{+}}$
$L_{+1} = K^{\dot{+}+}$
$2L_0 = H + M$.

~~diagonal~~

Choose $\mathbb{Q} = Q^1_- + \tilde{S}^{2\dot{-}}$

$$\Rightarrow \{\mathbb{Q}, \tilde{Q}_{1\dot{-}}\} = \bar{L}_{-1} + R^- \equiv \hat{L}_{-1}$$

$$\{\mathbb{Q}, S_2^-\} = \bar{L}_{+1} - R^+ = \hat{L}_{+1}$$

$$\{\mathbb{Q}, \mathbb{Q}^+\} = 2(\bar{L}_0 - R) \equiv \hat{L}_0$$

Note that $\widehat{sl(2)}$ is NOT a subalgebra of conformal alg.

* Cohomology classes of local ops

@ origin

$$\{\mathbb{Q}, \mathcal{O}(0)\} = 0. \qquad \mathcal{O}(0) \neq \{\mathbb{Q}_i, *\}$$

$(E \neq 0)$

$$\Rightarrow \begin{cases} E - (j_1 + j_2) - 2R = 0 \\ r + j_1 - j_2 = 0 \end{cases}$$

$\Rightarrow$ Schur operators!

@ away from origin: translation gen. by $sl(2) \times \widehat{sl(2)}$

$$\mathcal{O}(z, \bar{z}) = e^{zL_{-1} + \bar{z}\hat{L}_{-1}} \underset{\substack{\uparrow \\ \text{Schur op.}}}{\mathcal{O}(0)} e^{-zL_{-1} - \bar{z}\hat{L}_{-1}},$$

Cohomology class $[\mathcal{O}(z, \bar{z})]_{\mathbb{Q}} \Longrightarrow \mathcal{O}(z)$!

For a Schur op in the spin-k rep of $sl(2)_R$.

$$\mathcal{O}^{I_1 \cdots I_{2k}}. \qquad I_i = 1, 2 \rightarrow \text{Schur op: } \mathcal{O}^{11\cdots 1}(0)$$

$$\rightarrow \mathcal{O}(z, \bar{z}) = u_{I_1}(\bar{z}) \cdots u_{I_{2k}}(\bar{z})\, \mathcal{O}^{I_1 \cdots I_{2k}}(z, \bar{z})$$

w/ $u_I(\bar{z})_i = (1, \bar{z})$.

<u>example</u> Free hypermultiplet $\Rightarrow$ $\begin{pmatrix} Q, \tilde{Q} \\ \lambda_\alpha \tilde{\lambda}_\alpha \end{pmatrix}$

$Q(z, \bar{z})$

Schur ops $\Rightarrow Q, \tilde{Q}$ "

$SU(2)_R$ doublet : $(Q, \tilde{Q}^\dagger)$

$$q(z) := [\, Q(z, \bar{z}) + \bar{z}\, \tilde{Q}^*(z, \bar{z}) \,]_{\square}$$

$$\tilde{q}(z) := [\, \tilde{Q}(z, \bar{z}) - \bar{z}\, Q^*(z, \bar{z}) \,]_{\square}$$

$$q(z)\tilde{q}(\omega) \simeq Q(z, \bar{z})\tilde{Q}(\omega, \bar{\omega}) + \bar{z}\, \tilde{Q}^*(z, \bar{z})\tilde{Q}(\omega, \bar{\omega})$$

$$- \bar{\omega}\, Q(z, \bar{z}) Q^*(\omega, \bar{\omega}) + \bar{z}\bar{\omega}\, \tilde{Q}^*(z, \bar{z}) Q^*(\omega, \bar{\omega})$$

$$\sim \bar{z} \cdot \frac{1}{|z-\omega|^2} - \bar{\omega} \cdot \frac{1}{|z-\omega|^2} = \frac{1}{z-\omega}.$$

$$q(z)\, q(\omega) \sim 0 \qquad \tilde{q}(z)\tilde{q}(\omega) \sim 0$$

$$\tilde{q}(z)\, q(\omega) \sim -\frac{1}{z-\omega}$$

"

