Lecture 1.

Why did Hawking investigate Euclidean analytic continuation?

Bardeen - Carter - Hawking, 1973

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Phi\, dQ + \Omega\, dJ$$

M : mass  Q : charge  J : angular momentum

A : event horizon area

$\kappa$ : surface gravity

$\Phi$ : electrostatic potential

$\Omega$ : angular frequency

" 1st law of black hole thermodynamics "

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example )

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_2^2$$

$$f(r) = 1 - \frac{2M}{r}$$

$$r_+ = 2M$$

$$A = 4\pi r_+^2 = 16\pi M^2$$

$$\kappa = \frac{1}{2}\left|\frac{df}{dr}\right|_{r=r_+} = \frac{1}{4M}$$

$$\delta A = 32\pi M\,\delta M = \frac{8\pi}{\kappa}\,\delta M$$

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Hawking, 1971 $\delta A \geq 0$

(assuming energy conditions)

$\rightarrow$ Bekenstein, 1974 $\delta A \propto \delta S$

2nd law of thermodynamics ?

---

Hawking, 1975

Conclusion : $\langle n_\omega \rangle \propto \dfrac{1}{e^{2\pi\omega/\kappa} - 1}$

Hence, $T = \dfrac{\kappa}{2\pi}$

Hence, $S = \dfrac{A}{4}$

Let's revisit logical steps of Hawking, 1975.

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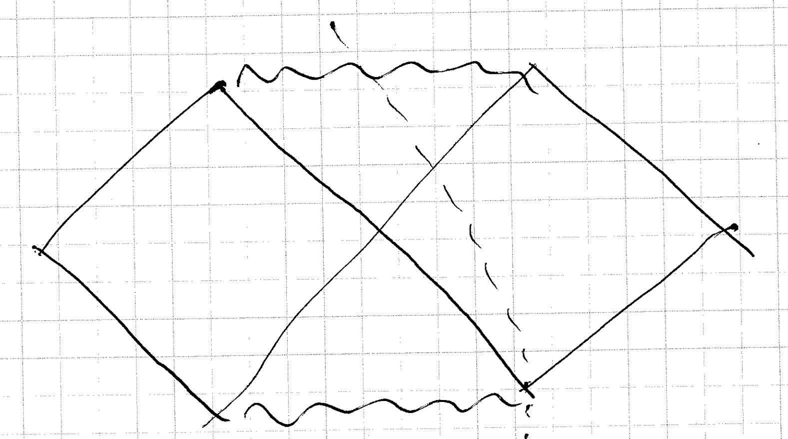
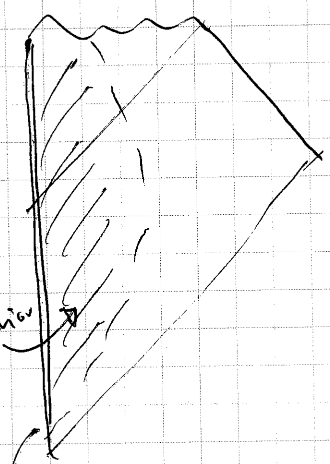
Penrose diagram



cut off



star interior

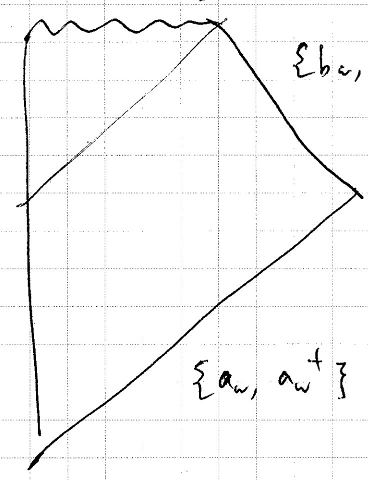
→ there is a time-like boundary.

Conceptually, important.





$\{c_\omega, c_\omega^\dagger\}$



$\{b_\omega, b_\omega^\dagger\}$

$\{a_\omega, a_\omega^\dagger\}$

"Bogoliubov transformation"

from

$$\phi = \sum_i \left[ a_i f_i + a_i^\dagger f_i^* \right]$$

to

$$\phi = \sum_i \left[ b_i p_i + b_i^\dagger p_i^* + c_i g_i + c_i^\dagger g_i^* \right]$$

$$p_i = \sum_j \left[ \alpha_{ij} f_j + \beta_{ij} f_j^* \right]$$

$$g_i = \sum_j \left[ \gamma_{ij} f_j + \eta_{ij} f_j^* \right]$$

or equivalently,

$$b_i = \sum_j \left[ \alpha_{ij}^* a_j - \beta_{ij}^* a_j^\dagger \right]$$

$$c_i = \sum_j \left[ \gamma_{ij}^* a_j - \eta_{ij}^* a_j^\dagger \right]$$

Normalization condition:

$$\sum_j \left( |\alpha_{ij}|^2 - |\beta_{ij}|^2 \right) = 1$$

We start from the a-vacuum: $a_\omega |0\rangle = 0$.

$$\langle n_\omega \rangle = \langle b^\dagger_\omega b_\omega \rangle = \sum_{\omega'} |\beta_{\omega\omega'}|^2$$

measured at future infinity

due to the a-vacuum condition

$$= \frac{\sum_{\omega'} |\beta_{\omega\omega'}|^2}{\sum_{\omega'} \left( |\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega'}|^2 \right)}$$

$$\overset{\circ}{\simeq} \frac{1}{e^{\frac{2\pi\omega}{\kappa}} - 1}$$

$$\boxed{\frac{|\alpha_{\omega\omega'}|^2}{|\beta_{\omega\omega'}|^2} \simeq e^{\frac{2\pi\omega}{\kappa}}} \quad (*)$$

Why $(*)$?

This is due to the red-shift!

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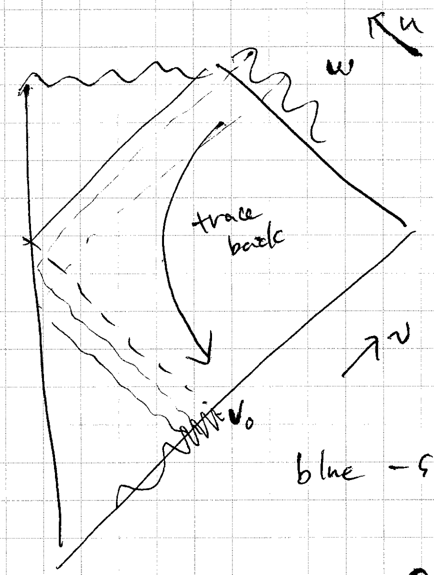


blue-shifted; by $v \sim e^{\kappa u}$

$$p_\omega \sim e^{i\omega u}$$

$$\sim e^{i\omega \frac{1}{\kappa} \ln(v_0 - v)} \qquad (v < v_0)$$

$\xrightarrow{\text{Fourier transform}}$ $p_\omega = \sum_{\omega'} \left[ \alpha_{\omega\omega'} \underbrace{f_{\omega'}}_{\sim e^{i\omega' v}} + \beta_{\omega\omega'} f^*_{\omega'} \right]$

$\xrightarrow{\text{Several calculations}}$ $\dfrac{|\alpha_{\omega\omega'}|^2}{|\beta_{\omega\omega'}|^2} \simeq e^{\frac{2\pi\omega}{\kappa}}$

∴ Red-shift is the origin of thermal radiation.

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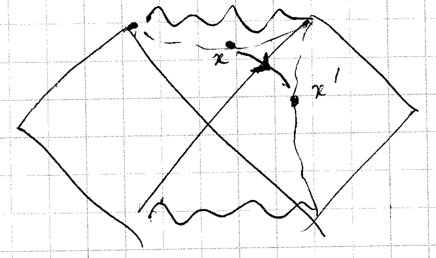
However, there exists more conceptual explanation!

Hartle and Hawking, 1976




Conclusion:

$$(\text{probability to emit a particle with energy } E) = e^{-\frac{2\pi E}{\kappa}} \times (\text{probability to absorb a particle with energy } E)$$

Step 1.

$$\underbrace{K(x,x')}_{\text{"Propagator"}} = i\int_0^\infty dW\, e^{-im^2 W} F(W,x,x')$$

where $F(W,x,x') = \underbrace{\int dx[w]\, e^{\frac{i}{4}\int_0^W g(\dot{x},\dot{x})\,dw}}_{\text{path-integral}}$





Step 2.

$$K(x,x') = -\frac{i}{4\pi^2}\frac{1}{\underbrace{s(x,x')}_{\text{"geodesic distance"}} + i\epsilon}$$

Step 3.

Analytic structure allows that one can integrate $\int_{-\infty}^{\infty} dt$

$$\to \int_{-\infty}^{\infty} d(t - 4\pi M i)$$

Step 4.

$$S_E(\vec{R},\vec{R}') = \int_{-\infty}^{\infty} dt\, e^{-iEt} K(0,\vec{R}';t,\vec{R})$$

$$= \int_{-\infty}^{\infty} dt\, e^{-iEt} K(t,\vec{R};0,\vec{R}')$$

$$= e^{-\pi E/\kappa}\underbrace{\int_{-\infty}^{\infty} dt\, e^{-iEt} K\left(t-\frac{i\pi}{\kappa},\vec{R};0,\vec{R}'\right)}_{(**)}$$

$|S_E|^2$ = emission probability

$|(**)|^2$ = absorption probability

$$\therefore \frac{|\text{emission}|}{|\text{absorption}|} = e^{-\frac{2\pi E}{\kappa}} = e^{-\frac{E}{T}}$$

Step 1.

$$F(W,x,x') = \int \underbrace{dx[w]}_{\text{Sum-over all paths}} e^{\frac{i}{4}\int_0^W g(\dot x,\dot x)\,dw}$$

Since W is not observable, we need to integrate it out

$$K(x,x') = i\int_0^\infty dw \, \underbrace{e^{-im^2 W}}_{\text{if massive scalar field}} F(W,x,x')$$

The problem is. "How to constraint $F(W, x, x')$".

Hartle and Hawking used the trick of Wick-rotation;

$$\rightarrow \quad F(\Omega, x_E, x_E') = \int \delta x[w] \, e^{\underbrace{-\frac{1}{4}\int_0^\Omega g_E(\dot x_E, \dot x_E)\,dw}_{\text{"real"}}}$$

This function behaves better.

