Lecture 1 QFT of IR divergence ~~move onto separate~~ IR 4

- Infrared Catastrophe

$|\text{hard}\rangle\ |\cancel{\text{Safe}}\ \text{photon}\rangle$? → how to describe it?

Photon emitted / absorbed ⇒ Heisenberg picture photon operator $A^\mu_H(\vec{x},t)$

$\hookrightarrow \vec{A}_H(\vec{x},t) \to \vec{A}_{in}(\vec{x},t)\quad t\to -\infty$

$\to \vec{A}_{out}(\vec{x},t)\quad t\to +\infty$ } transfer by S-matrix

Why 'soft' gauge boson special?
Spacetime symmetry
history
- old QED
- strongly
- QCD

[* S-matrix $S|\alpha\rangle_{out} = |\alpha\rangle_{in}$ $\quad S^\dagger = S \Rightarrow {}_{out}\langle\beta|S^\dagger = \langle\beta|_{in}$ or ${}_{out}\langle\beta| = {}_{in}\langle\beta|S$

$S_{\beta\alpha} \equiv {}_{out}\langle\beta|\alpha\rangle_{in} = {}_{in}\langle\beta|S|\alpha\rangle_{in}$ $\quad S a^\dagger_{out} S^\dagger = a^\dagger_{in}$ or $a_{in} = S a_{out} S^\dagger$]

$A_0 = 0.\ \vec{\nabla}\cdot\vec{A} = 0 \Rightarrow \vec{A}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2E(k)}} \sum_{\lambda=1}^{2} \left(\vec{\epsilon}(\vec{k},\lambda)\, a(\vec{k},\lambda)\, e^{+ik\cdot x} + \vec{\epsilon}^*(\vec{k},\lambda)\, a^\dagger(\vec{k},\lambda)\, e^{-ik\cdot x}\right)$ $\quad$ ($-+++$)

$= \int \frac{d^4k}{(2\pi)^4} (2\pi)^{5/2} \sqrt{2E(k)} \sum_{\lambda=1}^{2} \left(\theta(k_0)\,\delta(k^2)\,\vec{\epsilon}(\vec{k},\lambda)\, a(\vec{k},\lambda)\, e^{-ik\cdot x} + h.c.\right)$

$\partial_\nu F^{\nu\mu} = \partial_\nu(\partial^\nu A^\mu - \partial^\mu A^\nu) = J^\mu \Rightarrow \quad \Box \vec{A}_H = \vec{J}_{tr}(x) \quad \vec{\nabla}\cdot\vec{J}_{tr} = 0$ (transverse part)

$\vec{J}_{tr} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^{5/2}} \sum_{\lambda=1}^{2} \vec{\epsilon}(\vec{k},\lambda)\left(\tilde{J}(\vec{k},\lambda)\, e^{-ik\cdot x} + \tilde{J}^*(\vec{k},\lambda)\, e^{ik\cdot x}\right)$

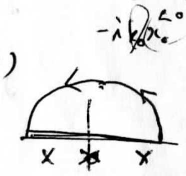
"retarded Green function" $\Delta_R(x) = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{-k^2 + i\epsilon k_0}$ $\quad \Delta_R(x) = 0\quad x^0 < 0$

"advanced Green function" $\Delta_A(x) = -\int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik\cdot x}}{-k^2 + i\epsilon k_0}$ $\quad \Delta_A(x) = 0\quad x^0 > 0$

$\Box\Delta_R(x) = \Box\Delta_A(x) = \delta^4(x)$

$k^2 + i\epsilon k_0 = (k_0 + i\epsilon)^2 - \vec{k}^2$

$k_0 = \pm|\vec{k}| + i\epsilon$



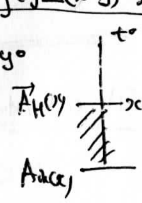
(Cauchy's theorem)

$\Rightarrow \vec{A}_H(x) = \vec{A}_{in}(x) + \int d^4y\, \Delta_R(x-y)\, \vec{J}_{tr}(y) = \vec{A}_{out}(x) + \int d^4y\, \Delta_A(x-y)\, \vec{J}_{tr}(y)$

$\frac{1}{k^2 + i\epsilon k_0} = P\left(\frac{1}{k^2}\right) - i\pi\,\epsilon(k_0)\,\delta(k^2)$, $\quad \epsilon(k_0) = \theta(k_0) - \theta(-k_0)$

$\Rightarrow \vec{A}_{out}(x) = \vec{A}_{in}(x) + \int d^4y\, \Delta(x-y)\, \vec{J}_{tr}(y)$ $\quad \boxed{\Delta(x) = \Delta_R(x) - \Delta_A(x)}$

$= 0$ for $x^0 < y^0$ $\quad\quad = 0$ for $x^0 > y^0$



$\Rightarrow$ $\cancel{\frac{1}{k^2 + i\epsilon k_0} - \frac{1}{k^2 + i\epsilon k_0} = -i\pi(\theta(k_0) - \theta(-k_0))\delta(k^2)}$

[$\frac{1}{k_0 - |\vec{k}| \pm i\epsilon} = P\left(\frac{1}{k_0 - |\vec{k}|}\right) \mp i\pi\,\delta(k_0 - |\vec{k}|)$

$\frac{1}{k_0 - |\vec{k}| + i\epsilon} - \frac{1}{k_0 + |\vec{k}| + i\epsilon} = \frac{2|\vec{k}|}{(k_0 + i\epsilon)^2 - |\vec{k}|^2}$]

$-i\pi\,\delta(k_0 - |\vec{k}|)$

$\Delta(x) = -\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}\left[\left(P\left(\frac{1}{k^2}\right) - i\pi\,\epsilon(k_0)\,\delta(k^2)\right) - \left(P\left(\frac{1}{k^2}\right) + i\pi\,\epsilon(k_0)\,\delta(k^2)\right)\right]$

$= i\int \frac{d^4k}{(2\pi)^3} e^{-ik\cdot x}\,\epsilon(k_0)\,\delta(k^2)$

$$\boxed{a_{out}(\vec{k},\lambda) = a_{in}(\vec{k},\lambda) + \frac{i}{\sqrt{2E(k)}}\tilde{J}(\vec{k},\lambda) = S^\dagger a_{in}(\vec{k},\lambda)\, S}$$

$\cancel{(2\pi)^{3/2}\sqrt{2E}}$

$\because$

$\int d^4y\, \Delta(x-y)\, \vec{J}_{tr} = \int d^4y\, i\int \frac{d^4k'}{(2\pi)^3} e^{-ik'\cdot(x-y)}\,\epsilon(k'_0)\,\delta(k'^2)\,\frac{1}{2}\int \frac{d^4k}{(2\pi)^{5/2}} \sum_\lambda \vec{\epsilon}\left(\tilde{J}\, e^{-ik\cdot y} + \tilde{J}^*\, e^{ik\cdot y}\right)$

$= \frac{i}{2}\int \frac{d^4k'\, d^4k}{(2\pi)^4 (2\pi)^{3/2}} \sum_\lambda \vec{\epsilon}\,\tilde{J}\, e^{-ik'\cdot x}\,\epsilon(k'_0)\,\delta(k'^2)\, \cancel{\int d^4y}\,(2\pi)^4\,\delta^4(k'-k) + h.c. = \frac{i}{2}\int \frac{d^4k}{(2\pi)^{3/2}} \sum_\lambda \vec{\epsilon}\,\tilde{J}\, e^{-ik\cdot x}\,\underset{\theta(k_0)-\theta(-k_0)}{\epsilon(k_0)}\,\delta(k^2) + \vec{\epsilon}\,\tilde{J}^*\, e^{ik\cdot x}\,\underset{\theta(-k_0)-\theta(k_0)}{\epsilon(-k_0)}\,\delta(k^2)$

$\theta(-k_0)$
$\vec{k} \to -\vec{k}$
$k_0 \to -k_0$

$= \frac{i}{2}\int \frac{d^4k}{(2\pi)^{3/2}} \sum_\lambda \vec{\epsilon}(\vec{k},\lambda)\,\tilde{J}(\vec{k},\lambda)\, e^{-ik\cdot x}\,\theta(k_0)\,\delta(k^2) - \vec{\epsilon}(-\vec{k},\lambda)\,\tilde{J}(-\vec{k},\lambda)\, e^{ik\cdot x}\,\theta(k_0)\,\delta(k^2)$

$+ \vec{\epsilon}(-\vec{k},\lambda)\,\tilde{J}^*(-\vec{k},\lambda)\, e^{-ik\cdot x}\,\theta(k_0)\,\delta(k^2) - \vec{\epsilon}(\vec{k},\lambda)\,\tilde{J}^*(\vec{k},\lambda)\, e^{ik\cdot x}\,\theta(k_0)\,\delta(k^2)$

$= i\int \frac{d^4k}{(2\pi)^{3/2}} \sum_\lambda \theta(k_0)\,\delta(k^2)\,\vec{\epsilon}(\vec{k},\lambda)\,\tilde{J}(\vec{k},\lambda)\, e^{-ik\cdot x} + h.c.$

L2

$$a_{out}(\vec{k},\lambda) = S^\dagger a_{in}(\vec{k},\lambda) S = a_{in}(\vec{k},\lambda) + \frac{i}{\sqrt{2E(k)}}\tilde{J}(\vec{k},\lambda)$$

$$\Rightarrow S = \exp\left(i\int\frac{d^3k}{\sqrt{2E(k)}}\sum_{\lambda=1}^{2}\left(\tilde{J}(\vec{k},\lambda)\, a^\dagger_{in}(\vec{k},\lambda) + \tilde{J}^*(\vec{k},\lambda)\, a_{in}(\vec{k},\lambda)\right)\right)$$

$\Big)\; e^{B} A e^{-B} = A + [B, A]$

Baker–Campbell–Hausdorff

$$= C \exp\left(i\int\frac{d^3k}{\sqrt{2E(k)}}\sum_{\lambda=1}^{2}\tilde{J}(\vec{k},\lambda)\, a^\dagger_{in}(\vec{k},\lambda)\right)\exp\left(i\int\frac{d^3k}{\sqrt{2E(k)}}\sum_{\lambda=1}^{2}\tilde{J}^*(\vec{k},\lambda)\, a_{in}(\vec{k},\lambda)\right)$$

~~(A) coherent state~~

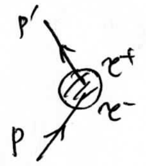
$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^{A} e^{B}$

$$C = \exp\left(-\frac{1}{2}\int\frac{d^3k}{2E(k)}\sum_{\lambda=1}^{2}|\tilde{J}(\vec{k},\lambda)|^2\right)$$

$\left(\frac{1}{2}\right)^N$

$$S|0\rangle_{in} = C\exp\left(i\int\frac{d^3k}{\sqrt{2E(k)}}\sum_{\lambda=1}^{2}\tilde{J}(\vec{k},\lambda)\, a^\dagger_{in}(\vec{k},\lambda)\right)|0\rangle \Rightarrow \text{coherent state}$$

Example: classical massive charged particle



$$x^\mu(\tau) = \frac{p^\mu}{m}\tau \quad \tau < \tau_- \qquad\qquad J^\mu(x) = e\int d\tau \frac{dx^\mu}{d\tau}\delta^4(x^\mu - x^\mu(\tau))$$
$$= \frac{p'^\mu}{m}\tau \quad \tau > \tau_+ \qquad\qquad \Big\downarrow F.T$$

$+ - - -$

$$\tilde{J}^\mu(k) = \frac{e}{(2\pi)^{3/2}}\int d\tau \frac{dx^\mu}{d\tau} e^{-ik\cdot x}$$

$$\int_{-\infty}^{\tau_-} d\tau \frac{p^\mu}{m} e^{-ik\cdot\frac{p}{m}\tau} = \frac{p^\mu}{-ik\cdot p} e^{-i\frac{k\cdot p}{m}\tau_-} \xrightarrow[k\to 0]{} i\frac{p^\mu}{k\cdot p}$$

$$\int_{\tau_+}^{+\infty} d\tau \frac{p'^\mu}{m} e^{-ik\cdot\frac{p'}{m}\tau} = \frac{p'^\mu}{ik\cdot p'} e^{-i\frac{k\cdot p'}{m}\tau_+} \xrightarrow[k\to 0]{} -i\frac{p'^\mu}{k\cdot p'}$$

$$\Rightarrow \tilde{J}^\mu(k) = \frac{ie}{(2\pi)^{3/2}}\left(\frac{p^\mu}{k\cdot p} - \frac{p'^\mu}{k\cdot p'}\right)M(k) \qquad M(k)\to 1 \text{ as } k\to 0$$

$k\cdot\tilde{J} = 0 \;(\Leftrightarrow \partial J = 0) \Rightarrow$ transverse

$$\tilde{J}^\mu(k)\tilde{J}^*_\mu(k) = \cancel{\sum_{\lambda}} \frac{e^2}{(2\pi)^3}\left(\frac{p}{k\cdot p} - \frac{p'}{k\cdot p'}\right)^2 |M|^2 = -\sum_{\lambda=1}^{2}|\tilde{J}(k)|^2$$

$$C = \exp\left(\frac{e^2}{2(2\pi)^3}\underbrace{\int\frac{d^3k}{2E(k)}\frac{1}{|\vec{k}|^2}}_{\text{log divergence}}\left(\frac{p}{E(p) - \hat{k}\cdot\vec{p}} - \frac{p'}{E(p') - \hat{k}\cdot\vec{p}'}\right)^2 |M(k)|^2\right) \sim e^{-\infty} = 0\,!$$

IR catastrophe

non-photon emission . .

