Lecture notes on Bootstrap.
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Unitarity bound (1602.07982)

Conformal algebra of the Euclidean conformal group $SO(d+1,1)$ is:

$[D, P_\mu] = P_\mu$

$[D, K_\mu] = -K_\mu$

$[K_\mu, P_\nu] = 2\delta_{\mu\nu} D - M_{\mu\nu}$

.....

The notation for generators here are slightly different from what physicists are used to.

physicists: $U = e^{i\omega_A T^A}$

mathematician $U = e^{\omega_A T_A}$ ← our notation

In our case, suppose U is unitary, hence $U^+ U = 1$, the generators T^A needs to be anti-Hermitian.

neglect the $M_{\mu\nu}$ in the algebra for the moment, K_μ and P_μ are like lowering and raising operators of Harmonic Oscillator

take an operator $\mathcal{O}^a(0)$ placed at $x^\mu = 0$, we have

$$[M_{\mu\nu}, \mathcal{O}^a(0)] = (S_{\mu\nu})_b{}^a \mathcal{O}^b(0)$$

Let us diagonalize the dilatation operator, so that operators has definite scaling dimension

$$[D, \mathcal{O}^a(0)] = \Delta \mathcal{O}^a(0)$$

translation generator acts on the operator as

$$[P^\mu, \mathcal{O}(x)] = \partial^\mu \mathcal{O}(x)$$

which integrates to

$$\mathcal{O}(x) = e^{x\cdot P}\, \mathcal{O}(0)\, e^{-x\cdot P}$$

Exercise: Show that $[D, \mathcal{O}^a(x)] = (x^\mu \partial_\mu + \Delta)\, \mathcal{O}^a(x)$.

Hint: eqn (40) of Arxiv: 1602.07982

suppose an operator satisfies

$$[K_\mu, \mathcal{O}(0)] = 0,$$

we call it a "primary operator".

Let us define a state to be

$$|\mathcal{O}\rangle = \mathcal{O}(0)|\Omega\rangle$$ where $|\Omega\rangle$ is the vacuum.

From $[K_\mu, \mathcal{O}(0)] = 0$ we get $K_\mu|\mathcal{O}\rangle = 0$

$$[D, \mathcal{O}(0)] = \Delta \mathcal{O}(0) \iff D|\mathcal{O}\rangle = \Delta|\mathcal{O}\rangle$$

states with the form of

$P_\mu|\mathcal{O}\rangle$, $P_\mu P_\nu|\mathcal{O}\rangle$,

are call "descendants" of $|\mathcal{O}\rangle$

When Conformal symmetry is preserved in a theory, a conformal primary and its descendants forms a representation of the conformal group. They behave collectively in the theory. In others words, once the correlation function of the primaries is calculated, the correlation fn. of descendants are fully fixed by the symmetry.

it is easy to see that

$$D(P_\mu|O\rangle) = [D, P_\mu]|O\rangle + P_\mu D|O\rangle$$
$$= (\Delta_O + 1) P_\mu|O\rangle$$

so that $P_\mu|O\rangle$ is an eigenstate of D with eigenvalue Δ_O+1

In radical quantization, we have

$$P_\mu^\dagger = K_\mu$$

To check the unitarity of the theory, Let's calculate the norm of $P_\mu|O\rangle$, with $|O\rangle$ being a conformal primary

$$\langle O|K_\nu P_\mu|O\rangle = \langle O|[K_\mu, P_\mu]|O\rangle$$
$$= \langle O|2D\delta_{\mu\nu} - 2M_{\mu\nu}|O\rangle$$
$$= 2\Delta\,\delta_{\mu\nu}\langle O|O\rangle$$

Unitarity requires that $\Delta > 0$.

Remark: More precisely speaking, we are imposing unitarity in Lorentzian signature, since our mother nature has Lorentzian signature.

Unitarity of a certain representation means all the generators of the group in such a representation is anti-Hermitian, so that (the map of) the group element is unitary such a representation. This is clearly NOT what we are requiring here. clearly

$$\tilde{D}_{\alpha\beta} = \langle\alpha|\hat{D}|\beta\rangle = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix} \qquad \alpha, \beta = |O\rangle, P_\mu|O\rangle \cdots\cdots$$

Dilation $\tilde{D}_{\alpha\beta}$ is Hermitian, so that

$$e^{\tau\tilde{D}_{\alpha\beta}} \text{ is not unitary} \quad (e^{\tau\hat{D}})^{+} e^{\tau\hat{D}} = e^{2\tau\hat{D}} \neq 1$$

However, after Wick rotation $\tau = it$,

$$e^{it\tilde{D}_{\alpha\beta}} \text{ is unitary.}$$

Now consider the case when the primary $\mathcal{O}^a$ carrying Euclidean index

$$\langle\mathcal{O}_a|K_\mu P_\nu|\mathcal{O}^b\rangle = \langle\mathcal{O}_a|2D\delta_{\mu\nu}\delta_a{}^b - 2M_{\mu\nu}|\mathcal{O}^b\rangle$$

with $a, b \in R_\mathcal{O}$

$\mu\nu \in V$ ← ; vector representation of the Euclidean group

$$M_{\mu\nu}|\mathcal{O}^b\rangle = \frac{1}{2}L_{\mu\nu}{}^{\alpha\beta}(S_{\alpha\beta})_a{}^b|\mathcal{O}^a\rangle$$

$(L_{\mu\nu})^{\alpha\beta} = 2\,\delta_{[\mu}^{\alpha}\delta_{\nu]}^{\beta}$ ← Euclidean generator in V

$S_{\alpha\beta}$ ← — — in $R_\mathcal{O}$

$$\langle \mathcal{O}_a | M_{\mu\nu} | \mathcal{O}^b \rangle = \frac{1}{2} (L_{\mu\nu}{}^{\alpha\beta}) (S_{\alpha\beta})_a{}^b \equiv K_X{}^Y$$

$X = \mu a$
$Y = \nu b$.

Unitarity requires

$$\Delta \geq \text{max-eigenvalue} \left[K_X{}^Y \right]$$

$$K = \tilde{L} \cdot \tilde{S} = \frac{1}{2} \left[(\tilde{L} + \tilde{S})^2 - \tilde{L}^2 - \tilde{S}^2 \right]$$

$$= \frac{1}{2} \left[- \text{Cas}(V \otimes R_{\mathcal{O}}) + \text{Cas}(V) + \text{Cas}(R_{\mathcal{O}}) \right]$$

$\tilde{L} = L_{\mu\nu} \otimes \mathbb{1}_a{}^b$
$\tilde{S} = \mathbb{1}_{\mu\nu} \otimes S_a{}^b$

So that in a proper basis, K is block diagonal.

take $R_{\mathcal{O}} = V_l$, hence $\mathcal{O}^a$ has spin l. $\mathcal{O}^a = \mathcal{O}^{(\mu_1 \cdots \mu_l)} - \text{trace}$

we have $V \otimes R_{\mathcal{O}} = V \otimes V_l = V_{l-1} \oplus \cdots$

(this is simply spin decomposition in d-dimensional Euclidean space)

Remembe the Casimir $\text{Cas}(V_l) = l(l-d+2)$, so that V_{l-1} gives us the lowest eigen value

$$\Delta \geq \frac{1}{2} \left[-\text{Cas}(V_{l-1}) + \text{Cas}(V) + \text{Cas}(V_l) \right]$$

$$= l + d - 2 \qquad \text{Unitarity bound!}$$

Exercise: Calculate the norm of $P_\mu P^\mu |\mathcal{O}\rangle$, with $\mathcal{O}$ being a scalar operator.

In general, unitarity bound is

$\Delta = 0$ Identity operator

or $\Delta \geq \begin{cases} \frac{d-2}{2} & l=0 \\ l+d-2 & l>0. \end{cases}$

It is very easy to remember since these are precisely the value corresponding to free theory:

$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial_\mu \phi$

$[\mathcal{L}] = d$ $[\partial_\mu] = 1 \Rightarrow [\phi] = \frac{d-2}{2}$

also $[\phi \partial_{\mu_1} \cdots \partial_{\mu_l} \phi] = l + d - 2$

For superconformal groups please check 1612.00809
"multiplets of superconformal symmetry in diverse dimensions"

Correlation functions 1602.07982

a conformal primary transforms as

$$\mathcal{O}^a(x) \to \Omega(x)^\Delta \, S[R(x)]_b{}^a \, \mathcal{O}^b(x')$$

where $\frac{\partial x'_\mu}{\partial x_\nu} = \Omega(x') R^\mu{}_\nu(x')$ with $R^\mu{}_\nu(x') \in SO(d)$

Using scaling (dilatation), Rotation and translation.
we get for scalars

$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2) \rangle = \frac{C}{|x_1 - x_2|^{\Delta_1+\Delta_2}}$$

Exercise: Check that inversion $I : x^\mu \to \frac{x^\mu}{x^2}$

requires $\Delta_1 = \Delta_2$ otherwise $C = 0$.

One can also get

$$\langle \mathcal{O}_1(x_1)\, \mathcal{O}_2(x_2)\, \mathcal{O}_3(x_3) \rangle = \frac{f_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}\, x_{23}^{\Delta_2+\Delta_3-\Delta_1}\, x_{31}^{\Delta_3+\Delta_1-\Delta_2}}$$

Similar consideration tells us

$$\langle J^\mu(x_1)\, J_\nu(x_2) \rangle = C_J \frac{I^\mu{}_\nu(x-y)}{|x-y|^{2\Delta_J}}$$

$$I^\mu{}_\nu = \delta^\mu{}_\nu - \frac{2x^\mu x_\nu}{x^2}$$

For higher spin cases.

$$\langle J^{\mu_1 \cdots \mu_\ell}(x)\, J_{\nu_1 \cdots \mu_\ell}(0) \rangle = \frac{I^{(\mu_1}{}_{\nu_1} \cdots I^{\mu_\ell)}{}_{\nu_\ell} - \text{traces}}{x^{2\Delta}}$$

Another important 3pt function is

$$\langle \phi^1(x_1)\, \phi^2(x_2)\, J^{\mu_1 \cdots \mu_\ell}(x_3) \rangle = f_{\phi_1\phi_2 J} \frac{(Z^{\mu_1} \cdots Z^{\mu_\ell} - \text{traces})}{x_{12}^{\Delta_1+\Delta_2-\Delta_3+\ell}\, x_{23}^{\Delta_2+\Delta_3-\Delta_1-\ell}\, x_{31}^{\Delta_3+\Delta_1-\Delta_2+\ell}}$$

$$Z^\mu = \frac{x_{13}^\mu}{x_{13}^2} - \frac{x_{23}^\mu}{x_{23}^2}$$

Exercise 1 Show that $\partial_\mu J^\mu = 0$. fixs. $\Delta_J = d-1$

by acting ∂_μ on $\langle J_\mu J^\nu \rangle$.

Conserved currents saturate the unitarity bound. Their scaling dimension would not be renormalized.

Exercise 2. Show that $\langle \phi(x_1)\, \phi(x_2)\, J^{\mu_1 \cdots \mu_l}(x_3) \rangle = 0$ if $l =$ odd.

Notice the two scalars are identical.