One can prove

$$\frac{\partial F}{\partial \Omega} = \Box^2 F$$

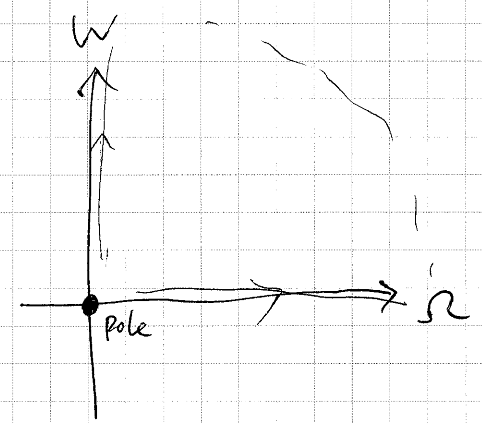




In order to justify the Wick rotation, we require

① $F(0, x_E, x_E') = \delta(x_E, x_E')$

② $F$ must vanish as $x_E \to \infty$.




$$\longleftrightarrow \quad i\frac{\partial F}{\partial W} = -\Box^2 F$$

with

$$\lim_{W\to 0} F(W, x, x') = \delta(x,x')$$

For small $W$,
$$S(W,x,x') = \frac{1}{4}\int_0^W g_{\mu\nu}\frac{dx^\mu}{dw}\frac{dx^\nu}{dw}\,dw$$

$$\simeq \frac{1}{4W}\underbrace{\int_0^1 g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,d\tau}_{\text{geodesic distance}} \qquad d\tau = \frac{dw}{dW}$$

$$\simeq \frac{S(x,x')}{4W}$$

For large $W$, assuming analyticity,

we estimate $S(W,x,x') \sim \dfrac{1}{W^2}$.

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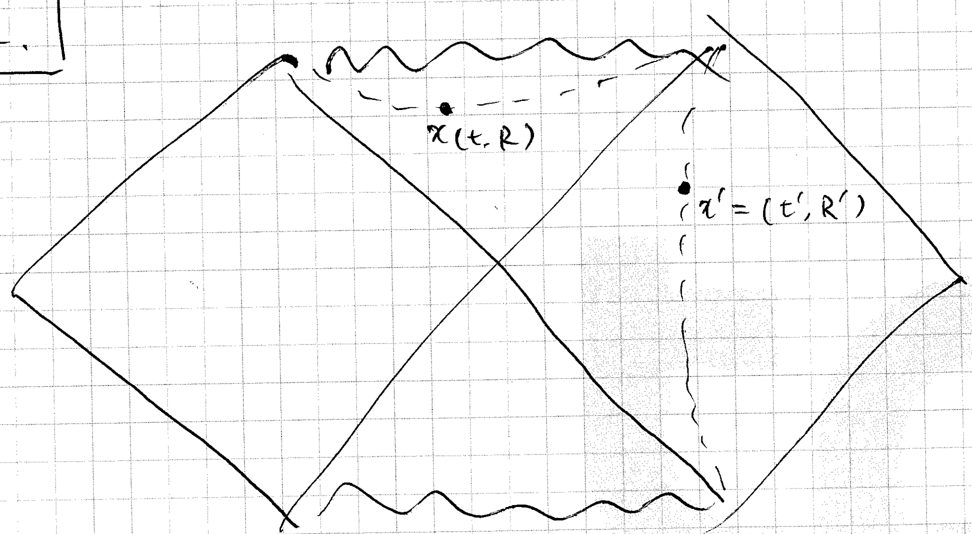
The correct form is then

$$F(w,x,x') = \frac{i}{(4\pi w)^2}\, e^{\frac{i s(x,x')}{4w}} .$$

$$\therefore\ K(x,x') = i\int_0^\infty dw\, \underbrace{e^{-\epsilon/w}}_{\text{regulator } (\epsilon\to 0)} \cdot \frac{i}{(4\pi w)^2}\, e^{\frac{i s(x,x')}{4w}}$$

$$= -\frac{i}{4\pi^2}\,\frac{1}{s(x,x')+i\epsilon}$$

Step 2.




$$K(x,x') = K(t,\vec{R};\ \underbrace{t'}_{=0},\vec{R}') = K(0,\vec{R}';\ t,\vec{R}')$$

fixing $R'$, integrate over $t$.

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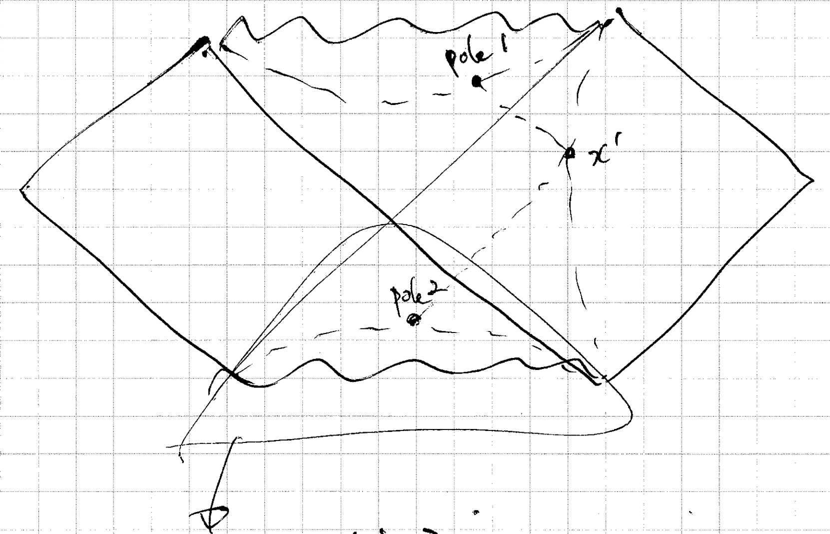
Where are poles?

$\rightarrow$ $\underline{s(x,x')=0}$ i.e., null geodesics.




What is this?

$$ds^2 = \left(\frac{2M}{r}-1\right)dt^2 - \left(\frac{2M}{r}-1\right)^{-1}dr^2 + r^2 d\Omega^2$$

$$x = 4M\sqrt{\frac{2M}{r}-1}$$

$$ds^2 = -\frac{r^4}{16M^4}dx^2 + \frac{x^2}{16M^2}dt^2 + r^2 d\Omega^2$$

$t \to -i\tau$

$$ds_E^2 = -\left(\frac{r^4}{16M^4}dx^2 + \frac{x^2}{16M^2}d\tau^2\right) + r^2 d\Omega^2 .$$

$$\simeq -\underbrace{\left(dx^2 + x^2\frac{d\tau^2}{16M^2}\right)}_{D^2} + r^2 d\Omega^2$$

$x\to 0$ (horizon)

$D^2 \rightarrow \Delta\tau = 8\pi M$

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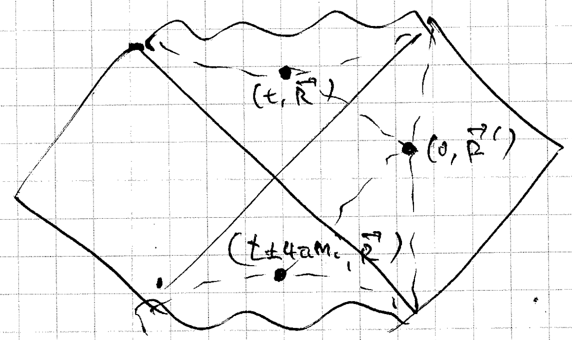
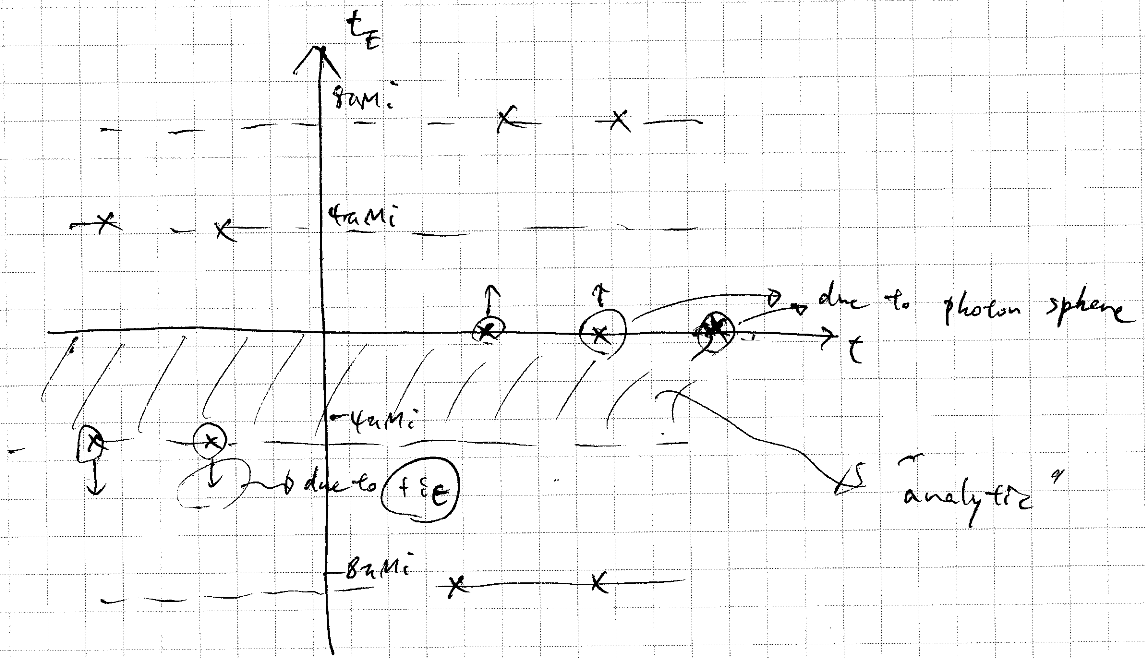
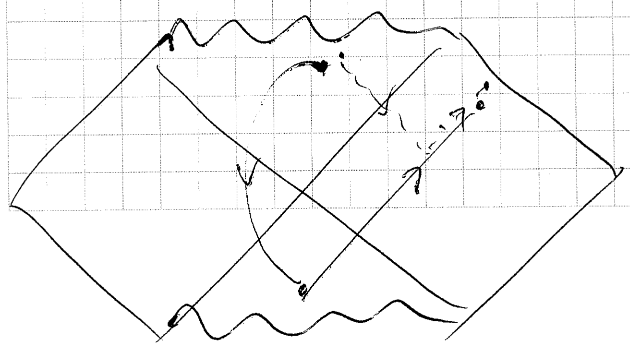



Analytic structures




$$\therefore \int_{-\infty}^{\infty} dt(\cdots) = \int_{-\infty}^{\infty} d(t-4aMi)(\cdots)$$

"Boltzmann factor"







"Conclusion"

"Euclidean Analytic Continuation" is deeply related to Hawking radiation.