⟨ ~~QFT point of view~~ ⟩ Resolution: F. Bloch, A. Nordsieck, Phys. Rev. 52 (1937) 570

Idea: finite resolution of photo detector : one can't see the existence of photon with $|\vec{p}| < \Delta$

$\Rightarrow$ "inclusive" sum of $\forall$ soft photon radiation

$$P_n(\vec{q},\lambda;\Delta) = \frac{1}{n!}\int_{|\vec{k}_i|<\Delta} d^3k_1\cdots d^3k_n \sum_{\{\lambda_i\}}\left|{}_{out}\langle\vec{q},\lambda;\vec{k}_1\lambda_1\cdots\vec{k}_n\lambda_n|0\rangle_{in}\right|^2$$

$$= \frac{|C|^2}{n!}\int_{|\vec{k}_i|<\Delta} d^3k_1\cdots d^3k_n\sum_{\{\lambda_i\}}\left|{}_{in}\langle\vec{q}\lambda;\vec{k}_1\lambda_1\cdots\vec{k}_n\lambda_n|\frac{i^{n+1}}{(n+1)!}\left(\int\frac{d^3k}{\sqrt{2E(k)}}\sum\tilde{J}(\vec{k},\lambda)a^\dagger_{in}(\vec{k},\lambda)\right)^{n+1}|0\rangle\right|^2$$

$1 = \sum\frac{1}{n!}\int\prod d^3k_i|k\rangle\langle k|$

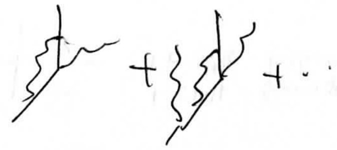
$$= |C|^2\frac{|\tilde{J}(\vec{q},\lambda)|^2}{2E(q)}\frac{1}{n!}\int_{|\vec{k}_i|<\Delta} d^3k_1\cdots d^3k_n\sum_{\{\lambda_i\}}\left|{}_{in}\langle\vec{k}_1\lambda_1\cdots\vec{k}_n\lambda_n|\frac{(A_J^\dagger(\Delta))^n}{n!}|0\rangle_{in}\right|^2 \qquad A_J(\Delta) = \int_{|\vec{k}|<\Delta}\frac{d^3k}{\sqrt{2E(k)}}\sum_\lambda\tilde{J}^*(\vec{k},\lambda)a_{in}(\vec{k},\lambda)$$

$$= |C|^2\frac{|\tilde{J}(\vec{q},\lambda)|^2}{2E(q)}\sum_\alpha\left|{}_{in}\langle\alpha|\frac{(A_J^\dagger(\Delta))^n}{n!}|0\rangle_{in}\right|^2 \sim |C|^2\frac{|\tilde{J}(\vec{q},\lambda)|^2}{2E(q)}\frac{1}{(n!)^2}\underbrace{{}_{in}\langle 0|A_J^n(\Delta)(A_J^\dagger(\Delta))^n|0\rangle_{in}}$$

$$= |C|^2\frac{|\tilde{J}(\vec{q},\lambda)|^2}{2E(q)}\frac{1}{n!}\left(\int_{|\vec{k}|<\Delta}\frac{d^3k}{2E(k)}\sum_\lambda|\tilde{J}(\vec{k},\lambda)|^2\right)^n \qquad = n!\left(\int_{|\vec{k}|<\Delta}\frac{d^3k}{2E(k)}\sum_\lambda|\tilde{J}(\vec{k},\lambda)|^2\right)^n$$

$$\Rightarrow P_{tot}(q,\lambda;\Delta) = \sum_{n=0}^{\infty} P_n(q,\lambda;\Delta) = \exp\left[-\int \frac{d^3k}{2E(k)} \sum_{\lambda} |f(k,\lambda)|^2\right] \frac{|f(q)|^2}{2E(q)} \exp\left[\int_{|k|<\Delta} \frac{d^3k}{2E(k)} \sum_\lambda |f(k,\lambda)|^2\right]$$

$$= \exp\left(-\int_{|k|>\Delta} \frac{d^3k}{2E(k)} \sum_\lambda |f(k,\lambda)|^2\right) \frac{|f(q,\lambda)|^2}{2E(q)}$$



QFT point of view

Scalar: $i(2\pi)^4 e(2p^\mu+q^\mu) \frac{-i}{(2\pi)^4} \frac{1}{(p+q)^2+m^2-i\epsilon} \xrightarrow{q\to 0} \frac{e p^\mu}{p\cdot q - i\epsilon}$

Spin-1/2: $\bar u(p,\sigma)(-(2\pi)^4 e\gamma^\mu) \frac{-i}{(2\pi)^4} \frac{-i(\not p+\not q)+m}{(p+q)^2+m^2-i\epsilon} \xrightarrow{q\to 0} \frac{e\cdot p^\mu}{p\cdot q - i\epsilon}$

$\gamma^\mu \not p = \not p\gamma^\mu + p^\mu \quad i\not p = m$ on-shell

$\rightarrow \frac{e p^\mu}{-p\cdot q - i\epsilon}$

$$\Rightarrow M^\mu_{\beta\alpha}(q) = M_{\beta\alpha} \sum_n \frac{\eta_n e_n p_n^\mu}{p_n\cdot q - i\eta_n\epsilon} \qquad \eta_n = \begin{cases} +1 & \text{outgoing} \\ -1 & \text{incoming} \end{cases}$$

the same factor in semi-classical analysis + "factorization"

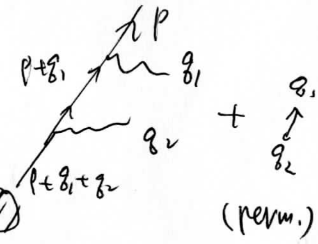
* How can we make sure IR divergence doesn't come from external charged particles?

S. Weinberg. Phys. Rev. 135 (1964) B1049

Gauge invariance = charge conservation $\quad q_\mu M^\mu_{\beta\alpha} = M_{\beta\alpha}\left(\sum_n \eta_n e_n\right) = 0$

for gravity $M^{\mu\nu}_{\beta\alpha}(q) = M_{\beta\alpha} \sum_n \frac{\eta_n f_n p_n^\mu p_n^\nu}{p_n\cdot q - i\eta_n\epsilon}$

$q_\mu M^{\mu\nu}_{\beta\alpha} = M_{\beta\alpha} \sum_n \eta_n f_n p_n^\nu \qquad f_n \sim$ universal! $(= \sqrt{8\pi G_N})$



$$= \left(\frac{\eta e p^{\mu_1}}{p\cdot q_1 - i\eta\epsilon}\right)\left(\frac{\eta e p^{\mu_2}}{p\cdot(q_1+q_2) - i\eta\epsilon}\right) + \left(\frac{\eta e p^{\mu_1}}{p\cdot q_2 - i\eta\epsilon}\right)\left(\frac{\eta e p^{\mu_2}}{p\cdot(q_1+q_2)-i\eta\epsilon}\right)$$

(perm.)

$$= \left(\frac{\eta e p^{\mu_1}}{p\cdot q_1 - i\eta\epsilon}\right)\left(\frac{\eta e p^{\mu_2}}{p\cdot q_2 - i\eta\epsilon}\right) \qquad \frac{\cancel{e}\, p\cdot(q_1+q_2)}{p\cdot q_1\; p\cdot q_2}$$

$$\rightarrow \frac{1}{(p\cdot q_1 - i\eta\epsilon)} \frac{1}{p\cdot(q_1+q_2) - i\eta\epsilon} \frac{1}{p\cdot(q_1+q_2+q_3)-i\eta\epsilon} \cdots + \text{perm.} = \frac{1}{p\cdot q_1 - i\eta\epsilon} \frac{1}{p\cdot q_2 - i\eta\epsilon} \frac{1}{p\cdot q_3 - i\eta\epsilon} \cdots$$

$$\therefore M^{\mu_1\cdots\mu_N}_{\beta\alpha}(q_1\cdots q_N) = M_{\beta\alpha} \prod_{r=1}^{N} \left(\sum_n \frac{\eta_n e_n p_n^{\mu_r}}{p_n\cdot q_r - i\eta_n\epsilon}\right)$$

- Virtual soft photons $\qquad \lambda: m_\gamma$

$$\Rightarrow \frac{1}{N!\,2^N}\left[\frac{1}{(2\pi)^4} \sum_{nm} e_n e_m \eta_n \eta_m J_{nm}\right]^N \qquad J_{nm} = -i(p_n\cdot p_m) \int_{\lambda\le|q|\le\Lambda} \frac{d^4q}{(q^2-i\epsilon)(p_n\cdot q - i\eta_n\epsilon)(-p_m\cdot q - i\eta_m\epsilon)}$$

incoming/outgoing

4 poles: $q^0 = |q| - i\epsilon, \; -|q| + i\epsilon$

$\vec v_n\cdot\vec q - i\eta_n\epsilon \quad \vec v_m\cdot\vec q + i\eta_m\epsilon$

$$\Rightarrow M^\Lambda_{\beta\alpha} = M^\lambda_{\beta\alpha} \exp\left[\frac{1}{2(2\pi)^4} \sum_{nm} e_n e_m \eta_n \eta_m J_{nm}\right]$$

$\sum_\alpha$

n outgoing, m incoming: $\eta_n = +1 \quad \eta_m = -1$

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$$J_{nm} = -i(p_n\cdot p_m)\int_{\lambda\le|\vec q|\le\Lambda} \frac{d^4q}{(q^2-i\epsilon)(p_n\cdot q - i\eta_n\epsilon)(-p_m\cdot q - i\eta_m\epsilon)}$$

$\vec p_n\cdot\vec q = p_n^0\, v_n$ $\qquad \vec v_n = \frac{\vec p_n}{p_0}$

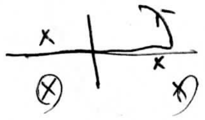
$q_0^2-\vec q^{\,2} = (\vec v_n - \vec q)\cdot\vec q$

$p_m q = p_m^0(\vec v_n-\vec q)\cdot\vec q$

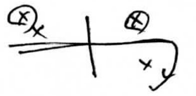
Poles: $q^0 = |\vec q| - i\epsilon \quad -|\vec q| + i\epsilon \quad \vec v_n\cdot\vec q - i\eta_n\epsilon \quad \vec v_m\cdot\vec q + i\eta_m\epsilon$

