November 6, 2018

QFT on curved spacetimes (ver.1)
: Lecture note prepared for K-HEP School

Miok Park

Korea Institute for Advanced Study,
85 Hoegiro, Dongdaemun-gu, Seoul 02455,
Republic of Korea

miokpark@kias.re.kr

References

- Gautam Mandal, Lecture note for THE 8TH ASIAN WINTER SCHOOL ON STRINGS, PARTICLES AND COSMOLOGY, "Introduction to Hawking radiation", (https://www.icts.res.in/program/asian8/talks)
- V. Mukhanov and S. Winitzki, "Introduction to Quantum Effects in Gravity", Cambridge University Press
- Paul Townsend, "Black holes: Lecture notes", gr-qc/9707012

Contents

1 Introduction **2**

2 Quantization of fields **3**
- 2.1 In flat spacetimes . . . 3
- 2.2 In curved spacetimes . . . 4

3 Quantum Vacuum in Minkowski spacetimes **5**

4 Quantum Vacuum in Cosmology **5**

5 Quantum Vacuum in Black Holes Spacetimes **8**
- 5.1 Rindler spacetime . . . 8
- 5.2 Unruh Effect . . . 9
- 5.3 Hawking Pair . . . 11
- 5.4 Kruskal Vacuum and Boulware vacuum . . . 15

6 Hawking Radiation from black hole formation **16**

1 Introduction

Let us consider a functional integral involving the spacetime metric $g_{\mu\nu}$ and other fields ϕ_i

$$\mathcal{Z} = \int \mathcal{D}g_{\mu\nu}\mathcal{D}\phi \; e^{iS[g_{\mu\nu},\phi]} \tag{1.1}$$

where we assume the action to be

$$S = \frac{S_1}{G_N} + S_2, \tag{1.2}$$

$$S_1 = \int d^4x\sqrt{g}\Big(R + \frac{1}{2}\partial^\mu\phi_1\partial_\mu\phi_1 + V(\phi_1)\Big), \tag{1.3}$$

$$S_2 = \int d^4x\sqrt{g}\Big(\frac{1}{2}\partial^\mu\phi_2\partial_\mu\phi_2 + V(\phi_2)\Big). \tag{1.4}$$

Here the scalar field ϕ_1 in S_1 couples to the gravitational field with the coupling constant G_N and the scalar field ϕ_2 in S_2 is a purely quantum theory on curved spacetime. When we take $G_N \to 0$ limit, there may exist a saddle point (or multiple saddle points). Around any saddle point $\bar{g}$ and $\bar{\phi}$, each field is expanded as

$$g = \bar{g} + \sqrt{G_N}h, \qquad \phi_1 = \bar{\phi}_1 + \sqrt{G_N}\phi_1 \tag{1.5}$$

where $\bar{g}$ and $\bar{\phi}_1$ satisfy

$$\frac{\delta S_1}{\delta \bar{g}_{\mu\nu}} = 0, \qquad \frac{\delta S_1}{\delta \bar{\phi}_1} = 0. \tag{1.6}$$

Then the full quantum gravity action and its functional integral are approximated to

$$S \simeq \frac{S_1^{(0)}[\bar{g},\bar{\phi}_1]}{G_N} + S_1^{(2)}(\bar{g},\bar{\phi}_1,h,\delta\phi_1) + S_2(\bar{g}_1,\phi_2) + \cdots, \tag{1.7}$$

$$\mathcal{Z} \simeq e^{i\frac{S_1^{(0)}[\bar{g},\bar{\phi}]}{G_N}} \int \mathcal{D}h_{\mu\nu}\mathcal{D}\delta\phi_1\mathcal{D}\phi_2 \; e^{i\left(S_1^{(2)}[\bar{g},\bar{\phi}_1,h_{\mu\nu},\delta\phi_1] + S_2[\bar{g},\phi_2] + \cdots\right)}. \tag{1.8}$$

In this situation, h and $\delta\phi_1$ in $S_1^{(2)}$ can be treated as the same way of the quantum field ϕ_2 in S_2. Thus, even though we do not know yet how to handle the full quantum gravity theory, we can approximately study the quantum behavior under a gravitational interaction by the semi-classical approximation.

2 Quantization of fields

2.1 In flat spacetimes

Firstly let us review the free scalar field theory in flat spacetimes, $\bar{g}_{\mu\nu} = \bar{\eta}_{\mu\nu}$ as follows

$$S = \frac{1}{2}\int d^4x \Big[\eta^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - m^2\phi^2\Big], \tag{2.1}$$

$$\delta_\phi S \ \rightarrow \ \nabla^2\phi + m^2\phi = 0 \tag{2.2}$$

and construct the field operator expansion in momentum space

$$\hat{\phi}(t,\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2}}\left(v^*_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}}\hat{a}_{\mathbf{k}} + v_{\mathbf{k}}(t)e^{-i\mathbf{k}\cdot\mathbf{x}}\hat{a}^\dagger_{\mathbf{k}}\right). \tag{2.3}$$

The canonical momenta operator is defined and yielded as

$$\hat{\pi}(t,\mathbf{y}) = \frac{\partial\hat{\phi}(t,\mathbf{y})}{\partial t} = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}}\frac{1}{\sqrt{2}}\left(\dot{v}^*_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{y}}\hat{a}_{\mathbf{k}} + \dot{v}_{\mathbf{k}}(t)e^{-i\mathbf{k}\cdot\mathbf{y}}\hat{a}^\dagger_{\mathbf{k}}\right). \tag{2.4}$$

Then we postulate the following commutation relations

$$[\hat{\phi}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{y})] = i\delta(\mathbf{x}-\mathbf{y}), \qquad [\hat{\phi}(t,\mathbf{x}),\hat{\phi}(t,\mathbf{y})] = [\hat{\pi}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{y})] = 0 \tag{2.5}$$

in terms of the time-independent operators $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$

$$[\hat{a}_{\mathbf{k}},\hat{a}^\dagger_{\mathbf{k}'}] = \delta(\mathbf{k}-\mathbf{k}'), \qquad [\hat{a}_{\mathbf{k}},\hat{a}_{\mathbf{k}'}] = [\hat{a}^\dagger_{\mathbf{k}},\hat{a}^\dagger_{\mathbf{k}'}] = 0. \tag{2.6}$$

which are annihilation and creation operators respectively.