Operator Product Expansion (1602.07982)

$$O_i(x_1)\, O_j(x_2) = \sum_k C_{ijk}(x_{12}, \partial_2)\, O_k(x_2)$$

the summation runs over primaries while ∂_2 takes care of the descendants

make an ansatz for C_{ijk}

$$C_{ijk}(x, \partial) = |x|^{\Delta_k - \Delta_i - \Delta_j} \left(1 + \#_1 x_\mu \partial^\mu + \#_2 x^\mu x^\nu \partial_\mu \partial_\nu + \#_3 x^2 \partial^2 + \cdots \right)$$

preform OPE in 3-pt function.

$$\langle O_i(x_1)\, O_j(x_2)\, O_k(x_3) \rangle$$

$$= \sum_{k'} C_{ijk'}(x_{12}, \partial_2) \langle O_{k'}(x_2)\, O_k(x_3) \rangle$$

$$= C_{ijk}(x_{12}, \partial_2)\, x_{23}^{-\Delta_k} = \frac{f_{ijk}}{x_{12}^{\eta_3}\, x_{23}^{\eta_1}\, x_{31}^{\eta_2}}$$

$\eta_1 = \Delta_2 + \Delta_3 - \Delta_1$

This eqn. helps us fix the unknown coefficients in $C_{ijk}(x.\partial)$

Ex. In case of scalar operators. show that

$$\#_1 = \frac{1}{2} \qquad \#_2 = \frac{\Delta+2}{8(\Delta+1)} \qquad \#_3 = -\frac{\Delta}{16(\Delta-\frac{d-2}{2})(\Delta+1)}$$

In four pt function

$$\langle \phi(x_1)\, \phi(x_2)\, \phi(x_3)\, \phi(x_4) \rangle$$

$$= \sum_{\mathcal{O}\mathcal{O}'} f_{\phi\phi\mathcal{O}}\, f_{\phi\phi\mathcal{O}'}\, C_A(x_{12}, \partial_2)\, C_B(x_{34}, \partial_4)\, \underbrace{\frac{I_{AB}(x_{24})}{x_{24}^{2\Delta_{\mathcal{O}}}}\, \delta_{\mathcal{O}\mathcal{O}'}}_{=\langle \mathcal{O}^A(x_2)\, \mathcal{O}'^B(x_4)\rangle}$$

$$\equiv \frac{1}{x_{12}^{2\Delta_\phi}\, x_{34}^{2\Delta_\phi}} \sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}}^2\, g_{\Delta_{\mathcal{O}} l_{\mathcal{O}}}(u, v)$$

This should be viewed as the defination of $g_{\Delta,l}(u,v)$

$g_{\Delta,l}$ are called conformal blocks. Notice it is a function fully fixed by conformal symmetry. They do not carry dynamical information specific to a certain theory. In other words $g_{\Delta,l}$ are the same for all CFT's. While the spectrum of operators: $\Delta_{\mathcal{O}}$ and OPE coefficients $f_{\phi\phi\mathcal{O}}$ depends on which theory you are considering.

Conformal block are the basis of conformal bootstrap. We need a more effecient way to calculate them.

Remember conformal generators has a representation in terms of differentials.

$d = x^\mu \partial_\mu$

$k_\mu = 2x_\mu (x\cdot\partial_\mu) - x^2 \partial_\mu$

$P_\mu = \partial_\mu$

$M_{\mu\nu} = x_\nu \partial_\mu - x_\mu \partial_\nu$

they form the algebra of $SO(d+1,1)$

$L_{\mu\nu} = M_{\mu\nu}$

$L_{-1,0} = D$

$L_{0,\mu} = \frac{1}{2}(P_\mu + k_\mu)$

$L_{-1,\mu} = \frac{1}{2}(P_\mu - k_\mu)$.

there exist a quadratic Casimir given by

$$C = -\frac{1}{2} L^{ab} L_{ab} = \Delta(\Delta - d) + l(l+d-2) \equiv \lambda_{\Delta,l}$$

here Δ and l are the scaling dimension and spin of the primary. Casimir commutes all the conformal generators, so that all the states belonging to the same multiplet have the same eigenvalue when C acts on them.

In radical quantization 4pt function has another way of being understood

$$\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle = \langle\Omega| R\{\phi(x_3)\phi(x_4)\} R\{\phi(x_1)\phi(x_2)\} |\Omega\rangle.$$

assuming $|x_3|, |x_4| > |x_1|, |x_2|$

Define a projector

$$|\mathcal{O}| = \sum_{\alpha,\beta = \mathcal{O}, P\mathcal{O}, PP\mathcal{O}} |\alpha\rangle (\mathcal{N}^{-1})_{\alpha\beta} \langle\beta|$$

where $\mathcal{N}_{\alpha\beta} = \langle\alpha|\beta\rangle$

Notice $|\mathcal{O}|\,|\mathcal{O}| = |\mathcal{O}|$. "projectors"

$$\langle\phi\phi\phi\phi\rangle = \sum_{\mathcal{O}} \langle\Omega|\, \phi(x_3)\phi(x_4)\, |\mathcal{O}|\, \phi(x_1)\phi(x_2)\, |\Omega\rangle$$

$$= \sum_{\mathcal{O}} f^2_{\phi\phi\mathcal{O}} \frac{1}{x_{12}^{2\Delta}\, x_{34}^{2\Delta}}\, g_{\Delta_{\mathcal{O}}, l_{\mathcal{O}}}(u,v)$$ The radical ordering symbol R has being omitted

In the following derivation $\hat{L}_{ab}$ means operators acting on Hilbert space, while $L_{1,ab}$ mean the differential operators

($\hat{D}$ acts on Hilbert space , $D = x^\mu \partial_\mu$)

Define $\mathrm{Diff}_{3,4} = -\frac{1}{2}(L_3^{ab} + L_4^{ab})(L_{3ab} + L_{4ab})$

we have

$$\mathrm{Diff}_{3,4} \langle\Omega|\, \hat{\phi}(x_1)\hat{\phi}(x_2)\, |\mathcal{O}|\, \hat{\phi}(x_3)\hat{\phi}(x_4)\, |\Omega\rangle$$

$$= \langle\Omega|\, \hat{\phi}(x_1)\hat{\phi}(x_2)\, |\mathcal{O}|\, \hat{C}\, \hat{\phi}(x_3)\hat{\phi}(x_4) |\Omega\rangle$$

$$= \lambda_{\Delta,l} \langle\Omega|\, \hat{\phi}(x_1)\hat{\phi}(x_2)\, |\mathcal{O}|\, \hat{\phi}(x_3)\hat{\phi}(x_4) |\Omega\rangle$$

we have used

$$\Leftarrow (L_{ab,3} + L_{ab,4})\, \phi(x_3)\phi(x_4)|\Omega\rangle$$

$$= [\hat{L}_{ab}, \hat{\phi}(x_3)]\, \hat{\phi}(x_4)|\Omega\rangle + \hat{\phi}(x_3)\, [\hat{L}_{ab}, \hat{\phi}(x_4)]|\Omega\rangle$$

$$= \hat{L}_{ab}\, \hat{\phi}(x_3)\, \hat{\phi}(x_4) |\Omega\rangle$$

Clear $\langle\Omega|\phi\phi|\mathcal{O}|\phi\phi|\Omega\rangle$ is an eigen function of the differential operator $\mathrm{Diff}_{3,4}$

plug in the exact form of $L_{ab,3}$ and $L_{ab,4}$ (in terms of X_3, ∂_3 and X_4, ∂_4), we get the following differential equation for conformal block

$$\tilde{D}\, g_{\Delta,l}(u,v) = \lambda_{\Delta,l}\, g_{\Delta,l}(u,v)$$

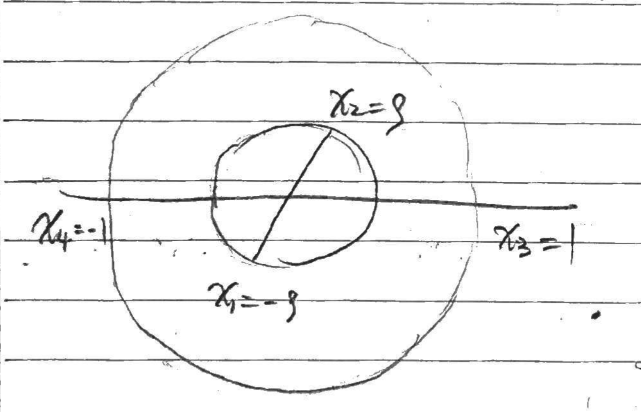
$$\hat{D} = 2\left(z^2(1-z)\partial_z^2 - z^2\partial_z\right) + z \leftrightarrow \bar{z}$$
$$+2(d-2)\frac{z\bar{z}}{z-\bar{z}}\left((1-z)\partial_z - (1-\bar{z})\partial_{\bar{z}}\right)$$

$u = z\bar{z}$
$v = (1-z)(1-\bar{z})$

Conformal block can be calculated by solving such a 2nd order PDE.

We need the boundary conditions.

It is better to use a different coordinate system.



ρ is related to the usual cross ratio by

$$z = \frac{4\rho}{(1+\rho)^2}$$

we can define $\rho \equiv re^{i\theta}$.

The conformal block is given by

$$g_{\Delta,l}(r,\theta) = \langle \psi(\vec{n}, r=1) | |O| | \psi(\vec{n}', r)\rangle$$
$$= \langle \psi(\vec{n}) | |O| e^{\tau\hat{D}} | \psi(\vec{n}')\rangle$$
$$= \langle \psi(\vec{n}) | |O| r^{\hat{D}} | \psi(\vec{n}')\rangle$$

$$|\psi(\vec{n}, r)\rangle = \phi(\vec{n}, r)\,\phi(-\vec{n}, r)|\Omega\rangle$$
$$= e^{\tau D}\phi(\vec{n}, 1)\,\phi(-\vec{n}, 1)|\Omega\rangle$$

in our case $\vec{n} = (1, 0, 0, \cdots)$
$\vec{n}' = (\cos\theta, \sin\theta, 0, \cdots)$

we can now act $r^{\hat{D}}$ to the left

the contribution of an operator

$$(P^2)^{\#} P^{(\mu_1} \cdots P^{\mu_j)} |\Omega\rangle - \text{traces} = |m, j\rangle^{\mu_1 \cdots \mu_j}$$

(# of P's ; spin)

is.

$$r^{\Delta+m} \langle \psi(\vec{n}) | m, j\rangle^{\mu_1 \cdots \mu_j} {}_{\mu_1 \cdots \mu_j}\langle m, j | \psi(\vec{n}')\rangle$$

it is easy to argue that

$$\langle \psi(\vec{n}) | m, j\rangle^{\mu_1 \cdots \mu_j} \propto n^{(\mu_1} \cdots n^{\mu_j)} - \text{traces.}$$

So that the conformal block is given by

$$g_{\Delta, l}(r, \theta) = \sum_{m=2,4 \cdots} B_{m,j}\, r^{\Delta+m} C_j^{\frac{d-2}{2}}(\cos\theta) \qquad (*)$$

where $C_j^{\frac{d-2}{2}}(\vec{n}, \vec{n}') \propto (n^{\mu_1} \cdots n^{\mu_j} - \text{traces})(n'_{\mu_1} \cdots n'_{\mu_j} - \text{traces})$

is the Gegenbauer polynomial

Remarks: The leading contribution comes from the conformal primary operator. We should treat (*) as an Ansatz plug it into the differential Eqn. we could solve for conformal blocks.

In 2D and 4D, conformal blocks could be expressed in terms of hypergeometric functions.

$$g^{(2d)}_{\Delta,l} = k_{\Delta+l}(z)\, k_{\Delta-l}(\bar{z}) + z \leftrightarrow \bar{z}$$

$$g^{4d}_{\Delta,l} = \frac{z\bar{z}}{z-\bar{z}} \left[k_{\Delta+l}(z)\, k_{\Delta-l-2}(\bar{z}) - z \leftrightarrow \bar{z} \right]$$

where $k_\beta \equiv x^{\beta/2}\, {}_2F_1\left(\frac{\beta}{2}, \frac{\beta}{2}, \beta, x\right)$.

Conformal bootstrap

Conformal bootstrap depends on the fact that four pt. function can be calculated in two different OPE channel. The result should be the same in the region where both OPE's converge:

$$\langle \phi(x_1)\, \phi(x_2)\, \phi(x_3)\, \phi(x_4) \rangle = \langle \phi(x_1)\, \phi(x_2)\, \phi(x_3)\, \phi(x_4) \rangle$$

(left: x_1 contracted with x_2, x_3 with x_4; right: x_1 with x_4, x_2 with x_3)

$$\Rightarrow \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}}^2\, g_{\Delta,l}(u,v) = \frac{1}{x_{14}^{2\Delta_\phi} x_{23}^{2\Delta_\phi}} \sum_{\mathcal{O}'} f_{\phi\phi\mathcal{O}'}^2\, g_{\Delta,l}(v,u)$$

(Remember $u = v|_{x_1 \leftrightarrow x_3}$)

$$\sum_{\mathcal{O}} f_{\phi\phi\mathcal{O}}^2 \left(v^{\Delta_\phi} g_{\Delta,l}(u,v) - u^{\Delta_\phi} g_{\Delta,l}(v,u) \right) = 0$$

$\equiv F_{\Delta,l}(u,v)$ — Convolved conformal block.

Q: For a fixed Δ_ϕ, can we impose an arbitrary large gap, so that $\Delta > \Delta_+$ for all scalar operators?

A: I do not know.

Q: What about unitary CFT?