This is the strong motivation to investigate Euclidean quantum gravity to understand Black Hole Thermodynamics.

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Lecture 2.

Entropy and topology of Euclidean black holes

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{\left(1-\frac{2M}{r}\right)} + r^2 d\Omega^2$$

$$x = 4M\sqrt{1-\frac{2M}{r}}$$

$$ds^2 = -\frac{x^2}{16M^2}dt^2 + \frac{r^4}{16M^4}dx^2 + r^2 d\Omega^2$$

$$t = -i\tau$$

$$ds_E^2 = \frac{x^2}{16M^2}d\tau^2 + \frac{r^4}{16M^4}dx^2 + r^2 d\Omega^2$$

$$x \to 0 \quad (r \to 2M)$$

$$ds_E^2 \simeq dx^2 + \underbrace{\frac{x^2}{16M^2}d\tau^2}_{= x^2 d\theta} + 4M^2 d\Omega^2$$

$$\therefore \Delta\theta = 2\pi = \frac{\Delta\tau}{4M}$$

$$\therefore \Delta\tau = 8\pi M = \frac{1}{T}$$

If the period is not $\frac{1}{T}$, then there appears a "cusp" singularity.
"conical"

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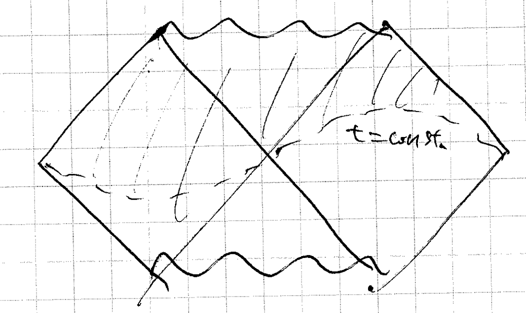


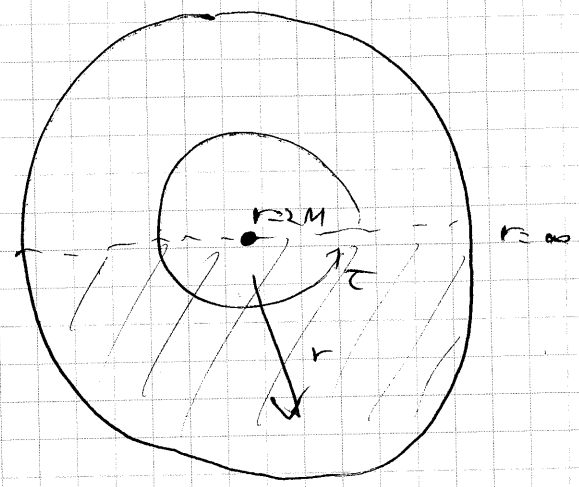


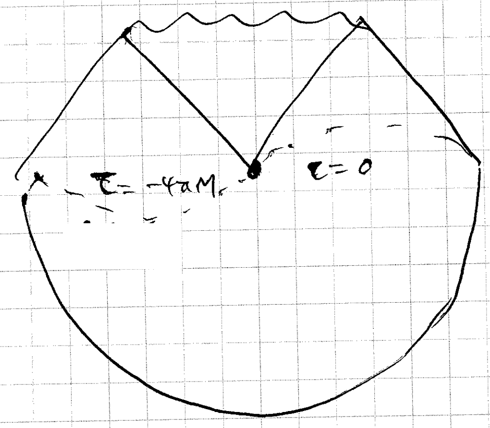


"What is the entropy cost of a black hole?"

Let us first fix $T$. Then we consider the partition function.



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Micro canonical Ensemble

$$dS = \frac{dE}{T} \qquad P_i = \frac{1}{Z} \qquad Z = e^{S}$$

$$S = \ln n$$

Canonical Ensemble

$$dF = -S\,dT \qquad P_i = \frac{e^{-\beta E_i}}{Z} \qquad Z = e^{-F/T}$$

$$\beta = \frac{1}{T} \qquad F = E - ST$$

$$Z = \mathrm{Tr}\, e^{-\beta H} = \int Dg\, e^{-S_E}$$

$$= e^{-F/T} \simeq \sum_{\text{inst.}} e^{-S_E^{\text{inst}}}$$

Steepest descent approximation

$$\therefore\ S_E^{\text{inst.}} = \frac{F}{T} = \frac{E}{T} - S$$

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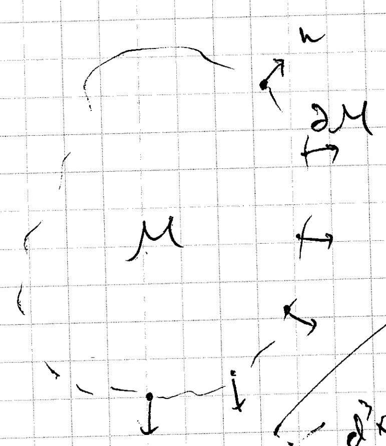
$$\boxed{ds_E^2 = f\,d\tau^2 + \frac{dr^2}{f} + r^2 d\Omega^2}$$

$$S_E = -\frac{1}{16\pi}\int_M \sqrt{+g}\; d^4x \underbrace{(R - 2\Lambda)}_{\text{Vanished for } \Lambda=0 \text{ Schwarzschild}}$$

$$\underbrace{-\frac{1}{8\pi}\int_{\partial M} K \sqrt{+h}\; d^3x}_{\text{“Gibbons-Hawking boundary term”}}$$

$K$: extrinsic curvature at $\partial M$

"Gibbons-Hawking boundary term" = see (***)




$$\boxed{d^3x = d\theta\, d\varphi\, d\tau}$$

$$\boxed{K = \frac{1}{r^2}\frac{d}{dr}\left(r^2\sqrt{f}\right)}$$

$$\boxed{\sqrt{h} = \sqrt{f}\, r^2 \sin\theta}$$

$$-\frac{1}{8\pi}\int_{\partial M} K\sqrt{h}\, d^3x = -\frac{1}{8\pi}\cdot 4\pi \cdot 8\pi M\left[\sqrt{f}\,\frac{d}{dr}\left(r^2\sqrt{f}\right)\right]$$

$$= -4\pi M\left[\sqrt{f}\,\frac{d}{dr}\left(r^2\sqrt{f}\right)\right]$$

By the way, where is the boundary?

→ "At infinity"

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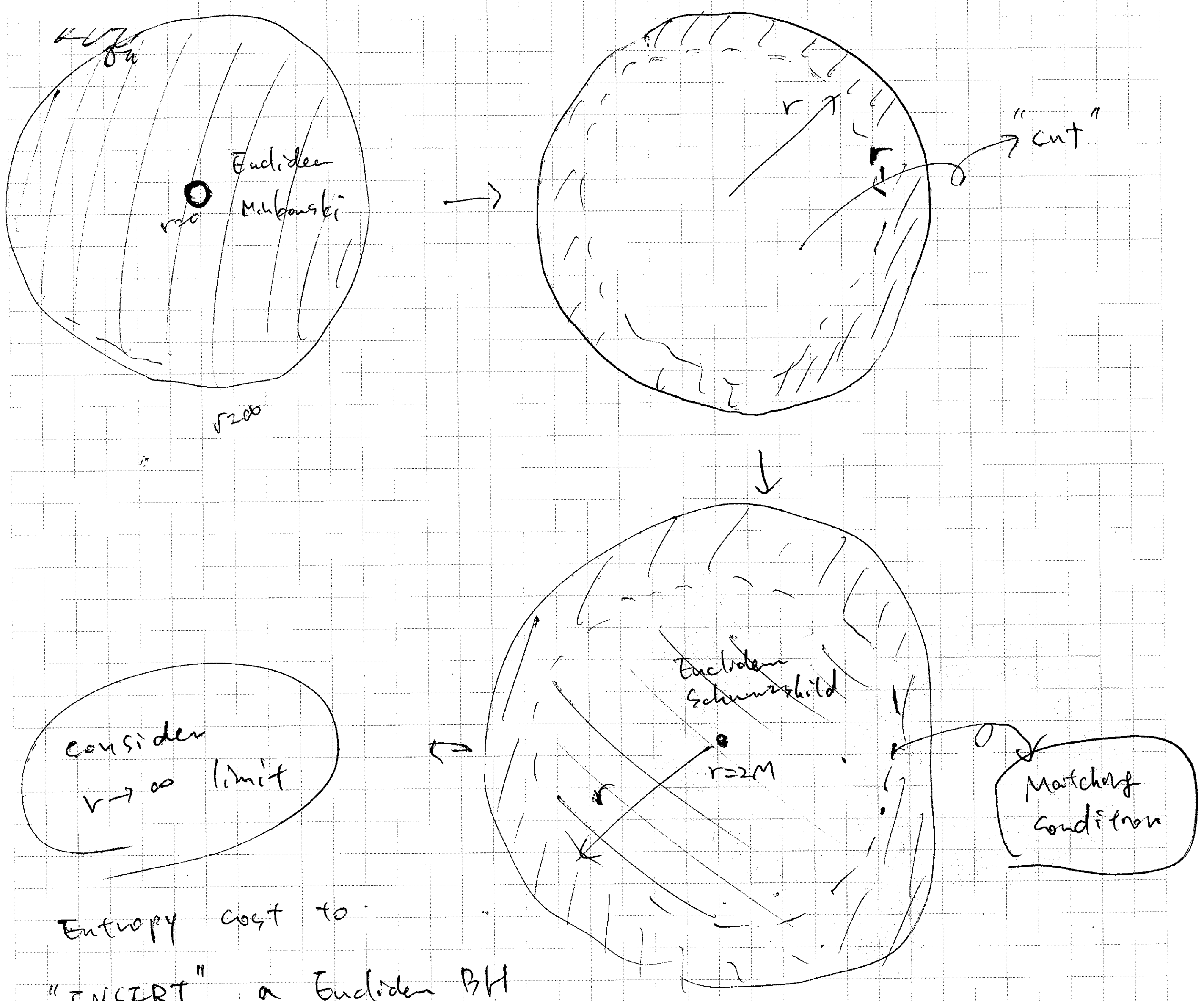
$$-\frac{1}{8\pi}\int_{\partial M} K\sqrt{h}\, d^3x = -4\pi M\left(2rf + \frac{r^2}{2}f'\right)$$

$$\xrightarrow[r\to\infty]{} \infty$$

However, we can compare this with Minkowski




Entropy cost to "INSERT" a Euclidean BH into Euclidean Minkowski.

