

Introduction to Two-dimensional Tensor Network Algorithms

Hyun-Yong Lee (ISSP)

Tensor Network States in APCTP (2019/10/16)

Outline

— Introduction

- Overview on Tensor Network Applications
- Frustrated Quantum Spin Systems

— Algorithms for Optimization

- Exact Constructions
- Numerical Optimizations

— Algorithms for Measurement

- Corner Transfer Matrix Renormalization Group

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Overview

❖ Applications of Tensor Network in Manybody systems

(1) Classical Statistical Mechanics (60s)

e.g.) Corner Transfer Matrix Renormalization Group (**CTMRG**)

Tensor Renormalization Group (**TRG**)

Tensor Network Renormalization Group (**TNR**)

Loop-optimization TNR (**Loop TNR**)

(2) Quantum Manybody States

e.g.) Matrix Product States (**MPS**) - **Y.-J Kao & Miles**

Multiscale Entanglement Renormalization Ansatz (**MERA**)

Projected Entangled Pair States (**PEPS**)

☞ (in general) Tensor Product States (**TPS**)

(3) Machine Learning, Classical Field Theory, AdS/CFT, etc...

Overview

❖ Doing TPS means

Step1) **Optimization**: finding tensors suitable for purpose

(a) **Exact construction** ← “Human” & “Physics”

e.g.) AKLT, SU(2) RVB, Quantum Dimer, Loop Gas, etc...

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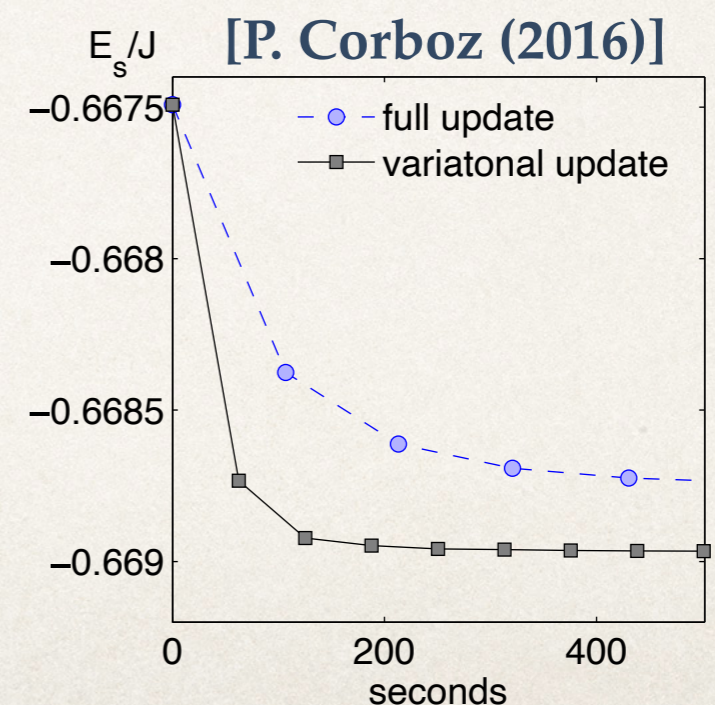
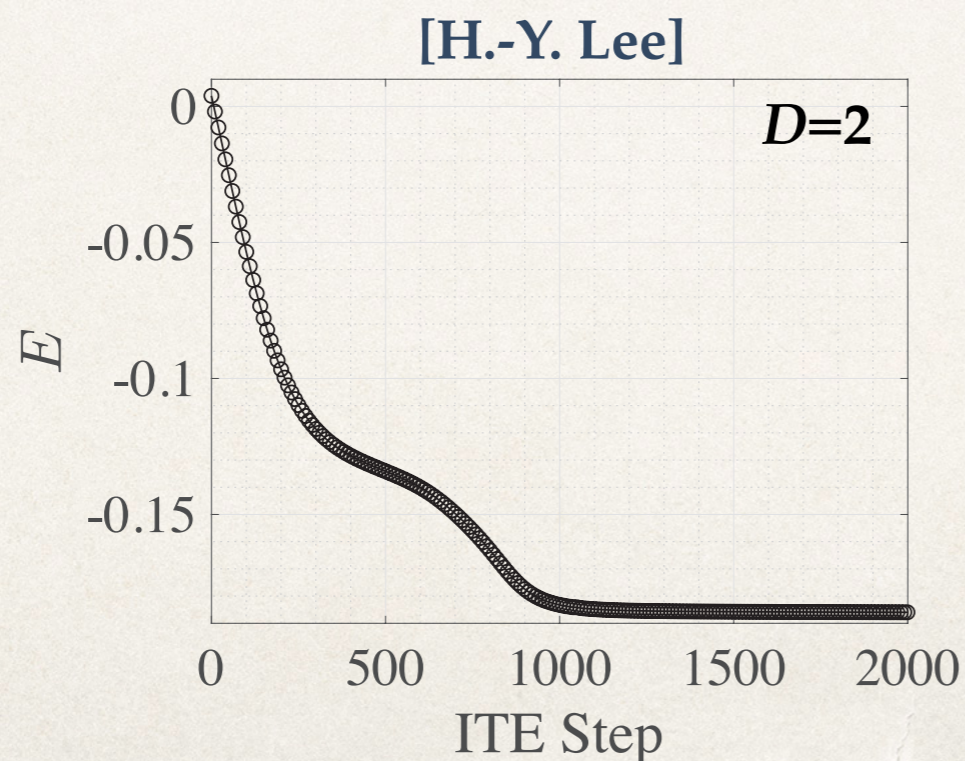
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Frustration in Spin System

❖ Competing configurations by geometrical frustration

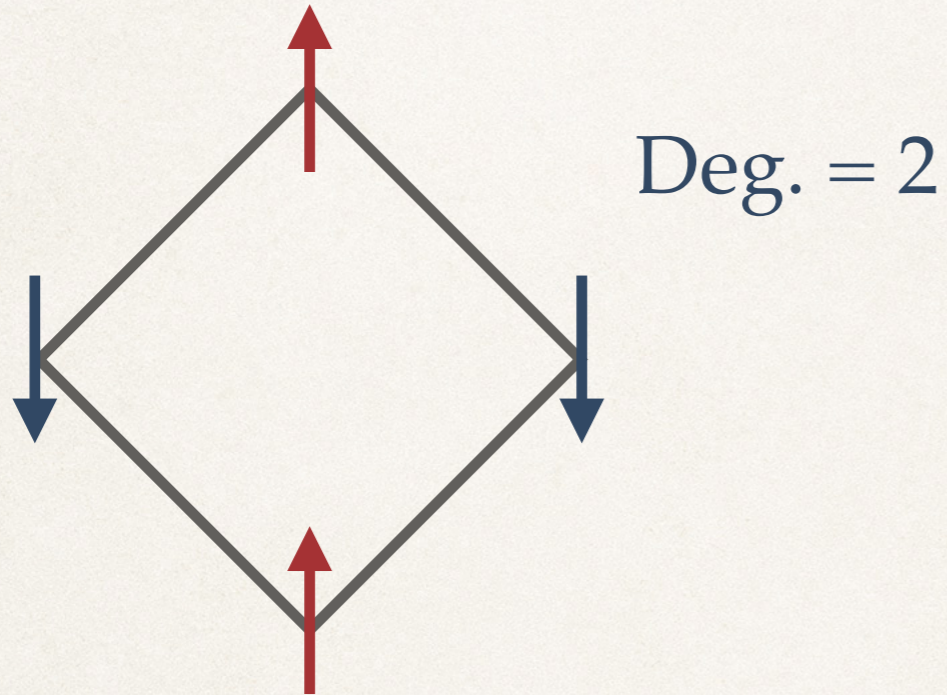
⇒ Antiferromagnetic model: $H = \sum_{\langle ij \rangle} \sigma_i \sigma_j \quad (\sigma_i = \pm 1)$