For Hermitian operators, their OPE's are real numbers, $f_{\phi\phi O}^2 > 0$

Suppose we could find a point μ, ν, such that

See Appendix A of 0807.0004 for explination.

$$\begin{cases} F_{0,0} > 0 & \text{Identity operator} \\ F_{\Delta, l=0} > 0 & \text{for } \Delta_{l=0} > \Delta_+ \\ F_{\Delta, l} > 0 & \text{for } \Delta_l > \Delta_{unitarity} \end{cases}$$

Then there is no way such that the crossing can be satisfied with positive $f_{\phi\phi O}^2 > 0$.

We have to release one of the conditions. we conclude that

" Any unitary CFT containing a scalar operator with Δ_ϕ, must contain a scalar operator whose scaling dimension is lower than Δ_+"

In practical, we usually search for linear functional such that

$\alpha(F_{0,0}) = 1$ (normalization)

$\alpha(F_{\Delta, l=0}) > 0$ for $\Delta_0 > \Delta_+$

$\alpha(F_{\Delta; l}) > 0$ for $\Delta > \Delta_{unitarity}$

First bootstrap paper
0807.0004

Let us just check the following functional.

$\alpha_1[F_{\Delta l}(u,v)] = F(0.5, 0.55) - F(0.5, 0.4)$

$\alpha_2[F_{\Delta,l}(u,v)] = F(0.5, 0.6) - F(0.43, 0.35)$

in 2d, and bootstrap Δ_+ at $\Delta_\phi = 1/8$.

```
In[•]:= Quit[]
```

```
In[•]:= k[β_, z_] := z^(β/2) Hypergeometric2F1[β/2, β/2, β, z];
```

The conformal block:

```
In[•]:= g[Δ_, L_][z_, zb_] := k[Δ + L, z] k[Δ - L, zb] + k[Δ + L, zb] k[Δ - L, z];
```

```
In[•]:= F[Δϕ_, Δ_, L_][z_, zb_] := 1/((z zb)^Δϕ - ((1 - z) (1 - zb))^Δϕ)
          (((1 - z) (1 - zb))^Δϕ g[Δ, L][z, zb] - (z zb)^Δϕ g[Δ, L][1 - z, 1 - zb]);
```

our functional:

```
In[•]:= vector[h_] := {h[0.5, 0.55] - h[0.5, 0.4], h[0.5, 0.6] - h[0.43, 0.35]};
```

normalize the vector for better display:

```
In[•]:= normalizeF[Δϕ_, Δ_, L_] := Module[
          {v = vector[F[Δϕ, Δ, L]],
           λ = If[L > 0, 1 - L/20, 1 - (Δ - L)/10]},
          λ v/Norm[v]
        ];
```

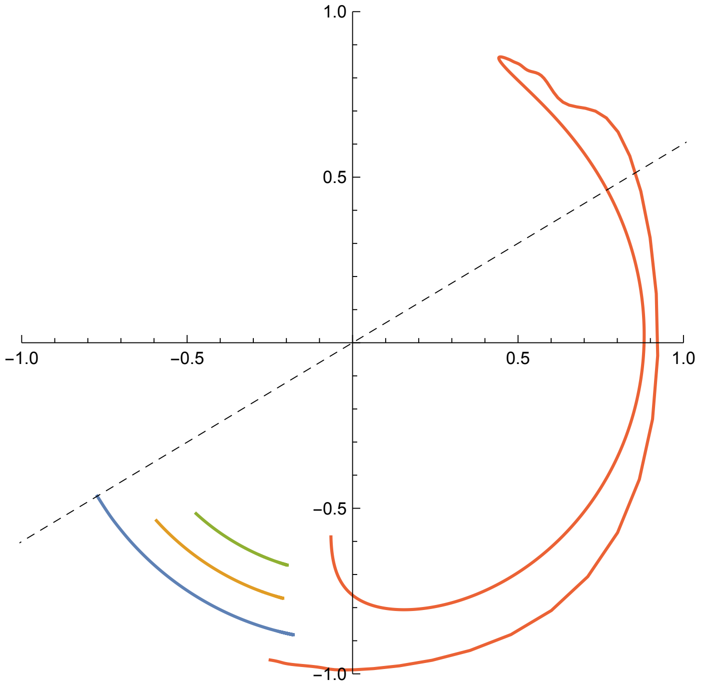
```
In[•]:= Flist[L_, Δmin_] := Module[{Δϕ = 0.125},
          Table[normalizeF[Δϕ, Δ, L],
           {Δ, Δmin, Δmin + 4, 1/20}
          ]
        ];


        Flist[0, Δmin_] := Module[{Δϕ = 0.125},
          Table[normalizeF[Δϕ, Δ, 0],
           {Δ, Δmin, Δmin + 4, 1/100}
          ]
        ];
```

```
In[•]:= stressTensorVector = normalizeF[0.125, 2, 2];
```

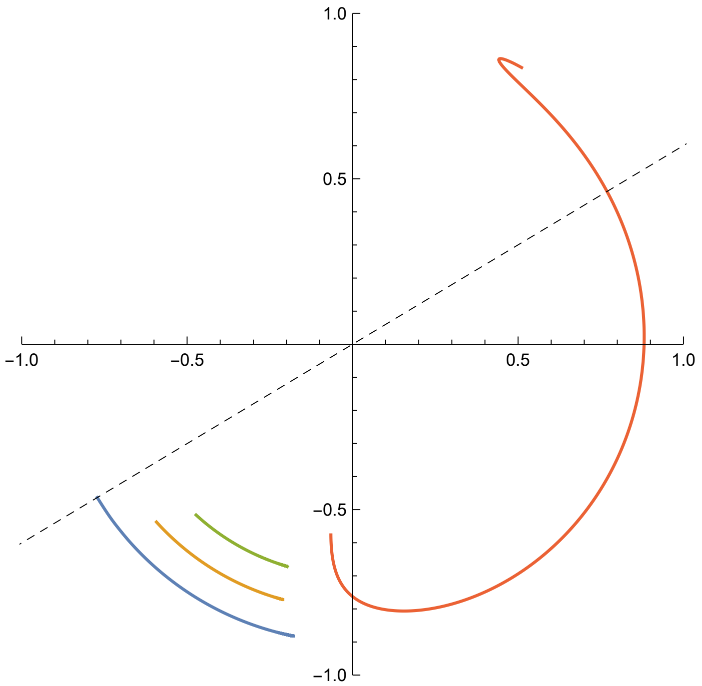
In[•]:=
```
Show[ListPlot[{Flist[2, 2], Flist[4, 4], Flist[6, 6], Flist[0, 0.1]},
  AxesOrigin → {0, 0}, Joined → True, AspectRatio → 1,
  InterpolationOrder → 4, PlotRange → {{-1, 1}, {-1, 1}}],
 Graphics[{Dashed, Line[{-2 stressTensorVector, 2 stressTensorVector}]}]]
```

Out[•]=

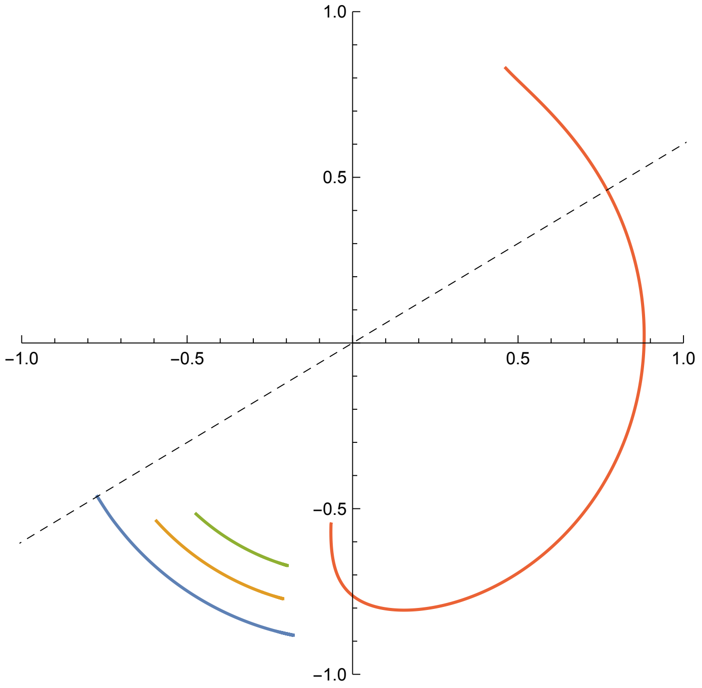


In[•]:=
```
Show[ListPlot[{Flist[2, 2], Flist[4, 4], Flist[6, 6], Flist[0, 0.2]},
  AxesOrigin → {0, 0}, Joined → True, AspectRatio → 1,
  InterpolationOrder → 4, PlotRange → {{-1, 1}, {-1, 1}}],
 Graphics[{Dashed, Line[{-2 stressTensorVector, 2 stressTensorVector}]}]]
```

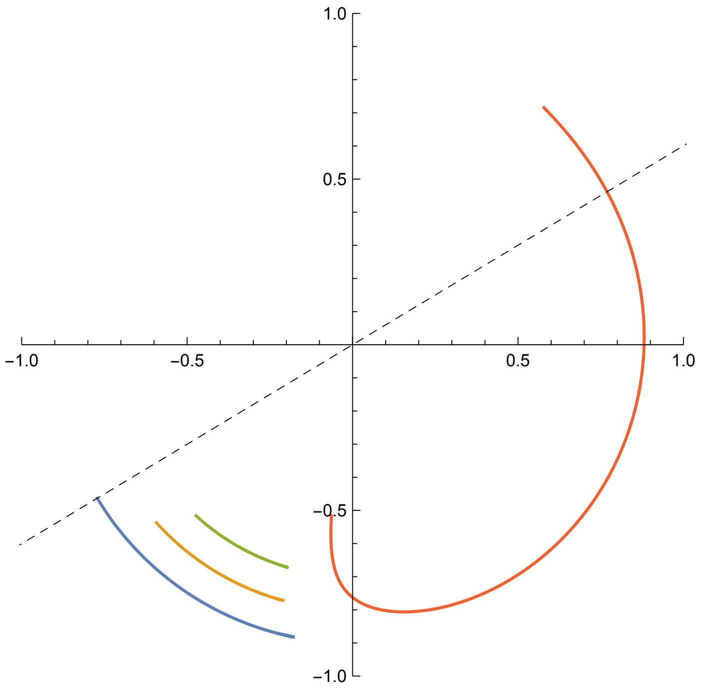
Out[•]=



In[∘]:=
```
Show[ListPlot[{Flist[2, 2], Flist[4, 4], Flist[6, 6], Flist[0, 0.5]},
  AxesOrigin → {0, 0}, Joined → True, AspectRatio → 1,
  InterpolationOrder → 4, PlotRange → {{-1, 1}, {-1, 1}}],
 Graphics[{Dashed, Line[{-2 stressTensorVector, 2 stressTensorVector}]}]]
```

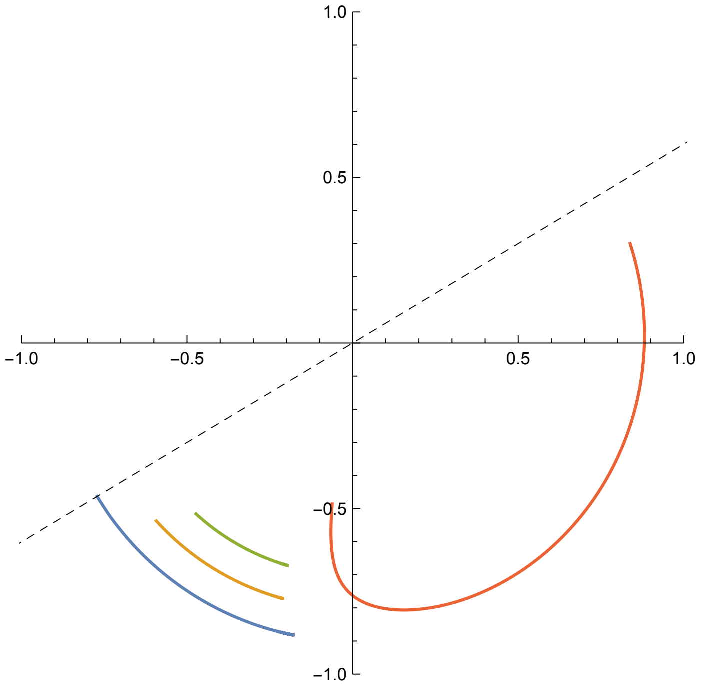
Out[∘]=


In[∘]:=
```
Show[ListPlot[{Flist[2, 2], Flist[4, 4], Flist[6, 6], Flist[0, 0.8]},
  AxesOrigin → {0, 0}, Joined → True, AspectRatio → 1,
  InterpolationOrder → 4, PlotRange → {{-1, 1}, {-1, 1}}],
 Graphics[{Dashed, Line[{-2 stressTensorVector, 2 stressTensorVector}]}]]
```