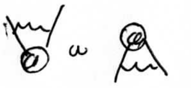
i) 

n outgoing $\eta_n = +1$
m incoming $\eta_m = -1$



n incoming $\eta_n = -1$
m outgoing $\eta_m = +1$



 $\eta_n = \eta_m = \pm 1$

$$\Rightarrow J_{nm} = -\pi(p_n\cdot p_m)\int_{\lambda\le|\vec q|\le\Lambda}\frac{d^3q}{|\vec q|^3(E_n-\hat q\cdot\vec p_n)(E_m-\hat q\cdot\vec p_m)}$$

$$\Rightarrow J_{nm} = -\pi(p_n\cdot p_m)\int_{\lambda\le|\vec q|\le\Lambda}\frac{d^3q}{|\vec q|^3(E_n-\hat q\cdot\vec p_n)(E_m-\hat q\cdot p_m)}$$

$$-\frac{4i\pi^3}{\beta_{nm}}\ln\left(\frac{\Lambda}{\lambda}\right), \quad \beta_{nm} = \sqrt{1-\frac{m_n^2 m_m^2}{(p_n\cdot p_m)^2}}$$

"velocity"

$\underbrace{\qquad}$ no contribution in taking absolute value of J_{nm}

$$\mathrm{Re}\,J_{nm} = -\pi(p_n\cdot p_m)\int_{\lambda\le|\vec q|\le\Lambda}\frac{d^3q}{|\vec q|^3(E_n-\hat q\cdot\vec p_n)(E_m-\hat q\cdot\vec p_m)}$$

$$= \frac{2\pi^2}{\beta_{nm}}\ln\left(\frac{1+\beta_{nm}}{1-\beta_{nm}}\right)\ln\left(\frac{\Lambda}{\lambda}\right)$$

$$\Gamma^\lambda_{\beta\alpha} = |M^\lambda_{\beta\alpha}|^2 = |M^\Lambda_{\beta\alpha}|^2\exp\left(\frac{1}{(2\pi)^4}\sum_{nm} e_n e_m\eta_n\eta_m J_{nm}\right) = \Gamma^\Lambda_{\beta\alpha}\left(\frac{\lambda}{\Lambda}\right)^{A(\alpha\to\beta)}$$

$$A(\alpha\to\beta) = -\frac{1}{8\pi^2}\sum_{nm}\frac{e_n e_m\eta_n\eta_m}{\beta_{nm}}\ln\left(\frac{1+\beta_{nm}}{1-\beta_{nm}}\right) \ (>0)$$

– real soft photons

$$M^\lambda_{\beta\alpha}(\vec q_1, \lambda_1\cdots) = M^\lambda_{\beta\alpha}\prod_{r=1}^{N}\frac{1}{(2\pi)^{3/2}(2|\vec q_r|)^{1/2}}\sum_n\frac{\eta_n e_n(p_n\cdot\epsilon^*(\vec q_r,\lambda_r))}{p_n\cdot q_r}$$

$$\sum_{\lambda=\pm1}\epsilon_\mu(\vec q\lambda)\epsilon_\nu(\vec q\lambda) = \eta_{\mu\nu} + q_\mu c_\nu + q_\nu c_\mu$$

$$c^0 = \frac{1}{2|\vec q|} \quad \vec c = \frac{-\vec q}{2|\vec q|^2}$$

$$\to d\Gamma^\lambda_{\beta\alpha}(\vec q_1\ldots\vec q_N) = \Gamma^\lambda_{\beta\alpha}\prod_{r=1}^{N}\frac{1}{(2\pi)^3(2|\vec q_r|)}\sum_{nm}\frac{\eta_n\eta_m e_n e_m(p_n\cdot p_m)}{(p_n\cdot q_r)(p_m\cdot q_r)}$$

$$-\pi(p_n\cdot p_m)\int\frac{d^2\hat q}{(E_n-\hat q\cdot\vec p_n)(E_m-\hat q\cdot\vec p_m)} = \frac{2\pi}{\beta_{nm}}\ln\left(\frac{1+\beta_{nm}}{1-\beta_{nm}}\right)$$

$$= \Gamma^\lambda_{\beta\alpha}\frac{d\omega_1}{\omega_1}\cdots\frac{d\omega_N}{\omega_N}\underbrace{\left(-\frac{1}{8\pi^2}\sum_{nm}\frac{e_n e_m}{\beta_{nm}}\eta_n\eta_m\ln\left(\frac{1+\beta_{nm}}{1-\beta_{nm}}\right)\right)}_{A(\alpha\to\beta)}$$

$\Rightarrow$ $\Gamma_{\beta\alpha}|_E$ ~~cont~~ for $\lambda\le\omega_r\le E$. $\sum_r\omega_r\le E_T$

$$\Gamma^\lambda_{\beta\alpha}(E,E_T) = \Gamma^\lambda_{\beta\alpha}\sum_{N=0}^{\infty}\frac{A(\alpha\to\beta)^N}{N!}\int_{\substack{\lambda\le\omega_r\le E\\ \sum\omega_r\le E_T}}\prod_{r=1}^{N}\frac{d\omega_r}{\omega_r}$$

$$\theta\left(E_T-\sum_r\omega_r\right) = \frac{1}{\pi}\int_{-\infty}^{\infty}du\,\frac{\sin E_T u}{u}\,e^{iu\sum\omega_r}$$

$$= \Gamma^\lambda_{\beta\alpha}\frac{1}{\pi}\int_{-\infty}^{\infty}du\,\frac{\sin E_T u}{u}\exp\left[A(\alpha\to\beta)\int_\lambda^E\frac{d\omega}{\omega}e^{i\omega u}\right]$$

$$\hookrightarrow \int_\lambda^E\frac{d\omega}{\omega}\left[(e^{i\omega u}-1)+1\right]$$

$$= f\left(\frac{E}{E_T}; A(\alpha\to\beta)\right)\left(\frac{E}{\lambda}\right)^{A(\alpha\to\beta)}\Gamma^\lambda_{\beta\alpha}$$

$$f(x;A) = \frac{1}{\pi}\int_{-\infty}^{\infty}du\,\frac{\sin u}{u}\exp\left(A\int_0^x\frac{d\omega}{\omega}(e^{i\omega u}-1)\right) = 1-\frac{A^2\theta(x-\frac{1}{2})}{2}\int_{1-x}^{x}\frac{d\omega}{\omega}\ln\left(\frac{x}{1-\omega}\right)+$$

$$f(1,A)\simeq 1-\frac{\pi^2}{12}A^2$$

IR finite!

D. R. Yennie. S. C. Frautschi. H. Suura.
Ann. Phys. 13 (1961) 379 etc.

T. D. Lee Nauenberg Kinoshita

· Faddeev-Kulish. PS

5

Conceptual issue an inclusive sum of soft photons

- makes sense. since the state created by charged particle is not physical

- physical state is gauge invariant (more precisely BRST invariant) — closed but not exact.

↳ excitation of combination of operators (BRST) gauge invariant

- gauge invariant + unitary S-matrix can be defined in terms of such physical states

↳ transition between states at $t=-\infty$ and $t=+\infty$

$$\int \frac{D\phi}{V_{gauge}} e^{-S_0} \sim \int D\phi\, \delta(F^A(\phi)) Db_A Dc^\alpha e^{-S_0 - \int b_A \delta_\alpha F^A c^\alpha} \qquad [\delta_\alpha, \delta_\beta] = i f_{\alpha\beta}^{\ \ \gamma} \delta_\gamma$$

$$\sim \int D\phi\, DB_A Db_A Dc^\alpha e^{-S_0 - i\underbrace{\int B_A F^A(\phi)}_{S_1} - \underbrace{\int b_A c \delta_\alpha F^A) c^\alpha}_{S_2}}$$

$\delta\phi_i = -i\epsilon c^\alpha \delta_\alpha \phi_i \quad \rightarrow \delta S_0 = 0$

$\delta b_A = -\epsilon B_A$

$\delta c^\alpha = -\frac{1}{2}\epsilon c^\beta c^\gamma f_{\beta\gamma}^{\ \ \alpha}$

$\delta B_A = 0$

$$\frac{i}{2} f_{\beta\alpha}^{\ \ \gamma} \delta_\gamma F^A$$

$$\delta(S_1+S_2) = i\int B_A \cancel{(-i\epsilon c^\alpha \delta_\alpha \phi_i \partial_i F^A)} + \int (\epsilon B_A) \delta_\alpha F^A c^\alpha - \int b_A \delta_\beta \delta_\alpha F^A (-i\epsilon c^\beta) c^\alpha + \int b_A \delta_\alpha F^A \left(-\frac{1}{2}\epsilon c^\beta c^\gamma f_{\beta\gamma}^{\ \ \alpha}\right)$$

$$= 0$$

$$+ \delta(b_A F^A) = -\epsilon B_A F^A + b_A(-i\epsilon c^\alpha \delta_\alpha F^A) = i\epsilon (i B_A F^A + b_A c^\alpha \delta_\alpha F^A) = i\epsilon (S_1 + S_2) \qquad i\epsilon b_A c^\alpha$$

$$\epsilon \delta_F \langle \psi | \psi' \rangle = \epsilon \langle \psi | (B_A \delta F^A + b_A c^\alpha \delta_\alpha \delta F^A) | \psi' \rangle = \epsilon \langle \psi | \{Q_B, b_A \delta F^A\} | \psi' \rangle \overset{!}{=} 0 \Rightarrow Q |phys\rangle = 0$$

change in gauge fixing cond.

refs: V. Chung. Phys. Rev. 140B (1965) 1110

T. Kibble. Phys. Rev. 173 1527; 174 1882; 175 1624 (1968)

P. P. Kulish. L. D Faddeev. Theor. Math. Phys. 4 (1970) 745

$$H_{int} = -\int d^3x\, e \bar{\psi}^f \gamma^\mu \psi^f A_\mu^f$$

$$\psi^f = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(b_p^s u^s(p) e^{-ip\cdot x} + d_p^{s\dagger} v^s(p) e^{ip\cdot x} \right)$$

$$\bar{\psi}^f = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left(d_p^s \bar{v}^s(p) e^{-ip\cdot x} + b_p^{s\dagger} \bar{u}^s(p) e^{ip\cdot x} \right)$$

$$A_\mu^f(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \sum_\lambda \left(a_k^\lambda \epsilon_\mu^\lambda(k) e^{-ik\cdot x} + a_k^{\lambda\dagger} \epsilon_\mu^{\lambda *}(k) e^{ik\cdot x} \right)$$

$+ \cdots$

$\mapsto V_{ref}$

$$-e\int d^3x \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\frac{d^3p}{(2\pi)^3\sqrt{2\varepsilon_p}}\frac{d^3q}{(2\pi)^3\sqrt{2\varepsilon_q}} \quad \cancel{\vec{k}=\vec{p}+\vec{q}} : \dots$$

$$\Big(a_k b_p^\dagger b_q \epsilon_\mu \bar{u}(p)\gamma^\mu u(q) e^{-i(k-p+q)\cdot x} + a_k^\dagger b_p^\dagger b_q \epsilon_\mu^* \bar{u}(p)\gamma^\mu u(q) e^{i(k+p-q)\cdot x} \Rightarrow$$

$$+ a_k b_p^\dagger d_q^\dagger \epsilon_\mu \bar{u}(p)\gamma^\mu v(q) e^{-i(k-p-q)\cdot x} + a_k^\dagger b_p^\dagger d_q^\dagger \epsilon_\mu^* \bar{u}(p)\gamma^\mu v(q) e^{i(k+p+q)\cdot x}$$

$\to \omega_k = |\vec{k}| \geq 0$, $\varepsilon_p = \sqrt{p^2+m^2} > 0$: $\underbrace{\pm(\omega_k+\varepsilon_p+\varepsilon_q)t}_{\neq 0}$

$$+ a_k d_p b_q \epsilon_\mu \bar{v}(p)\gamma^\mu u(q) e^{-i(k+p+q)\cdot x} + a_k^\dagger d_p b_q \epsilon_\mu^* \bar{v}(p)\gamma^\mu u(q) e^{i(k-p-q)\cdot x}$$

$$+ a_k d_p d_q^\dagger \epsilon_\mu \bar{v}(p)\gamma^\mu v(q) e^{-i(k+p-q)\cdot x} + a_k^\dagger d_p d_q^\dagger \epsilon_\mu^* \bar{v}(p)\gamma^\mu v(q) e^{i(k-p+q)\cdot x}\Big)$$