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► Square

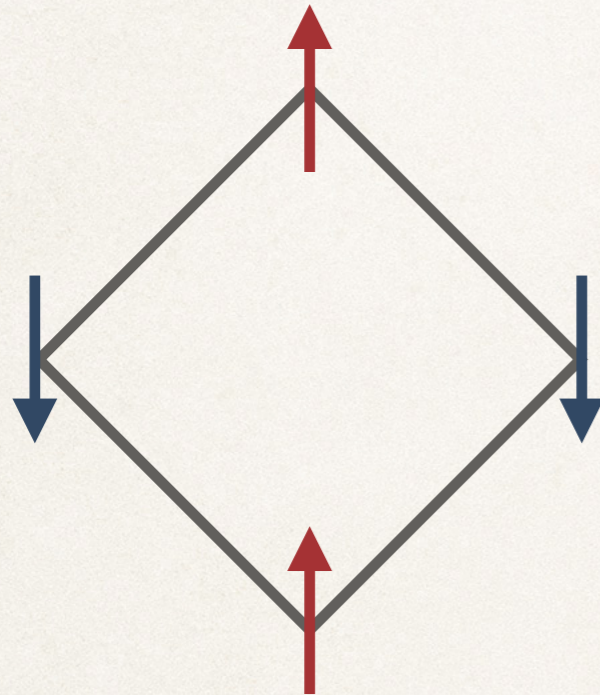


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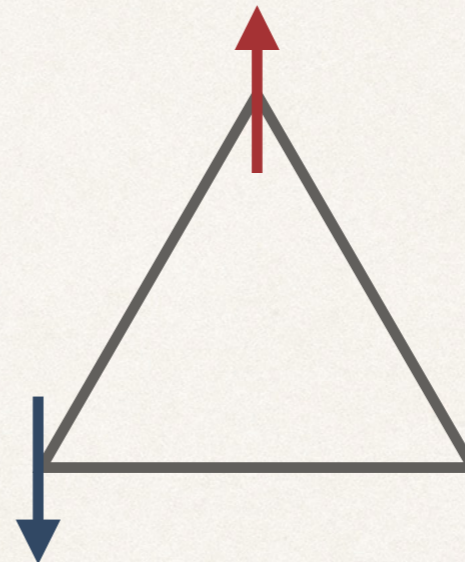
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Deg. = 2

▶ Triangle

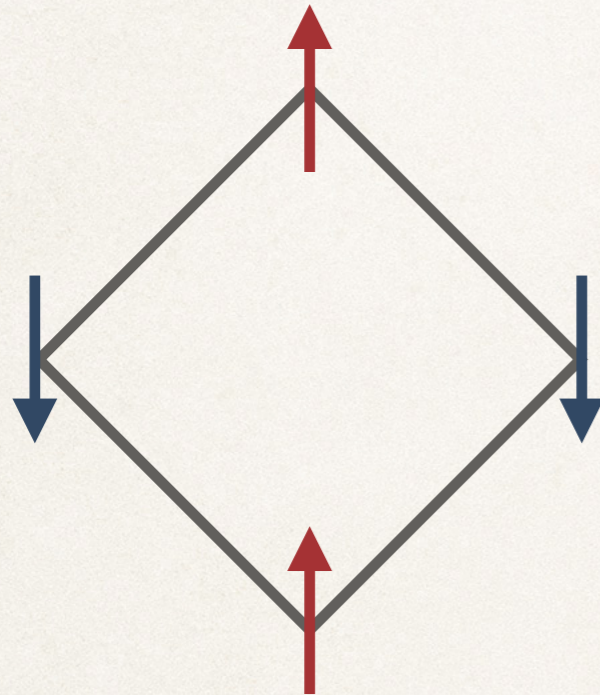


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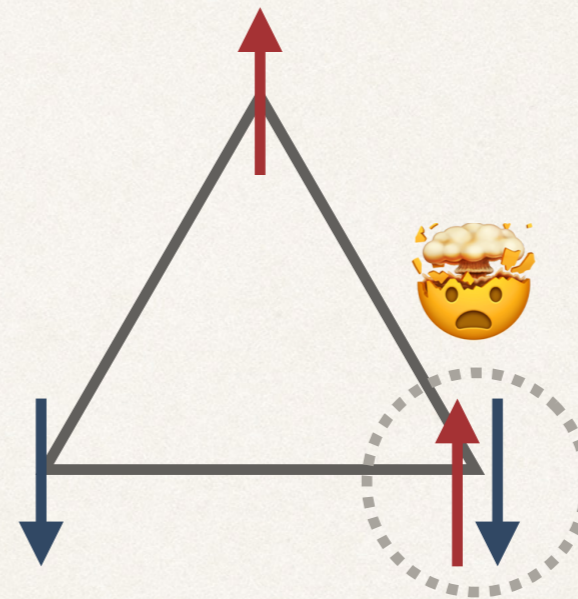
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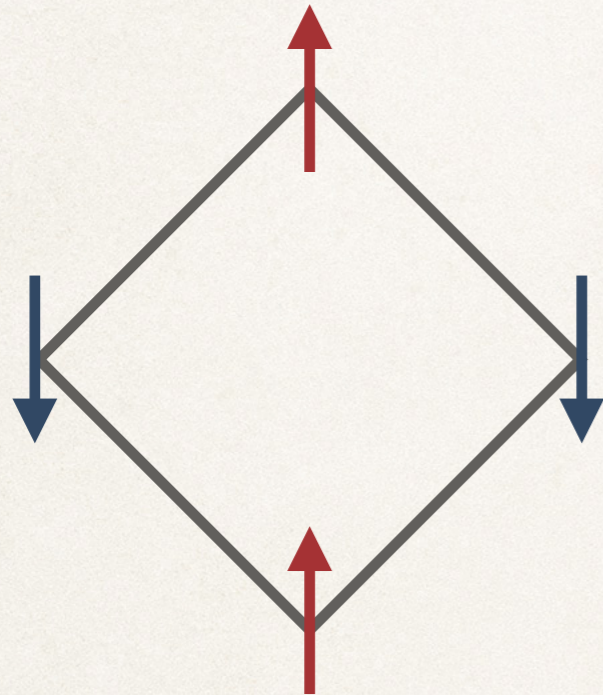
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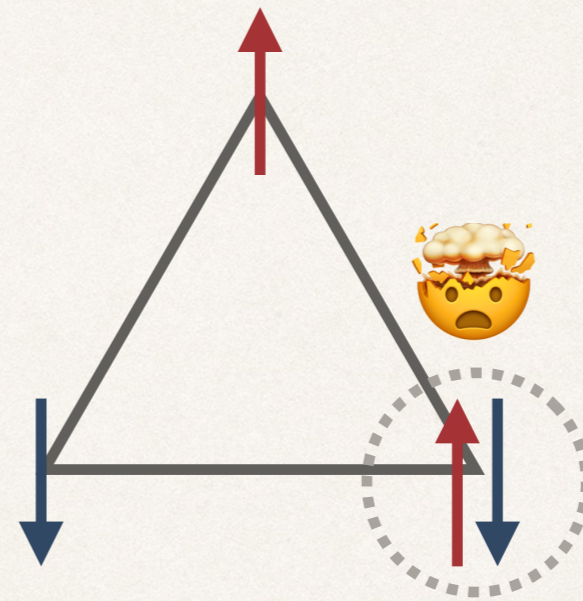