Out[∘]=


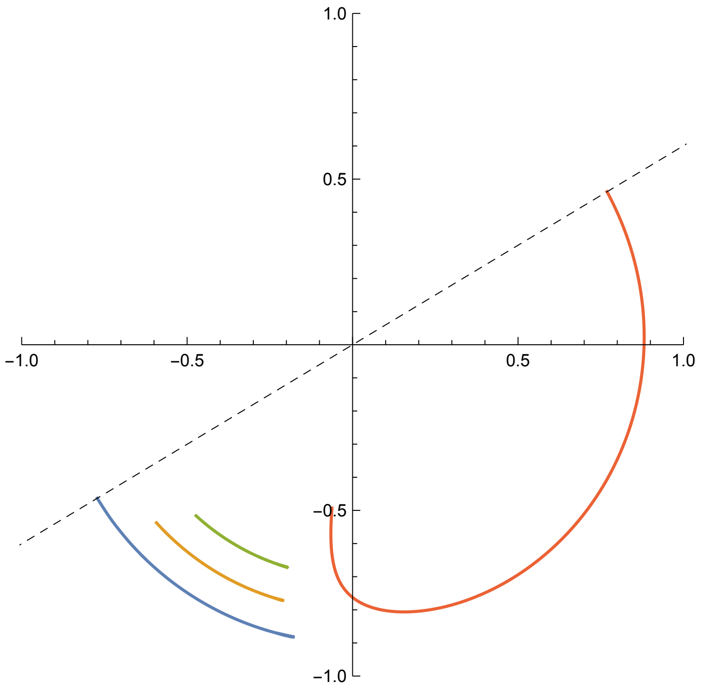
In[∘]:=
```
Show[ListPlot[{Flist[2, 2], Flist[4, 4], Flist[6, 6], Flist[0, 1.1]},
   AxesOrigin → {0, 0}, Joined → True, AspectRatio → 1,
   InterpolationOrder → 4, PlotRange → {{-1, 1}, {-1, 1}}],
  Graphics[{Dashed, Line[{-2 stressTensorVector, 2 stressTensorVector}]}]]
```

Out[∘]=



In[∘]:=
```
Show[ListPlot[{Flist[2, 2], Flist[4, 4], Flist[6, 6], Flist[0, 1.03]},
   AxesOrigin → {0, 0}, Joined → True, AspectRatio → 1,
   InterpolationOrder → 4, PlotRange → {{-1, 1}, {-1, 1}}],
  Graphics[{Dashed, Line[{-2 stressTensorVector, 2 stressTensorVector}]}]]
```

Out[∘]=



We can conclude that if a unitary CFT has a scalar operator with scaling dimension $\Delta_\phi = 1/8$, then there might be another scaling operators in the spectrum, whose scaling dimension is lower that

1.03.

Two dimensional ising model is solvable using Virasoro algebra. We know exactly the scaling dimension of the magnetization operator σ to be 1/8, and the scaling dimension of the thermal operator ϵ to be 1.

The test above shows that using $\Delta_\phi = 1/8$ as an input, the bound we obtained is very close to the exact value.

Notice we have not used Virasoro algebra in the calculation, the conformal blocks that we have used are fixed by sl(2)⊗sl(2) algebra. In higher dimensions, we do not have Virasoro in our hand, the above result suggests that this method could be generalized to higher space-time dimensions.

(After computer demenstration)

Date

The "2d1.example.nb" shows that searching for such a functional can help us ~~solve~~ carve out certained "excluded" region where there exist no ~~unitary~~ conformal field theory.

Choose a basis of such linear functional

$$\alpha = \sum_{m+n=\text{odd}}^{\Lambda} \partial_z^m \partial_{\bar{z}}^n \Big|_{z=\bar{z}=\frac{1}{2}}$$

one can convert this problem into some numerical problem

Before we proceed.

I mentioned that two point function is fixed up to a constant which is also true for 3pt function.

$$\langle \phi(x_1) \phi(x_2) \rangle = C \frac{1}{x_{12}^{2\Delta\phi}}$$

$$\langle \phi(x_1) \phi(x_2) \phi(x_3) \rangle = \frac{f_{\phi\phi\phi}}{x_{12}^{\eta_3} x_{23}^{\eta_1} x_{31}^{\eta_2}}$$

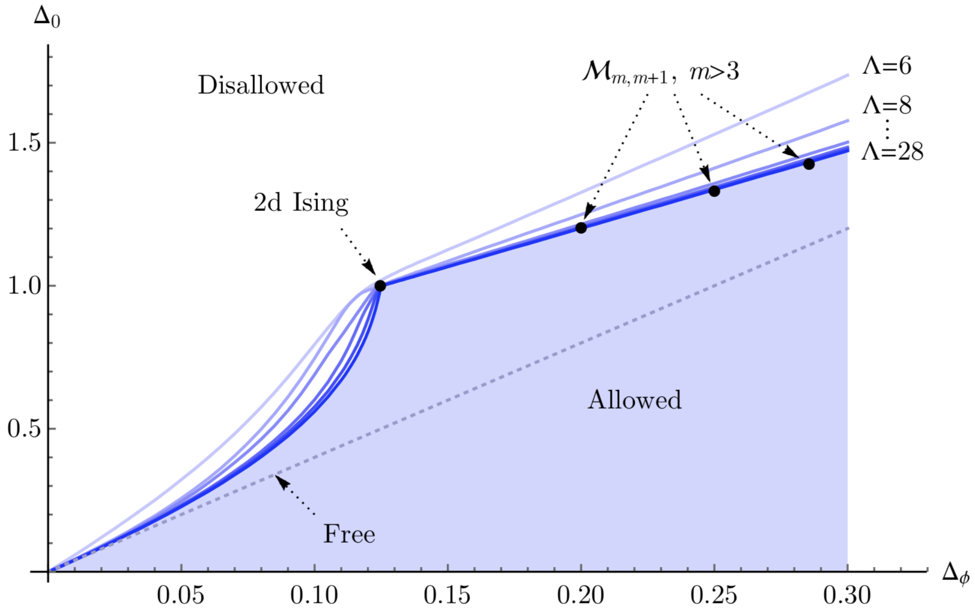
One can alway absorb C into the defination of ϕ, and normalize $\langle \phi\phi \rangle \sim 1$

After the normalization, $f_{\phi\phi\phi}$ is the physical data

Unitarity requires $f_{\phi\phi\phi}$ to be real after such normalization

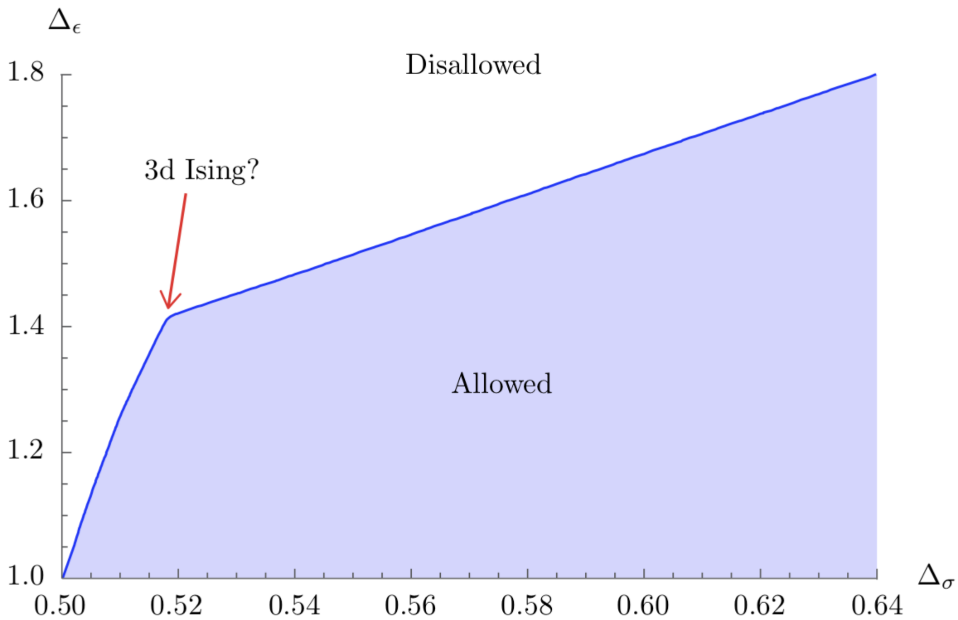
Short Review on Numerical Bootstrap Results

2D bootstrap with Z2 symmetry [arXiv:1602.07982]



1. The bounds converge as we increase the derivative truncation parameter Λ
2. All the minimal models appear along a straight line
3. 2D Ising model appear as a kink

A similar study in 3D gives [arXiv:1203.6064]



3D Ising model again appears as a kink. Notice in 3D we do not have Virasoro algebra. 3D Ising models is very very very very hard to solve. This is a non perturbative result.

The same plot in fractional dimension [arxiv: 1309.5089]

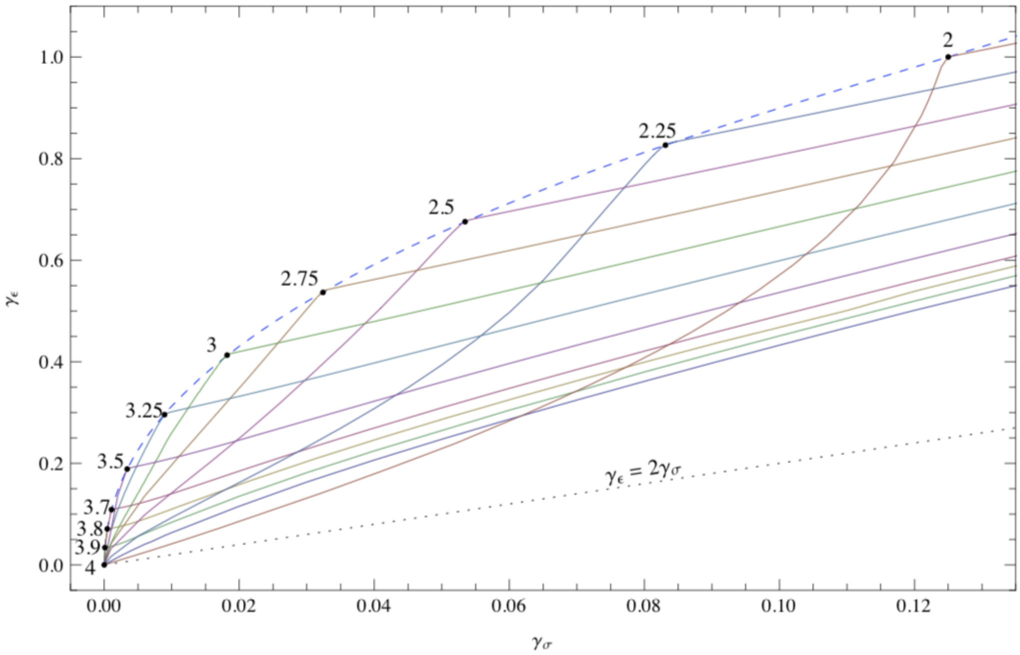


FIG. 1. Upper bounds on γ_ϵ as a function of γ_σ, plotted for $D = 2, 2.25, \ldots, 4$. For each $D < 4$, the bound shows a kink, where a CFT belonging to the Ising model universality class is conjectured to live (black dots, fitted by the blue dashed curve). An example of theories in the bulk of the allowed region are Gaussian models, where $\gamma_\epsilon = 2\gamma_\sigma$ (black dotted line).