$\to \vec{k} = \vec{p}+\vec{q}$

$$\omega_k - \varepsilon_p - \varepsilon_q = |\vec{k}| - \sqrt{p^2+m^2} - \sqrt{(\vec{p}-\vec{k})^2+m^2} \neq 0$$

$\because$ for $=0$,

$$k^2 + \cancel{p^2+m^2} - 2|\vec{k}|\sqrt{p^2+m^2} = k^2 + \cancel{p^2+m^2} - 2\vec{k}\cdot\vec{p}$$

$$k^2(p^2+m^2) = k^2p^2\cos\theta$$

$$p^2(1-\cos\theta) = -m^2$$

nonsense

$t\to\pm\infty$ Vanishing phase?

on-shell: $0 = (\not{p}-m)u^s(p) = \bar{u}^s(p)(\not{p}-m)$ $\qquad \bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left(\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right)u(p)$

$= (\not{p}+m)v^s(p) = \bar{v}^s(p)(\not{p}+m)$ $\qquad \bar{v}(p)\gamma^\mu v(p) = -\bar{v}(p')\left(\frac{(p'+p)^\mu}{2m} + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\right)v(p)$

$\bar{u}^r(p)u^s(p) = 2m\delta^{rs}$ $\qquad \bar{v}^r(p)v^s(p) = -2m\delta^{rs}$

$$\Rightarrow H_{int}^{asy} = \lim_{t\to|\infty|} H_{int} = -\lim_{|t|\to\infty} e\int d^3x \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\frac{d^3p}{(2\pi)^3\sqrt{2\varepsilon_p}}\frac{d^3q}{(2\pi)^3\sqrt{2\varepsilon_q}}$$

$$\times\Big(a_k b_p^\dagger b_q \epsilon_\mu \bar{u}(p)\gamma^\mu u(q) e^{-i(k-p+q)\cdot x} + a_k^\dagger b_p^\dagger b_q \epsilon_\mu^* \bar{u}(p)\gamma^\mu u(q) e^{i(k+p-q)\cdot x}$$

$$+ a_k d_p d_q^\dagger \epsilon_\mu \bar{v}(p)\gamma^\mu v(q) e^{-i(k+p-q)\cdot x} + a_k^\dagger d_p d_q^\dagger \epsilon_\mu^* \bar{v}(p)\gamma^\mu v(q) e^{i(k-p+q)\cdot x}$$

$$\varepsilon_p - \varepsilon_{p-k} = \sqrt{p^2+m^2} - \sqrt{(\vec{p}-\vec{k})^2+m^2} \simeq \frac{\vec{k}\cdot\vec{p}}{\varepsilon_p} \qquad |\vec{k}| \ll |\vec{p}|$$

$$\varepsilon_x \sim -e\int_{|\vec{k}|\simeq 0} \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\frac{d^3p}{(2\pi)^3\sqrt{2\varepsilon_p}}\frac{d^3q}{(2\pi)^3\sqrt{2\varepsilon_q}}(2\pi)^3\delta^3(\vec{k}-\vec{p}+\vec{q})\, a_k b_p^\dagger b_q \bar{u}(p)\gamma^\mu u(q) e^{-i\omega_k t} e^{i\frac{\vec{k}\cdot\vec{p}}{\varepsilon_p}t}$$

$$= e\int_{|\vec{k}|\simeq 0} \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\frac{d^3p}{(2\pi)^3 2\varepsilon_p} a_k b_p^\dagger b_p \frac{\epsilon_\mu p^\mu}{m}(2m) e^{-i\omega t} e^{i\frac{\vec{k}\cdot\vec{p}}{\varepsilon_p}t}$$

$$= e\int_{|\vec{k}|\simeq 0} \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}} a_k \epsilon_\mu e^{-i(\omega_k t - \vec{k}\cdot\vec{x})} \underbrace{\int d^3p \frac{p^\mu}{(2\pi)^3\varepsilon_p} b_p^\dagger b_p \int d^3x\, \delta^3\!\left(\vec{x}-\frac{\vec{p}}{\varepsilon_p}t\right)}_{J^\mu}$$

(the same situation as semi-classical analysis)

$$x = (t,\vec{x}) = \left(\frac{p^0}{\varepsilon_p}t, \frac{\vec{p}}{\varepsilon_p}t\right) = \frac{p\,t}{\varepsilon_p} \Rightarrow \underline{k\cdot x = \frac{p\cdot k\,t}{\varepsilon_p}}$$

$$H_{int}^{asy} = -e\int d^3x\, A_\mu^{soft}(x) J^\mu_{as}(x)$$

$$J^\mu_{as} = \int\frac{d^3p}{(2\pi)^3}\frac{p^\mu}{\varepsilon_p}\rho(p)\,\delta^3\!\left(\vec{x}-\frac{\vec{p}}{\varepsilon_p}t\right) \qquad \rho(p) = \sum_s b_p^{s\dagger}b_p^s - d_p^{s\dagger}d_p^s$$

(2)

$i\frac{d}{dt}U_{asy} = H^S_{asy} U_{asy}$ $\qquad U_{asy} = e^{-iH_0 t} Z(t) \rightarrow i\frac{d}{dt}Z(t) = H^I_{asy} Z(t)$

$H^S_{asy} = H_0 + H^{asy}_{int}$ $\qquad H^I_{asy} = e^{iH_0 t} H^S_{asy} e^{-iH_0 t}$

$$Z(t) = Te^{-i\int^t H^I_{asy} dt} = \exp\left(-i\int^t H^I_{asy}(t')dt' - \frac{1}{2}\int dt' \int^{t'} dt'' [H^I_{asy}(t'), H^I_{asy}(t'')] + \cdots\right)$$

Magnus expansion.

$[J^\mu_{asy}(x), J^\nu_{asy}(y)] = 0 \qquad [J^\mu_{asy}(x), A^f_\nu(y)] = 0 \qquad [A^f_\mu(x), A^f_\nu(y)] = -i\eta_{\mu\nu} D(x-y)$ $\quad D(x) = \frac{1}{2\pi}\epsilon(x^0)\delta(x^2)$

$$\Rightarrow [H^I_{asy}(x^0), H^I_{asy}(y^0)] = e^2\int d^3x\, d^3y\, [A^f_\mu(x), A^f_\nu(y)] J^\mu_{asy} J^\nu_{asy}(y)$$

$$= -ie^2\int d^3x\, d^3y \frac{1}{2\pi}\epsilon(x^0-y^0)\delta((x-y)^2)\frac{d^3p}{(2\pi)^3}\frac{p^\mu}{E_p}\rho(p)\delta^3(\vec{x}-\frac{\vec{p}}{E_p}x^0)\frac{d^3q}{(2\pi)^3}\frac{q^\mu}{E_q}\rho(q)\delta^3(\vec{y}-\frac{\vec{q}}{E_q}y^0)$$

$$= -\frac{ie^2}{2\pi}\int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{p\cdot q}{E_p E_q}\epsilon(x^0-y^0)\delta\left(\left(\frac{p}{E_p}x^0 - \frac{q}{E_q}y^0\right)^2\right)\rho(p)\rho(q)$$

$$t'_{1,2} = \frac{E_q x^0}{m^2 E_p}\left(p\cdot q \pm \sqrt{(p\cdot q)^2 - m^4}\right) \quad \delta\left(\frac{m^2}{E_p^2}x^{0\,2} + \frac{m^2}{E_q^2}y^{0\,2} - \frac{2p\cdot q}{E_p E_q}x^0 y^0\right) = \frac{1}{2\left|\frac{m^2}{E_q^2}y^0 - \frac{p\cdot q}{E_p E_q}x^0\right|}(\delta(y^0-t'_1) + \delta(y^0-t'_2))$$

$$= \frac{E_p E_q}{2x^0\sqrt{(p\cdot q)^2 - m^4}}(\delta(y^0-t'_1) + \delta(y^0-t'_2))$$

$$\left[[H^I_{asy}(x^0), H^I_{asy}(y^0)], \begin{matrix}J^\mu_{asy}\\ A_\mu^f\end{matrix}\right] = 0$$

$$Z(t) = e^{R(t)}e^{i\Phi(t)} = \exp\left(-i\int^t H^I_{asy}(t')dt' - \frac{1}{2}\int^t dt'\int^{t'} dt''[H^I_{asy}(t'), H^I_{asy}(t'')]\right)$$

$$ie\int\frac{d^3p}{(2\pi)^3}\frac{p^\mu}{E_p}\rho(p)\int_{k\simeq 0}\frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left(a_k\epsilon_\mu e^{-i\frac{p\cdot k}{E_p}t} + a_k^\dagger\epsilon_\mu^* e^{i\frac{p\cdot k}{E_p}t}\right)$$

$$= -e\int\frac{d^3p}{(2\pi)^3}\rho(p)\int_{k\simeq 0}\frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\frac{p^\mu}{p\cdot k}\left(a_k\epsilon_\mu e^{-i\frac{p\cdot k}{E_p}t} - a_k^\dagger\epsilon_\mu^* e^{i\frac{p\cdot k}{E_p}t}\right) \sim$$ anti Hermitian

= anti Hermitian

$$\Rightarrow \phi_{asy}(x) = U^\dagger_{asy}\phi_S(\vec{x})U_{asy} = Z^\dagger(t)\phi_I(x)Z(t)$$

$\sim$

$$e^{-R}\phi_I e^{R} = \int\frac{d^3p}{(2\pi)^3\sqrt{2E_p}}D(p,t)\sum_s\left(b(p,s)u^s(p)e^{-ip\cdot x} + d^\dagger(p,s)v^s(p)e^{ip\cdot x}\right)$$

$[\rho, d_p] = -(2\pi)^3\delta(\vec{p}-\vec{q})d_p$

$[\rho, b_p^\dagger] = -(2\pi)^3\delta(\vec{p}-\vec{q})b_p^\dagger$

$$D(p,t) = \exp\left(-e\int_{soft}\frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left(\sum_\lambda\frac{p\cdot\epsilon_\lambda}{p\cdot k}a_k e^{-i\frac{t k\cdot p}{E_p}} - \frac{p\cdot\epsilon_\lambda^*}{p\cdot k}a_k^\dagger e^{it k\cdot p/E_p}\right)\right)$$

Gauge invariance? Yes! $\quad \epsilon_\lambda \to \epsilon_\lambda + $

$$A_\mu = \int\frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left(a_k\epsilon_\mu e^{-ik\cdot x} + a_k^\dagger\epsilon_\mu^* e^{ik\cdot x}\right) \qquad A_\mu + \partial_\mu\Lambda$$

$$\Lambda = \int\frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\left(c_k e^{-ik\cdot x} + c_k^\dagger e^{ik\cdot x}\right) \quad \partial_\mu\Lambda = \int\frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}(-ik_\mu)\left(c_k e^{-ikx} - c_k^\dagger e^{ikx}\right)$$

$\sim$ ex ghost

$$\Delta a_k\epsilon_\mu \Rightarrow -ik_\mu c_k \qquad \Delta a_k^\dagger\epsilon_\mu^* \Rightarrow ik_\mu c_k^\dagger$$

$$\Delta D(p,t) = \exp\left(-e\int\frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}\sum_k\left(-ic_k e^{-i\frac{k\cdot p}{E_p}t} - ic_k^\dagger e^{i\frac{k\cdot p}{E_p}t}\right)\right) = \underline{e^{ie\Lambda}} \rightarrow \text{cancelled by } \underline{e^{-ie\Lambda}}\,\phi$$

8

See. e.g. G. Dvali, C. Gomez, D. Lüst, PLB 753 (2016) 173, 1509.2114

C. Gómez, R. Letschka, S. Zell, 1807.07079

1601.00921 S. Hawking, M. J. Perry, A. Strominger, PRL 116 (2016) 231301

A. ~~Averin, G. Dvali, C. Gomez, D. Lüst~~ ~~JHEP 1606 (2016) 088~~

M. Mirbabayi, M. Porrati, PRL 117 (2016) 211301, 1607.03120

$\xleftarrow{\text{QED process}} |\alpha\rangle|0\rangle \xrightarrow{\text{"dressing"}} |\alpha\rangle$

hard sector, ρ ~~charged~~, soft sector

$\lim_{t\to\infty} |\beta\rangle_{out} Z(t)|0\rangle \longleftarrow |\beta\rangle_{out}|0\rangle \xleftarrow{\text{QED process}} |\alpha\rangle_{in}|0\rangle \xrightarrow[\text{dressing } t\to-\infty]{} \lim |\alpha\rangle_{in} Z(t)|0\rangle$

dressing

S-matrix

depends ρ. $\quad Z(t) \sim e^{-e\sum(f(t)a_k + f^*(t)a_k^\dagger)} \sim e^{-\frac{e^2}{2}|f|^2}\, e^{ef a^\dagger}\, e^{-ef a}$

$Z(t)|0\rangle \sim$ coherent state $= |f(t)\rangle$ ← depending on hard process, determined by (correlation to physics ~~deep~~ inside BH?)