Now we check the mode expansion (2.3) satisfies orthonormal condition so that the field operator $\hat{\phi}$ properly get operator valued. Plugging the field operator (2.3) into the equation of motion (2.2), the mode functions are required to satisfy

$$\ddot{v}_{\mathbf{k}} + \omega_k^2 v_{\mathbf{k}} = 0 \tag{2.7}$$

where $w_k^2 = k^2 + m^2$. The commutation relations with (2.3) and (2.4) are consistent with (2.5) only when the following normalization condition is satisfied

$$\dot{v}_{\mathbf{k}}(t)v^*_{\mathbf{k}}(t) - v_{\mathbf{k}}(t)\dot{v}^*_{\mathbf{k}}(t) = 2i. \tag{2.8}$$

This condition is indeed the Wronskian of the independent complex solutions $v_{\mathbf{k}}(t)$ and $v^*_{\mathbf{k}}(t)$

$$W[v_k, v_k^*] = \dot{v}_{\mathbf{k}}(t)v^*_{\mathbf{k}}(t) - v_{\mathbf{k}}(t)\dot{v}^*_{\mathbf{k}}(t). \tag{2.9}$$

On the other hand, if $W[x_1,x_2] = 0$ this indicates the matrix

$$\begin{pmatrix} \dot{x}_1(t) & x_1(t) \\ \dot{x}_2(t) & x_2(t) \end{pmatrix} \tag{2.10}$$

is degenerate for each t.

The norm of the field operator is defined by Klein-Gordon inner product of the mode function such as

$$(v_1, v_2) = -i \int d^3\mathbf{x}[v_1(t,\mathbf{x})\dot{v}_2^*(t,\mathbf{x}) - \dot{v}_1(t,\mathbf{x})v_2^*(t,\mathbf{x})] \tag{2.11}$$

but this does not produce a genuine inner product space since it can be positive or negative, depending on values of v_1 and $\dot{v}_1$ (or v_2 and $\dot{v}_2$). This definition is deeply related to the Klein-Gordon charge density which is

$$(\phi, \phi) = i \int d^3\mathbf{x} \left[\phi^* \frac{\partial}{\partial t}\phi - \phi\frac{\partial}{\partial t}\phi^*\right] = Q \tag{2.12}$$

where Q can be negative. We identify the subspace of positive-energy solutions as the physical space of state vectors and restrict the Klein-Gordon inner product to this subspace. The general solution for (2.7) is obtained as

$$v_{\mathbf{k}}(t) = \frac{1}{\sqrt{\omega_{\mathbf{k}}}}(A_{\mathbf{k}}e^{i\omega t} + B_{\mathbf{k}}e^{-i\omega t}), \tag{2.13}$$

where ω is positive with respect to t, and from (2.8) the constants of integration $A_{\mathbf{k}}$ and $B_{\mathbf{k}}$ must obey

$$|A_{\mathbf{k}}|^2 - |B_{\mathbf{k}}|^2 = 1. \tag{2.14}$$

The Hamiltonian is calculated as

$$\hat{H} = \int d^3\mathbf{k}\omega_k \left[A_{\mathbf{k}}^* B_{\mathbf{k}}^* \hat{a}_{\mathbf{k}}\hat{a}_{-\mathbf{k}} + A_{\mathbf{k}}B_{\mathbf{k}}\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{-\mathbf{k}}^\dagger + \left(|A_{\mathbf{k}}|^2 + |B_{\mathbf{k}}|^2\right)\left(\hat{a}_{\mathbf{k}}^\dagger \hat{a}_{\mathbf{k}} + \frac{1}{2}\delta^{(3)}(0)\right)\right] \tag{2.15}$$

where $A_{\mathbf{k}}B_{\mathbf{k}} = 0$ is required. Together with (2.14), the coefficients are determined as

$$A_{\mathbf{k}} = e^{i\delta_{\mathbf{k}}}, \qquad B_{\mathbf{k}} = 0 \tag{2.16}$$

where $\delta_{\mathbf{k}}$ can set to be zero, and then the mode function becomes

$$v_k(t) = \frac{1}{\sqrt{w_k}}e^{i\omega_k t}. \tag{2.17}$$

2.2 In curved spacetimes

The free scalar field in curved spacetime is written as

$$S = \frac{1}{2}\int d^4x\sqrt{-\bar{g}}\left[\bar{g}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - m^2\phi^2\right] \tag{2.18}$$

and the Klein-Gordon inner product is defined as

$$(u, v) = -i \int d\Sigma^\mu \sqrt{g_\Sigma}\,[u(t,\mathbf{x})\partial_\mu v^*(t,\mathbf{x}) - (\partial_\mu u(t,\mathbf{x}))v^*(t,\mathbf{x})]. \tag{2.19}$$

While the mode function in (2.3) is uniquely fixed in flat spacetime as we have seen the result (2.17), in curved spacetime the mode function is not uniquely determined since there is no canonical choice to pick positive frequency solutions.

3 Quantum Vacuum in Minkowski spacetimes

In Minkowski spacetimes, there is a unique Poincare-invariant vacuum $|O_M\rangle$ such as

$$P_\mu|O_M\rangle = 0, \qquad J_{\mu\nu}|O_M\rangle = 0 \tag{3.1}$$

where P_μ is the translation generator and $J_{\mu\nu}$ is Lorentz transformation generator. Let us consider a mode expansion of a scalar field as follows

$$\hat{\phi} = \sum_{\vec{k}} \hat{a}_{\vec{k}} e^{-i\omega t + i\vec{k}\cdot\vec{x}} + c.c., \qquad \hat{a}_{\vec{k}}|O_M\rangle = 0 \tag{3.2}$$

where ω is a positive value with respect to t and the annihilation operator $\hat{a}_{\vec{k}}$ defines the Minkowski vacuum. Under a coordinate transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, the field operator is rewritten as

$$\hat{\phi} = \sum_{\vec{k}'} \hat{a}_{\vec{k}'} e^{-i\omega' t' + i\vec{k}'\cdot\vec{x}'} + c.c. \tag{3.3}$$

where the new annihilation operator $\hat{a}_{\vec{k}'}$ satisfies

$$\hat{a}_{\vec{k}'}|O_M\rangle = U_\Lambda^\dagger \hat{a}_{\vec{k}} U_\Lambda |O_M\rangle = 0. \tag{3.4}$$

Thus there exists a unique vacuum in Minkowski spacetimes.