Compare with $\in$ - expansion

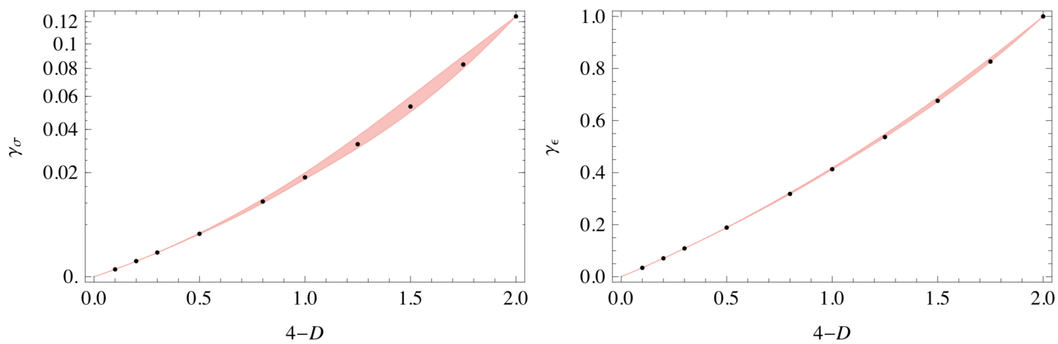


FIG. 2. Black dots: The anomalous dimensions corresponding to the kinks in Fig. 1. Red bands: The same dimensions determined by Borel-resumming the ε-expansion series [31]. Since $\gamma_\sigma = O(\varepsilon^2)$, we use a square root scale on the γ_σ axis.

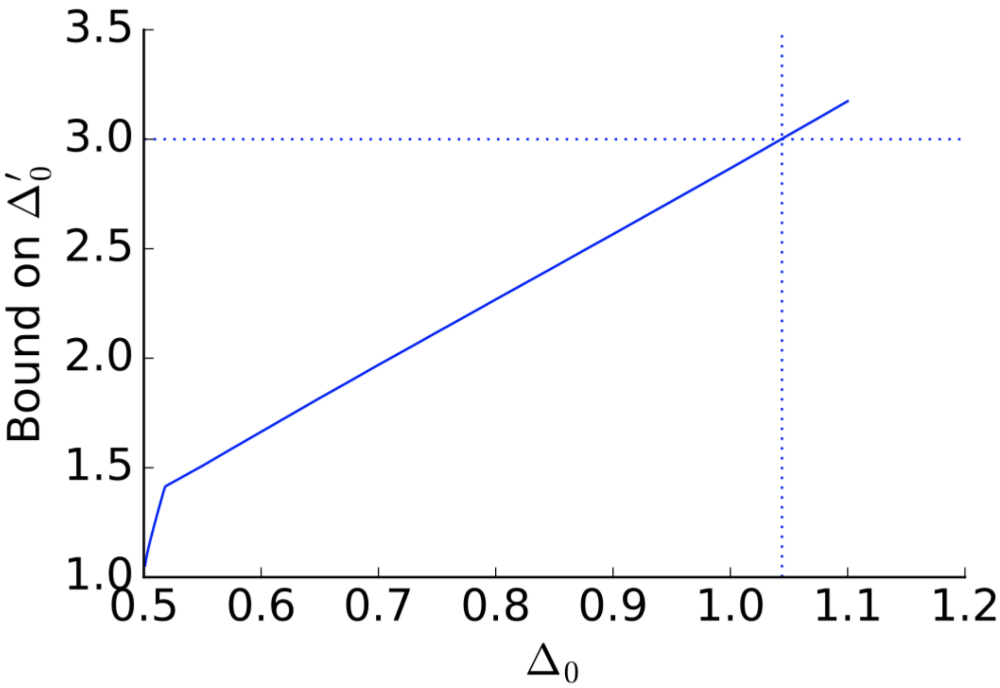
Feymann loops calculations :

"Critical exponents from seven loop strong coupling $\lambda\phi^4$ theory in three dimensions" Hagen Kleinert

this requires calculating thousands of Ferymann Diagrams.

Another problem is that the series you get does not converge, proper resummartion method is necessary.

[arXiv:1602.07295]



At wider range, the plot intersect with $\Delta_0' = 3$ at around $\Delta_0 \sim 1.04$.
This is a general bound for ANY 2nd order phase transition.

There must exist an operator invariant under any global symmetry, and have scaling dimension Δ>1.04.

In terms of critical exponents, this corresponds to

ν>0.51

Certains lattice simulation results has being excluded by this number.

It is hard for lattice simulation to tells so called weakly first order phase transtion from 2nd order phase transition.

[arXiv : 1602.07295]

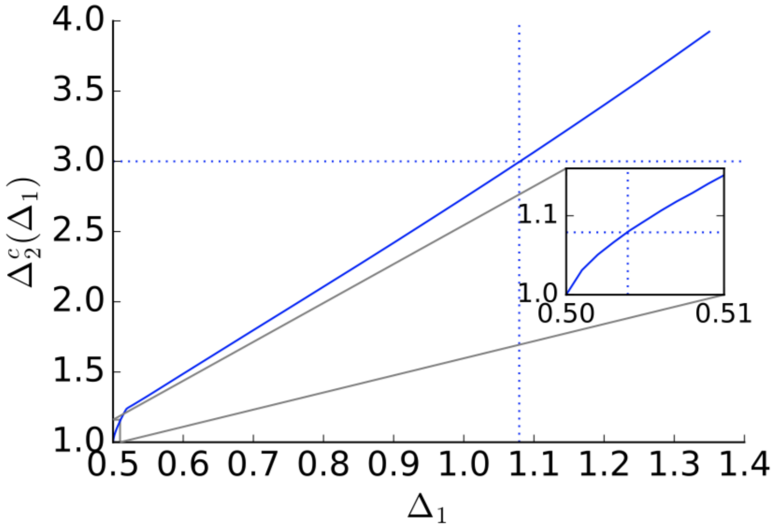


FIG. 1. The upper bound on the scaling dimension Δ_2^c of the lowest dimensional charge two scalar operator appearing in $O_1 \times O_1$ OPE as a function of Δ_1. The same bound applies to $O_2 \times O_2 \sim O_4$.

conformal bootstrap result can be used to constrain symmetry enhancement on lattice.

take $Z_n \to U(1)$ as an example

A recent hot topic in condensed matter physics is the phase transition from Neel phase to so VBS phase.

which could be studies by simulating so called J-Q models using quantum Monte Carlo method, the models has a IR fixed point with SU $(N) \times U(1)_b$ symmetry, where the $U(1)_b$ is the topological U(1) flavor symmetry mentioned by Dongmin yesterday.

Depends on the type of latticed used in the simulation, only some subgroup of U(1) is preserved. Z_4 on square lattice, Z_3 on honey-comb lattice, Z_2 on rectangular lattice and so on.

Suppose the CFT contains an operator with U(1) charge q=2 which is relevant (Δ<3), on rectangular lattice, it requres extra fine tuning to reach the fixed point.

From the bootrap, we notice that, for $Z_n \to U(1)$ enhancement to happen, $\Delta_{\text{VBS}} > 1.02$. Or in terms of critical exponents $\eta_{\text{VBS}} > 1.02$. (This is a big number).

A simulation on square lattice was done in [PRL108.137201].

N=2, $\eta_{\mathrm{VBS}} = 0.20\,(2)$
N=3, $\eta_{\mathrm{VBS}} = 0.42\,(3)$
N=4, $\eta_{\mathrm{VBS}} = 0.64\,(5)$

suppose you put these models on rectangular lattice, all of them should undergos 1st order phase transition.

[PRL108.137201] shows that
N≥4 we have 2nd order phase transtion
which N=2,3 case we have 1st order phase transtion.

The N=4 case is slightly in tenstion with bootstrap result.

It was argued that the $\mathrm{SU}\,(2) \times U(1)_b$ models has IR symmetry enhancement to SO(5). This has been rules out by another bootsrap study.

The famous Ising bootstrap island [arXiv:1406.4858] [arXiv:1603.04436]:

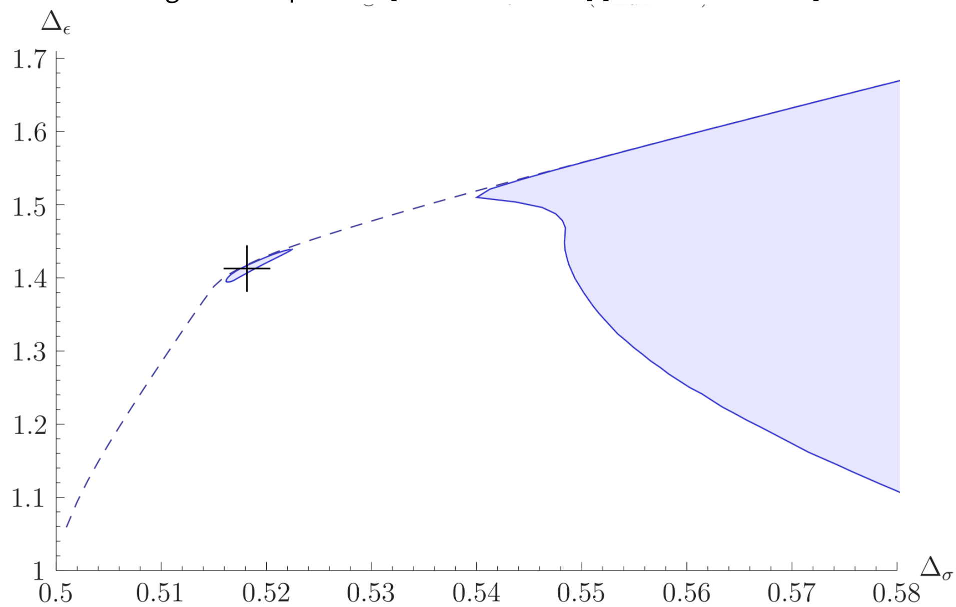


Figure 2: Allowed region of $(\Delta_\sigma, \Delta_\epsilon)$ in a $\mathbb{Z}_2$-symmetric CFT$_3$ where $\Delta_{\sigma'} \geq 3$ (only one $\mathbb{Z}_2$-odd scalar is relevant). This bound uses crossing symmetry and unitarity for $\langle\sigma\sigma\sigma\sigma\rangle$, $\langle\sigma\sigma\epsilon\epsilon\rangle$, and $\langle\epsilon\epsilon\epsilon\epsilon\rangle$, with $n_{\max} = 6$ (105-dimensional functional), $\nu_{\max} = 8$. The 3D Ising point is indicated with black crosshairs. The gap in the $\mathbb{Z}_2$-odd sector is responsible for creating a small closed region around the Ising point.

[arXiv : 1603.04436]

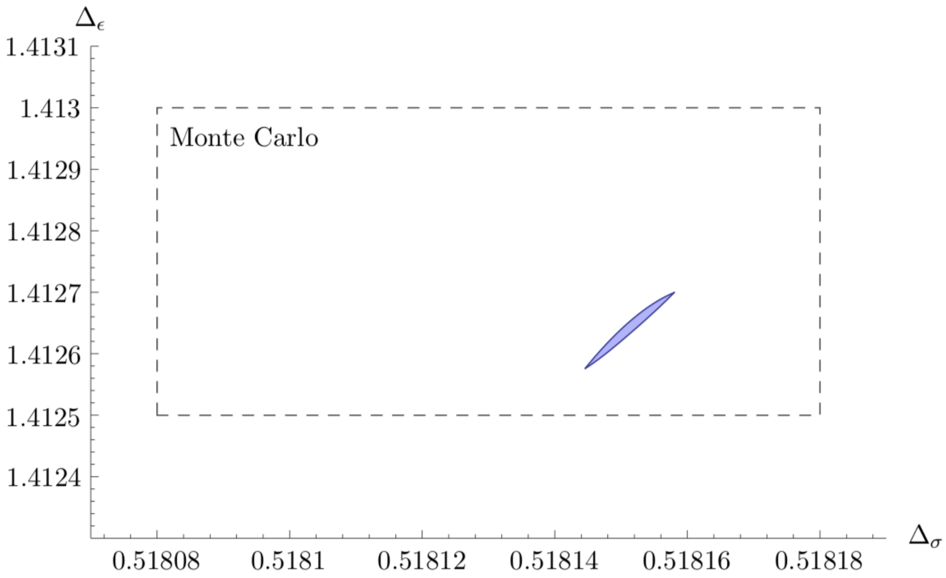


Fig. 30. Bound on $(\Delta_\sigma, \Delta_\epsilon)$ in a unitary 3d CFT with a $\mathbb{Z}_2$ symmetry and two relevant scalars σ, ϵ with $\mathbb{Z}_2$ charges $-, +$. The bound comes from studying crossing symmetry of $\langle\sigma\sigma\sigma\sigma\rangle$, $\langle\sigma\sigma\epsilon\epsilon\rangle$, $\langle\epsilon\epsilon\epsilon\epsilon\rangle$, and is computed with $\Lambda = 43$ using SDPB. The allowed region is the blue sliver. The dashed rectangle shows the 68% confidence region for the current best Monte Carlo determinations.