Deg. = 2

$N \rightarrow \infty$

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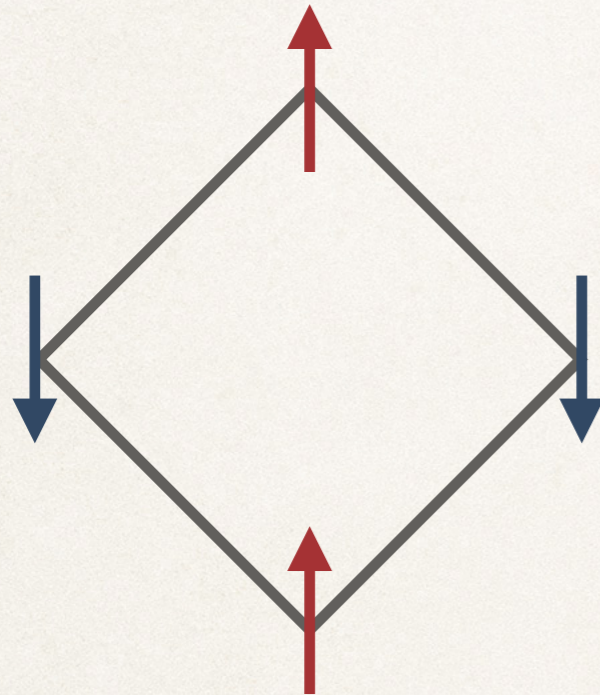
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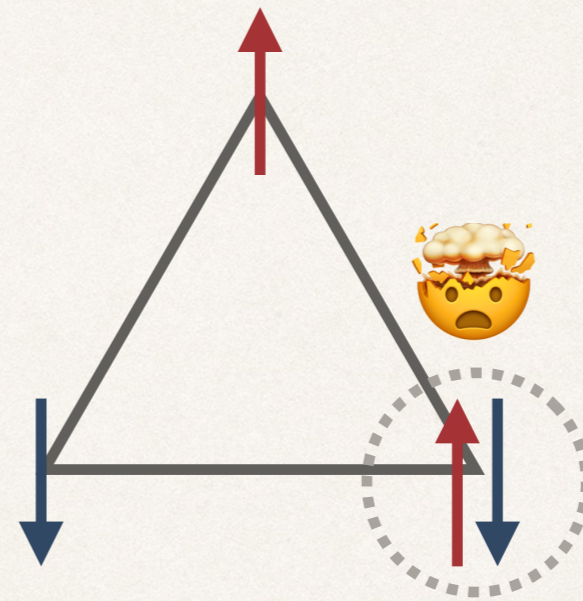


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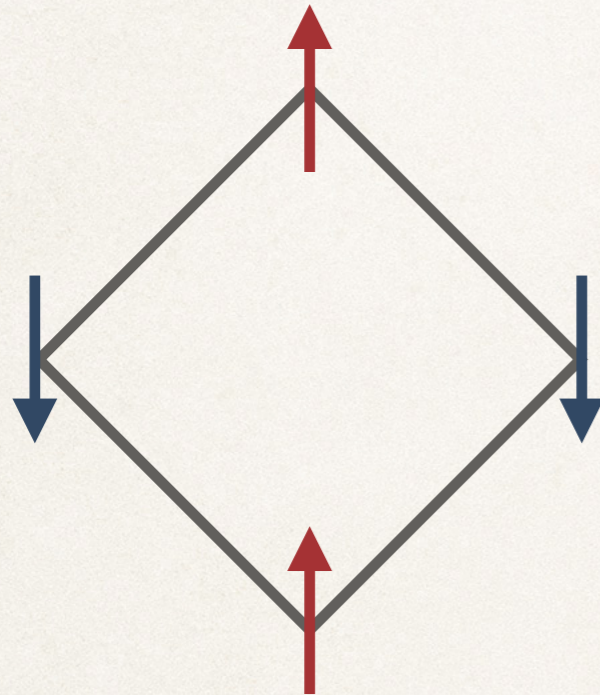
: Enormous quantum fluctuation and entanglement

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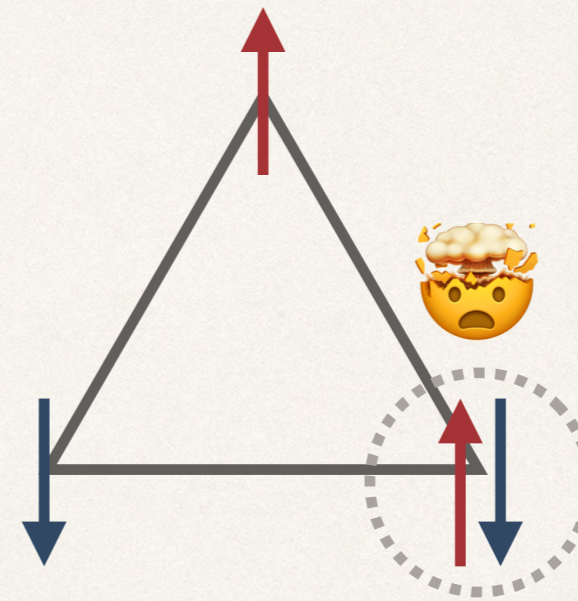


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**Exotic Quantum
Ground State?**

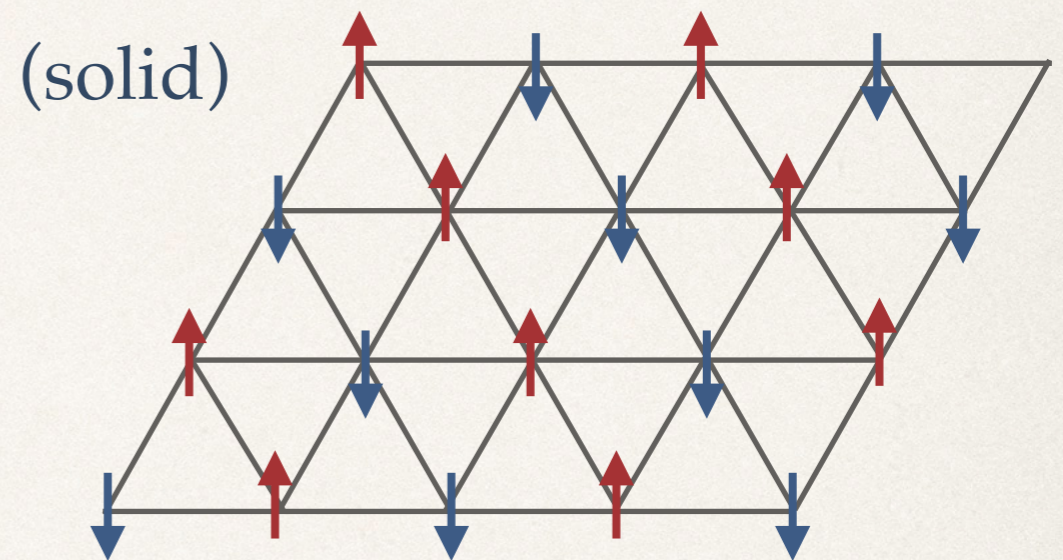
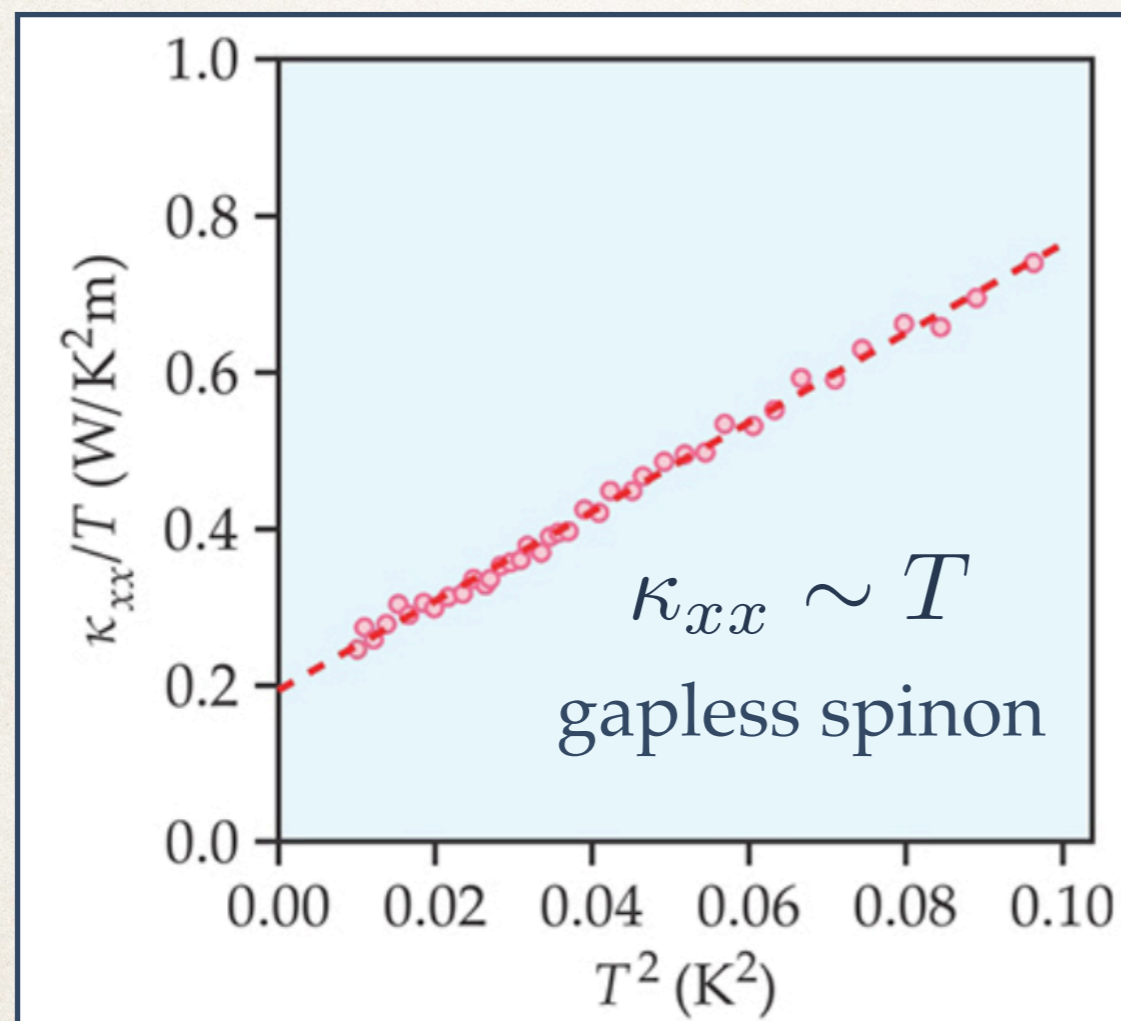
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❖ Novel Phases of Matter