4 Quantum Vacuum in Cosmology

The spatially flat Friedmann universe is conformally equivalent to the Minkowski metric as follows

$$ds^2 = -dt^2 + a^2(t)\delta_{ik}dx^i dx^k \tag{4.1}$$
$$= a^2(\eta)[-d\eta^2 + \delta_{ik}dx^i dx^k] = a^2(\eta)\eta_{\mu\nu}dx^\mu dx^\nu \tag{4.2}$$

where the conformal time is introduced as

$$\eta(t) \equiv \int^t \frac{dt}{a(t)}. \tag{4.3}$$

Let us consider again a free scalar field theory

$$S = \frac{1}{2}\int \sqrt{-g}d^4x \left[g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - m^2\phi^2\right] \tag{4.4}$$

and take the spacetime metric (4.2) which is $g^{\alpha\beta} = a^{-2}\eta^{\alpha\beta}$ and $\sqrt{-g} = a^4$. Then the action explicitly takes a form of

$$S = \frac{1}{2}\int d^3\mathbf{x}d\eta\, a^2\left[-\phi'^2 + (\nabla\phi)^2 - m^2a^2\phi^2\right], \tag{4.5}$$

and if redefining a field

$$\chi \equiv a(\eta)\phi, \tag{4.6}$$

the action is rewritten as

$$S = \frac{1}{2}\int d^3\mathbf{x}d\eta \left[-\chi'^2 + (\nabla\chi)^2 - \left(m^2a^2 - \frac{a''}{a}\right)\chi^2\right]. \tag{4.7}$$

Performing the Fourier Transformation to the momentum space

$$\chi(\eta, \mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \chi_{\mathbf{k}}(\eta) e^{i\mathbf{k}\cdot\vec{x}}, \tag{4.8}$$

the complex Fourier modes $\chi_{\mathbf{k}}(\eta)$ is required to satisfy the equation of motion

$$\chi''_{\mathbf{k}} + \omega_k^2(\eta)\chi_{\mathbf{k}} = 0 \tag{4.9}$$

where $\omega_k^2(\eta) = k^2 + m_{\text{eff}}^2 = k^2 + m^2 a(\eta)^2 - \frac{a''}{a}$ and $k \equiv |\mathbf{k}|$. The general solution would take a form of

$$\chi_{\mathbf{k}}(\eta) = \frac{1}{\sqrt{2}} \left[a_{\mathbf{k}} v_k^*(\eta) + a_{\mathbf{k}}^\dagger v_k(\eta) \right], \tag{4.10}$$

and plugging this to (4.8) the field solution yields as

$$\chi(\eta, \mathbf{x}) = \frac{1}{\sqrt{2}} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[a_{\mathbf{k}} v_k^*(\eta) + a_{-\mathbf{k}}^\dagger v_k(\eta) \right] e^{i\mathbf{k}\cdot\mathbf{x}} \tag{4.11}$$

$$= \frac{1}{\sqrt{2}} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[a_{\mathbf{k}} v_k^*(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + a_{\mathbf{k}}^\dagger v_k(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \right]. \tag{4.12}$$

Inserting this to the equation of motion the mode function should obey the following differential equation

$$v_k'' + \omega_k^2 v_k = 0. \tag{4.13}$$

For the field solutions to get operator valued, they should satisfy the commutation relation such as (2.5) and (2.6), and this automatically requires for the mode functions to be

$$W[v_k, v_k^*] \equiv v_k' v_k^* - v_k v_k'^* = 2i\text{Im}(v'v*). \tag{4.14}$$

Here the Wronskian (4.14) is time-independent and becomes non-zero if and only if v_k and v_k' are linearly independent solutions. We take $\text{Im}(v'v^*) = 1$ which can be always chosen if $W \neq 0$.

Indeed, this normalization condition (4.14) is not enough to determine the solutions. Their linear combination can also become a solution

$$u_k(\eta) = \alpha_k v_k(\eta) + \beta_k v_k^*(\eta) \tag{4.15}$$

where α_k and β_k are time-independent complex values and obey the condition

$$|\alpha_k|^2 - |\beta_k|^2 = 1 \tag{4.16}$$

that is consistent with the normalization condition for $u_k(\eta)$ to be $\text{Im}(u'u*) = 1$. Then, with the new mode functions u_k and u_k^* the field solution can be written as

$$\chi(\eta, \mathbf{x}) = \frac{1}{\sqrt{2}} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \left[b_{\mathbf{k}} u_k^*(\eta) e^{i\mathbf{k}\cdot\mathbf{x}} + b_{\mathbf{k}}^\dagger u_k(\eta) e^{-i\mathbf{k}\cdot\mathbf{x}} \right]. \tag{4.17}$$

Taking operator values to the field solutions, the annihilation operators $\hat{a}_{\mathbf{k}}$ and $\hat{b}_{\mathbf{k}}$ and the creation operators $\hat{a}_{\mathbf{k}}^\dagger$ and $\hat{b}_{\mathbf{k}}^\dagger$ are related by the following Bogoliubov transformation

$$\hat{a}_{\mathbf{k}} = \alpha_k^* \hat{b}_{\mathbf{k}} + \beta_k \hat{b}_{-\mathbf{k}}^\dagger, \qquad \hat{a}_{\mathbf{k}}^\dagger = \alpha_k \hat{b}_{\mathbf{k}}^\dagger + \beta_k^* \hat{b}_{-\mathbf{k}}, \tag{4.18}$$

$$\hat{b}_{\mathbf{k}} = \alpha_k \hat{a}_{\mathbf{k}} - \beta_k \hat{a}_{-\mathbf{k}}^\dagger, \qquad \hat{b}_{\mathbf{k}}^\dagger = \alpha_k^* \hat{a}_{\mathbf{k}}^\dagger - \beta_k^* \hat{a}_{-\mathbf{k}}, \tag{4.19}$$

where the annihilation operators define each vacuum as

$$\hat{a}_{\mathbf{k}}|O_a\rangle = 0, \qquad \hat{b}_{\mathbf{k}}|O_b\rangle = 0. \tag{4.20}$$