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$$-\frac{1}{8\pi}\int_{\partial M} \underset{\downarrow}{\underbrace{K_0}} \sqrt{h}\, d^3x = -\frac{1}{8\pi}\cdot \underbrace{\left(\frac{2}{r}\right)}_{=K} 4\pi r^2 \,\Delta\tau_M$$

Minkowski

$$d\tau_M^2 + \cancel{dr^2} + \cancel{r^2 d\Omega^2} = f d\tau^2 + \cancel{\frac{dr^2}{f}} + \cancel{r^2 d\Omega^2}$$

$$\Delta\tau_M = \sqrt{f}\,\Delta\tau = \sqrt{f}\cdot 8\pi M$$

: matching condition

$$= - \quad r \cdot \sqrt{f}\cdot 8\pi M$$

$$\therefore\ -\frac{1}{8\pi}\int_{\partial M} K \sqrt{h}\, d^3x + \frac{1}{8\pi}\int_{\partial M} K_0 \sqrt{h}\, d^3x$$

$8\pi M r\sqrt{f}$

$$= -4\pi M\left(2rf + \frac{r^2}{2} f'\right) + $$

$$\underset{r\to\infty}{=} -4\pi M\left(\cancel{2r} - 4M + M - \cancel{2r} + 2M\right)$$

$$= +4\pi M^2$$

$$f = 1 - \frac{2M}{r}$$

$$f' = +\frac{2M}{r^2}$$

$$\sqrt{f} = \sqrt{1-\frac{2M}{r}} \simeq 1 - \frac{M}{r}$$

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$$\therefore\ S_E = -\frac{1}{8\pi}\int_{\partial M} [K]\sqrt{h}\, d^3x$$

$$= 4\pi M^2$$

$$= \frac{E - ST}{T}$$

$$= 8\pi M^2 - S$$

$$\boxed{\therefore\ S = 4\pi M^2} = \frac{A}{4}$$

**Note** dS space

$$S_E = -\frac{A_H}{4}$$ → Area of cosmological horizon

$$= \frac{E - ST}{T} \underset{(E=0)}{=} -S$$

$$\boxed{\therefore\ S = \frac{A_H}{4}}$$

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Is there any generic reason why the GH boundary term gives the entropy cost ?

See hep-th/9409195.

For large r, one may approximate (2D + angular directions)

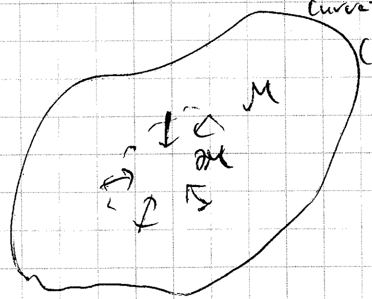
(Approximately)
In 2D, there is the Gauss-Bonnet theorem.

$$\int_M K\, dA + \int_{\partial M} k_g\, ds = 2\pi \chi(M)$$

$K$: Gaussian curvature (≃ Riemann curvature)

$k_g$: geodesic curvature (≃ extrinsic curvature at $\partial M$)

$\chi(M)$: Euler characteristic → topological invariant




For 4D, Asymptotically $\underbrace{S^1}_{\tau} \times \underbrace{S^2}_{\theta,\varphi}$



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$\partial M = S^1 \times S^2$

↑ ?

$M$

2 typical examples

$(\partial D^n = S^{n-1})$

① $\underset{(\tau, r)}{D^2} \times \underset{(\theta,\varphi)}{S^2}$ : Schwarzschild

② $\underset{(\tau)}{S^1} \times \underset{(r,\theta,\varphi)}{D^3}$ : Minkowski

Topology is different !

→ Non-trivial Euclidean action !

$\therefore$ (Entropy cost) ≃ (Topological cost)

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## More on topology

See Maldacena, JHEP 0304 (2003) 021.

Based on AdS/CFT, AdS is unitary.

However, for a given BH (eternal), correlations decay to zero. Then How to recover this?

Indeed, there should be a contribution by "periodically identified AdS". (= Euclidean AdS) Via the geometry, correlations will be recovered.

$$\langle \phi\phi \rangle \simeq \underbrace{\underbrace{\langle \phi\phi \rangle_1}_{\to 0} \underbrace{P_1}_{\approx 1}}_{BH} + \underbrace{\langle \phi\phi \rangle_2}_{\neq 0} \underbrace{P_2}_{\approx e^{-A}} + \cdots$$

$$\simeq \text{const} \times e^{-A}$$

"exponentially decreased, but non-zero"

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## Conversion of Hawking, 2005.

In the path integral, there are contributions by "trivial" topology (= periodically identified Minkowski, AdS) and "non-trivial" topology (= Sch, Sch-AdS, ⋮)

Information will be preserved via trivial topologies.

Q1. What is the Lorentzian meaning? "tunneling"

Q2. Is it generic? "Yes"

Q3. Is it enough to solve the information loss problem? "maybe not"

Please see my papers.

Sasaki and Yeom, 1404.1565

Chen, Domenech, Sasaki and Yeom, 1704.04020

Chen, Sasaki and Yeom, 1806.03766

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(***)

Thin-shell Approximation

$$ds_\pm^2 = -f_\pm(r)\,dt^2 + \frac{dr^2}{f_\pm(r)} + r^2 d\Omega_2^2$$

$$ds_{shell}^2 = -d\tau^2 + r^2(\tau)\,d\Omega_2^2$$

Israel junction conditions ($[*] = *|_+ - *|_-$)

① $[h_{ab}] = 0$ metric continuity

② $[K_{ab}] - h_{ab}[K] = -8\pi S_{ab}$

Einstein equation

induced energy momentum tensor

Perfect fluid form:

$$S^a{}_b = \begin{pmatrix} -\sigma & & \\ & \lambda & \\ & & \lambda \end{pmatrix}$$

$$K^a{}_b = \begin{pmatrix} \dot\beta_\pm/\dot r & & \\ & \beta_\pm/r & \\ & & \beta_\pm/r \end{pmatrix}$$

$\rightarrow$ $\epsilon_-\sqrt{\dot r^2 + f_-} - \epsilon_+\sqrt{\dot r^2 + f_+} = 4\pi r\sigma$

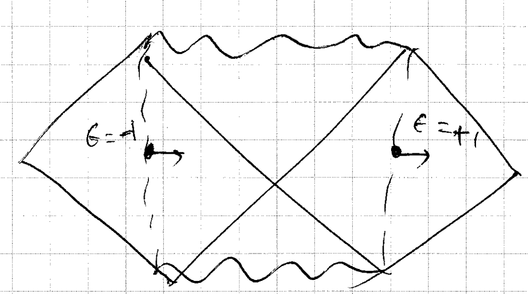
$\dot\sigma = -2\frac{\dot r}{r}(\sigma + \lambda)$

$\beta_\pm = \epsilon_\pm\sqrt{\dot r^2 + f_\pm}$

$\pm 1$, directs outward increasing/decreasing normal direction








$\rightarrow$ $K = \frac{\dot\beta}{\dot r} + \frac{2\beta}{r} = \frac{1}{r^2}\frac{d}{dr}(r^2\sqrt{f})$

$\dot r = 0$

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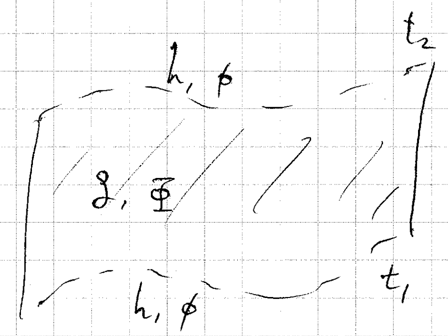
## Lecture 3.

Quantum cosmology : homogeneous analytic continuation

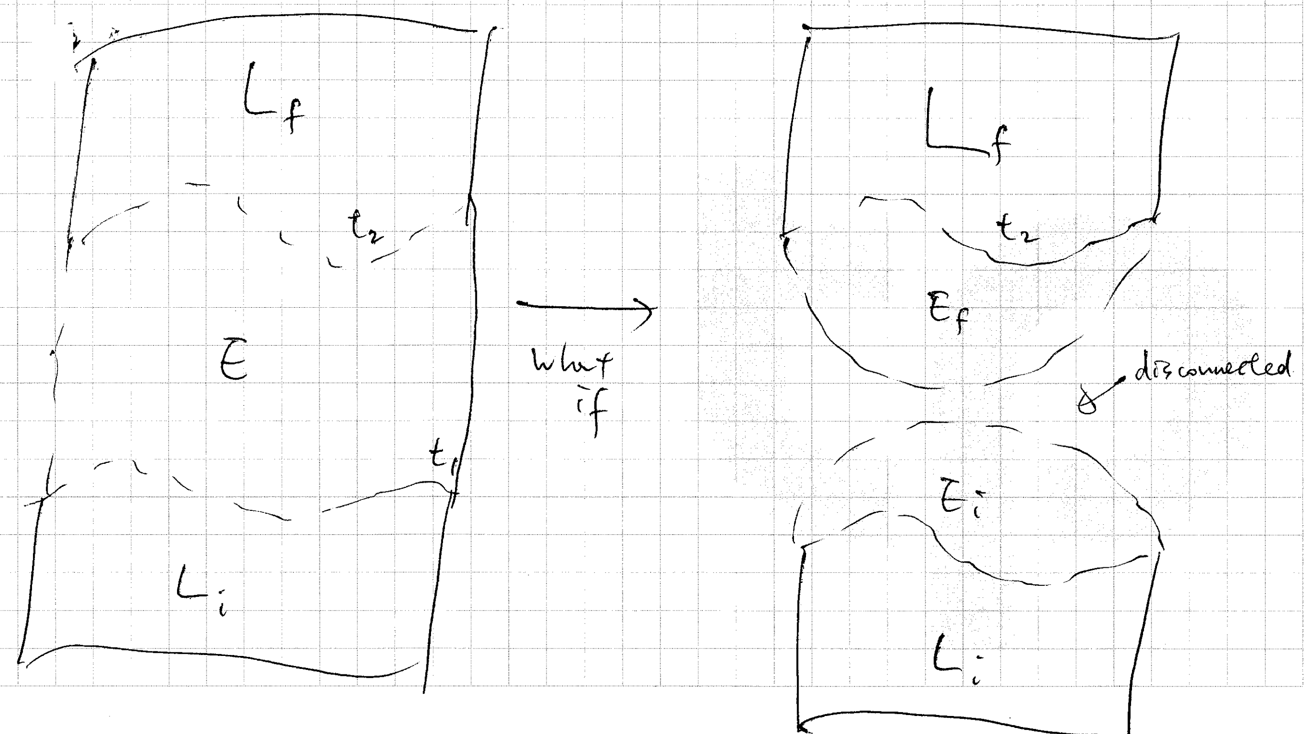
Hartle and Hawking, 1983

$\boxed{\text{Propagator}}$




$$\langle h, \phi [t_2] \,|\, h, \phi [t_1] \rangle = \int Dg\, D\Phi\; e^{iS[g,\phi]} = \int Dg\, D\Phi\; e^{-S_E[g,\phi]}$$

ground state(?)