$\Delta_\sigma = 0.5181489\,(10)$
$\Delta_\epsilon = 1.412625\,(10)$

Just for fun, let us check this number on “inverse symbolic calculator”.

In[33]:=
```
Feigen2 = 2.502907875095892822283902873218;
```
$$\frac{\pi}{\text{Gamma}[1/6]}\ \text{Feigen2}$$

Out[34]=
```
1.412624973231575784493604374302
```

Exercise:
1) Search wiki "the second Feigenbaum constant".
2) Search "Feigenbaum constant + renormalization".

It is not clear to me whether this means a connection between Ising model and chaos.

How to encode global symmetry

$$\langle \phi^i \phi^j \phi^k \phi^l \rangle = \frac{1}{x_{12}{}^{2\Delta_\phi} x_{34}{}^{2\Delta_\phi}} \Sigma_{I \in V\times V} P^{(I)}{}_{\text{ijkl}} \Sigma_{O\in I} \lambda^2{}_{\phi\phi O}\, g_{\Delta,l}(u, v)$$

Notice there is an extra summation over the irreps appearing in V×V.

For O(n) group, we have

$$P^{(S)}{}_{\text{ijkl}} = \frac{1}{n} \delta_{\text{ij}} \delta_{\text{kl}}$$
$$P^{(T)}{}_{\text{ijkl}} = \frac{1}{2} \delta_{\text{ik}} \delta_{\text{jl}} + \frac{1}{2} \delta_{\text{il}} \delta_{\text{jk}} - \frac{1}{n} \delta_{\text{ij}} \delta_{\text{kl}}$$
$$P^{(A)}{}_{\text{ijkl}} = \frac{1}{2} \delta_{\text{ik}} \delta_{\text{jl}} - \frac{1}{2} \delta_{\text{il}} \delta_{\text{jk}}$$

which tell us how to decompose reducible reps V×V into irreps. For example,

$$P^{(S)}{}_{\text{ijkl}}\, \phi^k \phi^l$$

is an O(n) singlet.

One can also check that

$$P^{(I)}{}_{\text{ijkl}}\, \delta_{\text{ik}} \delta_{\text{jl}} = \dim I.$$

Crossing equation is

$$\frac{1}{x_{12}{}^{2\Delta_\phi} x_{34}{}^{2\Delta_\phi}} \Sigma_{I \in V\times V} P^{(I)}{}_{\text{ijkl}} \Sigma_{O\in I} \lambda^2{}_{\phi\phi O}\, g_{\Delta,l}(u, v) = \frac{1}{x_{23}{}^{2\Delta_\phi} x_{14}{}^{2\Delta_\phi}} \Sigma_{I \in V\times V} P^{(I)}{}_{\text{kjil}} \Sigma_{O\in I} \lambda^2{}_{\phi\phi O}\, g_{\Delta,l}(v, u)$$

RHS is LHS with i<>j, 1<>3 flip. Remember $u = \frac{x_{12}{}^2 x_{34}{}^2}{x_{13}{}^2 x_{24}{}^2}, v = \frac{x_{23}{}^2 x_{14}{}^2}{x_{13}{}^2 x_{24}{}^2}$.

Let us define a matrix M by

$$P^{(R')}{}_{\text{kjil}} = \Sigma_R M_{R'R} P^{(R)}{}_{\text{ijkl}}$$

the crossing equation becomes

$$\Sigma_R \left(P^{(R)}{}_{\text{ijkl}} \Sigma_{O\in R} \lambda^2{}_{\phi\phi O}\, v^{\Delta_\phi} g_{\Delta,l}(u, v) \right) = \Sigma_{R'} M_{R'R} P^{(R)}{}_{\text{ijkl}} \Sigma_{O\in R'} \lambda^2{}_{\phi\phi O}\, u^{\Delta_\phi} g_{\Delta,l}(v, u)$$

$$P^{(R)}{}_{\text{ijkl}} \left(\Sigma_{O\in R} \lambda^2{}_{\phi\phi O}\, v^{\Delta_\phi} g_{\Delta,l}(u, v) - \Sigma_{R'} M_{R'R} \Sigma_{O\in R'} \lambda^2{}_{\phi\phi O}\, u^{\Delta_\phi} g_{\Delta,l}(v, u) \right) = 0 \qquad (1)$$

we have three independent equations. We have omitted the summation over operators in eqch irreps.

The numerical code works with
$F_{\pm,\Delta,l}(u,v) = v^{\Delta_\phi} g_{\Delta,l}(u,v) \pm u^{\Delta_\phi} g_{\Delta,l}(v,u)$.

This is because the derivatives acting on the diagonal (u=v) direction vanish for F_-. The off diagonal (u=- v) direction directive vanish for F_+. We need to get rid of these flat directions when doing numerics, otherwise the numerics becomes instable.

In eqn (1), make the replacement u<>v.

$$P^{(R)}{}_{ijkl}\Big(\Sigma_{O\in R}\, \lambda^2{}_{\phi\phi O}\, u^{\Delta_\phi} g_{\Delta,l}(v,u) - M_{R'R}\, \Sigma_{O\in R'}\, \lambda^2{}_{\phi\phi O}\, v^{\Delta_\phi} g_{\Delta,l}(u,v)\Big) = 0 \qquad (2)$$

(1)±(2) we get

$$(\Sigma_{O\in R}\, \lambda^2{}_{\phi\phi O}\, F_{\pm,\Delta,l}(u,v) \mp \Sigma_{R'}\, M_{R'R}\, \Sigma_{O\in R'}\, \lambda^2{}_{\phi\phi O}\, F_{\pm,\Delta,l}(u,v)) = 0$$

which would be collectively written as

$$\big(1 \mp M^T\big).\begin{pmatrix} \Sigma_{O\in S}\, \lambda^2{}_{\phi\phi O}\, F_{\pm,\Delta,l}\,(u,v) \\ \Sigma_{O\in T}\, \lambda^2{}_{\phi\phi O}\, F_{\pm,\Delta,l}\,(u,v) \\ \Sigma_{O\in A}\, \lambda^2{}_{\phi\phi O}\, F_{\pm,\Delta,l}\,(u,v) \end{pmatrix} = 0$$

This is basically the crossing equation.
For O(n)

$$M = \begin{pmatrix} \frac{1}{n} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{2-n-n^2}{n^2} & -\frac{2-n}{2n} & -\frac{-2-n}{2n} \\ -\frac{-1+n}{n} & \frac{1}{2} & \frac{1}{2} \end{pmatrix};$$

```
IdentityMatrix[3] - Transpose[M] // RowReduce // MatrixForm
IdentityMatrix[3] + Transpose[M] // RowReduce // MatrixForm
```

$$\begin{pmatrix} 1 & \frac{-2-n}{n} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & -\frac{2(-1+n)}{n} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

So that we have all together three crossing equations.

$$\Sigma_{O\in S}\,\lambda^2{}_{\phi\phi O}\begin{pmatrix} F_{-,\Delta,l}(u,v) \\ 0 \\ F_{+,\Delta,l}(u,v) \end{pmatrix} + \Sigma_{O\in T}\,\lambda^2{}_{\phi\phi O}\begin{pmatrix} 0 \\ 1 \\ \frac{-2-n}{n} F_{+,\Delta,l}(u,v) \end{pmatrix} + \Sigma_{O\in A}\,\lambda^2{}_{\phi\phi O}\begin{pmatrix} -\frac{2(-1+n)}{n} F_{-,\Delta,l}(u,v) \\ F_{-,\Delta,l}(u,v) \\ F_{+,\Delta,l}(u,v) \end{pmatrix} = 0$$

Exercise: Derive the crossing equation for SU(N) group, with $\phi^I \in$ Adj.

[arXiv:1307.6856]

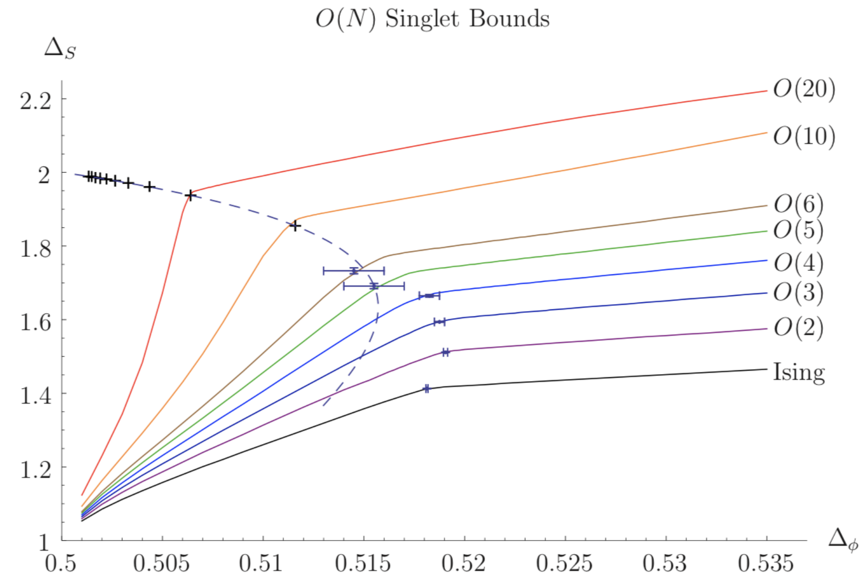
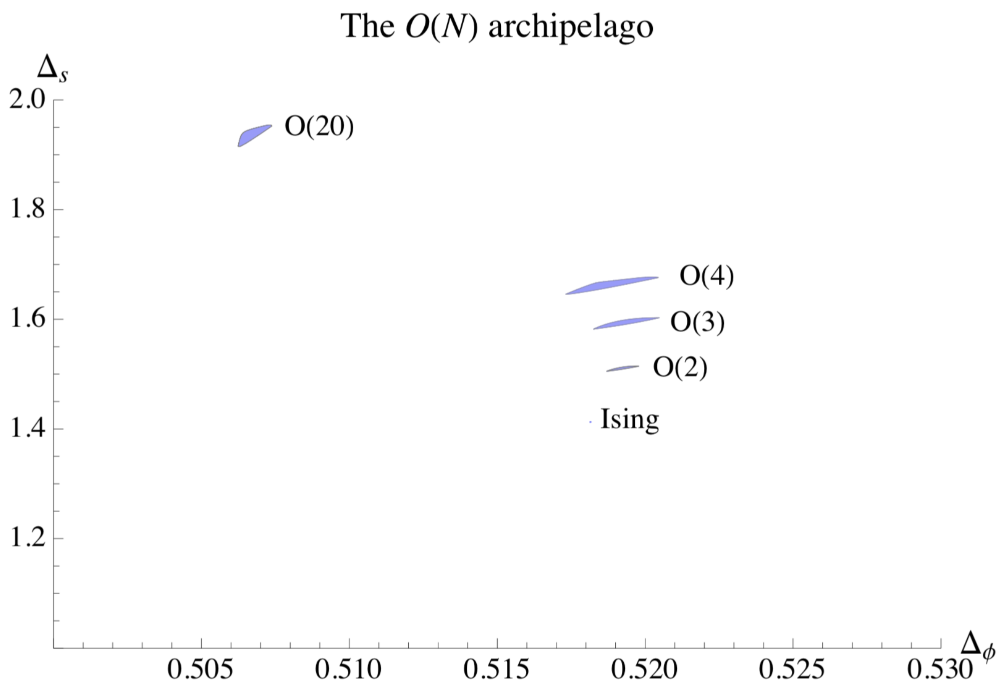
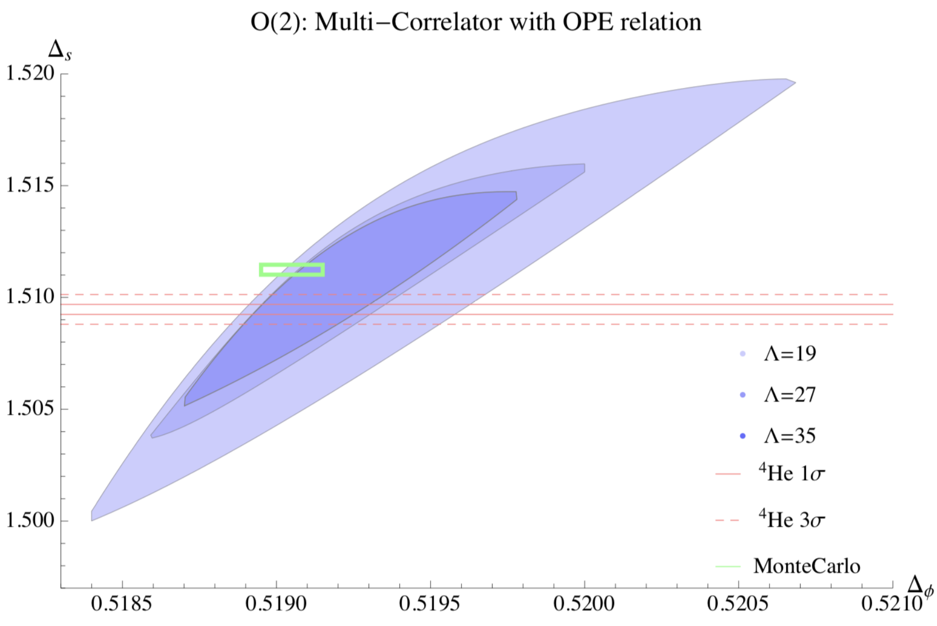