Let us take a-particle number operator $\hat{N}^{(a)}_{\mathbf{k}} = \hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}$ to b-vacuum

$$\begin{aligned}\langle O_b|\hat{N}^{(a)}_{\mathbf{k}}|O_b\rangle &= \langle O_b|\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}|O_b\rangle \\ &= \langle O_b|(\alpha_k\hat{b}^\dagger_{\mathbf{k}} + \beta^*_k\hat{b}_{-\mathbf{k}})(\alpha^*_k\hat{b}_{\mathbf{k}} + \beta_k\hat{b}^\dagger_{-\mathbf{k}})|O_b\rangle \\ &= \langle O_b|(\beta^*_k\hat{b}_{-\mathbf{k}})(\beta_k\hat{b}^\dagger_{-\mathbf{k}})|O_b\rangle = |\beta_k|^2\delta^3(0)\end{aligned} \tag{4.21}$$

where the expectation value for the particle number in mode $\mathbf{k}$ is non-zero. To drop the delta function, we use the mean density of the a-particles in the mode $\mathbf{k}$ as follows

$$n_{\mathbf{k}} = |\beta_k|^2 \tag{4.22}$$

and calculate the total mean density of all particles

$$n = \int d^3\mathbf{k}|\beta_k|^2. \tag{4.23}$$

Two vacua are related to

$$\begin{aligned}|O_b\rangle &= \left[\prod_{\mathbf{k}} \frac{1}{|a_k|^{1/2}} \exp\left(\frac{\beta_k}{2\alpha_k}\hat{a}^\dagger_{\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}}\right)\right]|O_a\rangle \\ &= \prod_k \frac{1}{|a_k|^{1/2}}\left(\sum_{n=0}^{\infty}\left(\frac{\beta_k}{2\alpha_k}\right)^n |n^a_{\mathbf{k}}, n^a_{-\mathbf{k}}\rangle\right).\end{aligned} \tag{4.24}$$

Physical vacuum can be chosen by the preference set of the mode functions that describe the "actual" physical vacuum and particles.

Quantum mechanical analogy

This particle creation, the time dependent metric case (4.2), has an analogy with the quantum mechanics as follows. The equation of motion for the mode function (4.13), which is

$$\frac{d^2 v_k}{d\eta^2} + \omega_k^2 v_k = 0,$$

is compared to the stationary Schrodinger equation for a particle in an one-dimensional potential such as

$$\frac{d^2\psi}{dx^2} + (E - V(x))\psi = 0 \tag{4.25}$$

by replacing $v_k \to \psi$, $\eta \to x$, and $\omega_k^2 \to E - V(x)$. Which indicates that the particle creation mechanism in expanding universe can be understood by quantum-mechanical potential barrier problem. When an incident wave function meets the potential barrier at some position of x the wave function is split to the transmission and the reflection part. Normalizing the transmitted wave to unity ($T = 1$) the incident wave probability ($|\alpha|^2$) and the reflected wave probability ($|\beta|^2$) satisfy the conservation of probability that $|\alpha|^2 = |\beta|^2 + 1$, which is analogous to (4.16). Here the transmitted wave can be compared to the initial vacuum fluctuation and the reflected wave can correspond to "the particle creation" that occurs in the expanding universe .

5 Quantum Vacuum in Black Holes Spacetimes

5.1 Rindler spacetime

Let us start with an observer in the two-dimensional Minkowski spacetime

$$ds^2 = -dt^2 + dx^2. \tag{5.1}$$

and consider another constantly accelerating observer with the acceleration α which is defined as

$$\eta_{AB} d\ddot{x}^A d\ddot{x}^B = \alpha^2 \tag{5.2}$$

in the inertial frame (t, x). The trajectory in the inertial frame is described as

$$X^\mu = \frac{1}{\alpha}(\sinh \alpha\tau, \cosh \alpha\tau) \tag{5.3}$$

which means

$$t(\tau) = \frac{1}{\alpha}\sinh \alpha\tau, \qquad x(\tau) = \frac{1}{\alpha}\cosh \alpha\tau. \tag{5.4}$$

We would like to find a metric in the accelerating frame whose coordinate is (η, ξ) and takes a form of

$$ds^2 = \Omega^2(\eta, \xi)[-d\eta^2 + d\xi^2]. \tag{5.5}$$

Then the trajectory can be expressed in terms of η and ξ as follows

$$t(\eta, \xi) = \frac{1}{a}e^{a\xi}\sinh a\eta, \qquad x(\eta, \xi) = \frac{1}{a}e^{a\xi}\cosh a\eta \tag{5.6}$$

where $\alpha = ae^{-a\xi}$, and the metric in the accelerated frame is written as

$$ds^2 = e^{2a\xi}[-d\eta^2 + d\xi^2] \tag{5.7}$$

where $-\infty < \eta < \infty$ and $-\infty < \xi < \infty$. This metric is known as Rindler spacetime.

In a lightcone coordinate, the Minkowski spacetime and the Rindler spacetime become

$$ds^2 = -dUdV, \tag{5.8}$$
$$= -e^{2a\xi}dudv \tag{5.9}$$

where

$$U = t - x = -\frac{1}{a}e^{-au}, \tag{5.10}$$
$$V = t + x = \frac{1}{a}e^{au}, \tag{5.11}$$
$$u = \eta - \xi, \tag{5.12}$$
$$v = \eta + \xi. \tag{5.13}$$

The Rindler coordinates covers only one quarter of the Minkowski spacetimes, which is $U < 0$ and $V > 0$, see Fig. (1)

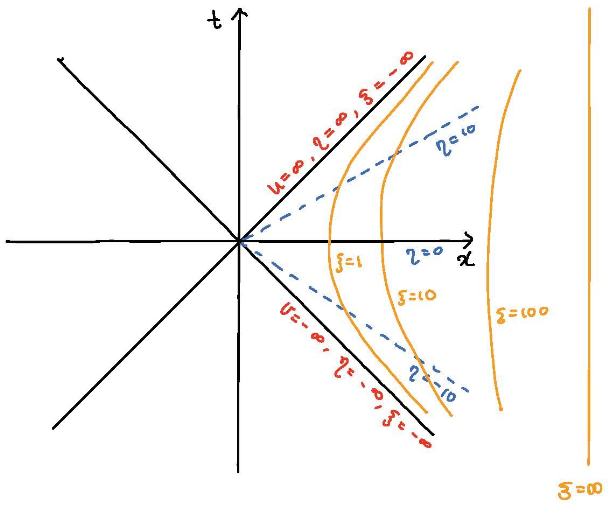

Figure 1: Rindler spactimes covers only one quarter of the Minkowski spacetimes

Rindler spacetime from Black hole spacetime

The Rindler spacetime also can appear near horizon of black hole spacetimes. Let us consider the Schwarzschild black hole spacetime

$$ds^2 = -\left(1 - \frac{2M}{R}\right) dT^2 + \left(1 - \frac{2M}{R}\right)^{-1} dR^2 + R^2 d\Omega^2 \tag{5.14}$$

and change a variable as follows

$$R - 2M = \frac{X^2}{8M}. \tag{5.15}$$

Taking the expansion near the horizon at $R \approx 2M$ and fixing angular values, the metric takes a form of

$$ds^2 \approx -(\kappa X)^2 dT^2 + dX^2, \tag{5.16}$$

which is the two dimensional Rindler spacetime that is equivalent to (5.7) by a proper coordinate transformation.