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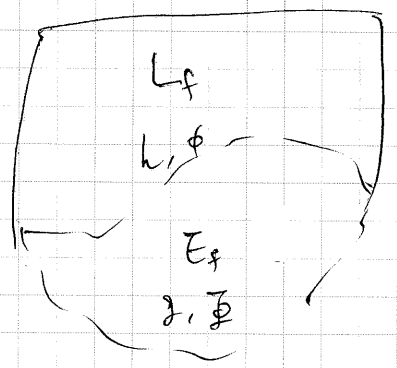
One can define

$$\Psi[h,\phi] = \int \mathcal{D}g\,\mathcal{D}\phi\; e^{-S_E}$$

where we sum-over all geometries with $(h, \phi)$ as their only boundary.




Then this wave function has the only future boundary.

$\rightarrow$ No (initial) boundary

$$\mathcal{H}\Psi = 0$$

Wheeler-DeWitt equation

$\rightarrow$ gives a consistent boundary condition of WDW equation.

"Hartle-Hawking proposal" or

"No-boundary proposal"

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## More calculations

Approximation 1. Mini-superspace model

$$ds_E^2 = N^2 d\tau^2 + a^2 d\Omega_3^2$$

$a = a(\tau)$

$d\Omega_3^2 = d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)$

$N = 1$ (fixed)

Approximation 2. Steepest-Descent Approx.

$$\int \mathcal{D}g\,\mathcal{D}\Phi\; e^{-S_E} \simeq \sum_{\text{inst.}} e^{-S_E^{\text{inst.}}}$$

Is this fine?

Recently there is a debating between

$$\begin{pmatrix} \text{Turok} \\ \text{Lehners} \\ \vdots \end{pmatrix} \longleftrightarrow \begin{pmatrix} \text{Hartle} \\ \text{Hertog} \\ \text{Halliwell} \\ \vdots \end{pmatrix}$$

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$$S_E = -\int d^4x \sqrt{g} \left( \frac{R}{16\pi} - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right)$$

$$= 2\pi^2 \int d\tau \left( -\frac{3}{8\pi}(a\dot{a}^2 + a) + \frac{1}{2}a^3\dot{\phi}^2 + a^3 V(\phi) \right)$$

Equs. of motion

$$\ddot{\phi} = -3\frac{\dot{a}}{a}\dot{\phi} + V'$$

$$\ddot{a} = -\frac{8\pi}{3} a \left(\dot{\phi}^2 + V\right)$$

$$\dot{a}^2 = 1 + \frac{8\pi}{3} a^2 \left( \frac{\dot{\phi}^2}{2} - V \right)$$

$$\longrightarrow \quad S_E^{inst.} = 4\pi^2 \int d\tau \left[ a^3 V - \frac{3}{8\pi} a \right]$$





The simplest solution

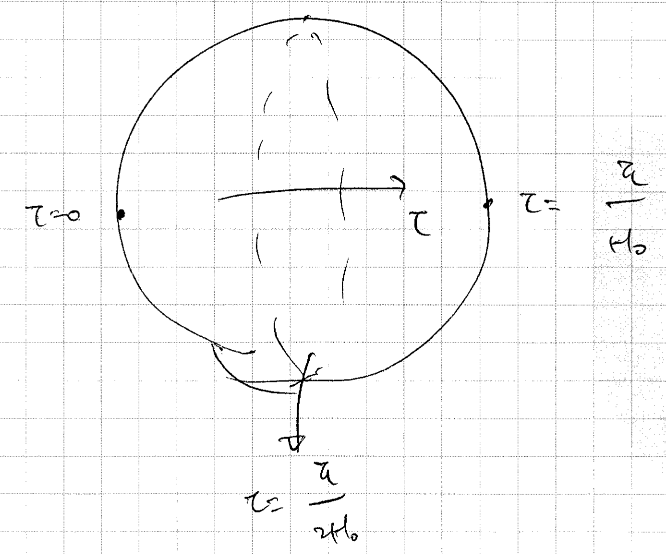
$V = V_0 \qquad V' = 0$

$$\longrightarrow \quad 8\pi V_0 = \frac{3}{\ell^2} = 3H_0^2$$

$$\dot{a}^2 = 1 - H_0^2 a^2$$

$a(0) = 0$ : compactness

$$\longrightarrow \quad \boxed{a = \frac{1}{H_0}\sin H_0 \tau}$$



$$S_E = 4\pi^2 \int_0^{\pi/H_0} d\tau \left( \frac{1}{H_0^3} \cdot \sin^3 H_0\tau \cdot \frac{3}{8\pi} H_0^2 - \frac{3}{8\pi} \cdot \frac{1}{H_0} \sin H_0\tau \right)$$

$$= \frac{4\pi^2}{H_0} \cdot \frac{3}{8\pi} \int_0^{\pi/H_0} d\tau \left( \sin^3 H_0\tau - \sin H_0\tau \right)$$

$$= -\frac{3\pi}{2H_0} \int_0^{\pi/H_0} d\tau \, \sin H_0\tau \cos^2 H_0\tau$$

$$= \frac{3\pi}{2H_0^2} \int_0^{\pi/H_0} d(\cos H_0\tau) \cos^2 H_0\tau$$

$$= \frac{\pi}{2H_0^2} \cos^3 H_0\tau \Big|_0^{\pi/H_0}$$

$$= -\frac{\pi}{H_0^2} = -\pi l^2 = -\frac{A_h}{4}$$

$$S_E = \frac{F}{T} = \frac{E - ST}{T} = -S$$

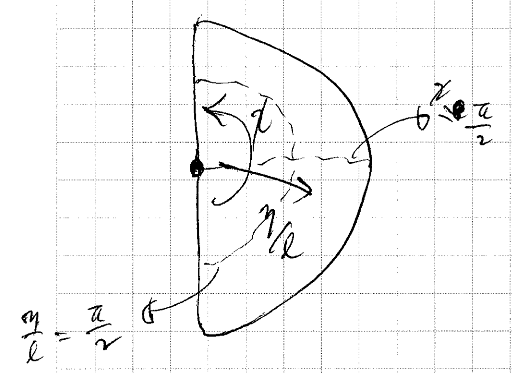
$$\therefore S = +\frac{A_h}{4}$$





Two presentation

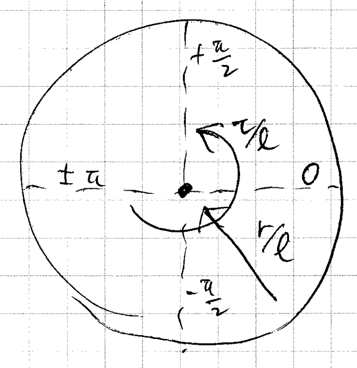
$$a(\eta) = l \sin\frac{\eta}{l} \qquad ds_E^2 = d\eta^2 + a^2(\eta)\, d\chi^2 + a^2(\eta) \sin^2\chi \, d\Omega_2^2$$




$$r = l \sin\frac{\eta}{l} \sin\chi$$

$$\tan\frac{\tau}{l} = \tan\frac{\eta}{l} \cos\chi$$

$$ds_E^2 = \left(1 - \frac{r^2}{l^2}\right) d\tau^2 + \left(1 - \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2 d\Omega^2$$



In principle, one can Wick-rotate at any surface. However, due to the Cauchy-Riemann condition, the manifold after the Wick-rotation is complex-valued in general, unless the time variation is zero.

So, the typical Wick-rotations are

① $\frac{\eta}{\ell} = \frac{\pi}{2}$ or equivalently $\frac{\tau}{\ell} = \pm\frac{\pi}{2}$

② $\chi = \frac{\pi}{2}$ or equivalently $\tau/\ell = 0, \pm\pi$

After the Wick-rotation,

① $a(t) = \frac{1}{H_0}\cosh H_0 t$

$$ds^2 = -dt^2 + a^2 d\Omega_3^2$$

closed-homogeneous universe (k=+1)

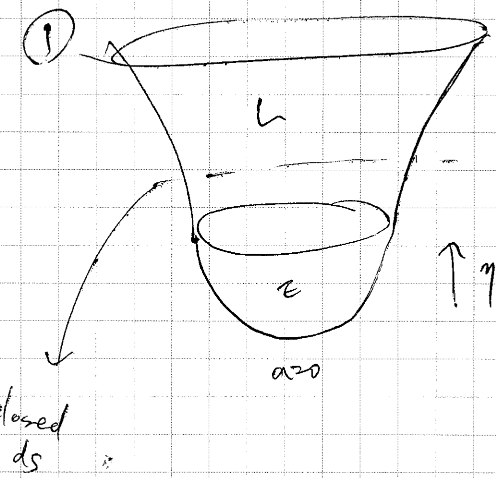




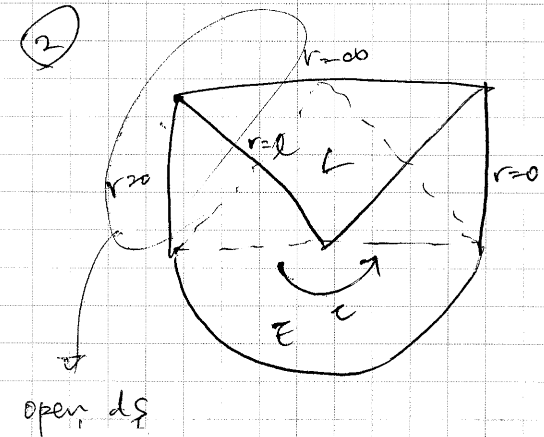
② $ds^2 = d\chi^2 - a^2 dt^2 + a^2\cosh^2 t\, d\Omega^2$

or

$$ds^2 = -\left(1-\frac{r^2}{\ell^2}\right)dt^2 + \frac{dr^2}{\left(1-\frac{r^2}{\ell^2}\right)} + r^2 d\Omega^2$$

① 


homogeneous

② 

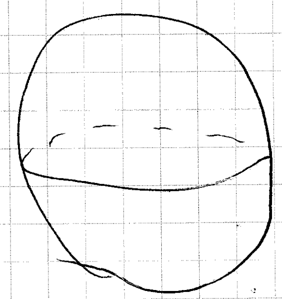

inhomogeneous

The probability?

$$\Psi \simeq e^{-S_E}$$

$$P = |\Psi|^2 = e^{-2\,\mathrm{Re}\,S_E}$$

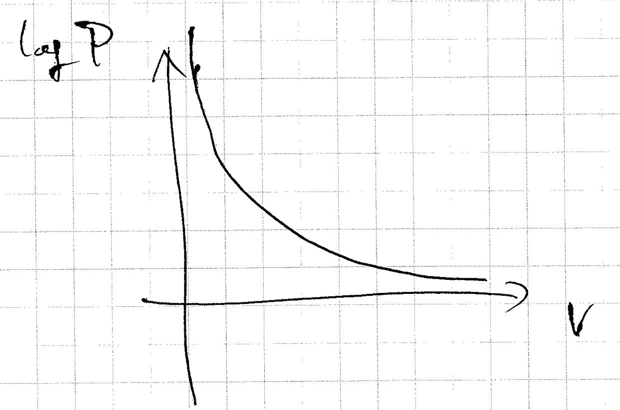
$\times 2$ or



$\times 1$

$$P \simeq e^{+\frac{3}{8V_0}}$$

badly diverges!