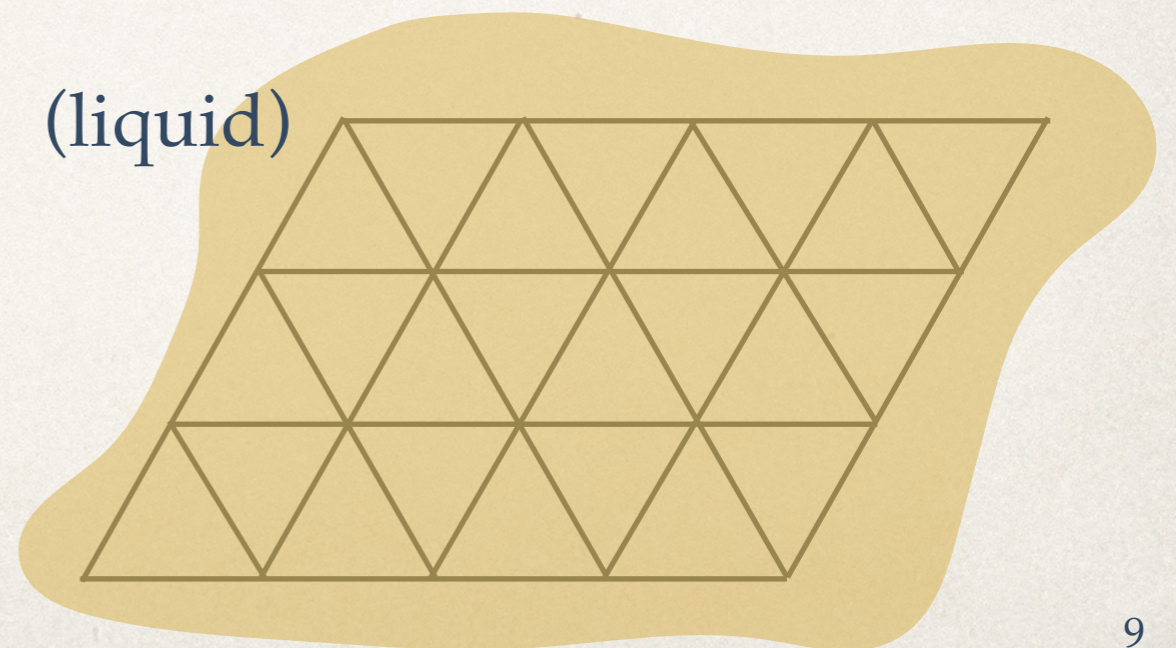
⇒ Quantum Spin Liquids on Frustrated Lattice

- ▶ No spontaneous symmetry-breaking down to Zero temperature

[Yamashita et al. (2010)]



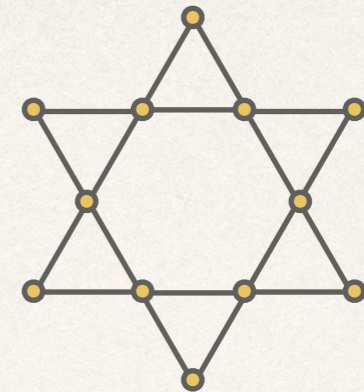
(liquid)



Frustration in Spin System

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Ex3) Kagome Antiferromagnets

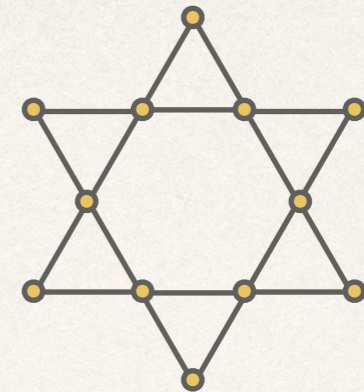


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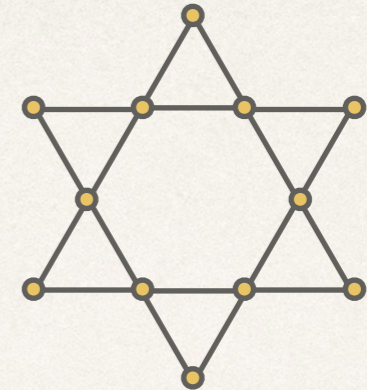


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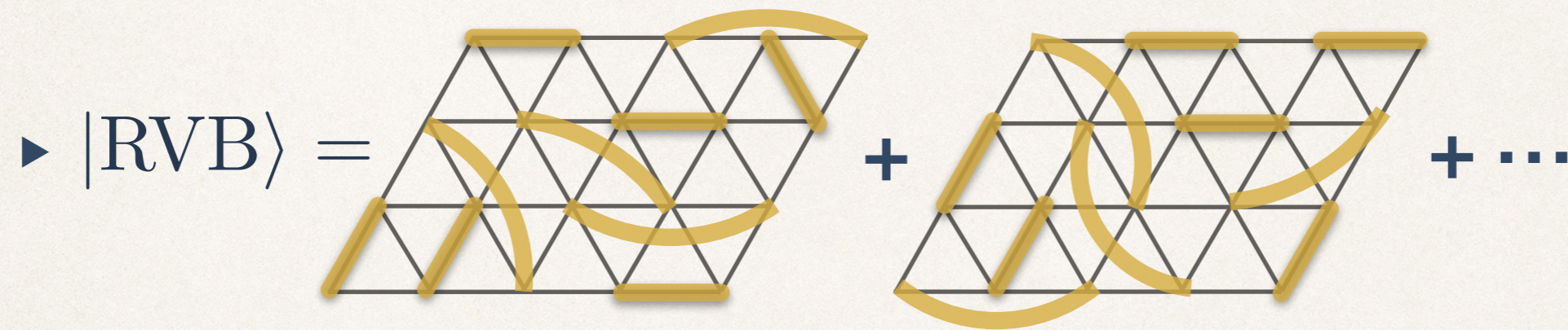
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Ex1) SU(2) Resonating Valence Bond State [Anderson (1987)]