5.2 Unruh Effect

For a massless scalar field in a lightcone coordinate, the standard mode expansion is written as

$$\hat{\phi} = \int_0^\infty \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} [e^{-i\omega U} \hat{a}_\omega + e^{i\omega U} \hat{a}^\dagger_\omega] + \text{(left-moving)} \tag{5.17}$$

$$= \int_0^\infty \frac{d\Omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\Omega}} [e^{-i\Omega u} \hat{b}_\Omega + e^{i\Omega u} \hat{b}^\dagger_\Omega] + \text{(left-moving)} \tag{5.18}$$

where the Minkowski vacuum and the Rindler vacuum are defined as $\hat{a}_\omega |O_M\rangle = 0$ and $\hat{b}_\Omega |O_R\rangle = 0$ respectively. These operators satisfy the commutation relation

$$[\hat{a}_\omega, \hat{a}^\dagger_{\omega'}] = \delta(\omega - \omega'), \qquad [\hat{b}_\Omega, \hat{b}^\dagger_{\Omega'}] = \delta(\Omega - \Omega'), \tag{5.19}$$

and the operators $\hat{a}_\omega, \hat{a}^\dagger_\omega$ and $\hat{b}_\Omega, \hat{b}^\dagger_\Omega$ are related by Bogoliubov transformation

$$\hat{b}_\Omega = \int_0^\infty d\omega[\alpha_{\Omega\omega}\hat{a}_\omega - \beta_{\Omega\omega}\hat{a}^\dagger_\omega] \tag{5.20}$$

where the inverse Bogoliubov transformation is not defined since the Rindler spacetime covers only the quarter of the Minkowski spacetime. Replacing $\hat{b}_\Omega$ and $\hat{b}^\dagger_\Omega$ in (5.19) with (5.20), the normalization condition for the Bogoliubov coefficient is obtained

$$\int_0^\infty d\omega(\alpha_{\Omega\omega}\alpha^*_{\Omega'\omega} - \beta_{\Omega\omega}\beta^*_{\Omega'\omega}) = \delta(\Omega - \Omega'). \tag{5.21}$$

Plugging (5.20) into (5.18) and comparing it with (5.17),

$$\frac{1}{\sqrt{\omega}}e^{-i\omega U} = \int_0^\infty \frac{d\Omega'}{\sqrt{\Omega'}}(\alpha_{\Omega'\omega}e^{-i\Omega' u} - \beta^*_{\Omega'\omega}e^{-i\Omega' u}) \tag{5.22}$$

and multiplying $e^{\pm i\Omega u}$ both side and integrating with respect to u, we obtain

$$\int_{-\infty}^\infty du\frac{1}{\sqrt{\omega}}e^{-i\omega U \pm i\Omega u} = \int_{-\infty}^\infty du\int_0^\infty \frac{d\Omega'}{\sqrt{\Omega'}}(\alpha_{\Omega'\omega}e^{i(\Omega\mp\Omega')u} - \beta^*_{\Omega'\omega}e^{i(\Omega\pm\Omega')u}). \tag{5.23}$$

Using the following relation

$$\int_{-\infty}^\infty e^{i(\Omega-\Omega')u}du = 2\pi\delta(\Omega - \Omega'), \tag{5.24}$$

the Bogoliubov coefficients take a form of

$$\alpha_{\Omega\omega} = \frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^\infty du\; e^{-i\omega U + i\Omega u}, \tag{5.25}$$

$$\beta^*_{\Omega\omega} = \frac{1}{2\pi}\sqrt{\frac{\Omega}{\omega}}\int_{-\infty}^\infty du\; e^{-i\omega U - i\Omega u}. \tag{5.26}$$

Let us change variables

$$t \equiv e^{-au}, \tag{5.27}$$

then (5.26) is yielded as

$$\alpha_{\Omega\omega} = \frac{1}{2\pi a}\sqrt{\frac{\Omega}{\omega}}\int_0^\infty dt\; t^{-\frac{i\Omega}{a}-1}e^{\frac{i\omega}{a}t} = \frac{1}{2\pi a}\sqrt{\frac{\Omega}{\omega}}\; b^{-s}\int_0^\infty dy\; y^{s-1}e^{-y}, \tag{5.28}$$

$$= \frac{1}{2\pi a}\sqrt{\frac{\Omega}{\omega}}\; e^{-s\ln b}\int_0^\infty dy\; y^{s-1}e^{-y} \tag{5.29}$$

where

$$b = -\frac{i\omega}{a}, \qquad s = -\frac{i\Omega}{a}. \tag{5.30}$$

If we do the analytic continuation

$$b = -\frac{i\omega}{a} + \epsilon, \qquad s = -\frac{i\Omega}{a} + \epsilon, \qquad \epsilon > 0 \tag{5.31}$$

we can use the following definition

$$\Gamma(s) = \int_0^\infty dy\, y^{s-1} e^{-y}, \qquad \mathrm{Re}(s) > 0 \tag{5.32}$$

$$\ln b = \ln(|b| e^{i\theta}). \tag{5.33}$$

Then the each coefficients are calculated to

$$\alpha_{\Omega\omega} = \frac{1}{2\pi a}\sqrt{\frac{\Omega}{\omega}} e^{\frac{i\Omega}{a}\ln|\frac{\omega}{a}| + \frac{\Omega\pi}{2a}} \Gamma\left(-\frac{i\Omega}{a}\right), \tag{5.34}$$

$$\beta_{\Omega\omega} = -\frac{1}{2\pi a}\sqrt{\frac{\Omega}{\omega}} e^{\frac{i\Omega}{a}\ln|\frac{\omega}{a}| - \frac{\Omega\pi}{2a}} \Gamma\left(-\frac{i\Omega}{a}\right), \tag{5.35}$$

and they yield the following relationship

$$|\alpha_{\Omega\omega}|^2 = e^{\frac{2\pi\Omega}{a}} |\beta_{\Omega\omega}|^2. \tag{5.36}$$