$V_0 = 0$ is exponentially preferred!



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Problem 1

The Euclidean wave function prefers $\begin{pmatrix} \Lambda = 0 \\ G = 0 \\ \vdots \end{pmatrix}$.

If we consider Euclidean wormholes, then one may consider probabilities between different constants of nature according to Sydney Coleman's argument.

However, our world is $\begin{pmatrix} \Lambda \neq 0, & \Lambda > 0 \\ G \neq 0, & G > 0 \\ \vdots & \end{pmatrix}$!

Problem 2

Our univers experienced more than 50 e-foldings.

However, the wave function does not prefer large e-foldings.

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Hartle-Hawking-Hertog generalized to

"fuzzy instantons",

complex-valued

$$\begin{pmatrix} a \\ \phi \\ \dot{a} \\ \dot{\phi} \end{pmatrix} \longrightarrow \begin{pmatrix} a^{re} & a^{im} \\ \phi^{re} & \phi^{im} \\ \dot{a}^{re} & \dot{a}^{im} \\ \dot{\phi}^{re} & \dot{\phi}^{im} \end{pmatrix}$$

No boundary condition:

$$a^{re}(0) = a^{im}(0) = 0$$

$\rightarrow$ (consistency)

$$\dot{a}(0) = 1$$

$$\dot{\phi}^{re}(0) = \dot{\phi}^{im}(0) = 0$$

the only d.o.f.

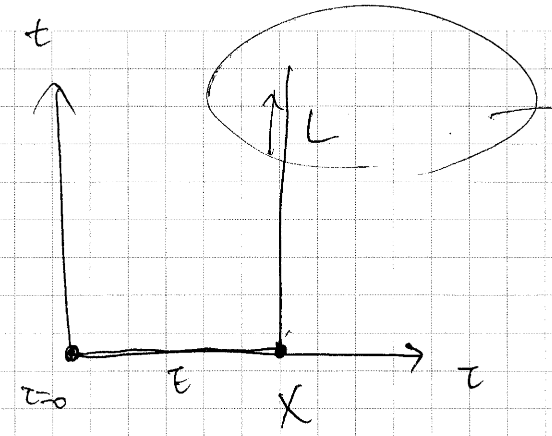
$\phi^{re}(0)$ and $\phi^{im}(0)$

or

$$|\phi(0)|\, e^{i\theta}$$








$\rightarrow$ Impose the reality condition

$$\boxed{\begin{aligned} \mathrm{Im}\, a &\to 0 \\ \mathrm{Im}\, \phi &\to 0 \end{aligned}}$$

By tuning $(\theta, X)$, we impose the boundary condition

$$(\mathrm{Im}\, a \to 0, \quad \mathrm{Im}\, \phi \to 0)$$

If these conditions are satisfied,

$$\Psi \cong A e^{iS} = e^{-S_E^{re} + i S_E^{im}}$$

$$|D S^{re}|^2 \ll |D S^{im}|^2$$

derivation w.r.t. a or $\phi$

$\rightarrow$ probability slowly varies

$\rightarrow$ classicality!

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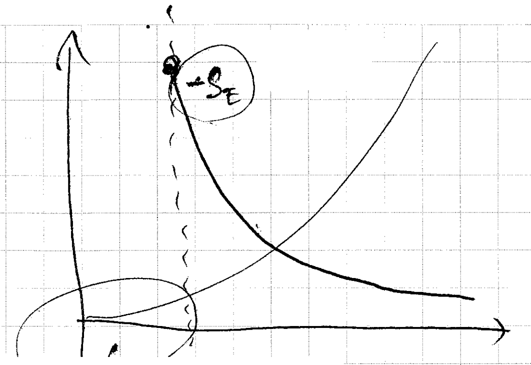


$V = \frac{1}{2} m^2 \phi^2$

No classicalization!

Why? $\quad V = V_0 + \frac{1}{2} m^2 \phi^2$

$$\ddot{\phi}_{r,i} + 3H\dot{\phi}_{r,i} + m^2 \phi_{r,i} = 0$$ (in Lorentzian signatures)

$$\phi_r \simeq A_+ e^{-\alpha_+ t} + A_- e^{-\alpha_- t}$$

$$\phi_i \simeq B_+ e^{-\alpha_+ t} + B_- e^{-\alpha_- t}$$

$$\alpha_\pm = \frac{3H_0 \pm \sqrt{9H_0^2 - 4m^2}}{2}$$

If $\frac{m}{H_0} < \frac{3}{2}$, by tuning $\begin{pmatrix} A_+ \simeq 0 \\ B_- \simeq 0 \end{pmatrix}$,

$$\frac{|\phi_i|}{|\phi_r|} \simeq e^{-\sqrt{9H_0^2 - 4m^2}\, t} \to 0$$

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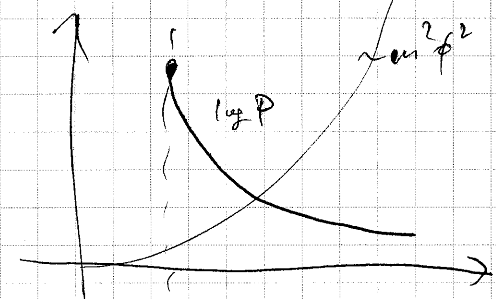
However, if $\frac{m}{H_0} > \frac{3}{2}$, then

$$\frac{|\phi_i|}{|\phi_r|} \simeq \left| e^{-i\sqrt{4m^2 - 9H_0^2}\, t} \right|$$

$\simeq O(1)$ : not classicalized.

Therefore, $V_0$ should be large enough for classicalities.




This may explain that why our universe (classical) requires inflaton (non-vanishing $V$). However, still disfavor large e-foldings.

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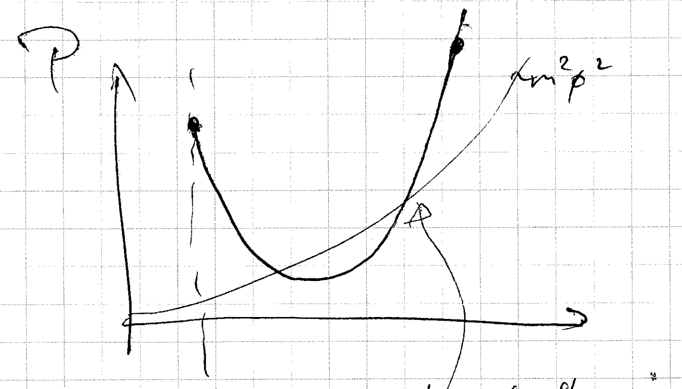
Top-down approach of HHH :

$$P \simeq e^{-2\,\mathrm{Re}\,S_E + 3N}$$

Volume weighting




However, this is an ad-hoc explanation.

See my papers :

Hwang and Yeom, 1311.6872 (Review of alternatives)

Hwang, Kim, Lee, Sahlmann and Yeom, 1207.0359 (Large N)

Hwang, Kim and Yeom, 1404.2800 (2-field)

Sasaki, Yeom and Zhang, 1307.5948 (modified gravity)

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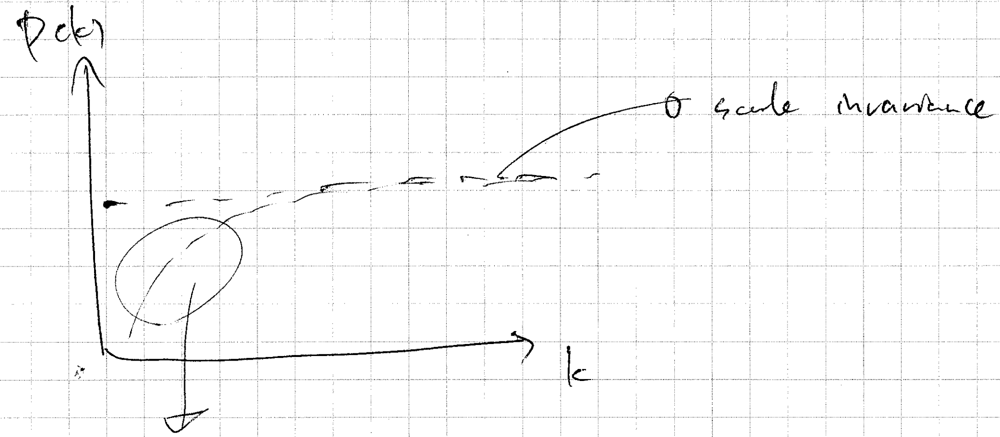
Observational consequencies ?