~~Consider~~ ~~Suppose we are sitting on the null infinity.~~

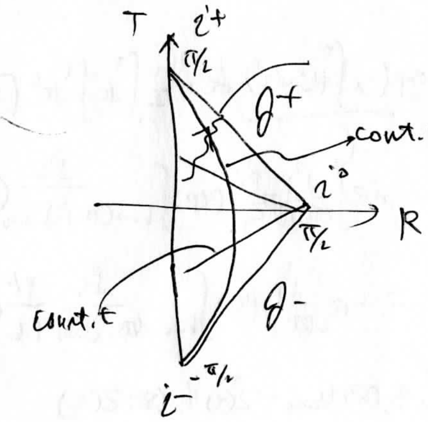
* Penrose diagram → ~~to see~~ conformal transformed spacetime revealing causal structure more evidently

$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2 \qquad T+R = \arctan(t+r)$

$\qquad T-R = \arctan(t-r)$

$= \frac{1}{\cos^2(T+R)\cos^2(T-R)}\left(-dT^2 + dR^2 + \left(\frac{\sin 2R}{2}\right)^2 d\Omega_2^2\right)$

$|T \pm R| < \frac{\pi}{2} \quad R \geq 0$

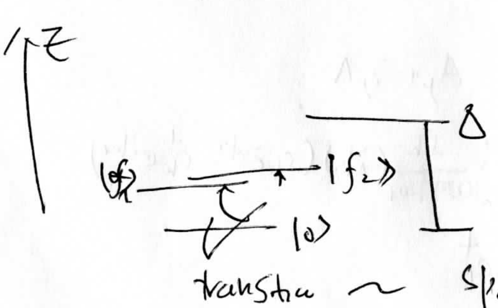


light-like geodesic → $\pi/4$ slope line.

→ Suppose 1) we are sitting on the null infinity

2) there is no massless charged particle.

⇒ detail information deep inside null infinity is contained in "current" f(t), but softness of photon make us to be ignorant of it



⇒ looks as if degenerate vac.

transition ~ spontaneously broken symmetry?

difference: undetectable soft photon (soft photon theorem?)

- Gauge fixing

for a review. A. Strominger 1703.05448.

gauge sym. ~ redundancy of physical deg. of freedom.

→ choose one 'orbit' of the physical d.o.f. through the gauge fixing cond. $G(A_\mu)=0$

Lorentz group representation ⇒ 2 physical d.o.f but 4 manifest d.o.f

In fact. A_0 ~ lagrange multiplier (no dynamics) → constraint

Conservation quantity of the system ⇒ generator of gauge transformation.

A_μ + conj. momenta

→ constraint + gauge fixing ⇒ 2+2 phase space

$$\mathcal{L} = -\frac{1}{2g^2}\left(E_k \dot{A}_k - \frac{1}{2}(E_k^2+B_k^2) + A_0 C\right)$$

$E_k = -F_{0k}$

$B_k = \epsilon_{ijk}F_{jk}$

$\underbrace{C = \partial_k E_k + [A_k, E_k]}_{\text{constraint = generator}}$ for pure YM

(first class)

$$H_T = h(q_1 \cdots q_f; p_1 \cdots p_f) + \sum_{m=1}^{r} \lambda_m \chi_m$$

$[\chi_i, \chi_j] = f_{ijk}\chi_k$

+ gauge fixing ϕ_m $m=1 \cdots r$ $[\phi_i, \phi_j] = 0$

→ by canonical transformation. $P_m = \phi_m$ $m = 1 \cdots r$

$$\prod_{i=1}^{f-r} DQ_i^* DP_i^* = \prod_{i=1}^{f-r} DQ_i^* DP_i^* \prod_{m=1}^{r} DQ_m DP_m \delta(P_m)\, \underbrace{\delta(Q_m - f_m(Q_i^*, P_i^*))}$$ ← solving constraint

$$= \prod_{m=1}^{r} \delta(\chi_m) \frac{\partial(\chi_1 \cdots \chi_r)}{\partial(Q_1 \cdots Q_r)} = \prod_{m=1}^{r} \delta(\chi_m) \det[\chi_m, \phi_n]$$

$\underbrace{\delta(\chi_m)}_{= \int D\lambda_m}$

constraint = gauge generator

gauge variation of χ_m

gauge fixing cond: $G(A_\mu) = 0$

↳ Faddeev-Popov det: $\mathrm{Det}\left[\frac{\delta G}{\delta A_\mu} \partial_\mu\right] \neq 0$

"residual gauge transformation"?

= the gauge transformation surviving the gauge fixing

In principle does not exist

But "emergent" residual gauge symmetry can exist if one just focus on some specific spacetime region. → mainly null infinity (LGT, Asymptotic gauge sym.)

Orthogonal to gauge symmetry. But some different type

ex $\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2\xi} B^2 + B\partial_\mu A^\mu - \bar{c}\partial^2 c$

BRST: $\delta A_\mu = \epsilon \partial_\mu c$ $\delta c = 0$ $\delta \bar{c} = \epsilon B$ $\delta B = 0$

LGT: $\delta A_\mu = \partial_\mu \epsilon$ $\delta c = 0$ $\delta \bar{c} = 0$ $\delta B = 0$

to see physics at $r\to\infty$. We adopt the "retarded time" $(u, r, z, \bar z)$ + "stereographic" angular variables

$u = t - r$

$z = \tan\frac{\theta}{2} e^{i\varphi}$

flat spacetime metric becomes

$$ds^2 = -du^2 - 2du\,dr + 2r^2\gamma_{z\bar z}\,dz\,d\bar z \qquad \gamma_{z\bar z} = \frac{2}{(1+|z|^2)^2}$$

$r = x_1^2 + x_2^2 + x_3^2$

$$z = \frac{x_1 + i x_2}{r + x_3}$$

$t = u + r$

$$\vec x = \frac{r}{1+z\bar z}\left(z+\bar z,\ i(\bar z - z),\ 1-|z|^2\right)$$

Why is it useful?

consider the free field expansion of the photon field.

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left(\epsilon_\mu^{\lambda}(k) a_\lambda(\vec k) e^{ik\cdot x} + \epsilon_\mu^{\lambda*}(k) a_\lambda^\dagger(\vec k) e^{-ik\cdot x}\right)$$

$$|z|^2 = \frac{x_1^2 + x_2^2}{(r+x_3)^2} = \frac{r^2 - x_3^2}{(r+x_3)^2} \qquad 1+|z|^2 = \frac{2r}{(r+x_3)^2}$$

$$\frac{1-|z|^2}{1+|z|^2} = \frac{(r+x_3)^2 - r^2 + x_3^2}{(r+x_3)^2 + r^2 - x_3^2} = \frac{2rx_3 + 2x_3^2}{2r^2 + 2rx_3} = \frac{2x_3(r+x_3)}{2r(r+x_3)}$$

Using $\displaystyle\lim_{r\to\infty} e^{-ik\cdot x} = -\frac{2\pi i}{\omega r} e^{i\omega u}\,\delta^2(\hat x - \hat k)$

$$= \frac{\delta^2(z_x - z_k)}{\gamma_{z\bar z}}$$

$$\hat x = \frac{1}{1+|z|^2}\left(z+\bar z,\ -i(z-\bar z),\ 1-|z|^2\right)$$

$$\int \frac{d\omega_k\, d^2 z\, \cancel{\gamma_{z\bar z}}}{(2\pi)^3\, 2\omega_k}\,\omega_k^2\left(\epsilon_\mu^{\lambda}(k)\, a_\lambda(\vec k)\,\frac{-2\pi i}{\omega r}\, e^{-i\omega u}\,\frac{\delta^2(z - z_k)}{\gamma_{z\bar z}} + \text{h.c.}\right)$$

$$\Rightarrow r + x_3 = r\left(1 + \frac{1-|z|^2}{1+|z|^2}\right) = \frac{2r}{1+|z|^2}$$

$$x_1 + i x_2 = \frac{2rz}{1+|z|^2}$$

$$x_1 - i x_2 = \frac{2r\bar z}{1+|z|^2}$$

$$= -\frac{i}{r}\frac{1}{8\pi^2}\int d\omega_k \left(\epsilon_\mu^{\lambda}(\hat x)\, a_\lambda(\omega\hat x)\, e^{-i\omega u} - \epsilon_\mu^{\lambda*}(\hat x)\, a_\lambda^\dagger(\omega\hat x)\, e^{i\omega u}\right)$$

$$\epsilon_+ = \frac{1}{\sqrt2}(\bar z, 1, -i, -\bar z)$$

$$\epsilon_- = \frac{1}{\sqrt2}(z, 1, i, -z)$$

$$\partial_z x^\mu = \partial_z\left(r\frac{z+\bar z}{1+|z|^2},\ r\frac{-i(z-\bar z)}{1+|z|^2},\ r\frac{1-|z|^2}{1+|z|^2}\right)$$

$$dz = \frac{1}{2}\frac{e^{i\varphi}}{\cos^2\frac{\theta}{2}}d\theta + i\tan\frac{\theta}{2}e^{i\varphi}d\varphi$$

$$d\bar z = \frac{e^{-i\varphi}}{2\cos^2\frac{\theta}{2}}d\theta - i\tan\frac{\theta}{2}e^{-i\varphi}d\varphi$$

$A_\mu dx^\mu = A_z dz + A_{\bar z} d\bar z + \dots$

$\epsilon_-\,\partial_z x = \epsilon_+^*\,\partial_z x = r\sqrt{\gamma_{z\bar z}}$

$\epsilon_-^*\,\partial_z x = \epsilon_+\,\partial_z x = 0$

$$|dz|^2 = \frac{1}{4\cos^4\frac{\theta}{2}}\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$\boxed{\begin{aligned} \Rightarrow A_z &= -i\sqrt{\gamma_{z\bar z}}\int_0^\infty \frac{d\omega}{8\pi^2}\left(a_+(\omega\hat x)e^{-i\omega u} + a_-^\dagger(\omega\hat x)e^{i\omega u}\right) \\ A_{\bar z} &= -i\sqrt{\gamma_{z\bar z}}\int_0^\infty \frac{d\omega}{8\pi^2}\left(a_-(\omega\hat x)e^{-i\omega u} - a_+^\dagger(\omega\hat x)e^{i\omega u}\right)\end{aligned}}$$

$$\Rightarrow d\Omega_2^2 = \frac{4\,dz\,d\bar z}{(1+|z|^2)^2}$$

$$\because 1+|z|^2 = 1 + \frac{\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} = \frac{1}{\cos^2\frac{\theta}{2}}$$

helicity structure is evident!

$$\partial_z x^\mu = \gamma_{z\bar z}\left(0,\ \frac{(1-\bar z^2)}{2r},\ \frac{-i(1+\bar z^2)}{2r},\ -\frac{\bar z}{2r}\right)$$

$$e^{ik\cdot x} = 4\pi\, e^{-i\omega(u+r)}\sum_l i^l j_l(\omega r)\sum_m Y_l^m(\hat k)\, Y_l^{m*}(\hat x)$$

~~Lecture 2~~ Condition for residual symmetry

e.g. Lorenz gauge $G = \partial \cdot A = 0 \Rightarrow \frac{\delta G}{\delta A_\mu} \partial_\mu \varepsilon = \Box \varepsilon = 0$

~~under the boundary~~ condition, since we are interested in null infinity ($r \to \infty$)

let $\varepsilon(x) = \varepsilon^{(0)}(u, z, \bar z) + \frac{1}{r}\varepsilon^{(1)}(u, z, \bar z) + \frac{1}{r^2}\varepsilon^{(2)}(u, z, \bar z) + \dots$

$$0 = \Box \varepsilon = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, g^{\mu\nu} \partial_\nu \varepsilon) = -\partial_u \partial_r \varepsilon + \frac{1}{r^2}\partial_r [r^2(\partial_r \varepsilon - \partial_u \varepsilon)] + \frac{2}{r^2 \gamma_{z\bar z}} \partial_z \partial_{\bar z} \varepsilon$$

$$= -\frac{2}{r}\partial_u \varepsilon^{(0)} + \frac{1}{r^2}\left(\frac{2}{\gamma_{z\bar z}} \partial_z \partial_{\bar z} \varepsilon^{(0)}\right) + \dots \Rightarrow$$

if we impose $\lim_{r\to\infty} \varepsilon(x) = 0$, only trivial sol. $\varepsilon = 0$ allowed $\to$ No r^0 term.