$\Rightarrow$ <u>meromorphic!</u>

<u>OPE</u>:
$$O_1(z, \bar{z})\, O_2(0) = \sum_k \lambda_{12k} \frac{\bar{z}^{R_1+R_2-R_k}}{z^{h_1+h_2-h_k}\, \bar{z}^{\bar{h}_1+\bar{h}_2-\bar{h}_k}} O_k$$

$$h = \frac{E+j_1+j_2}{2} \qquad \bar{h} = \frac{E-j_1-j_2}{2} = R \text{ for Schur}$$

$$= \sum_{k\, \text{Schur}} \lambda_{12k} \frac{O_k(0)}{z^{h_1+h_2-h_k}} + \{\mathbb{Q}, \cdots]$$

4. SCFT / VOA corresp.

VOA → algebra of meromorphic ops
+ state / op correspondence

- $\mathcal{V}$ ⊃ states
- $a \longmapsto a(z)_{op} = Y(a,z) = \sum_n \frac{a_n}{z^{h+n}}$

state/op map.

$a_n \in \mathrm{End}(\mathcal{V})$

- Normal-ordered product $NO(a,b)|0\rangle := a_{-h_a} b_{-h_b}|0\rangle$

- χ : 4d $N=2$ SCFT ⟶ VOA
(2d chiral algebra)

$c_{2d} = -12\, c_{4d}$ SCFA → Vir

$k_{2d} = -1/2\, k_{4d}$ flavor sym G → $\widehat{\mathfrak{g}}_{k_{2d}}$

for $\chi_{\mathcal{V}} = \mathrm{Tr}_{\mathcal{V}}(-1)^F q^{L_0} = I_{\mathrm{Schur}}(q) = \mathrm{Tr}_{\mathcal{H}}(-1)^F q^{E-R}$

$L_0 = \frac{1}{2}(E + j_1 + j_2)$

$= \frac{1}{2}(E + E - 2R) = E - R.$

. Some Schur ops.

$\hat{\mathcal{C}}_{0(0,0)}$: stress-tensor multiplet. $\underline{\underline{J_{+\dot{+}}^{11}}}$ is Schur.
⇒ includes R-currents

$\hat{\mathcal{C}}_{0(j_1,j_2)}$: high-spin current. → absent for interacting thy.

- $\hat{B}_{\frac{1}{2}}$: free hyper
- $\hat{B}_1$: flavor-current. ⇒ moment map op M^{IJ}.

scalar, $SU(2)_R$ triplet.

$U(1)_r$ neutral,

M^{11} is the Schur op. | adj of G_F.

= absolutely protected.

- $\mathcal{D}_{0(0,0)} \oplus \bar{\mathcal{D}}_{0(0,0)}$ = free $\mathcal{N}=2$ vector
- $\mathcal{D}_{\frac{1}{2}(0,0)} \oplus \bar{\mathcal{D}}_{\frac{1}{2}(0,0)}$ = extra SUSY current.

eg) $\mathcal{N}=4$ has two ~~of them~~ copies.

⊕ Virasoro

Schur op in $\hat{C}_{0(0,0)}$ ⇒ $J^{11}_{+\dot{+}}$ of $SU(2)_R$ current.

$$J_R(z,\bar{z}) = u_I(\bar{z})\, u_J(\bar{z})\, J^{IJ}_{+\dot{+}}(z,\bar{z})$$

$$T(z) = \kappa\,[J_R(z,\bar{z})]_{\mathbb{Q}}$$

$$\underline{T(z)\,T(0) \sim ?} \quad \frac{c_{2d}}{2z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}$$

$$J_R(z,\bar{z})\, J_R(0,0) \sim -\frac{3c_{4d}}{2\pi^4 z^4} - \frac{1}{\pi^2}\frac{J_R(0,0)}{}$$

$$\left(-\frac{1}{\pi^2}\,\bar{z}\,\frac{u_I u_J J^{IJ}_{-\dot{-}}(0)}{z^3} + \frac{2}{\pi^2}\bar{z}\cdots\right)$$

↓ $\mathbb{Q}$-exact.