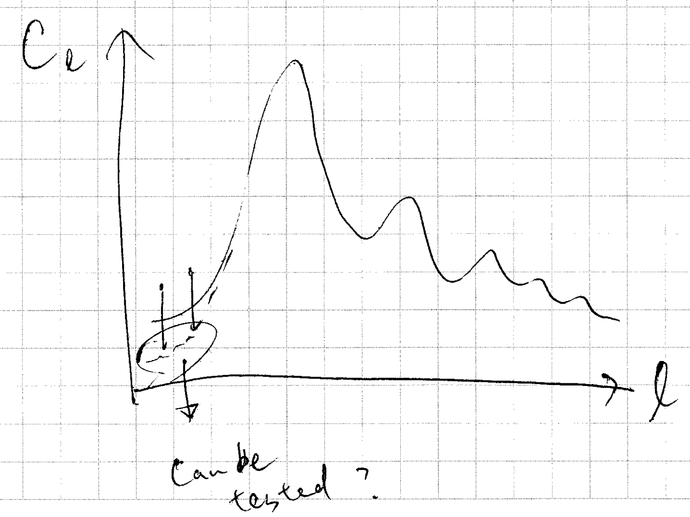
① K=+1 closed universe

Hallliwell and Hawking, 1985
Chen, Lin and Yeom, 1707.01471

② K=−1 open universe

White, Zhang and Sasaki, 1407.5816

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# Lecture 4.

Quantum cosmology : inhomogeneous analytic continuation

$$ds_E^2 = d\eta^2 + a^2(\eta)\left(d\chi^2 + \sin^2\chi\, d\Omega_2^2\right)$$

For pure dS, $a(\eta) = \ell \sin\frac{\eta}{\ell}$

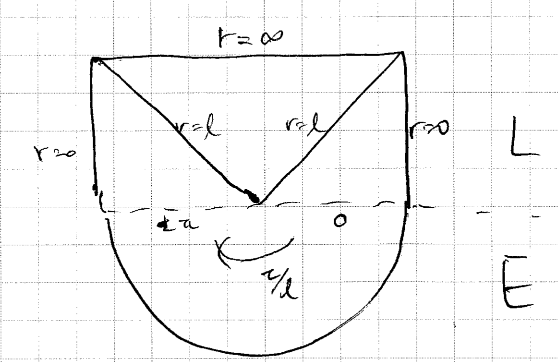
$$r = \ell \sin\frac{\eta}{\ell}\sin\chi$$

$$\tan\frac{\tau}{\ell} = \tan\frac{\eta}{\ell}\cos\chi$$

$$ds_E^2 = \left(1-\frac{r^2}{\ell^2}\right)d\tau^2 + \left(1-\frac{r^2}{\ell^2}\right)^{-1}dr^2 + r^2 d\Omega_2^2$$

Wick-rotation at $\chi = \frac{\pi}{2}$

$\rightarrow$ Wick-rotation at $\tau = 0, \pm\pi$



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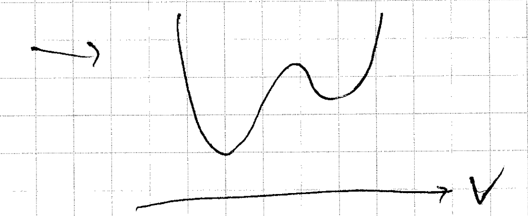
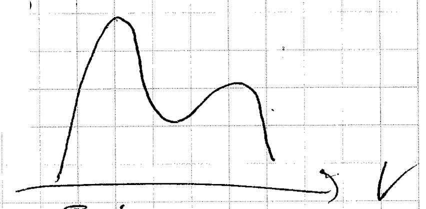
Lorentzian: $\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V' = 0$

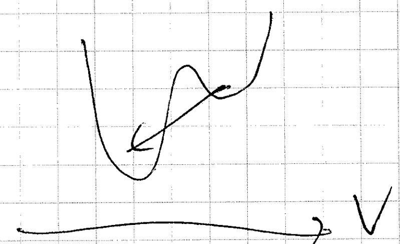
Euclidean: $\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - V' = 0$



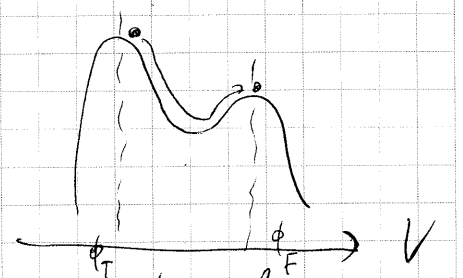

Lorentzian



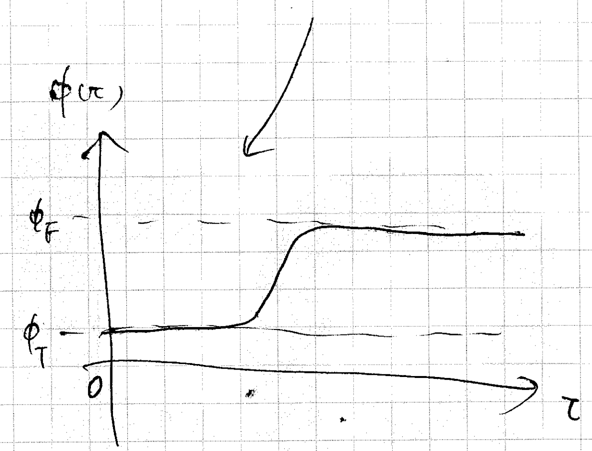

Euclidean

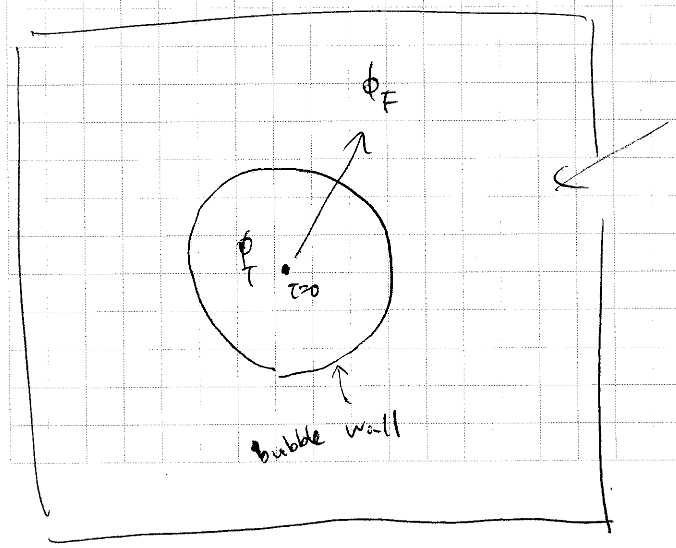



quantum tunneling




classical rolling

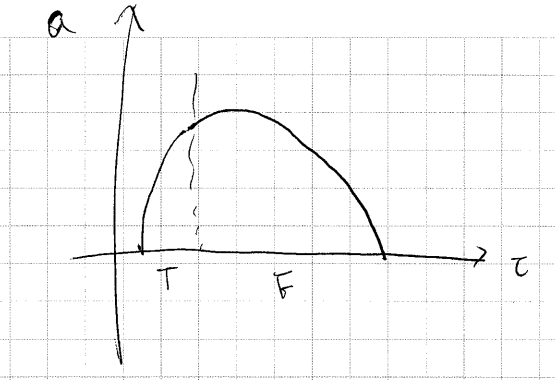



"Coleman-DeLuccia instantons"








$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - V' = 0$$

$$\ddot{a} + \frac{8\pi}{3} a(\dot{\phi}^2 + V) = 0$$

$a(0) = 0$

$\dot{a}(0) = 1$

$\dot{\phi}(0) = 0$

$\phi(0)$ : free parameter

Boundary condition: $\phi(\tau \to \tau_{max}) = \phi_F$

(at the same time, $\dot{\phi}(\tau_{max}) = a(\tau_{max}) = 0$ and $\dot{a}(\tau_{max}) = 1$ for asymptotic dS.)

By tuning $\phi(0)$, we obtain $\phi(\tau_{max}) = \phi_F$.

$$S_E = 4\pi^2 \int d\tau \left(a^3 V - \frac{3}{8\pi} a\right)$$

$$P \simeq e^{-B}$$

$$B = S_E(\text{solution}) - \underbrace{S_E(\text{background})}_{\phi = \phi_F}$$

integration: covers all period of $\tau$.

"Thin-shell approximation"

$$\boxed{\underbrace{\dot\phi}_{\text{inside/outside the wall}}\ \underbrace{\frac{\dot a}{a}}_{\text{at the wall}} \ll 1}$$

inside/outside the wall, $\dot\phi \simeq 0$

at the wall, $a \gg 1$

$$\dot a^2 = 1 + \frac{8\pi}{3} a^2 \left(\frac{\dot\phi^2}{2} - V\right)$$

$$\simeq 1 - \frac{8\pi}{3} a^2 V \quad \text{(for inside/outside the shell)}$$

$$\ddot\phi + 3\frac{\dot a}{a}\dot\phi - V' = 0 \qquad \therefore\ \dot a \simeq \sqrt{1 - \frac{8\pi a^2}{3} V}$$

$$\simeq \ddot\phi - V'$$

$$\dot\phi\ddot\phi - \dot\phi\frac{dV}{d\phi} = 0$$

$$= \frac{1}{2}\frac{d\dot\phi^2}{d\tau} - \frac{dV}{d\tau} \longrightarrow \frac{\dot\phi^2}{2} - V = \text{const.}$$

$$\therefore\ \frac{\dot\phi^2}{2} = V(\phi) - V(\phi_T)$$

$$\dot\phi = \sqrt{2(V(\phi) - V(\phi_T))}$$

(at the shell)