However, if we just allow "approximate" residual gauge symmetry.

i.e. $\Box \varepsilon = O\left(\frac{1}{r^2}\right)$ $\quad \partial_u \varepsilon^{(0)} = 0 \Rightarrow \varepsilon = \varepsilon(z, \bar z) + O\left(\frac{\varepsilon^{(1)}(u,z,\bar z)}{r}\right) \to$ $A_z = O(1)$ $\quad A_z dz = (\frac{1}{r}A_z)(r\, dz) + \dots$

$A_u = O(r^{-1})$ $\quad A_r = O(r^{-2})$

$O\left(\frac{1}{r^3}\right)$ $\quad \partial_z \partial_{\bar z} \varepsilon^{(0)} = 0$ $\qquad \varepsilon(z) + \bar\varepsilon(\bar z) \to$ holomorphy!

$\Rightarrow$ $i[Q_\varepsilon, A_{z/\bar z}] = \partial_{z/\bar z} \varepsilon(z, \bar z)$ $\qquad$ Point: ~~gauge fixing was already~~

what is Q_ε?

BRST $\to \delta(\text{ghost})$

Applying Noether theorem for gauge transformation.

but ~~in this case~~ in our case ~~$\delta A = 0$~~ $\delta(\text{ghost}) = 0$

$$J_N^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} \delta A_\nu + J^\mu \Lambda = -F^{\mu\nu} \partial_\nu \Lambda + J^\mu \Lambda = -(F^{\mu\nu}\Lambda)_{;\nu} + (F^{\mu\nu}{}_{;\nu} + J^\mu)\Lambda$$

if ~~$\Lambda = $ const.~~ $\qquad 0$ e.o.m

$$Q = -\int_S d^2 z\, r^2 \gamma_{z\bar z}\, \varepsilon(z, \bar z)\, F_{ru}$$

if const. $\to$ electric charge $\Rightarrow$ $\vec E$-flux over 2-dim surface

Maxwell's eq.

$g_{uu} = -1$ $\quad g_{ur} = -1$ $\quad g_{rr} = 0$ $\quad \begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$ $\quad r^2\begin{pmatrix} 0 & \gamma \\ \gamma & 0 \end{pmatrix}$ $\quad \frac{1}{r^2}\begin{pmatrix} 0 & \frac{1}{\gamma} \\ \frac{1}{\gamma} & 0 \end{pmatrix}$

$g^{uu} = 0$ $\quad g^{ur} = -1$ $\quad g^{rr} = 1$

$$\partial_\mu F^{\mu r} = \partial_u F^{ur} + \partial_z F^{zr} + \partial_{\bar z} F^{\bar z r} = +\partial_u F_{ru} + \partial_z (F_{\bar z r} - F_{\bar z u}) + \partial_{\bar z}(F_{zr} - F_{zu})$$

$$= \partial_u F_{ru} \mp \frac{1}{r^2 \gamma_{z\bar z}} \Big[\underbrace{\partial_z (F_{r\bar z} - F_{u\bar z}) + \partial_{\bar z}(F_{rz} - F_{uz})}\Big]$$

$A_z \sim A_z^{(0)} + \frac{1}{r}A_z^{(1)} \dots$

$$\partial_z\Big[(\underbrace{\partial_r A_{\bar z}}_{\sim \frac{1}{r^2}} - \underbrace{\partial_{\bar z} A_r}_{\sim \frac{1}{r^2}}) - (\partial_u A_{\bar z} - \underbrace{\partial_{\bar z} A_u}_{\sim \frac{1}{r}})\Big]$$

$$= \partial_u F_{ru} + \frac{1}{r^2 \gamma_{z\bar z}} \partial_u (\partial_z A_{\bar z} + \partial_{\bar z} A_z) = J^r$$

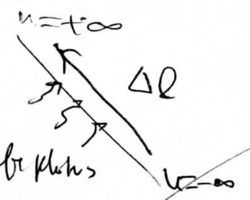
$\Rightarrow Q_\varepsilon(u=+\infty) - Q_\varepsilon(u=-\infty) = -\int_{-\infty}^{\infty} du \int d^2z\, r^2 \gamma_{z\bar z}\, \varepsilon\, \partial_u F_{ru}$

charge conservation

hard part

in the absence of massless charged particles

$= -\int_{-\infty}^{\infty} du \int d^2z\, r^2\gamma_{z\bar z}\, \varepsilon J^r + \int_{-\infty}^{\infty} du \int d^2z\, \varepsilon\, \partial_u (\partial_z A_{\bar z} + \partial_{\bar z} A_z)$



soft photons

created @ null infinity

$$\| $$

$$\int d^2z \lim_{\omega\to 0} \frac{\omega\sqrt{\gamma_{z\bar z}}}{8\pi}\Big(\partial_z\varepsilon(z,\bar z)\, a_+(\omega\hat x) + \partial_{\bar z}\varepsilon(z,\bar z)\, a_+^\dagger(\omega\hat x) + \partial_{\bar z}\varepsilon(z,\bar z)\, a_-(\omega\hat x) + \partial_z\varepsilon(z,\bar z)\, a_-^\dagger(\omega\hat x)\Big)$$

$$\partial_u A_z = -i\sqrt{\gamma_{z\bar z}}\int \frac{d\omega}{8\pi}\left(a_+(\omega\hat x)(-i\omega)e^{-i\omega u} - a_-^\dagger(\omega\hat x)(i\omega)e^{i\omega u}\right) = -\sqrt{\gamma_{z\bar z}}\int\frac{d\omega}{8\pi}\,\omega\left(a_+(\omega\hat x)e^{-i\omega u} + a_-^\dagger(\omega\hat x)e^{i\omega u}\right)$$

$$\int du\, \partial_u(A_z + A_{\bar z}) = -\sqrt{\gamma_{z\bar z}}\int\frac{d\omega}{8\pi}\,\omega\, \frac{2\pi\delta(\omega)}{2}\left(a_+(\omega\hat x) + a_-^\dagger(\omega\hat x)\right) + (+\leftrightarrow -)$$

$\omega \in [0, \infty]$ $\qquad \int du\, e^{i\omega u}$

- Connection to the origin of gauge symmetry

cf. Y. Hamada, M. Seo, G. Shiu PRD **96** (2017) 105013 1704.08773

JHEP **02** (2018) 046 1711.04668

$$J = \int d^2z\, (a_{+,z}^\dagger a_{+z} - a_z^\dagger a_{-z}) \qquad \delta Q_\varepsilon = \int d^2z\, g(z,\bar z)\left[(a_{+,z} + a_{+,z}^\dagger) + (a_{-,z} + a_{-z}^\dagger)\right]$$

$i[Q_\varepsilon, A_{z/\bar z}] = \partial_{z/\bar z}\varepsilon$

$[a_{\lambda,z}, a_{\lambda',z'}^\dagger] = \delta_{\lambda\lambda'}\delta(z-z')$ $\qquad [a, a] = [a^\dagger, a^\dagger] = 0$

do not commute with each other.

by absorbing the phase of $\partial_z\varepsilon$ to def. of creation/annihilation operator

$\frac{\omega\sqrt{\gamma_{z\bar z}}}{8\pi}|\partial_z\varepsilon|$

then, ~~————~~ what is the "natural closed algebra" describing soft photons @ null infinity?

$a + a^\dagger = X$ $\qquad [X, P] = \frac{i}{2}[a + a^\dagger, a - a^\dagger] = i$ $\qquad i(a_+ + a_-^\dagger)$

$i(a - a^\dagger) = P$

$[X_1 + X_2, P_1 - P_2] = 0$

$$\Delta P_\varepsilon = \int d^2z\, g(z,\bar z)\left[i(a_{+,z} - a_{+z}^\dagger) - i(a_{-,z} - a_{-z}^\dagger)\right]$$

$\pm i\,\partial_{z/\bar z}\varepsilon$ helicity dishipring

complex

$[\Delta Q_\varepsilon, \Delta P_\varepsilon] = 0$ $\quad [J, \delta Q] = i\delta P$ $\quad [J, \Delta P] = -i\Delta Q$: ISO(2) algebra

Representation of Lorentz group SO(1,3)

$i[J^{\mu\nu}, J^{\rho\sigma}] = \eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\sigma\mu}J^{\rho\nu} + \eta^{\sigma\nu}J^{\rho\mu}$

angular momentum: $J_1 = J_{23}$ $J_2 = J_{31}$ $J_3 = J_{12}$
(boost) $K_1 = J_{10}$ $K_2 = J_{20}$ $K_3 = J_{30}$

$[J_i, J_j] = i\epsilon_{ijk}J_k$
$[J_i, K_j] = i\epsilon_{ijk}K_k$
$[K_i, K_j] = -i\epsilon_{ijk}J_k$

$\vec{A} = \frac{1}{2}(\vec{J} + i\vec{K})$
$\vec{B} = \frac{1}{2}(\vec{J} - i\vec{K})$

$[A_i, A_j] = i\epsilon_{ijk}A_k$
$[B_i, B_j] = i\epsilon_{ijk}B_k$
non-compactness $[A_i, B_j] = 0$

$\sim SU(2) \times SU(2)$

$(0,$	$0)$	Scalar
$(\frac{1}{2}$	$0)$	left/right handed spinor
$(0$	$\frac{1}{2})$	
$(\frac{1}{2}$	$\frac{1}{2})$	vector (real field)
$(1$	$0)$	$F_{\mu\nu} = \pm\frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$
$(0$	$1)$	

$\Downarrow$

non compatibility btw unitarity & finite dim. representation
$\hookrightarrow$ as we observe.