⇒ $\kappa = -2\pi^2$

⇒ $\underline{c_{2d} = -12\, c_{4d}}$

∘ Flavor sym ⇝ affine symm.

$$\underline{\underline{M}}^{11} \rightsquigarrow M(z,\bar z) := u_I(\bar z)\, u_J(\bar z)\, M^{IJ}(z,\bar z)$$

$$J(z) = \kappa\,[M(z,\bar z)]_{\bar Q}.$$

$$M^{11}(z,\bar z) + \bar z M^{12}(z,\bar z) + \bar z M^{21}(z,\bar z) + \bar z^2 M^{22}(z,\bar z).$$

$$J^A_\mu(x)\, J^B_\nu(0) \sim \frac{3k_{4d}}{4\pi^4}\delta^{AB}\frac{x^2 g_{\mu\nu} - 2x_\mu x_\nu}{x^8} + \frac{2}{\pi^2}\frac{x_\mu x_\nu f^{ABC}\, x\cdot J^C(0)}{x^6} + \cdots$$

$$M^{A\,IJ}(x)\, M^{B\,KL}(0) \sim -\frac{3k_{4d}}{\cancel{3}\,48\pi^4}\frac{\varepsilon^{K(I}\varepsilon^{J)L}\delta^{AB}}{x^4} + \frac{\sqrt{2}}{4\pi^2}\frac{f M}{x^2}$$

$$M^A(z,\bar z)\, M^B(0,0) \sim -\frac{3k_{4d}\,\varepsilon^{12}\varepsilon^{12}\,\bar z^{\cancel{2}}}{48\pi^4\,16\, z^2 \bar z^{\cancel{2}}} + \frac{\sqrt{2}}{4\pi^2}\, i\, f^{ABC}\frac{M^C(0,0)}{z\cancel{\bar z}}\cdot\cancel{\bar z} + \cdots$$

$$\to \kappa = 2\sqrt{2}\pi^2$$

$$\Rightarrow J^A(z)\, J^B(w) \sim \frac{k_{2d}\,\delta^{AB}}{(z-w)^2} + i f^{ABC}\frac{J^C(w)}{z-w}.$$

$$\boxed{k_{2d} = -\frac{k_{4d}}{2}}$$

- HL generators are strong generators of VOA.
  (HL operators can never arise as a N.O of other ops)

- HL chiral alg ops are always Virasoro prim.

Ex: Free hypers

$$Q^I = \begin{pmatrix} Q \\ \tilde{Q}^* \end{pmatrix} \qquad \tilde{Q}^I = \begin{pmatrix} \tilde{Q} \\ -Q^* \end{pmatrix}$$

$$Q_{\hat{I}}(z,\bar{z}) = u_I(\bar{z})\, Q^I_{\hat{I}}(z,\bar{z})$$

$$q_{\hat{I}}(z) = [\, Q_I(z,\bar{z})\,]_{Q}$$

$$\Rightarrow \quad q_{\hat{I}}(z)\, q_{\hat{J}}(w) \sim \frac{\epsilon_{\hat{I}\hat{J}}}{z-w}$$

2) symplectic boson or β-γ ghost w/ $\lambda = 1/2$.

$$T(z) = \frac{1}{2}\epsilon^{\hat{I}\hat{J}}\, q_{\hat{I}}\, \overleftrightarrow{\partial}\, q_{\hat{J}}(z)$$

$SU(2)_R$ curr in 4d $\quad J_\mu^{IJ}(x) \sim \epsilon^{\hat{I}\hat{J}}\, Q^I_{\hat{I}}\, \partial_\mu Q^J_{\hat{J}}(x)$

$$T \cdot T \sim \frac{-1}{2z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}$$

$$c_{2d} = -1 \qquad (c_{4d} = 1/12)$$

Ex: Free vector

$$\bar{D}_{0}(\tfrac{1}{2},0) \quad \& \quad D_0(\tfrac{1}{2},0)$$

$$\lambda^1_+ \qquad\qquad \tilde{\lambda}^1_{\dot +}$$

$$\lambda(z,\bar z) := u_I(\bar z)\,\lambda_+^I(z,\bar z)$$

$$\tilde\lambda(z,\bar z) := u_I(\bar z)\,\tilde\lambda^I_{\dot +}(z,\bar z)$$

$$\lambda(z) := [\lambda(z,\bar z)]_{\bar Q}\ ,\quad \tilde\lambda(z) := [\tilde\lambda(z,\bar z)]_{\bar Q}.$$