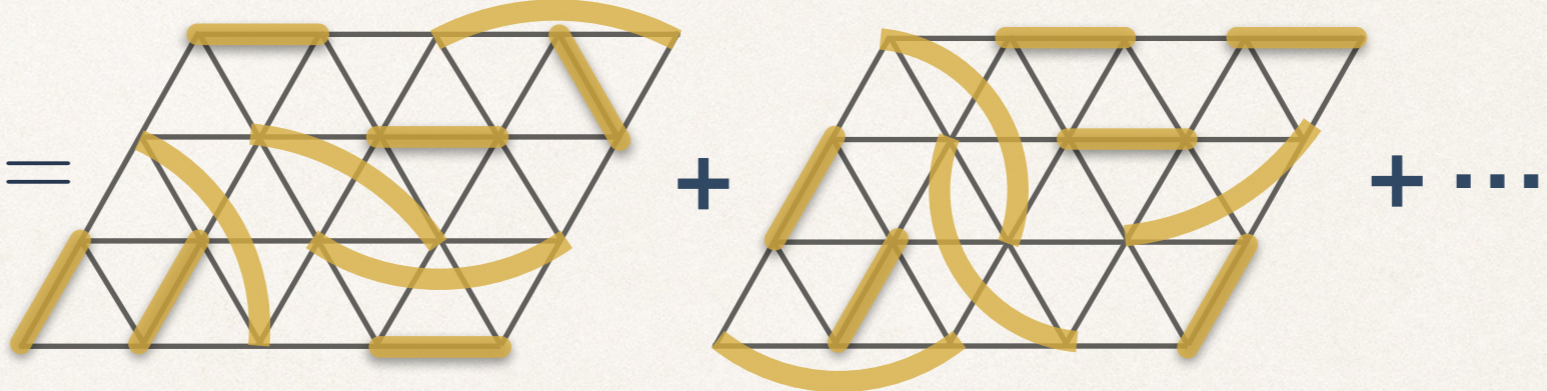
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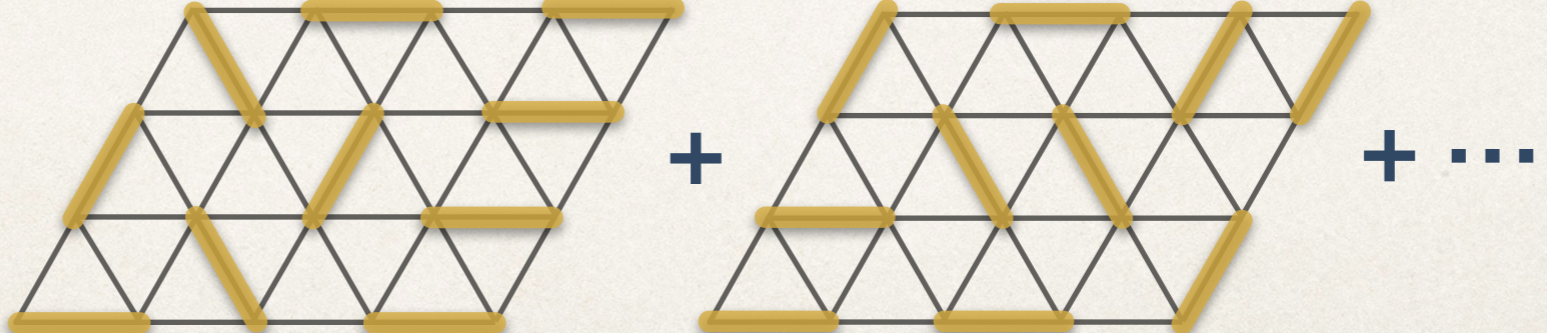
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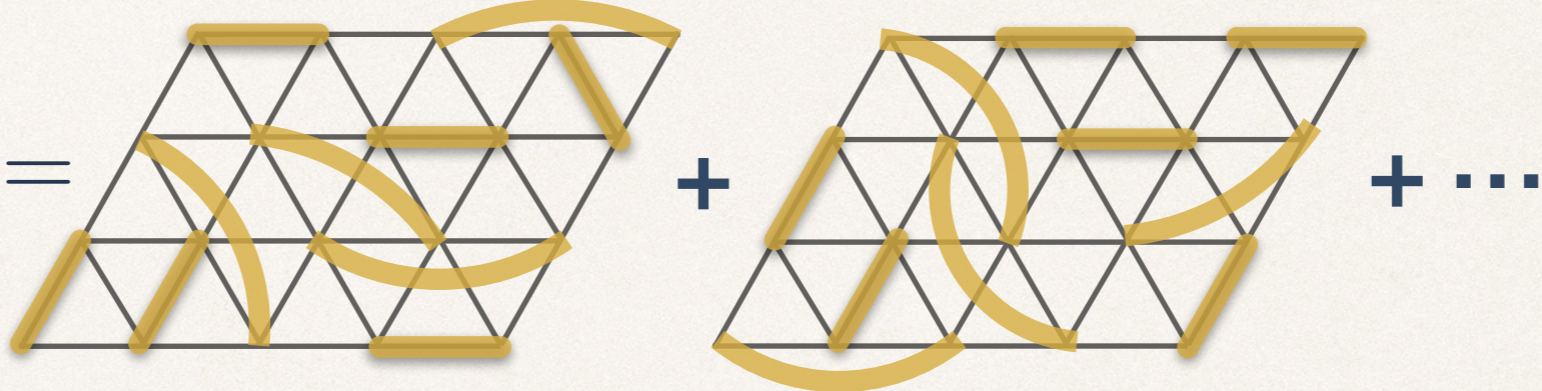
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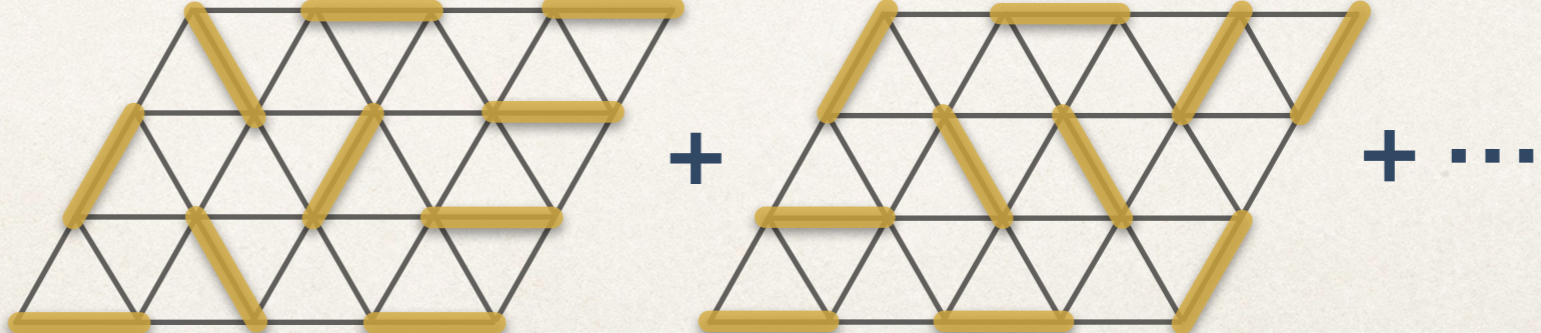
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Ex2) Quantum Dimer Liquid [Rokhsar, Kivelson (1988)]

$$\begin{aligned} \blacktriangleright H_{\text{QDM}} = & - \sum_p |\text{dimer}_p\rangle \langle \text{monomer}_p| + |\text{monomer}_p\rangle \langle \text{dimer}_p| \\ & + \sum_p |\text{dimer}_p\rangle \langle \text{dimer}_p| + |\text{monomer}_p\rangle \langle \text{monomer}_p| \end{aligned}$$