E. Wigner Ann. Math. 40 (1939) 149

$\Rightarrow$ little group: subgroup of Lorentz group that does not change the momentum

$\Rightarrow$ ~~?~~ since non-compactness comes from boost
we fix the specific frame

massive particle $\begin{pmatrix} M \\ \vec{0} \end{pmatrix} \Rightarrow SO(3)$: spin labels ~~particle~~ particle

massless $\begin{pmatrix} p \\ 0 \\ 0 \\ p \end{pmatrix} \Rightarrow$ $X = K_1 + J_2$
$Y = K_2 - J_1$
J_3

non-compactness remains

$[J_3, X] = iY$
$[J_3, Y] = -iX$
$[X, Y] = 0$

$\to$ 2D vector
$\Downarrow$
acts as if 2D momentum

a fixed fix at some specific value $\Rightarrow$ J_3 labels massless particle

S. Weinberg Phys. Rev. 135 (1964) B1049

$$J_2 + K_1 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&-1&0&0 \end{pmatrix} + \begin{pmatrix} 0&-1&0&0 \\ -1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix} = \begin{pmatrix} 0&-1&0&0 \\ -1&0&0&1 \\ 0&0&0&0 \\ 0&-1&0&0 \end{pmatrix} \quad -J_1 + K_2 = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&-1&0 \end{pmatrix} + \begin{pmatrix} 0&0&-1&0 \\ 0&0&0&0 \\ -1&0&0&0 \\ 0&0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&-1&0 \\ 0&0&0&0 \\ -1&0&0&1 \\ 0&0&-1&0 \end{pmatrix}$$

$$\Rightarrow \alpha(J_2 + K_1) + \beta(-J_1 + K_2) = \begin{pmatrix} 0&-\alpha&-\beta&0 \\ -\alpha&0&0&\alpha \\ -\beta&0&0&\beta \\ 0&-\alpha&-\beta&0 \end{pmatrix} \quad e^{X} = \begin{pmatrix} 1+\gamma&-\alpha&-\beta&-\gamma \\ -\alpha&1&0&\alpha \\ -\beta&0&1&\beta \\ \gamma&-\alpha&-\beta&1-\gamma \end{pmatrix} \quad e^{X}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \pm i \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -(\alpha \pm i\beta) \\ 0 \\ \pm i \\ -(\alpha \pm i\beta) \end{pmatrix}$$

$\gamma = \frac{1}{2}(\alpha^2 + \beta^2)$

$\Downarrow$

helicity universal

$(e^X)^\mu{}_\nu e^\nu(\vec{k}, \mp 1) = e^\mu(\vec{k}, \mp 1) - \frac{\alpha \pm i\beta}{\sqrt{2}}\frac{k^\mu}{|\vec{k}|}$

helicity distinguishing

Ch. 8 of Vol 1 QFT

gauge transformation?
without reference of matter
(not conquered yet)

$\sqrt{?}$

4) $\vec{E} = \varepsilon_0(\vec{\epsilon}_+ + \vec{\epsilon}_-)e^{-i\omega x}$ $\quad \vec{B} = \hat{k}\times\vec{E} = \varepsilon_0 i(\vec{\epsilon}_- - \vec{\epsilon}_+)e^{-i\omega x}$

$dF = d\tilde{F} \Rightarrow$ no massless charged particle

(14)

$P_\varepsilon = -\int_S d^2z\, r^2\gamma_{z\bar z}\,\varepsilon(z\bar z)\,\tilde{F}_{ru} = -i\int_S d^2z\,\varepsilon(z\bar z)\,F_{z\bar z}$

$F = dA, \quad \tilde{F} = d\tilde{A}$

$A \to A + d\Lambda \quad \tilde{A} \to \tilde{A} + d\tilde{\Lambda}$

$\Delta P_\varepsilon = -i\int_{-\infty}^{+\infty} du \int d^2z\, \varepsilon\, \partial_u(\partial_z A_{\bar z} - \partial_{\bar z}A_z)$

$= -\int d^2z \lim_{\omega\to 0}\frac{\omega\sqrt{\gamma_{z\bar z}}}{8\pi}\left[i(\partial_z\varepsilon\, a_+ - \partial_{\bar z}\varepsilon\, a_+^\dagger) - i(\partial_{\bar z}\varepsilon\, a_- - \partial_z\varepsilon\, a_-^\dagger)\right]$

Lecture 3 Gravity (~~asymptotic flat background~~ curved background → there may be a residual symmetry provided Faddeev-Popov det $\neq 0$

1 Asymptotic flat spacetime ↳ BMS symmetry ? the first ex. of ~~as~~ CGT

(H. Bondi, M.G.J. van der Burg, A.W.K. Metzner Proc. Roy. Soc. Lond. A 269 (1962) 21
R.K. Sachs " " 270 (") 103
Phys. Rev 128 (1962) 2851

- taking gauge: Bondi gauge $\quad g_{rr} = g_{rz} = g_{r\bar z} = 0$

$g^{AB}\delta g_{AB} = 0 \Rightarrow$ ~~g^{AB}~~ $\gamma^{AB}\partial\gamma_{AB} = \frac{1}{\det\gamma}\partial\det\gamma$ angular determinant is kept fixed

flat space time $\quad ds^2 = -du^2 - 2du\,dr + 2r^2\gamma_{z\bar z}\,dz\,d\bar z$

$\gamma_{z\bar z} = \frac{2}{(1+|z|^2)^2}$

→ ~~$ds^2 = e^{2\beta}\frac{V}{r}du^2$~~ ~~$\delta g_{rr} = 2\xi_{r;r}$~~

$\Gamma^u_{z\bar z} = \frac{2r}{(1+|z|^2)^2} \quad \Gamma^r_{z\bar z} = \frac{-2r}{(1+|z|^2)^2} \quad \Gamma^z_{rz} = \Gamma^{\bar z}_{r\bar z} = \frac{1}{r}$

$\Gamma^z_{zz} = \frac{-2\bar z}{1+|z|^2} \quad \Gamma^{\bar z}_{\bar z\bar z} = -\frac{2z}{1+|z|^2}$

$\xi_u = -\xi^u - \xi^r \quad \xi_r = -\xi^u$

$\xi_z = \frac{2r^2}{(1+|z|^2)^2}\xi^{\bar z} \quad \xi_{\bar z} = \frac{2r^2}{(1+|z|^2)^2}\xi^z$

1) $\delta g_{rr} = 2\xi_{r;r} = -2\xi^u_{,r} = 0 \Rightarrow \xi^u = \xi^u(u, z, \bar z)$

2) $\delta g_{rz} = \xi_{r;z} + \xi_{z;r} = -\xi^u_{;z} + \frac{2r^2}{(1+|z|^2)^2}\xi^{\bar z}_{;r} = -\left(\xi^u_{,z} + \cancel{\frac{2r}{(1+|z|^2)^2}\xi^{\bar z}}\right) + \frac{2r^2}{(1+|z|^2)^2}\left(\xi^{\bar z}_{,r} + \cancel{\frac{1}{r}\xi^{\bar z}}\right) = 0$

$\Rightarrow \xi^{\bar z}_{,r} = \frac{(1+|z|^2)^2}{2r^2}\xi^u_{,z} \Rightarrow \xi^{\bar z} = -\frac{1}{r}\gamma^{z\bar z}\xi^u_{,z} + f^A(u, z, \bar z)$

3) $g^{z\bar z}\delta g_{z\bar z} = \xi^{\bar z}_{;\bar z} + \xi^z_{;z} = \left(\xi^{\bar z}_{,\bar z} - \frac{2z}{1+|z|^2}\xi^{\bar z} + \frac{1}{r}\xi^r\right) + \left(\xi^z_{,z} - \frac{2\bar z}{1+|z|^2}\xi^z + \frac{1}{r}\xi^r\right)$

$= \left(-\frac{1}{r}\gamma^{z\bar z}\xi^u_{,z} + f^A\right)_{,\bar z} - \frac{2z}{1+|z|^2}\left(-\frac{(1+|z|^2)^2}{2r}\xi^u_{,z} + f^A\right) + \frac{1}{r}\xi^r$

$+ \left(-\frac{1}{r}\gamma^{z\bar z}\xi^u_{,\bar z} + f^A\right)_{,z} - \frac{2\bar z}{1+|z|^2}\left(-\frac{(1+|z|^2)^2}{2r}\xi^u_{,\bar z} + f^A\right) + \frac{1}{r}\xi^r$

$\rightarrow -\frac{1}{r}\gamma^{z\bar z}\xi^u{}_{,z\bar z} - \frac{1}{r}\cancel{(1+|z|^2)\bar z}\,\xi^u{}_{,\bar z} + f^A{}_{,\bar z} + \cancel{\frac{(1+|z|^2)}{r}}\bar z\,\xi^u{}_{,\bar z} - \frac{2z}{1+|z|^2}f^A + \frac{1}{r}\xi^r + c.c = 0$

$\Rightarrow -\frac{2}{r}\gamma^{z\bar z}\xi^u{}_{,z\bar z} + (f^A{}_{,\bar z} - \frac{2z}{1+|z|^2}f^A + \bar f^A{}_{,z} - \frac{2\bar z}{1+|z|^2}\bar f^A) + \frac{2}{r}\xi^r = 0$

$\Rightarrow \xi^r = \gamma^{z\bar z}\xi^u{}_{,z\bar z} - \frac{r}{2}\Big[\underbrace{(f^A{}_{,\bar z} - \frac{2z}{1+|z|^2}f^A) + (\bar f^A{}_{,z} - \frac{2\bar z}{1+|z|^2}\bar f^A)}_{\equiv d(u z \bar z)}\Big]$

In flat spacetime metric

for asymptotic-flat spacetime metric, gauge fixing cond. is corrected by $\mathcal{O}(\frac{f}{r})$

how "quickly" made?

$\xi^u = \xi^u(u, z, \bar z)$

$\xi^{\bar z} = -\frac{1}{r}\gamma^{z\bar z}\xi^u{}_{,z} + f^A(u z \bar z)$

$\xi^r = \gamma^{z\bar z}\xi^u{}_{,z\bar z} - \frac{r}{2}d(u z\bar z)$

$\delta g_{ur} = \mathcal{O}(r^{-2})$ $\quad (-\xi^u - \xi^r)_{,r} - \xi^u{}_{,u} = -(\xi^r{}_{,r} + \xi^u{}_{,u})$, $\quad \xi^u{}_{,r} = 0$

$= \frac{1}{2}d(u z\bar z) - \xi^u{}_{,u} = 0 \quad \Rightarrow \xi^u = f(z\bar z) + \frac{u}{2}d(z\bar z)$

$\delta g_{uu} = \mathcal{O}(r^{-1})$ $\quad (-\xi^u - \xi^r)_{,u} = 0 \rightarrow \xi^r{}_{,u} = -\xi^u{}_{,u} \Rightarrow \boxed{\gamma^{z\bar z}\xi^u{}_{,z\bar z u} - \frac{r}{2}d(u,z,\bar z)_{,u}} - \frac{1}{2}d(u z\bar z)$

$\mathcal{O}(r)$ $\qquad \mathcal{O}(r) = 0 \Rightarrow d_{,u} = 0 \Rightarrow d = d(z\bar z)$

i.e. $f^A, \bar f^A \sim u$ independent

$\delta g_{uz} = \mathcal{O}(1)$ $\quad (-\xi^u - \xi^r)_{,z} + \cancel{(\gamma_{z\bar z}\xi^{\bar z})_{,u}} = 0$

$= -\xi^u{}_{,z} - \cancel{\Gamma^u_{z z}\xi^z} - \xi^r{}_{,z} - \cancel{\Gamma^r_{z\bar z}\xi^{\bar z}} = \mathcal{O}(1)$

$\Rightarrow \mathcal{O}(r):\ d = 0$ $\qquad d_{,z} = 0 = d_{,\bar z}$ $\quad \delta g_{u\bar z}$

$\Rightarrow \cancel{f^A{}_{,\bar z} - \frac{2z}{1+|z|^2}f^A = 0}$

$\cancel{D^z D_z \xi^r - D^{\bar z}D_{\bar z}\xi^r = 0}$

$\Rightarrow \mathcal{O}(r):\ \xi^r = d = \bar d(z) + d(\bar z)$

$c + \bar c = 0$, $c, \bar c$ real $\quad c = \bar c = 0$

$\Gamma^z_{z\bar z} = 0$

$\therefore \xi^u = f(z\bar z)$ $\quad \xi^r = \gamma^{z\bar z}\xi^u{}_{,z\bar z} = D^z D_z f$

$\xi^z = -\frac{1}{r}D^z f + \bar f^A$

$\xi^{\bar z} = -\frac{1}{r}D^{\bar z} f + f^A$

$D_z \bar f^A + D_{\bar z} f^A = 0$

$D_z \bar f^A = 0$

$D_{\bar z} f^A = 0$

z-space cov. der.