$\cancel{\phi}$

$$\lambda(z)\cdot\tilde\lambda(w) = \left(\lambda^1_+(z,\bar z) + \bar z\,\lambda_+^2(z,\bar z)\right) \times \left(\tilde\lambda^1_{\dot +}(w,\bar w) + \bar w\,\tilde\lambda^2_{\dot +}(w,\bar w)\right) \sim \frac{1}{z^2}$$

$$\psi_\alpha \bar\psi_{\dot\alpha} \sim \frac{(x_\mu - y_\mu)\,\sigma^\mu_{\alpha\dot\alpha}}{|x-y|^4}$$

$$\sigma^\mu = (\sigma, \sigma) = (\sigma, iI)$$

$$\begin{pmatrix} x_3 + ix_4 & x_1 + ix_2 \\ x_1 - ix_2 & -x_3 + ix_4 \end{pmatrix}$$

$$\frac{-\bar z\cdot(\bar z - w) + \bar w\,(\cancel{z} - \cancel{w})}{|z-w|^4} = -\frac{1}{(z-w)^2}.$$

$$(z-w)^2(\bar z - \bar w)^2$$

$$\tilde\lambda := b(z) \qquad \lambda(z) := \partial c(z)$$

$\Rightarrow$ (b,c) ghost w/ wt $(1,0)$.

$$J^{IJ}_{\alpha\dot\alpha} \sim \lambda^{(I}_\alpha \tilde\lambda^{J)}_{\dot\alpha}(x)$$

$$C_{2d} = -2 \qquad C_{4d} = \frac{1}{6}$$

5. Conseq. for 4d physics (Unitary bounds)

⇒ Unitary bounds from VOA.

Note that VOA itself is non-unitary $c_{2d} = -12\, c_{4d}$, $k_{2d} = -\frac{1}{2} k_{4d}$.

$$f(z_i) = \langle \mathcal{O}_1^{I_1}(z_1, \bar{z}_1)\, \mathcal{O}_2^{I_2}(z_2, \bar{z}_2)\, \mathcal{O}_3^{I_3}(z_3, \bar{z}_3)\, \mathcal{O}_4^{I_4}(z_4, \bar{z}_4)$$

by construction, this is meromorphic.

1) $f(z_i)$ can be expanded in terms of $sl(2)$ conformal blocks

$$f(z_i) = \left(\frac{z_{24}}{z_{14}}\right)^{h_{12}} \left(\frac{z_{14}}{z_{13}}\right)^{h_{34}} \frac{1}{z_{12}^{h_1+h_2}\, z_{34}^{h_3+h_4}} \sum_{\ell=0}^{\infty} (-1)^{\ell} a_\ell\, g_\ell(z),$$

$$g_\ell(z) = \left(-\tfrac{1}{2} z\right)^{\ell-1} {}_2F_1(\ell, \ell; 2\ell; z).$$

$$\left(\begin{array}{ll} z_{ij} = z_i - z_j & z = \dfrac{z_{12} z_{34}}{z_{13} z_{24}} \\ h_{ij} = h_i - h_j & \end{array} \right)$$

2) $f(z_i)$ can be expanded in terms of $su(2,2|2)$ conformal blocks & 3-pt coeffs.

(1) = (2) ⇒ OPE coeffs appear in (2) can be determined.