Figure 2: Upper bounds on the dimension of the lowest dimension singlet S in the $\phi \times \phi$ OPE, where ϕ transforms as a vector under an $O(N)$ global symmetry group. Here, we show $N = 1, 2, 3, 4, 5, 6, 10, 20$. The blue error bars represent the best available analytical and Monte Carlo determinations of the operator dimensions (Δ_ϕ, Δ_S) in the $O(N)$ vector models for $N = 1, 2, 3, 4, 5, 6$ (with $N = 1$ being the 3D Ising Model). The black crosses show the predictions in Eq. (4.1) from the large-N expansion for $N = 10, 20, ..., 100$. In this expansion, Δ_ϕ has been determined to three-loop order, while Δ_S is at two-loop order. The dashed line interpolates the large-N prediction for $N \in (4, \infty)$.

[arxiv : 1504.07997]





O(2) vector model describes normal phase to superfluid phase transition, red lines are experimental measurement.

[arXiv:1211.2810]

$\sum_{O\in\phi\times\phi}\lambda_{\phi\phi O}{}^2\, F_{\Delta,l}(u,v)=0$

$\lambda_{O_0}{}^2\, F_{\Delta_0,l_0}(u,v)=-F_{0,0}(u,v)-\sum_O\lambda_O{}^2\, F_{\Delta,l}(u,v)$
where we used the normalization $\lambda_{\phi\phi\,\mathrm{Id}}=1$

We try to find a linear functional such that
$\alpha(F_{\Delta_0,l_0}(u,v))=1$
$\alpha(F_{\Delta,l}(u,v))\geq 0$ for each O in the $\sum_O$ ...

If such α exist, then there is an inequality

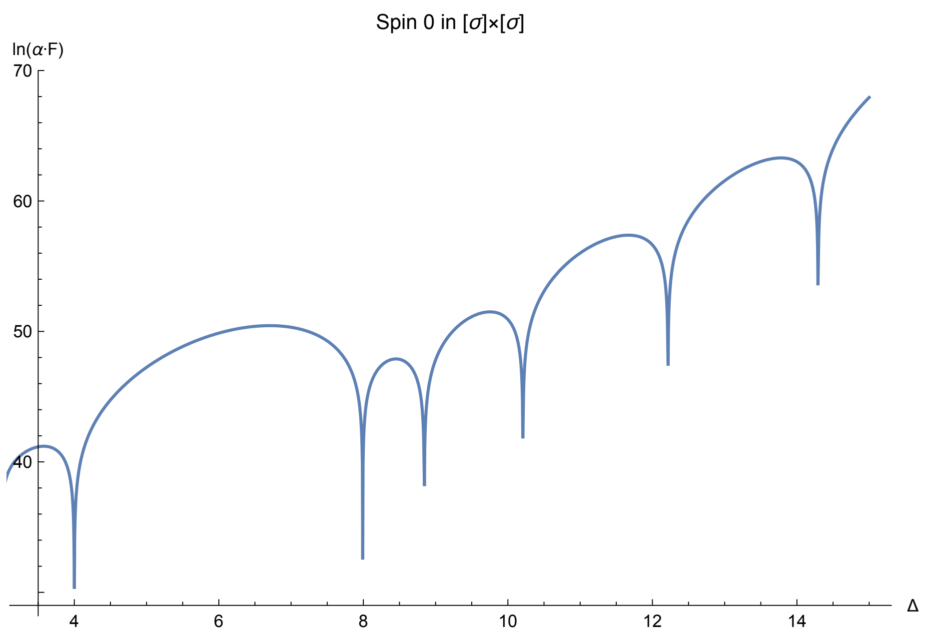
$\lambda_{O_0}{}^2=-\alpha(F_{0,0}(u,v))-\sum_O\lambda_O{}^2\,\alpha(F_{\Delta,l}(u,v))\leq-\alpha(F_{0,0}(u,v))$

We want to find the most restrictive bound, which minimize $-\alpha(F_{0,0}(u,v))$.

Such that α should satisfy the condition $\sum_O\lambda_O{}^2\,\alpha(F_{\Delta,l}(u,v))=0$.

On a physical theory, the spectrum is discrete. Remember that $\lambda_O{}^2>0$. The only way that the above eqn. can be satisfied is the $\alpha(F_{\Delta,l}(u,v))$=0 on some discrete choices of Δ, which correspondss to the physical spectrum.

This means we can read off the physical spectrum Δ from zeros in $\alpha(F_{\Delta,l}(u,v))$. This is called the **Extremal Functional Method**.

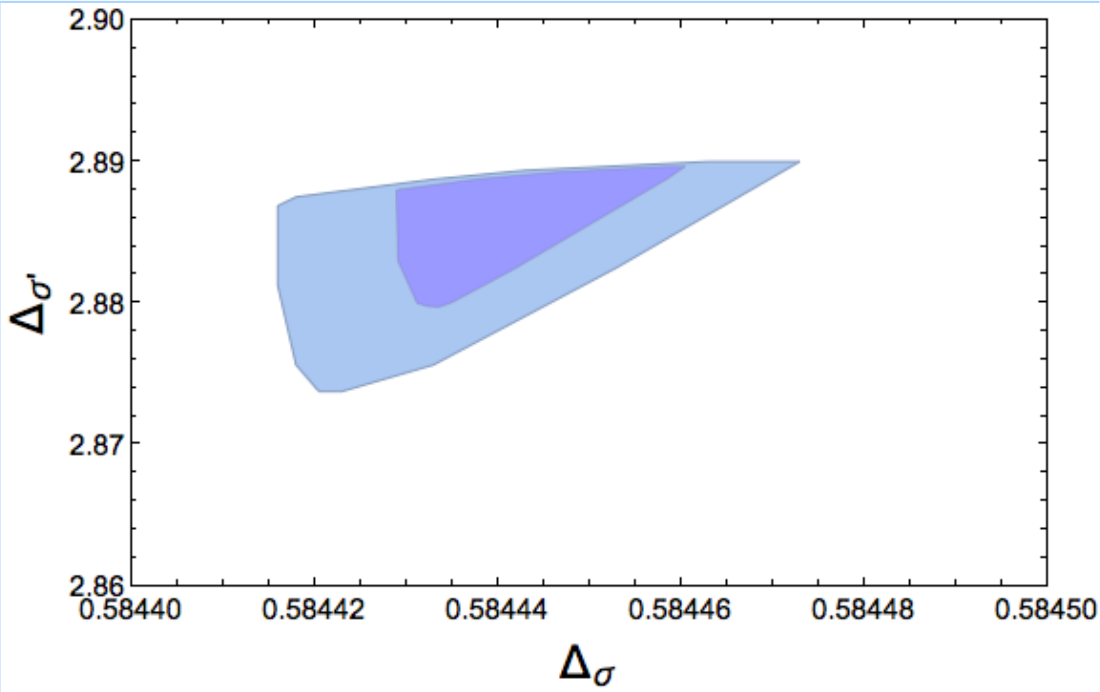


Zero: 4.000004175, 7.991361449, 8.843618529

The exact value are 4, 8, 9 ...

We can solve 2D Ising model without Virosora algebra!

[arXiv:1807.04434]



3D supersymmetric Ising model

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \sigma)^2 + \bar{\psi}\partial\!\!\!/\psi + \frac{\lambda_1}{2}\sigma\bar{\psi}\psi + \frac{\lambda_2^2}{8}\sigma^4.$$

The models contains Majorana fermions.

It was argued in arXiv:1301.7449 that this models has emergent supersymmetry and could be realized at the boundary of topological superconductor.

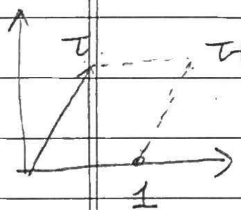
[Modular bootstrap : 0902.2790v2]

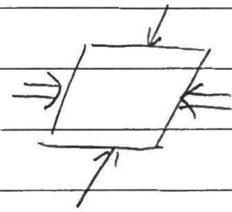
At $d=2$. the conformal algebra is enhanced to the infinite dimensional Virasoro algebra

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{c}{12}(n^2-n)\,\delta_{m+n,0}$$

$\{L_0, L_{-1}, L_{+1}\}$ forms an $SL(2)$ sub algebra. $Sl(2,R)\times Sl(2,R)$ is the global conformal algebra in 2d.

A torus is defined by the modular parameter τ





identify opposite edges we get a torus.

The torus partition function is

$$Z = \mathrm{tr}\left[e^{2\pi i \tau_1 P}\, e^{-2\pi \tau_2 H}\right]$$
$$= \mathrm{tr}\left[e^{2\pi i \tau_1(\hat{L}_0 - \hat{\bar{L}}_0)}\, e^{-2\pi i \tau_2(\hat{L}_0 + \hat{\bar{L}}_0)}\right]$$
$$= \mathrm{tr}\left[e^{2\pi i \tau \hat{L}_0}\, e^{2\pi i \bar{\tau} \hat{\bar{L}}_0}\right]$$
$$= \mathrm{tr}\left[q^{L_0 - \frac{c}{24}}\, \bar{q}^{\bar{L}_0 - \frac{c}{24}}\right]$$

$\hat{L}_0$ is L_0 on torus

L_0 is on plane

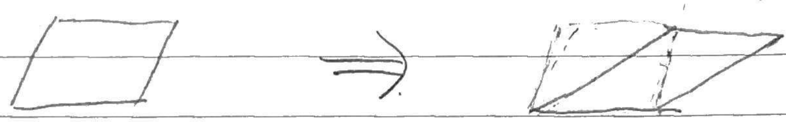
Remember Stress-Energy tensor transform under conformal trans

$$T_{cylinder}(w) = \left(\frac{\partial z}{\partial w}\right)^2 T(z) + \frac{c}{12} S(z,w)$$

Schwarzian

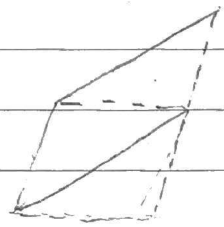
$$z = e^{w} \qquad T(z) = \sum_{n=z} L_n z^{-n}$$

Clearly $\tau \to \tau+1$ gives us the same torus.



T transformation

U transfomatision



$\tau \to \tau/_{\tau+1}$

also we have $S = T^{-1} U T^{-1}$ gives us $\tau \to -\frac{1}{\tau}$

S. T. U. transformation should leave the torus partition function invariant.

We could write down a Crossing eqn similar to the one from crossing symmety of CFT 4-pt function. This eqn again gives us constrains of the spectrum.

Charactors are generating functions of degenency of states ②
similar to
~~For~~ the partition function

we define.

$$\chi_h = \mathrm{tr}_h \left(q^{L_0 - \frac{c}{24}} \right)$$

tr_h: is taken over $|h\rangle$ and it descendants

Let us count the states.