Supertranslation

Superrotation (angular variable)

no way to make $c, \bar c$ pure imaginary which is real after holomorphicity

adiabatic mode S. Weinberg, Phys. Rev D67 (2003) 123504

"conservation outside the horizon" → the mode that does not change even after the end of inflation, going back to $\mathcal{O}(\varphi)$ system

Generic curved spacetime: No asymptotic flatness (S-matrix is not defined)

$\partial_z \frac{(1+|z|^2)^2}{2} = (1+|z|^2)\bar z$

$f = g^{z\bar z}g_z = \frac{(1+|z|^2)^2}{2}g_z$ $\qquad \cancel{(1+|z|^2)\bar z\, g_z} + \frac{(1+|z|^2)^2}{2}\partial_{\bar z} g_z - \frac{2\bar z}{1+|z|^2}\cancel{\frac{(1+|z|^2)^2}{2}}g_z = 0$

$\Rightarrow \partial_{\bar z} g_z = 0$

$\Rightarrow q \to 0$ mode symmetry : first order spatially homogeneity

$\cancel{h_{00} = -2\Phi(t)}$, $g_{00} = -1 - 2\Phi$. $g_{0i} = 0$ $g_{ij} = a^2 \delta_{ij} (1 - 2\Psi)$

$\Delta h_{\mu\nu}(x) = g'_{\mu\nu}(x) - g_{\mu\nu}(x)$ $g'_{\mu\nu}(x') = g_{\lambda\kappa}(x) \frac{\partial x^\lambda}{\partial x'^\mu} \frac{\partial x^\kappa}{\partial x'^\nu}$

$= g'_{\mu\nu}(x') - \epsilon^\lambda \partial_\lambda g_{\mu\nu}(x) - g_{\mu\nu}(x) = -\bar{g}_{\lambda\mu} \partial_\nu \epsilon^\lambda - \bar{g}_{\lambda\nu} \partial_\mu \epsilon^\lambda - \epsilon^\lambda \partial_\lambda \bar{g}_{\mu\nu}$

$\Delta h_{ij} = -\partial_i \epsilon_j - \partial_j \epsilon_i + 2 a \dot{a} \delta_{ij} \epsilon_0$

$\Delta h_{i0} = -\dot{\epsilon}_i - \partial_i \epsilon_0 + 2 \frac{\dot{a}}{a} \epsilon_i$ for FRLW metric $g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2 d\vec{x}^2$

$\Delta h_{00} = -2\dot{\epsilon}_0$

$h_{00} = -2\Phi(t)$ $h_{i0} = 0$ $h_{ij} = -2\delta_{ij} a^2(t) \Psi(t) + a^2(t) D_{ij}(t)$

$\Delta h_{00} = -2\dot{\epsilon}_0 \Rightarrow \epsilon_0(\vec{x}, t) = \epsilon(t) + \chi(\vec{x})$, $\Delta\Phi = \dot{\epsilon}$

$\Delta h_{i0} = -\dot{\epsilon}_i - \partial_i \epsilon_0 + 2 \frac{\dot{a}}{a} \epsilon_i \stackrel{!}{=} 0 \Rightarrow \dot{\epsilon}_i - 2\frac{\dot{a}}{a} \epsilon_i = a^2 \left(\frac{\epsilon_i}{a^2}\right)^{\cdot} = -\partial_i \epsilon_0 = -\partial_i \chi$

$\epsilon_i(\vec{x}, t) = a^2(t) f_i(\vec{x}) - a^2 \partial_i \chi(\vec{x}) \int \frac{dt}{a^2(t)}$

$\Delta h_{ij} = -a^2(\partial_i f_j + \partial_j f_i) - 2 a^2 \partial_i \partial_j \chi \int \frac{dt}{a^2} + 2 a \dot{a} \delta_{ij} (\epsilon + \chi) \to t$-dependent only

const $\to f_i = \omega_{ij} x^j$ $\Phi = 0$, $= 0 \to \chi = \text{const} \Rightarrow$ by choice of χ we can set $\chi = 0$

$\Rightarrow \epsilon_0(\vec{x}, t) = \epsilon(t)$ $\epsilon_i = a^2(t) \omega_{ij} x^j$

$= -a^2(\omega_{ij} + \omega_{ji}) + 2\delta_{ij} a \dot{a} \epsilon \stackrel{!}{=} -2a^2 \delta_{ij} \Delta\Psi + a^2 \Delta D_{ij}$

$\Rightarrow \Delta\Psi = \frac{1}{3}\omega_{ii} - H\epsilon$, $\Delta D_{ij} = -\omega_{ij} - \omega_{ji} + \frac{2}{3}\delta_{ij}\omega_{kk}$

Angular momentum memory effect

$T_{\mu\nu} = p g_{\mu\nu} + (\rho + p) u_\mu u_\nu$ $\delta T_{00} = -\bar{\rho} h_{00} + \delta\rho$

$\delta T_{i0} = \bar{p} h_{i0} + (\bar{\rho} + \bar{p})(\partial_i \delta u + \delta u_i^V)$

$\delta T_{ij} = \bar{p} h_{ij} + a^2(\delta_{ij} \delta p + \ldots)$

$\Delta p = \dot{\bar{p}} \epsilon$ $\delta\rho = \dot{\bar{\rho}} \epsilon$ $\delta u = -\epsilon$ $\bar{u}^0 = 1$ $\bar{u}^i = 0$

$u \cdot u = -1$

$(\bar{g}^{00} + h^{00})(1 + \delta u_0)^2 = -1$

$\Rightarrow (1 + h_{00})(1 - 2\delta u_0) = 1 \Rightarrow \delta u_0 = \frac{h_{00}}{2} = \dot{u}$

$\Delta T_{\mu\nu} = -\bar{T}_{\lambda\mu} \partial_\nu \epsilon^\lambda - \bar{T}_{\lambda\nu} \partial_\mu \epsilon^\lambda - \partial_\lambda \bar{T}_{\mu\nu} \epsilon^\lambda$

$\delta T_{ij} = -\bar{p}(\partial_j \epsilon_i + \partial_i \epsilon_j) + \partial_0 (a^2 \bar{p}) \delta_{ij} \epsilon_0$

$\delta T_{i0} = -\bar{p} \partial_0 \epsilon_i + \bar{p} \partial_i \epsilon_0 + 2\bar{p} \frac{\dot{a}}{a} \epsilon_i$

$\delta T_{00} = 2\rho \dot{\epsilon} + \dot{\bar{\rho}} \epsilon_0$

$\mathcal{R} = -\Psi_q + H \delta u_q$

$= -\frac{1}{3}\omega_{ii} + \cancel{H\epsilon} \cancel{- H\epsilon}$

(17)

Weyl tensor

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{1}{d-2}\left(g_{\mu\rho} R_{\nu\sigma} + g_{\nu\sigma} R_{\mu\rho} - g_{\mu\sigma} R_{\nu\rho} - g_{\nu\rho} R_{\mu\sigma} \right) - \frac{1}{(d-1)(d-2)} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right) R$$

cf) maximally symmetric space $R_{\mu\nu\rho\sigma} = k(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})$

$C = 0$

$k = 0$ the Minkowski

$-\frac{1}{L^2}$ AdS

$\frac{1}{L^2}$ dS

geodesic deviation

$C_{uAuB} = -\frac{1}{2r}\ddot{C}_{AB}$ $\quad r^2 \partial_{z\bar z}\partial_u^2 s^{\bar z} = -C_{uzuz} s^z$

$\cancel{C_{urur} = -\frac{1}{r^3}\left(2m + \frac{1}{4} C_{AB} N^{AB}\right)} + \left(\frac{1}{r^4}\right)$

$e_u = \partial_u \quad e_r = \partial_r \quad e_A = \frac{1}{r}\partial_A$

$C_{urA} = \frac{N_A}{r^4} + \mathcal{O}\left(\frac{1}{r^5}\right)$

$\Delta s^{\bar z} = \frac{\gamma^{z\bar z}}{2r} \Delta C_{\bar z z} s^z$

$$ds^2 = -U e^{2\beta} du^2 - 2e^{2\beta} du\, dr + r^2 \gamma_{AB} (d\theta^A - \mathcal{U}^A du)(d\theta^B - \mathcal{U}^B du)$$

$(N_{AB} N^{AB} \sim$ energy flux across $\mathcal{I}^+$

under Bondi gauge

$\beta = \frac{\beta_0}{r} + \frac{\beta_1}{r^2} + \frac{\beta_2}{r^3} \cdots$

$U = 1 - \frac{2m}{r} - \frac{2M}{r^2} + \cdots$

$\partial_u A_z$

$N_{AB} = \partial_u C_{AB}$ Bondi news tensor

$\gamma_{AB} = h_{AB} + \frac{1}{r} C_{AB} + \frac{1}{r^2} D_{AB} + \frac{1}{r^3} E_{AB} - \cdots$

$\mathcal{U}^A = \frac{1}{r^2} U^A + \frac{1}{r^3}\left(-\frac{2}{3} N^A + \frac{1}{16} D^A (C_{BC} C^{BC}) + \frac{1}{2} C^{AB} D^C C_{BC} \right) + \mathcal{O}$

$h^{AB} C_{AB} = 0 \quad D_{AB} = \frac{1}{4} C_{CD} C^{CD} + \mathcal{D}_{AB} \quad E_{AB} = \frac{1}{2} C_{CD} C^{CD} h_{AB} + \mathcal{E}_{AB} \cdots$

$T_{uu} = \frac{1}{r^2}\hat{T}_{uu}(u, \theta) + \mathcal{O}(r^{-3}) \quad T_{rr} = \frac{1}{r^4}\hat{T}_{rr}(u, \theta^A) + \frac{1}{r^5}\tilde{T}_{rr}(u, \theta^A) +$

$T_{uA} = \frac{1}{r^2}\hat{T}_{uA}(u, \theta^A) \cdots \quad T_{rA} = \frac{1}{r^3} T_{rA}(u, \theta^A) \quad T_{AB} = \frac{1}{r}\hat{T}(u, \theta^A) h_{AB} -$

$U_A = -\frac{1}{2} D^B C_{AB} \quad \beta_0 = 0 \quad \beta_1 = -\frac{1}{32} C_{AB} C^{AB} - \pi \hat{T}_{rr} \quad \beta_2 = -\frac{1}{12} C_{AB} D^{AB} - \frac{2\pi}{3}\tilde{T}_{rr} \quad D^A \mathcal{D}_{AB} = -8\pi \hat{T}_{rB}$

$\int du\, \partial_u Q$

$\dot m = -4\pi \hat{T}_{uu} - \frac{1}{8} N_{AB} N^{AB} + \frac{1}{4} D_A D_B N^{AB}$

$Q = \int d^2 z\, \gamma_{z\bar z}\, f m_B$

$\dot N_A = -8\pi \hat{T}_{uA} + \pi D_A \partial_u T_{rr} + D_A m + \frac{1}{4} D_B D_A D_C C^{BC} - \frac{1}{4} D_B D^B D^C C_{CA} + \frac{1}{4} D_B (N^{BC} C_{CA}) + \frac{1}{2} D_B N^{BC} C_{CA} \cdots$

Supertranslation

$Q = \int \xi\, Y^A N_A$ ~~$\xi = f du$~~

superrotation

$\delta m = f \dot m + \frac{1}{4} N^{AB} D_A D_B f + \frac{1}{2} D_A f D_B N^{AB} \cdots + Y^A D_A m +$

$\delta C_{AB} = f N_{AB} - 2 D_A D_B f + h_{AB} D^2 f \cdots \quad + \delta_Y C_{AB}$

$\partial_u Q$

$\delta N_A = f \dot N_A + 3 m D_A f + \quad + \delta_Y N_A$

symplectic structure

$[N_{\bar z\bar z}, C_{ww}] = 16\pi G i\, \gamma_{z\bar z} \delta^2(z-w)\delta(u-u')$

for details, G. Barnich, C. Troessaert JHEP 05 (2010) 062

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