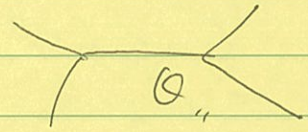
g) $J^A_{(z)} = \cancel{\ } \left[M^A(z,\bar z) \right]_Q$

$$\Rightarrow z_{12}^2 z_{34}^2 \langle J^A(z_1) J^B(z_2) J^C(z_3) J^D(z_4) \rangle$$

$$= f^{ABCD}(z) = \underbrace{\sum_{R \in \mathrm{Sym}^2(\mathrm{adj})}} P_R^{ABCD} f_R(z)$$

projector to R_n

In 4d

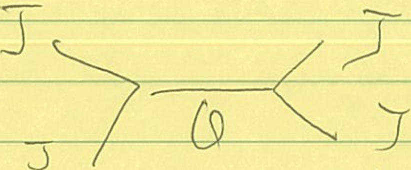


g) $f^{ABCD}(z) = \delta^{AB}\delta^{CD} + z^2 \delta^{AC}\delta^{BD} + \frac{z^2}{(1-z)^2}\delta^{AD}\delta^{BC} - \frac{z}{k_{2d}} f^{ACE} f^{BDE}$

$$- \frac{z}{k_{2d}(z-1)} f^{ADE} f^{BE}$$

$$f_{R_1} = \dim G_F + z^2\left(1 + \frac{1}{(1-z)^2}\right) + \frac{4 z^2 h^\vee}{k_{2d}(z-1)}$$

$$= \quad " \quad + \sum_{\ell = 0,1,\ldots} \frac{2^\ell (\ell+1) (\ell!)^2 \left(2(\ell+1)(\ell+2) k_{2d} - \ldots \right)}{k_{2d} (2\ell+1)!} \, g_{\ell+\ldots}$$



$\mathcal{O} \in A_\Delta(j,j)$, $\hat{C}_0(j,j)$, $\hat{C}_1(j,j)$, $\hat{B}_1$, $\hat{B}_2$, I

$\cancel{\sum} \ldots f_\mathcal{O}$ $\quad 0 \quad \lambda^2_{\hat{C}_0(\frac{1}{2},\frac{1}{2})} g_{\ell+2}$ $\quad \underline{-2\lambda^2_{\hat{C}_1(\frac{1}{2},\frac{1}{2})} g_{\ell+3}}$

$\hat{B}_1 : \lambda^2_{\hat{B}_1} g_1$

$\hat{B}_2 : (-2)\lambda^2_{\hat{B}_2} g_2$

$I_d : \lambda^2_{I_d} = \dim G_F$

Some are fixed : $\lambda^2_{\hat{C}_0(0,0)} = \frac{\dim G_F}{3 C_{4d}}$, $\lambda^2_{\hat{B}_1} = \frac{4h^\vee}{k_{2d}}$

(*) (2d) = (4d) in $\underline{R=1 \text{ channel}}$

$\Rightarrow$ $\lambda_{Id}^2 = \dim G_F$

$\lambda^2_{\hat{C}_{0(r,0)}} \to \lambda^2_{\hat{B}_2} = \frac{\#h^\vee}{k_{4d}} - 4$

$\lambda^2_{\hat{C}_{1(\frac{\ell}{2},\frac{\ell}{2})}} = \frac{2^{\ell+1}(\ell+2)((\ell+2)!)^2}{k_{2d}(2\ell+3)!}\left((\ell+2)(\ell+3)k_{4d} - 4h^\vee\right)$

$\uparrow$ ℓ odd

$\lambda^2_{\hat{C}_{0(0,0)}} = \frac{\dim G_F}{3c_{4d}}$

$\Rightarrow$ $\boxed{\frac{\dim G_F}{\cancel{3}c_{4d}} \geq \frac{24h^\vee}{k_{4d}} - 12}$ (1)

$\ell = 1$, $\lambda^2 \geq 0$ $\Rightarrow$ $\boxed{k_{4d} \geq \frac{1}{3}h^\vee}$

(*) ~~R~~ Other channel of R.