Let us take a number operator $\hat{N}_\Omega = \hat{b}^\dagger_\Omega \hat{b}_\Omega$ to the Minkowski vacuum

$$\langle \hat{N}_\Omega \rangle = \langle O_M | \hat{N}_\Omega | O_M \rangle = \langle O_M |\ \hat{b}^\dagger_\Omega \hat{b}_\Omega\ | O_M \rangle \tag{5.37}$$

$$= \langle O_M | \int_0^\infty d\omega [\alpha^*_{\Omega\omega} \hat{a}^\dagger_\omega - \beta^*_{\Omega\omega} \hat{a}_\omega] \int_0^\infty d\omega' [\alpha_{\Omega\omega'} \hat{a}_{\omega'} - \beta_{\Omega\omega'} \hat{a}^\dagger_{\omega'}] | O_M \rangle, \tag{5.38}$$

$$= \int d\omega |\beta_{\Omega\omega}|^2 \tag{5.39}$$

$$= \frac{1}{e^{\frac{2\pi\Omega}{a}} - 1} \delta(0) \tag{5.40}$$

where (5.21) and (5.36) were used in the last line. This is the mean number of particles with frequency Ω observed by the accelerated observer. The mean number density with the frequency Ω is written as

$$n_\Omega = \frac{\langle \hat{N}_\Omega \rangle}{V} = \frac{1}{e^{\frac{2\pi\Omega}{a}} - 1} \tag{5.41}$$

which is the Bose-Einstein distribution with the Unruh temperature

$$T \equiv \frac{a}{2\pi}. \tag{5.42}$$

5.3 Hawking Pair

Let us consider the Minkowski and Rindler spacetime

$$ds^2 = -dt^2 + dx^2 = e^{2a\xi}(-d\eta^2 + d\xi^2)$$

and the action for a massless scalar field

$$S = \frac{1}{2}\int dt dx \left\{-(\partial_t \phi)^2 + (\partial_x \phi)^2\right\} = \frac{1}{2}\int d\xi d\eta \left\{-(\partial_\eta \phi)^2 + (\partial_\xi \phi)^2\right\}. \tag{5.43}$$

Here we suppose the two patches of the Rindler spacetimes, which are denoted by the region "R" for $U < 0, V > 0$ and the region "L" for $U > 0, V < 0$, see Fig(2). The two regions are casually

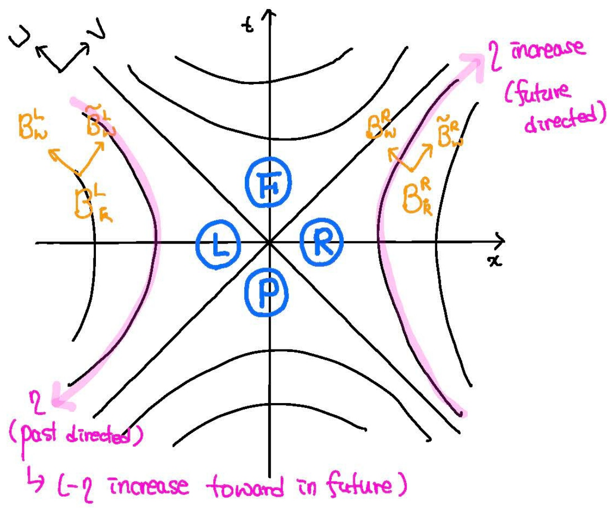

Figure 2: Two patches of Rindler spactimes

disconnected. The field operator which cover two regions can be expanded as

$$\hat{\phi} = \sum_k B_k^R \hat{b}_k^R + B_k^L \hat{b}_k^L + c.c \tag{5.44}$$

where the each mode function is split by the two independent modes depending on the sign of the momentum, which are the left-moving and right-moving. Namely, for the mode function B_k^R

$$B_k^R = e^{ik\xi - i\omega\eta},\ (k > 0) \longrightarrow e^{-i\omega(\eta-\xi)} = e^{-i\omega u} = \tilde{B}_\omega^R(u) \qquad : \text{right-moving} \tag{5.45}$$

$$(k < 0) \longrightarrow e^{-i\omega(\eta+\xi)} = e^{-i\omega v} = B_\omega^R(v) \qquad : \text{left-moving} \tag{5.46}$$

and for the mode function B_k^L

$$B_k^L = e^{ik\xi - i\omega\eta},\ (k > 0) \longrightarrow e^{i\omega(\eta+\xi)} = e^{i\omega v} = B_\omega^L(v) \qquad : \text{left-moving} \tag{5.47}$$

$$(k < 0) \longrightarrow e^{i\omega(\eta-\xi)} = e^{i\omega u} = \tilde{B}_\omega^L(u) \qquad : \text{right-moving} \tag{5.48}$$

Then the field operators are expanded as follows

$$\text{Minkowski :} \qquad \hat{\phi} = \sum_\omega A_\omega \hat{a}_\omega + \tilde{A}_\omega \hat{\tilde{a}}_\omega + \bar{A}_\omega \hat{a}_\omega^\dagger + \bar{\tilde{A}}_\omega \hat{\tilde{a}}_\omega^\dagger, \tag{5.49}$$

$$\text{Rindler}(R, L) : \qquad \hat{\phi} = \sum_\omega B_\omega^R \hat{b}_\omega^R + \tilde{B}_\omega^R \hat{\tilde{b}}_\omega^R + B_\omega^L \hat{b}_\omega^L + \tilde{B}_\omega^L \hat{\tilde{b}}_\omega^L + c.c. \tag{5.50}$$

for the Minkowski spacetime and the two patches of the Rindler spacetimes respectively. Accordingly, the each vacuum is defined as

$$\hat{a}_\omega |O_M\rangle = \hat{\tilde{a}}_\omega |O_M\rangle = 0, \tag{5.51}$$

$$\hat{b}_\omega^{L,R} |O_R\rangle = \hat{\tilde{b}}_\omega^{L,R} |O_R\rangle = 0 \tag{5.52}$$

and the following mode functions take the positive frequency with respect to T and η

$$\text{Minkowski :} \qquad A_\omega \sim e^{-i\omega V}, \qquad \tilde{A}_\omega \sim e^{-i\omega U} \tag{5.53}$$

$$\text{Rindler}(R, L) : \qquad B_\omega^R \sim e^{-i\omega v}\Theta(V), \qquad B_\omega^L \sim e^{i\omega v}\Theta(-V) \tag{5.54}$$

$$\tilde{B}_\omega^R \sim e^{-i\omega u}\Theta(-U), \qquad \tilde{B}_\omega^L \sim e^{i\omega u}\Theta(U). \tag{5.55}$$