Virsoro algebra.

level	L_0 eigen value	states
0	h	$\|h\rangle$
1	$h+1$	$L_{-1}\|h\rangle$
2	$h+2$	$L_{-2}\|h\rangle$, $L_{-1}^2\|h\rangle$
3	$h+3$	$L_{-3}\|h\rangle$, $L_{-2}L_{-1}^2\|h\rangle$ $L_{-1}^3\|h\rangle$
⋮	⋮	⋮
N	$h+N$	$P(N)$ states

$$\chi_h^{(Vir)} = q^{h-\frac{c}{24}}(1+q+2q^3+ \cdots P_N q^N + \cdots) = q^{h-\frac{c}{24}} \frac{1}{\prod_n (1-q^n)}$$

level	L_0	states
0	h	$\|h\rangle$
1	$h+1$	$L_{-1}\|h\rangle$
2	$h+2$	$L_{-1}^2\|h\rangle$
⋮	⋮	⋮
N	$h+N$	$L_{-1}^N\|h\rangle$

$$\chi^h_{SL(2)} = \mathrm{tr}_h^{SL(2)} q^{L_0 - \frac{c}{24}} = q^{h-\frac{c}{24}}(1+q+ \cdots q^N) = q^{h-\frac{c}{24}} \frac{1}{1-q}$$

For minimal models some descendants decouples.

For example 2d Ising model have $C_{(L)} = 1/2$

and ~~the~~ charactors for the vacunm

$$q^{-h+\frac{c}{24}} \chi_{h=0} = 1 + 0 \cdot q + q^2 + 2q^4 + \cdots$$

the missing of q term is because of the fact that the vacuum $|h=0\rangle$ is invariant under any transformation

one can :

compare this with $\chi^{Vir}_{h}(q)$ in generic case, it is easy to notice that some states are missing (decoupled)

by subtracting $\chi^{SL(2)}_{h.}(q)$ from $\chi^{Vir}(q)$.

it is easy to read out the SL(2) primary (quasi-primaries) here. in the vacuum multiplet.

$h = 0,\ 2,\ 4\ \ldots$

the ϵ operator has scaling dimension $h=\frac{1}{2}$.

$$q^{-h+\frac{c}{24}}\chi_{h=\frac{1}{2}} = 1+q+q^2+q^3+2q^4$$

So that we have quasi primaries with

$$h-\tfrac{1}{2} = 0,\ 4.$$

$h=\bar{h}$ for scalars. we get their scaling dimension to be

$$\Delta = 1,\ 4,\ 8,\ 9,\ \ldots$$

which are the spectrum observed in numerics.

Modular bootstrap. Date ③

arxiv: 0902.2790.v2

Charactor of Identity (vacumm)

$$\chi_0(\tau) = \frac{q^{-\frac{c-1}{24}}}{\eta(\tau)}(1-q)$$

when $c>1$, the charactor of a primary is,

$$\chi_h(\tau) = \frac{q^{h-\frac{c-1}{24}}}{\eta(\tau)}$$

where $\eta(\tau)$ is the Dedekind eta function $q = \exp(2\pi i\tau)$

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=0}^{\infty}(1-q^n)$$

The partition function is given by

$$Z = \chi_0(q)\chi_0(\bar{q}) + \sum_A \chi_{h_A}(q)\chi_{\bar{h}_A}(\bar{q})$$

where A labels the conformal primaries.

or equivalently

$$Z = \chi_0(q)\chi_0(\bar{q}) + \sum_{h,\bar{h}>0} d(h,\bar{h})\,\chi_h(q)\,\chi_{\bar{h}}(\bar{q})$$

where $d(h,\bar{h})$ denotes the degenerancy, or simply the number of conformal primaries with $(h,\bar{h})$.

Just like OPE^2, $d(h,\bar{h})$ is a positive (non-negative) number.

Modular invariance are consist of S, T and U duality.

T invariance: $\tau \to \tau+1$

$$\chi_h(\tau+1) = \chi_h(\tau)\, e^{i2\pi(h\theta-\frac{c}{24})}$$

$$\chi_h(q)\,\chi_{\bar h}(\bar q) \longrightarrow e^{2\pi i(h-\bar h)}\,\chi_h(q)\,\chi_{\bar h}(\bar q)$$

thus T invariance tells us that $(h-\bar h)\equiv j$ is an integer

S-invariance requires.

$$Z(\tau,\bar\tau) = Z\left(-\frac{1}{\tau}, -\frac{1}{\bar\tau}\right)$$

Dedekind $\eta(\tau)$ satisfies

$$\eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{1/2}\,\eta(\tau)$$

so that it is better to Defined

$$\hat Z = |\tau|^{1/2}|\eta(\tau)|^2 Z(\tau,\bar\tau)$$

$$= \overline{\hat\chi_0(\tau)}\,\hat\chi_0(\bar\tau) + \sum_{h,\bar h>0} \overline{\hat\chi_h(\tau)}\,\hat\chi_h(\tau)$$

the modified characters are.

$$\hat\chi_h(\tau) = (-i\tau)^{1/4}\,\eta(\tau)\,\chi_h(\tau)$$

so that $\overline{\hat\chi_h(\tau)}\,\hat\chi_h(\tau) = |\tau|^{1/2}\,\bar q^{h-\gamma} q^{h-\gamma}$ where $\gamma = \frac{c-1}{24}$

we have get rid of the complicated Dedekind function

~~It is convenient to change variable $\tau = i\exp(z)$~~

~~the S-invariance now requires the partition function to be invariant under $z \leftrightarrow -z$~~

~~Restrict our attention to $\tau = i\frac{\beta}{2\pi}$ with β to be inverse temperature, hence keep~~ ~~In fact $\beta/2\pi = \frac{1}{T}$~~

~~$\tau \leftrightarrow -\frac{1}{\tau} \Leftrightarrow \ldots \frac{2\pi}{\beta} \Leftrightarrow \frac{\beta}{2\pi}$~~

~~Remember $q=\bar{q}=\exp(-\beta)$~~

Restrict our attention to $\tau=\frac{i\beta}{2\pi}$ with β being inverse temperature, hence real valued.

under S trans.

$$\tau \leftrightarrow -\frac{1}{\tau} \quad \Longleftrightarrow \quad \frac{2\pi}{\beta} \leftrightarrow \frac{\beta}{2\pi}$$

$$q=\bar{q}=\exp(-\beta)$$

$$\overline{\hat{\chi}_{\bar{h}}(\bar{\tau})\,\hat{\chi}_h(\tau)} = \left(\frac{\beta}{2\pi}\right)^{1/2} \exp[-\beta(h+\bar{h}-2\nu)] \equiv g_{\hat{\Delta}}(\beta)$$

$\hat{\Delta}=h+\bar{h}-2\nu$, $h+\bar{h}=\Delta$ usual scaling dim.

$$\overline{\hat{\chi}_0(\bar{\tau})}\,\hat{\chi}_h(\tau) = \left(\frac{\beta}{2\pi}\right)^{1/2} \exp(+\beta 2\nu)\,(1-\exp(-\beta))^2 \equiv g_{Id}(\beta)$$

crossing eqn is

$$\hat{Z}\left(\frac{i\beta}{2\pi}\right) - \hat{Z}\left(i\frac{2\pi}{\beta}\right) = 0$$

Acting derivatives on the eqn. it is still valid.

$$(\beta\partial_\beta)^n\left[\hat{Z}\left(\frac{i\beta}{2\pi}\right) - \hat{Z}\left(\frac{i2\pi}{\beta}\right)\right] = 0 \qquad \text{at any } \beta$$

At $\beta=2\pi$ (the fixed points of S trans.)

$$(\beta\partial_\beta)^n \hat{Z}\left(\frac{i\beta}{2\pi}\right)\Big|_{\beta=2\pi} = (-1)^n(\beta\partial_\beta)^n \hat{Z}\left(\frac{i2\pi}{\beta}\right)\Big|_{\beta=2\pi}$$

so that

$$(\beta\partial_\beta)^n\left[\hat{Z}\left(\frac{i\beta}{2\pi}\right) - \hat{Z}\left(\frac{i2\pi}{\beta}\right)\right]\Big|_{\beta=2\pi} = (1-(-1)^n)(\beta\partial_\beta)^n \hat{Z}\left(\frac{i\beta}{2\pi}\right)\Big|_{\beta=2\pi}$$

which is trivially satisfied for $n=$ even.

Let us check the eqn. for $n=1$ & $n=3$.

$$\beta \partial_\beta \, g_{\hat{\Delta}}(\beta) = e^{-2\pi\hat{\Delta}} \left(\frac{1}{2} - 4\pi\hat{\Delta}\right) \equiv f_1(\hat{\Delta})$$

$$(\beta \partial_\beta)^3 g_{\hat{\Delta}}(\beta) = e^{-2\pi\hat{\Delta}} \left[\frac{1}{8}\left(1 - 52\pi\hat{\Delta} + 144\pi\hat{\Delta}^2 - 64\pi^3\hat{\Delta}^3\right)\right] \equiv f_3(\hat{\Delta})$$

$$(\beta\partial_\beta)\, g_{Id}(\beta) = b_1(\gamma)$$

$$(\beta\partial_\beta)^3 g_{Id}(\beta) = b_3(\gamma)$$

Crossing eqn. tells us that

$$\Sigma_A f_1(\hat{\Delta}) \exp(-2\pi\hat{\Delta}) = -b_1(\gamma) \qquad ①$$

$$\Sigma_A f_3(\hat{\Delta}) \exp(-2\pi\hat{\Delta}) = -b_3(\gamma) \qquad ②$$

$$①/② \Rightarrow \frac{\Sigma_A \; I_{31}(\hat{\Delta}_A) f_1(\hat{\Delta}_A) \exp(-2\pi\hat{\Delta}_A)}{\Sigma_B \; f_1(\hat{\Delta}_B) \exp(-2\pi\hat{\Delta}_B)} = K_{31}(\gamma)$$

we have defined $I_{31} = \frac{f_3}{f_1}$ $\quad K_{31} = \frac{b_3}{b_1}$

$$\Rightarrow \frac{\Sigma_A \left(I_{31}(\hat{\Delta}_A) - K_{31}(\gamma)\right) f_1(\hat{\Delta}_A) \exp(-2\pi\hat{\Delta}_A)}{\Sigma_B \; f_1(\hat{\Delta}_B) \exp(-2\pi\hat{\Delta}_B)} = 0$$

One can pick a gap $\hat{\Delta}_+$, such that

$I_{31} - K_{31} > 0 \qquad$ for $\hat{\Delta} > \hat{\Delta}_+$

$f_1(\hat{\Delta}) \qquad < 0.$

One can check explicitly that this is possible. $f_1(\hat{\Delta})$ is monotonic
and $I_{31} \approx 4\pi^2\hat{\Delta}^2$ for $\hat{\Delta} \gg 1$

All the terms on LHS are negative. This means that there must be an operator with $\hat{\Delta} < \Delta_+$

thus we have obtained an upper bound of the scaling dimension of the leading operator.

Δ_+ (no hat) depends on γ, hence dealy depends on central charge

Exercise: show that at $c \gg 1$,

$$\Delta_+ \simeq \frac{c_{total}}{12} + \delta_0$$

δ_0 is a constant of order $\mathcal{O}(1)$.

Comments:

① We have analysis the constraint from $n=1$ and $n=3$. Higher derivatives would give us further constraints. [1307.6562]

② We have consider the derivatives along the imaginary axis of τ. It is also possible to take into account the deriv. along $\mathrm{Re}(\tau)$ axis. [1608.06241]

③ In partition we have not include the contribution of charactors with $\chi_0(q)\,\chi_h(q)$ + c.c.

They corresponde to conserved currents with spin-h

After including their contributions, one can obtain a bound saturated by $(A_1)_1$ $(A_2)_1$ ... $(E_8)_1$

which are "minimal models" of corresponding Kac-Moody algebra.

[Bae, Lee, Song '17]

④ $\Delta_+ \approx \frac{C_{total}}{12} + \cdots$

The lightest BTZ black hole corresponds to $\Delta = \frac{C_{total}}{24}$

which clearly satisfies the bound.

The number $\frac{\Delta_+}{C_{total}}$ is now decreasing as numerics improves.

⑤ $(\tau \partial_\tau)^n \hat{\chi}_h(q) \Big|_{\tau=i} = e^{-\hat{h}} P_n(\hat{h})$

with $\hat{h} = 2\pi (h - \gamma')$

$4^n P_n(-\hat{h})$ is the "row polynomial of Sheffer triangle $S2[4,1]$"

I do not know why such a funny object appears.

⑥ For further story about the dual of pure Quantum Gravity in AdS_3, check 1610.05814