For $SU(N)$ $N \geq 3$ $\underline{k_{4d} \geq N}$ for $R = \underline{\text{adj}}$

$\underline{\lambda_{\hat{B}_2}^2 \geq 0}$ (2)

$\to$ Saturated: $(M \otimes M)_R = 0$

"Joseph ideal"

. Bound (2) saturated $\Rightarrow$

$c_{4d} = \dim G_F \cancel{\frac{24h^\vee}{k_{4d}}}$

$c_{4d} = \frac{k_{4d}\dim G_F}{24h^\vee - 12k_{4d}} = \frac{-2k_{2d}\dim G_F}{24(k_{2d}+h^\vee)}$

$\Rightarrow$ $c_{2d} = \frac{k_{2d}|G_F|}{k_{2d}+h^\vee}$ Sugawara central c

$$\boxed{T = \frac{1}{k_{2d}+h^\vee}(J^a J^a)}$$

$\Rightarrow$ Both cond. saturated for

"Deligne-exceptional series thys"

$$\underbrace{H_0,\ \overset{A_1}{H_1},\ \overset{A_2}{H_2}}_{\text{(Argyres-Douglas)}},\ D_4,\ E_6,\ E_7,\ E_8$$

$\wedge$ G_2 $\wedge$ F_4 ?

D3-brane probing F-thy singularities w/ constant dilaton

$\Rightarrow$ Higgs branch $= \overline{\mathcal{O}_{min}(G)}$

minimal nilp. orbit of G

$\Rightarrow$ 1-inst. mod. space of G

Chiral alg. $\Rightarrow$ $\begin{cases} MM(2,5) = Vir_{c=-\frac{22}{5}} \\ \widehat{su(2)}_{-4/3},\ \widehat{su(3)}_{-3/2} \\ \widehat{so(8)}_{-2},\ (E_6)_{-3},\ (E_7)_{-4},\ (E_8)_{-6} \end{cases}$

3. ~~Higgs branch & Macdonald index~~

Schur ~~ops~~ $\cancel{\hat{B}_R}$ $\hat{B}_1$

$\langle TTJJ \rangle$ & $\langle TTTT \rangle$

gives more bounds. g) $\boxed{c_{4d} \geq \frac{11}{30}}$

saturated for the H_0 thy!

The Minimal $\mathcal{N}=2$ SCFT!

6. Higgs branch / Macdonald index

Recall: Schur ops $\ni$ $\hat{B}_R$ $\to$ Higgs. } Hall-Littlewood.
$D_{R(0,j_2)}$
$\bar{D}_{R(j_1,0)}$
$\hat{C}_{R(j_1,j_2)}$

Higgs branch chiral rig $= \{\hat{B}_R\}$ $R_H = \mathbb{C}[\mathcal{M}_H]$
$\leftarrow$ 'commutative'

$\mathcal{M}_H = \mathrm{Spec}\, R_H$

$\hat{B}_R$: $h = R$, $j_1 = j_2 = 0$, $r = 0$

$D_{R(0,j_2)}$ $h = R + j_2 + 1$ $r = j_2 + \frac{1}{2}$
$= R + r + \frac{1}{2}$.

$\bar{D}_{R(j_1,0)}$ $h = R + j_1 + 1$, $r = -j_1 - \frac{1}{2}$
$= R - r + \frac{1}{2}$.

$\hat{C}_{R(j_1,j_2)}$ $h = R + j_1 + j_2 + 2$. $r = j_2 - j_1$

$$\mathcal{V} = \bigoplus_{h,R,r} \mathcal{V}_{h,R,r}.$$

- N.O. product preserves h, r but NOT R.
- $\exists$ filtration of R

· Li-filtration

Any VOA has a filtration.

$$V = F^0 V \supset F^1 V \supset F^2 V \supset \cdots$$

$$F^p V = \{ a_{-i-h_a} b \mid a \in V,\ b \in F^{p-i},\ i \geq 1 \}$$

$$gr_F V = \bigoplus_{i \geq 0} \left(F^i V / F^{i+1} V \right) = \bigoplus_{i \geq 0} \bigoplus_{h} X^{(i)}_h.$$

Zhu's C_2-algebra: $\boxed{R_V = F^0 V / F^1 V.}$ ⇒ generated by the elems w/o ∂

associated variety. $\boxed{X_V = \mathrm{Spec}(R_V)}$

($\mathrm{MaxSpec}(R_V)$)

or $\mathrm{Spec}((R_V)_{red})$ remove nilrads

Conjecture: $\boxed{X_V = \mathcal{M}_{Higgs}}$

(R_V = HL ring.)