By the Bogoliubov transformation, the annihilation operator in the Minkowski spacetime is expressed by the linear combination of the operators in the two patches of the Rindler spacetimes

$$\hat{a}_\omega = \sum_{\omega'} \alpha^R_{\omega\omega'} \hat{b}^R_\omega + \bar{\beta}^R_{\omega\omega'} \hat{b}^{R\dagger}_\omega + \alpha^L_{\omega\omega'} \hat{b}^L_\omega + \bar{\beta}^L_{\omega\omega'} \hat{b}^{L\dagger}_\omega \tag{5.56}$$

and similarly can be done for $\hat{\tilde{a}}_\omega$, $\hat{\tilde{b}}_\omega$'s. Thus the Minkowski vacuum can be defined as

$$\hat{a}_\omega|O_\omega\rangle \;\longrightarrow\; \sum_{\omega'} \left(\alpha^R_{\omega\omega'} \hat{b}^R_\omega + \bar{\beta}^R_{\omega\omega'} \hat{b}^{R\dagger}_\omega + \alpha^L_{\omega\omega'} \hat{b}^L_\omega + \bar{\beta}^L_{\omega\omega'} \hat{b}^{L\dagger}_\omega \right) |O_M\rangle = 0. \tag{5.57}$$

Here we would like to find the relation between $|O_M\rangle$ and $|O_R\rangle$. To do so, let us use the following trick. Pretending the operators to be

$$\hat{b}^{R\dagger} = x, \qquad \hat{b}^R = \frac{\partial}{\partial x}, \qquad \hat{b}^{L\dagger} = y, \qquad \hat{b}^L = \frac{\partial}{\partial y}, \tag{5.58}$$

(5.57) can be written as

$$\left(x + \frac{\partial}{\partial x} + y + \frac{\partial}{\partial y} \right) \psi(x,y) = 0. \tag{5.59}$$

This first order differential equation, however, is not solvable. So we take a different approach as follows. Our observation is that $e^{-i\omega' V}$ is analytic in the lower half plane of V when $\omega' > 0$ and $V \to -i\infty$. The mode function can be written as

$$B^R_\omega = e^{-i\omega v}\Theta(V) = (aV)^{-i\omega'/a}\,\Theta(V), \tag{5.60}$$

and let us do the analytic continuation of mode function B^R_ω

$$(aV)^{-i\omega'/a} = B^R_\omega, \qquad (V > 0) \tag{5.61}$$

$$= (-aV)^{-i\omega'/a} = (e^{-i\pi} aV)^{-i\omega'/a} = e^{-\pi\omega a} e^{-i\omega v} = e^{-\pi a} \bar{B}^L_\omega, \qquad (V < 0). \tag{5.62}$$

Then we can construct a new mode function that covers the half of the Minkowski spacetime which is $U < 0, -\infty < V < \infty$

$$F^R_\omega = B^R_\omega + e^{-\pi\omega/a} \bar{B}^L_\omega. \tag{5.63}$$

Like the same way, other mode functions can be generated. The field operator in the Minkowski spacetime is expanded to

$$\hat{\phi} = \sum F^R_\omega \hat{b}'^R_\omega + F^L_\omega \hat{b}'^L_\omega + \tilde{F}^R_\omega \hat{\tilde{b}}'^R_\omega + \tilde{F}^L_\omega \hat{\tilde{b}}'^L_\omega + c.c. \tag{5.64}$$

with new annihilation/creation operators and new mode functions, which are

$$F^R_\omega = B^R_\omega + e^{-\pi\omega/a} \bar{B}^L_\omega, \qquad \tilde{F}^R_\omega = \tilde{B}^R_\omega + e^{\pi\omega/a} \bar{\tilde{B}}^L_\omega, \tag{5.65}$$

$$F^L_\omega = B^L_\omega + e^{-\pi\omega/a} \bar{B}^R_\omega, \qquad \tilde{F}^L_\omega = \tilde{B}^L_\omega + e^{\pi\omega/a} \bar{\tilde{B}}^R_\omega. \tag{5.66}$$

Then the Minkowski vacuum can be re-defined as

$$\hat{\tilde{b}}'^{(L,R)}_\omega |O_M\rangle = 0, \qquad \hat{b}'^{(L,R)}_\omega |O_M\rangle = 0. \tag{5.67}$$

The new mode function (5.64) refers the bogoliubov transformation of the operator in the following form

$$\hat{b}^R_\omega = \alpha_\omega \hat{b}'^R_\omega + \beta_\omega \hat{b}'^{L\dagger}_\omega, \tag{5.68}$$

and due to the normalization condition which are required to satisfying the commutation relation the new annihilation operators become

$$\hat{b}^R_\omega = \frac{1}{\sqrt{2\sinh\frac{\beta\omega}{2}}}\left(e^{\beta\omega/4}\hat{b}'^R_\omega + e^{-\beta\omega/4}\hat{b}'^{L\dagger}_\omega\right), \tag{5.69}$$

$$\hat{b}'^R_\omega = \frac{1}{\sqrt{2\sinh\frac{\beta\omega}{2}}}\left(e^{\beta\omega/4}\hat{b}^R_\omega - e^{-\beta\omega/4}\hat{b}^{L\dagger}_\omega\right), \tag{5.70}$$

where we used $\beta = \frac{2\pi}{a}$. Now we apply the previous same trick again as follows

$$\left(\hat{b}^R_k - e^{-\frac{\beta\omega_k}{2}}\hat{b}^{L\dagger}_k\right)|O_M\rangle = 0 \qquad \longrightarrow \qquad \left(\frac{\partial}{\partial y} - \gamma x\right)\psi(x,y) = 0 \tag{5.71}$$

where $b^R_k = b^R_\omega + \tilde{b}^R_\omega$ and $\gamma = e^{-\frac{\beta\omega_k}{2}}$ and similarly

$$\left(\hat{b}^L_k - e^{-\frac{\beta\omega_k}{2}}\hat{b}^{R\dagger}_k\right)|O_M\rangle = 0 \qquad \longrightarrow \qquad \left(\frac{\partial}{\partial x} - \gamma y\right)\psi(x,y) = 0. \tag{5.72}$$