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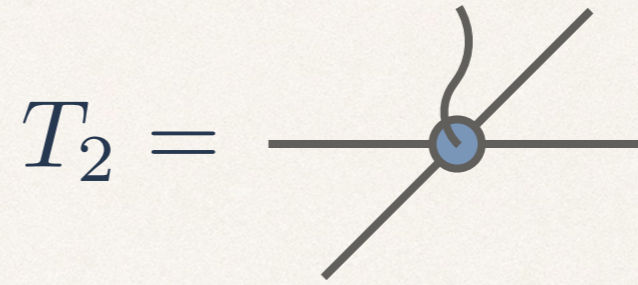
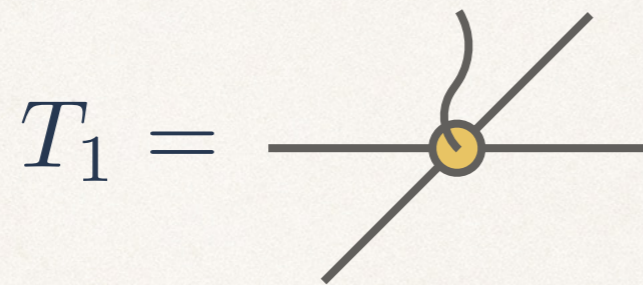
- Corner Transfer Matrix Renormalization Group

Optimization - Exact Construction

❖ Strategy

Step1) Find Tensors suitable for purpose

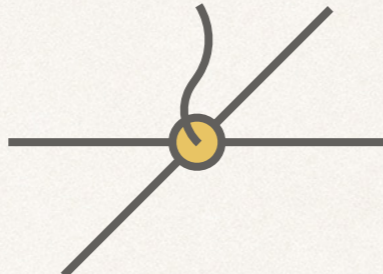
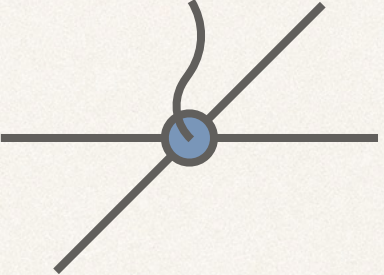
e.g.)



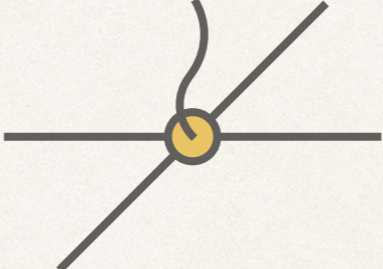
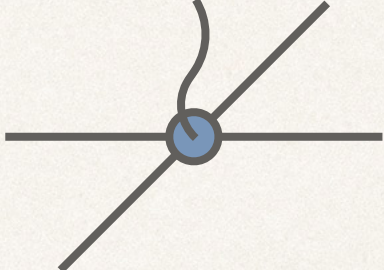
Optimization - Exact Construction

❖ Strategy

Step1) Find Tensors suitable for purpose

e.g.) $T_1 =$  $T_2 =$ 

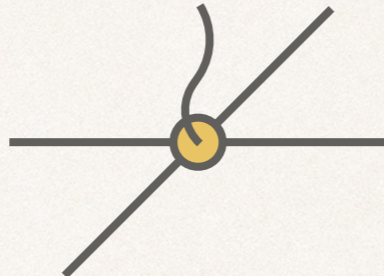
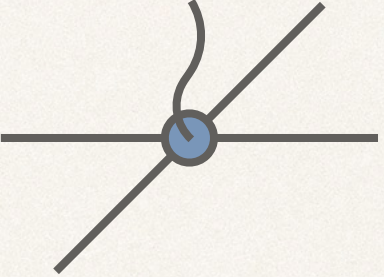
Step2) Mix tensors, and find best superposition

e.g.) $T(c) =$  $+ c$ 

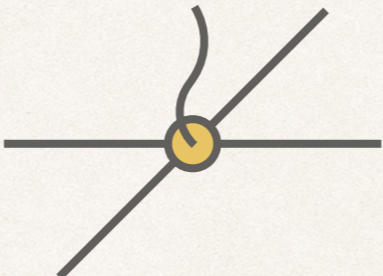
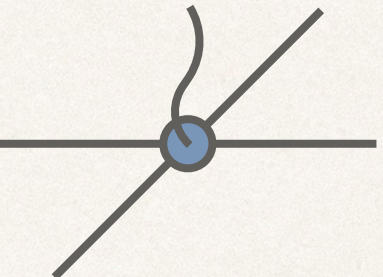
Optimization - Exact Construction