(= Higgs branch chiral ring somehow)

Conjecture ⊗

$$\boxed{\begin{aligned} \chi_V^{(ref)} &\equiv \sum_{i,h} \dim X^{(i)}_h\, \xi^h t^{h-i} \\ &= I_{Mac}(\xi, t) \end{aligned}}$$ [JS]

⊗ Use Feigin's "PBW-filtration"

(· $PV^0 \subset PV^1 \subset PV^2 \subset \cdots$

$PV^r = \{ \underbrace{X_{-i-h_1} \cdots}_{s} |\Omega\rangle \mid s \leq r \}$)

⇝ $gr_P V = \bigoplus_{h,i} \bigoplus V^{(h)}_i.$

Example (A_1, A_{2n}) AD thy $\rightsquigarrow$ $Vir_{c=MM(2,2n+3)}$

$$c = -\frac{22}{5},\ -\frac{68}{7},\ -\frac{46}{3},\ \cdots$$

$$c_{4d} = \frac{11}{30}\ \text{: universal bd.}$$

· $(A_1, A_2) = H_0$ thy

$(L_{-4} + \alpha (L_{-2})^2)|0\rangle$ is null.

$$R_V = \mathrm{Span}\{(L_{-2})^k\} \Big/ \{(L_{-2})^2\} = \mathbb{C}[t]/\langle t^2 \rangle$$

$\therefore$ $M_H = \mathrm{Specm}\, R_V = \{0\}$ pt.

no Higgs branch!

· ~~Macdonald~~

(A_1, D_{2n+1}) thy. $\to$ $\widehat{su(2)}_{-\frac{4n}{2n+1}}$

$$R_V = \mathrm{Span}\{J^+_{-1}, J^0_{-1}, J^-_{-1}\} \Big/ \ldots$$

$j^+ \quad j^0 \quad j^-$

$$\begin{cases} L_{-2} \sim J^+_{-1} J^-_{-1} + (J^0_{-1})^2 & (L_{-2})^{n+1} \sim 0. \\ L_{-3} \end{cases}$$

$N_A = j^A (j^+ j^-)^n$ is null.

$$\therefore\ (R_V)_{red} = \mathbb{C}[j^+, j^-, j^0] \Big/ \langle j^+ j^- + (j^0)^2 \rangle$$

$$M_H \quad \underbrace{\mathrm{Spec}(R_V)_{red}}_{\overset{\shortparallel}{M_H}} = \{x, y, z \mid xy + z^2 = 0\} = \mathbb{C}^2/\mathbb{Z}_2.$$

7. Future direction

- $\{$4d $\mathcal{N}=2$ SCFTs$\}$ $\longleftrightarrow$ $\{$VOAs$\}$

  NOT 1 to 1. ..

  Classify $\mathcal{N}=2$ SCFTs ?

- How much does VOA determine the 4d CFT?
- Macdonald grading. (Can be made intrinsic to the VOA ?)
- Other SUSY ptn fns? Lens index seems possible
- $\forall$ VOAs from $\mathcal{N}=2$ SCFT. $\exists$ null vector in the Verma module

  $\Rightarrow$ LMDE on $I_{\text{Schur}} = \chi_{\text{vac}}$.

  $\Rightarrow$ Other modules?

  Surface defect ?

- Defects. (surface, domain wall !
- Other dim? $\mathcal{N}=(2,0)$ in 6d

  $\mathcal{N}=(0,4)$ in 2d !! (NOT done !)

  $\mathcal{N}=4$ in 3d. $\rightsquigarrow$ Top. QM. analog

  $\mathcal{V}=H^*_{\mathbb{Q}}$.

- Other VOA / QFT correspondence ...

  4mfd $\longleftrightarrow$ VOA.

  3d $\mathcal{N}=4$ QFT , .. intersecting domain walls ..