Then two first order differential equations are solved as

$$\psi(x,y) = const.\ e^{\gamma xy}. \tag{5.73}$$

Taking this to the operator values, the Minkowski vacuum can be constructed from the Rindler vacuum by applying an entangled pair of the creation operators in the R and L regions

$$|O_M\rangle = \exp\left[\sum_k e^{-\beta\omega_k/2}(b^{R\dagger}_k b^{L\dagger}_k)\right]|O_R\rangle \tag{5.74}$$

$$= \prod_k \sum_{n_k=0}^{\infty} \frac{e^{-\beta\omega_k n_k/2}}{n_k!}(b^{R\dagger}_k)^{n_k}(b^{L\dagger}_k)^{n_k}|O_R\rangle \tag{5.75}$$

$$= \prod_k \sum_{n_k=0}^{\infty} e^{-\beta\omega_k n_k/2}|n_k, n_k\rangle \tag{5.76}$$

$$= \sum_{\{n_k\}} e^{-\beta/2\sum_k \omega_k n_k}|\{n_k\},\{n_k\}\rangle \tag{5.77}$$

where $n^L_k = n^R_k\ \forall k$. Thus the entangled pair has the same Hilbert space

$$|O_M\rangle = \sum_{\text{all states}} e^{-\beta E_i/2}|i\rangle_L|i\rangle_R \tag{5.78}$$

where

$$|i\rangle_R = \prod_k \frac{(b^{R\dagger}_k)^{n_k}}{\sqrt{n_k!}}|O_R\rangle \tag{5.79}$$

$$|i\rangle_L = \prod_k \frac{(b^{L\dagger}_k)^{n_k}}{\sqrt{n_k!}}|O_R\rangle. \tag{5.80}$$

5.4 Kruskal Vacuum and Boulware vacuum

Let us consider the Schawrzschild black hole spacetime

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega \tag{5.81}$$

and change variables

$$r_* = r - 2M + 2M \ln\left(\frac{r}{2M} - 1\right). \tag{5.82}$$

This naturally lead us to the tortoise lightcone coordinates

$$ds^2 = \left(1 - \frac{2M}{r}\right)(-dt^2 + dr_*^2) = \left(1 - \frac{2M}{r}\right) d\tilde{u} d\tilde{v} \tag{5.83}$$

where

$$\tilde{u} \equiv t - r_*, \qquad \tilde{v} \equiv t + r_*. \tag{5.84}$$

In this background spacetime, the action for the massless scalar field is written as

$$S[\phi] = \frac{1}{2} \int d^4x \sqrt{-g}\, g^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta}. \tag{5.85}$$

When considering the asymptotic observer near the infinity, the metric approaches

$$ds^2 \to -d\tilde{u} d\tilde{v} = -dt^2 + dx^2, \qquad (r \to \infty) \tag{5.86}$$

and one of the field solutions takes a form of

$$\phi \propto e^{-i\Omega\tilde{u}} = e^{-i\Omega(t - r_*)} \tag{5.87}$$

which describes a right-moving positive-frequency mode with respect to time t. With the metric (5.86) the field operator is constructed as

$$\hat{\phi} = \int_0^\infty \frac{d\Omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\Omega}} \left[e^{-i\Omega\tilde{u}} \hat{b}_\Omega + e^{i\Omega\tilde{u}} \hat{b}_\Omega^\dagger \right] + (\text{left moving}) \tag{5.88}$$

and the vacuum for this asymptotic observer is defined as

$$\hat{b}_\Omega |O_B\rangle = 0 \tag{5.89}$$

which is known as Boulware vacuum.

Now let us introduce new variables

$$u = -4M \exp\left(-\frac{\tilde{u}}{4M}\right), \qquad v = 4M \exp\left(\frac{\tilde{v}}{4M}\right), \tag{5.90}$$

and plug this to the metric (5.83).

$$ds^2 = -\frac{2M}{r(u,v)} \exp\left(1 - \frac{r(u,v)}{2M}\right) du dv \tag{5.91}$$

which is known as Kruskal-Szekeres coordinates and regular at $r = 2M$ while the the Schawrzschild metric is singular at $r = 2M$. Taking the vicinity of the horizon, the metric behaves as

$$ds^2 \longrightarrow -du dv = -dT^2 + dR^2 \tag{5.92}$$

where

$$u \equiv T - X, \qquad v \equiv T + X. \tag{5.93}$$

With the metric (5.92), the field operator is expanded as

$$\hat{\phi} = \int_0^\infty \frac{d\omega}{(2\pi)^{1/2}} \frac{1}{\sqrt{2\omega}} \left[e^{-i\omega u} \hat{b}_\omega + e^{i\omega u} \hat{b}_\omega^\dagger \right] + (\text{left moving}) \tag{5.94}$$

and the vacuum for the observer near the event horizon is defined as

$$\hat{a}_\omega |O_K\rangle = 0 \tag{5.95}$$

which is called Kruskal vacuum.

When the remote observer takes the number operator $\hat{N}_\Omega = \hat{b}_\Omega^\dagger \hat{b}_\Omega$ to the Kruskal vacuum, from the his point of view the Kruskal vacuum contains the particles with the following the thermal spectrum

$$\langle \hat{N}_\Omega \rangle \equiv \langle O_K | \hat{b}_\Omega^\dagger \hat{b}_\Omega | O_K \rangle = \frac{1}{e^{2\pi\Omega/\kappa} - 1} \delta(0), \tag{5.96}$$

corresponding to the temperature

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{8\pi M}. \tag{5.97}$$

This situation has a mathematical similarity with the accelerated observer in the Minkowski spacetime as follows

Accelerated observer	Schwarzschild spacetime
Minkowski vacuum $\lvert O_M\rangle$	Kruskal vacuum $\lvert O_K\rangle$
Rindler vacuum $\lvert O_R\rangle$	Boulware vacuum $\lvert O_B\rangle$
Acceleration a	Surface gravity κ
$u = -a^{-1}\exp(-a\tilde{u})$	$u = -\kappa^{-1}\exp(-\kappa\tilde{u})$
$v = a^{-1}\exp(a\tilde{v})$	$v = \kappa^{-1}\exp(\kappa\tilde{v})$

6 Hawking Radiation from black hole formation

will be updated