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$$B_{\text{outside}} = S_E(\text{solution} \mid a > \bar a) - S_E(\text{background} \mid a > \bar a) = 0$$

$$B_{\text{shell}} = S_E(\text{solution} \mid a = \bar a) - S_E(\text{background} \mid a = \bar a)$$

$$B_{\text{inside}} = S_E(\text{solution} \mid a < \bar a) - S_E(\text{background} \mid a < \bar a)$$

$$B_{\text{shell}} = 4\pi^2 \int d\tau \left(a_{sol}^3 V - \frac{3}{8\pi} a_{sol}\right) - 4\pi^2 \int d\tau \left(a_{bg}^3 V_F - \frac{3}{8\pi} a_{bg}\right)$$

$$= 4\pi^2 \int \frac{d\phi}{\dot\phi}\left(\bar a^3 V(\phi) - \frac{3}{8\pi}\bar a\right) - 4\pi^2 \int \frac{d\phi}{\dot\phi}\left(\bar a^3 V_F - \frac{3}{8\pi}\bar a\right)$$

$$= 4\pi^2 \int \frac{d\phi}{\dot\phi}\, \bar a^3 (V - V_F)$$

$$= 4\pi^2 \bar a^3 \underbrace{\int_{\phi_T}^{\phi_F} \frac{V - V_F}{\sqrt{2(V - V_T)}}\, d\phi}_{\text{a constant only depends on } V(\phi).}$$

$$\equiv 2\pi^2 \bar a^3 \sigma$$

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$$B_{\text{inside}} = 4\pi^2 \int d\tau \left(a_{sol}^3 V_T - \frac{3}{8\pi} a_{sol}\right) - 4\pi^2 \int d\tau \left(a_{bg}^3 V_F - \frac{3}{8\pi} a_{bg}\right)$$

$$= 4\pi^2 \int_0^{\bar a} \frac{da_{sol}}{\dot a_{sol}} \left(a_{sol}^3 V_T - \frac{3}{8\pi} a_{sol}\right)$$

$$- 4\pi^2 \int_0^{\bar a} \frac{da_{bg}}{\dot a_{bg}} \left(a_{bg}^3 V_F - \frac{3}{8\pi} a_{bg}\right)$$

$$= 4\pi^2 \int_0^{\bar a} da \frac{\left(-\frac{3}{8\pi} a\right)\left(1 - \frac{8\pi V_T}{3} a^2\right)}{\sqrt{1-\frac{8\pi a^2}{3} V_T}}$$

$$- 4\pi^2 \int_0^{\bar a} da \frac{\left(-\frac{3}{8\pi}a\right)\left(1-\frac{8\pi V_F}{3}a^2\right)}{\sqrt{1-\frac{8\pi a^2}{3}V_F}}$$

$$= -\frac{3\pi}{2}\int_0^{\bar a} da\, a \left(\sqrt{1-\frac{8\pi a^2}{3}V_T} - \sqrt{1-\frac{8\pi a^2}{3}V_F}\right)$$

$$= -\frac{3\pi}{4}\int_0^{\bar a} da^2 \left(\sqrt{1-\frac{8\pi V_T}{3}a^2} - \sqrt{1-\frac{8\pi a^2}{3}V_F}\right)$$

$$= -\frac{3\pi}{2}\left(\frac{1}{8\pi V_T}\left(1-\left(1-\frac{8\pi V_T}{3}\bar a^2\right)^{\frac32}\right) - \frac{1}{8\pi V_F}\left(1-\left(1-\frac{8\pi V_F}{3}\bar a^2\right)^{\frac32}\right)\right)$$



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$$= -\frac{3}{16}\left(\frac{1}{V_T}\left(1-\left(1-\frac{8\pi V_T}{3}\bar a^2\right)^{\frac32}\right) - \frac{1}{V_F}\left(1-\left(1-\frac{8\pi V_F}{3}\bar a^2\right)^{\frac32}\right)\right)$$

$$\therefore B = -\frac{3}{16}\left(\frac{1}{V_T}\left(1-\left(1-\frac{8\pi V_T}{3}\bar a^2\right)^{\frac32}\right) - \frac{1}{V_F}\left(1-\left(1-\frac{8\pi V_F}{3}\bar a^2\right)^{\frac32}\right)\right) + 2\pi^2\sigma\bar a^3$$

$$\frac{\partial B}{\partial \bar a} = 0 = -\frac{3}{16 V_T}\left(\frac{3}{2}\right)\left(1-\frac{8\pi V_T}{3}\bar a^2\right)^{\frac12}\cdot\left(-\frac{8\pi V_T}{3}\right)2\bar a$$

stationary condition

$$- \frac{3}{16V_F}\left(\frac32\right)\left(1-\frac{8\pi V_F}{3}\bar a^2\right)^{\frac12}\left(-\frac{8\pi V_F}{3}\right)2\bar a$$

$$+ 6\pi^2\sigma\bar a^2$$

$$= \frac{3\pi\bar a}{2}\left(\sqrt{1-\frac{8\pi V_T}{3}\bar a^2} + \sqrt{1-\frac{8\pi V_F}{3}\bar a^2} + 4\pi\sigma\bar a\right) = 0$$

$$\therefore \boxed{\sqrt{1-\frac{8\pi V_T}{3}\bar a^2} - \sqrt{1-\frac{8\pi V_F}{3}\bar a^2} = 4\pi\sigma\bar a}$$

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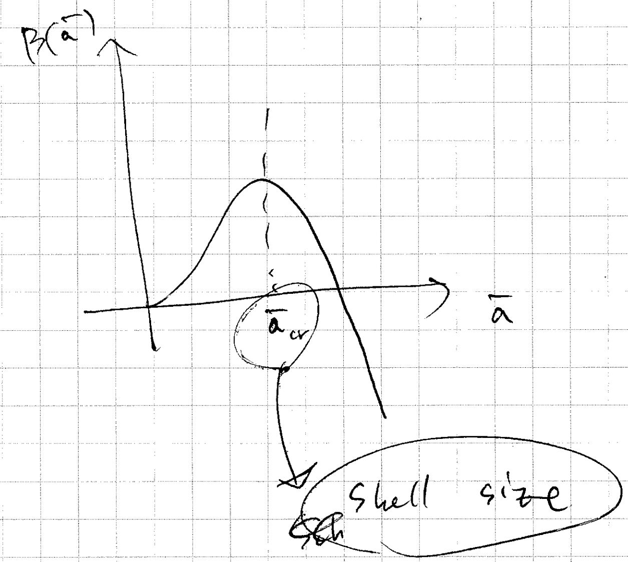
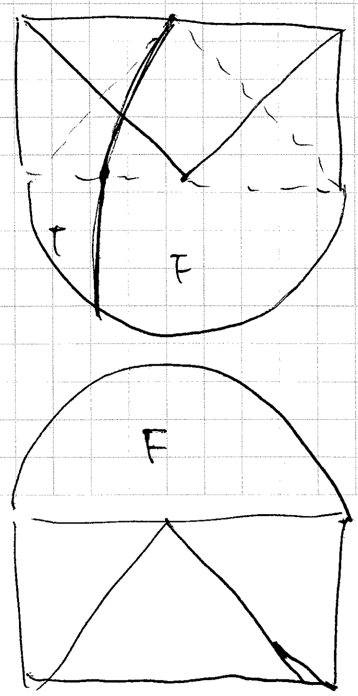
Note that Israel's junction equation;

$$\epsilon_- \sqrt{\dot r^2 + f_-} - \epsilon_+ \sqrt{\dot r^2 + f_+} = 4\pi r \sigma$$

$\longrightarrow$ $\dot r = 0$ limit

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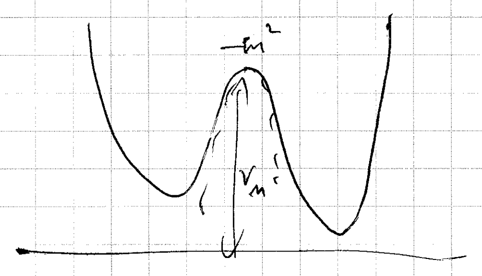
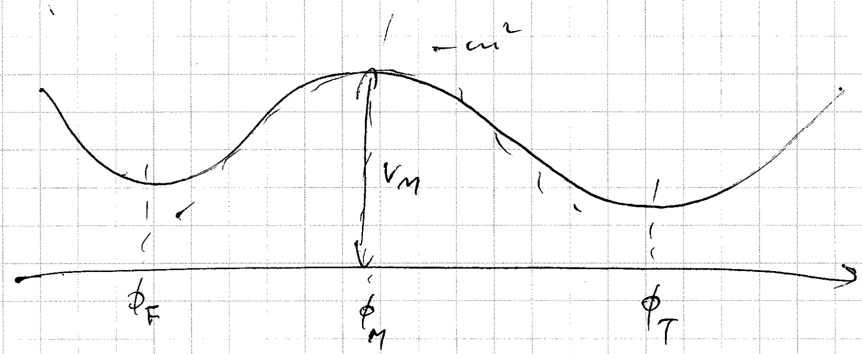
Thin-shell approx :



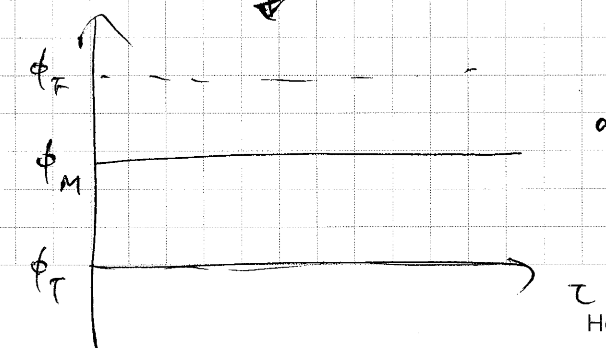

$$\frac{m^2}{V_m} \gg 1$$




$$\frac{m^2}{V_M} \ll 1$$

?


Shell becomes thicker




approaches to the local maximum

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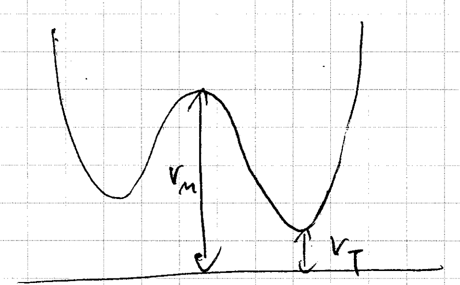
The infinitely thick-shell limit:

$e^{-2B}$ where

$$2B = -\frac{3}{8V_M} + \frac{3}{8V_T} > 0$$




"Hawking-Moss instantons"

In general, $P_{CDL} > P_{HM}$.

As $\frac{m^2}{V_M}$ decreases, $P_{CDL} \cong P_{HM}$.

If $\frac{m^2}{V_M}$ is very big, then there may be oscillating instantons. See

Hackworth and Weinberg, hep-th/0410142

Lee, Lee and Yeom, 1206.7040

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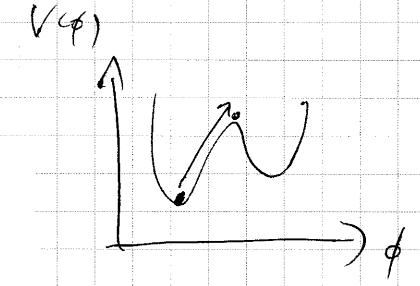
What is the interpretation of HM instantons?




Does the 'entire' universe tunnel to local maximum?

→ "Thermal interpretation" (Linde, Linde, Mezhlumian)
gr-qc/9306035

Hawking temperature of dS: $\boxed{T \propto H}$

→ $\phi$ random walks with thermal fluctuations (up to the Hubble patch)

→ Langevin equation

→ Fokker-Planck equation for probability distribution

→ Stationary limit: HM instantons

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