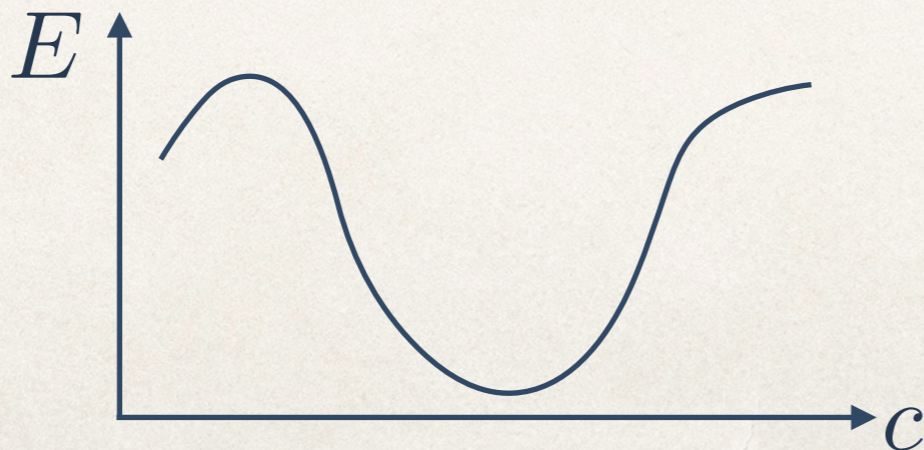
❖ Strategy

Step1) Find Tensors suitable for purpose

e.g.) $T_1 =$  $T_2 =$ 

Step2) Mix tensors, and find best superposition

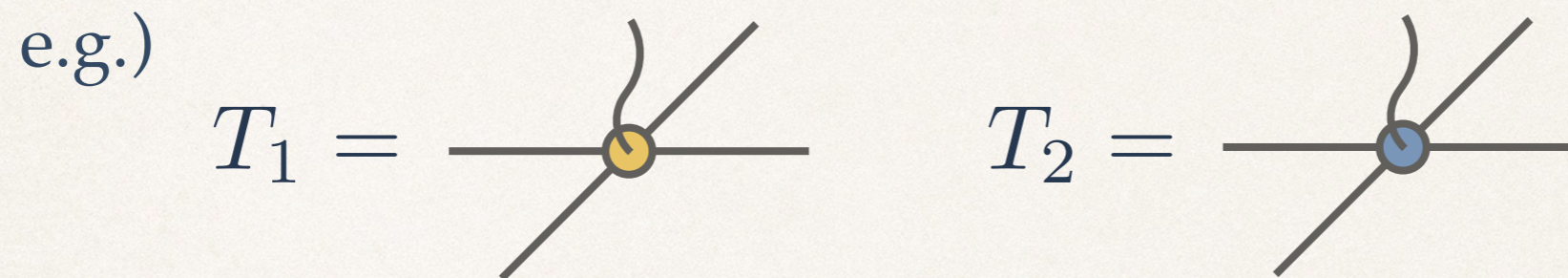
e.g.) $T(c) =$  $+ c$ 



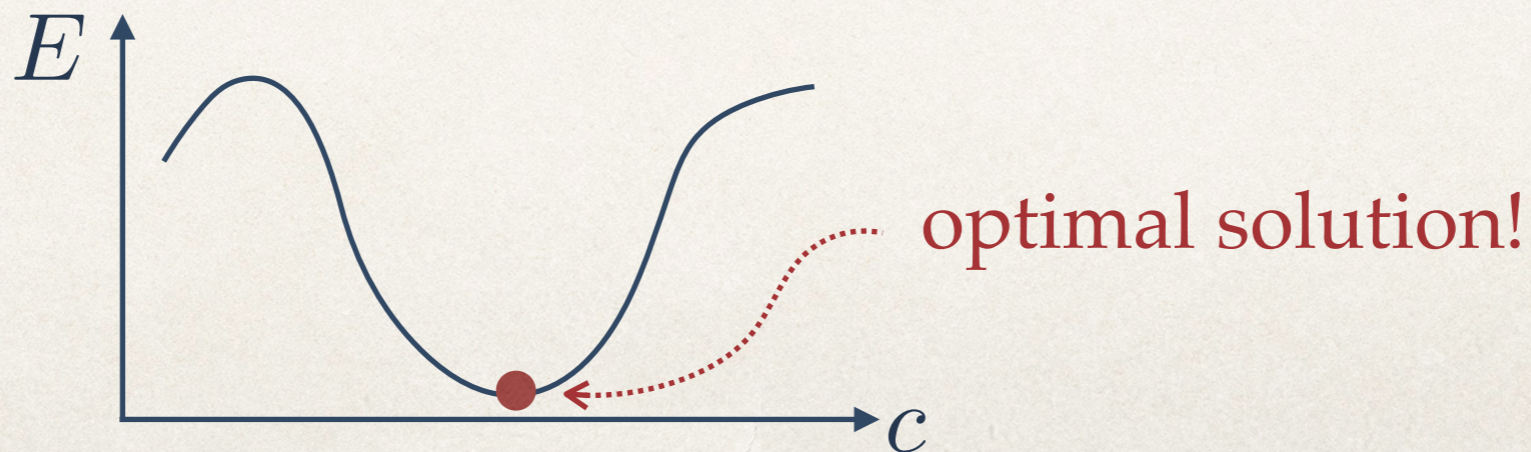
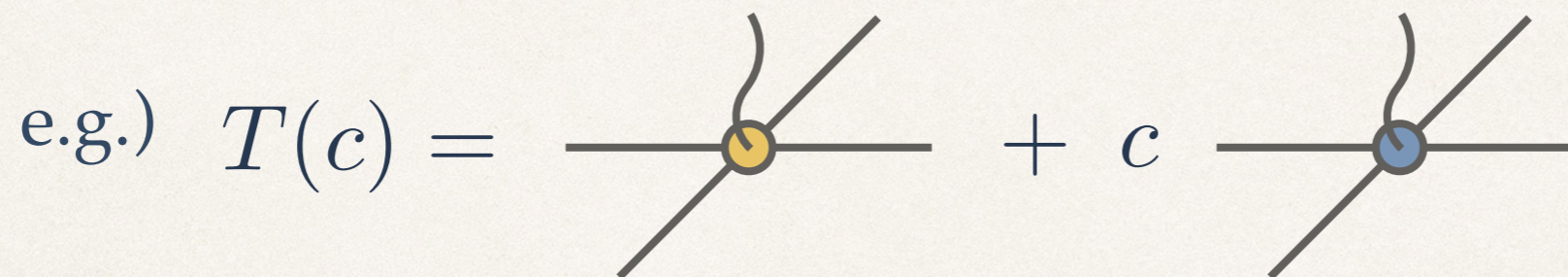
Optimization - Exact Construction

❖ Strategy

Step1) Find Tensors suitable for purpose



Step2) Mix tensors, and find best superposition



Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

$$\blacktriangleright |RK\rangle = \left| \begin{array}{cccc} \text{---} & & & \\ | & & & \\ | & & & \\ | & & & \end{array} \right\rangle + \left| \begin{array}{cccc} \text{---} & & & \\ | & & & \\ | & & & \\ | & & & \end{array} \right\rangle + \left| \begin{array}{cccc} & & & \\ | & & & \\ | & & & \\ | & & & \end{array} \right\rangle + \dots$$



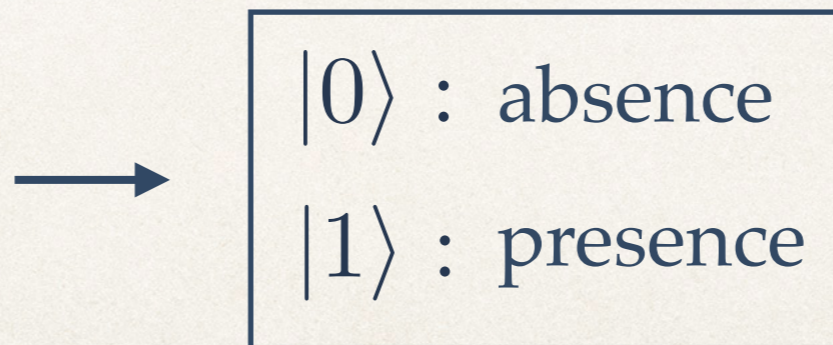
$|0\rangle$: absence
 $|1\rangle$: presence

Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

$$\blacktriangleright |RK\rangle = \left| \begin{array}{cccc} \text{---} & & & \\ | & & & \\ \text{---} & & & \\ | & & & \\ \text{---} & & & \\ | & & & \end{array} \right\rangle + \left| \begin{array}{cccc} \text{---} & & & \\ | & & & \\ \text{---} & & & \\ | & & & \\ \text{---} & & & \\ | & & & \end{array} \right\rangle + \left| \begin{array}{cccc} \text{---} & & & \\ | & & & \\ \text{---} & & & \\ | & & & \\ \text{---} & & & \\ | & & & \end{array} \right\rangle + \dots$$

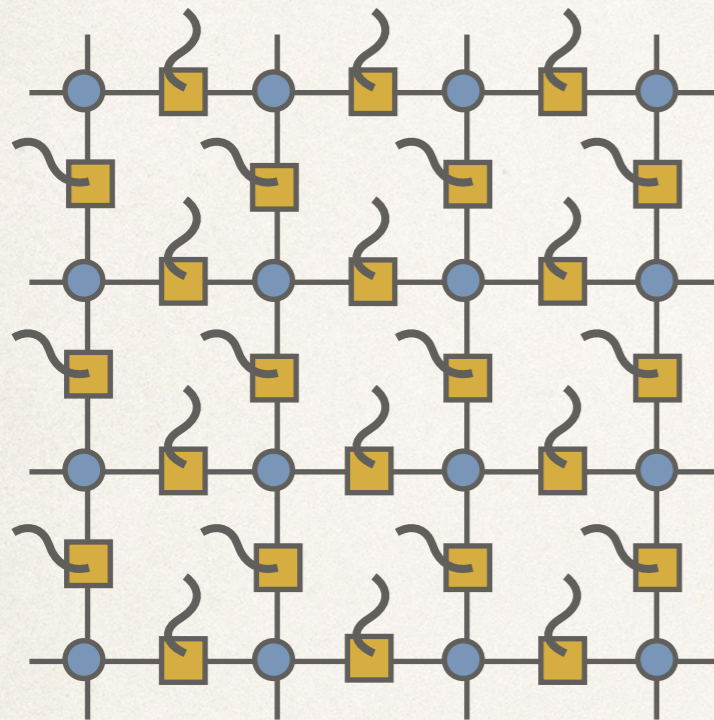
- \blacktriangleright Constraints:
 - a. one dimer at each vertex
 - b. equal weight superposition
 - c. orthogonality $\langle \begin{array}{|c|} \hline \square \\ \hline \end{array} \mid \begin{array}{|c|} \hline \square \\ \hline \end{array} \rangle = 0$



Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

▶ TPS ($d_p = d_v = 2$)



orthogonality



Index
0: absence
1: presence

Optimization - Exact Construction

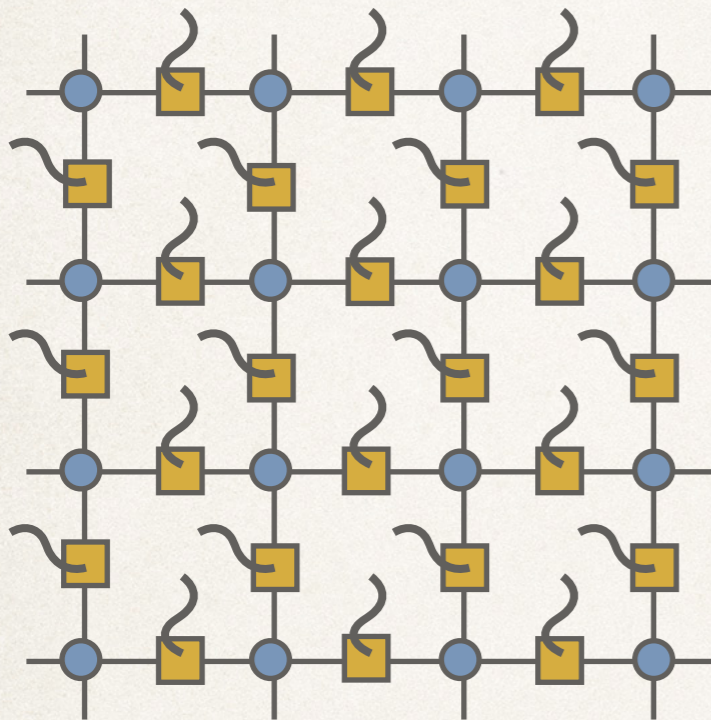
❖ Example 1: Quantum Dimer Liquid

orthogonality

Index
0: absence
1: presence

▶ TPS ($d_p = d_v = 2$)

▶ Tensors (non-zero elements)



$$i \text{---} \overset{p}{\square} \text{---} j : 0 \text{---} \overset{0}{\square} \text{---} 0 = 1 \text{---} \overset{1}{\square} \text{---} 1 = 1$$

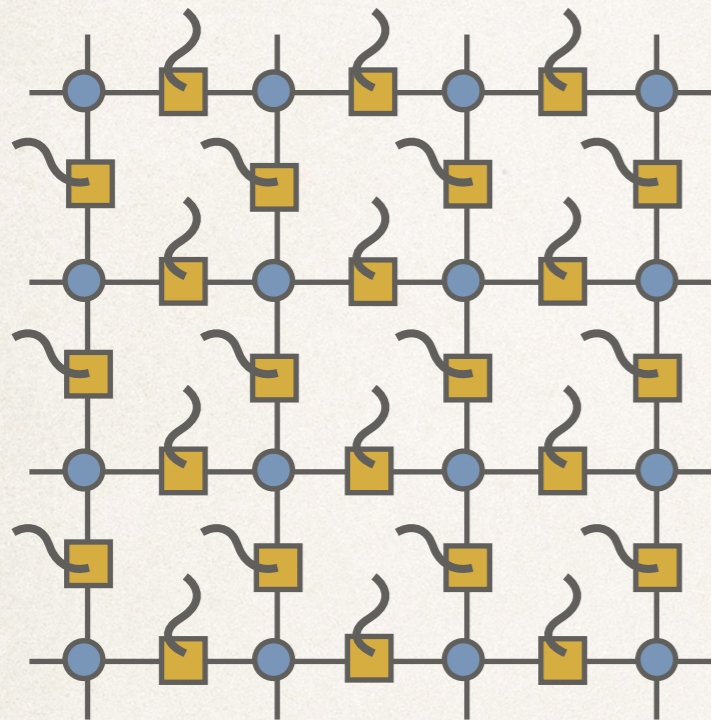
Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

orthogonality

Index
0: absence
1: presence

▶ TPS ($d_p = d_v = 2$)



▶ Tensors (non-zero elements)

$$i \begin{array}{c} p \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} j : 0 \begin{array}{c} 0 \\ \text{---} \end{array} 0 = 1 \begin{array}{c} 1 \\ \text{---} \end{array} 1 = 1$$

$$\left(T_{ij}^p : T_{00}^0 = T_{11}^1 = 1 \right)$$

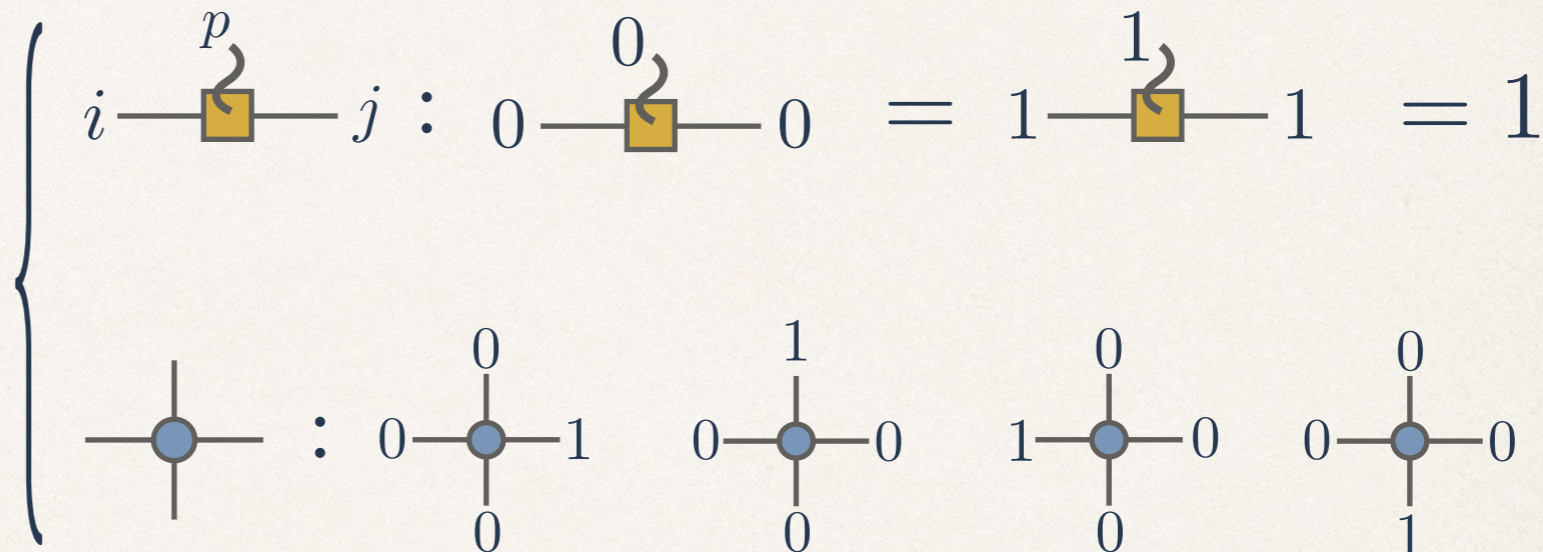
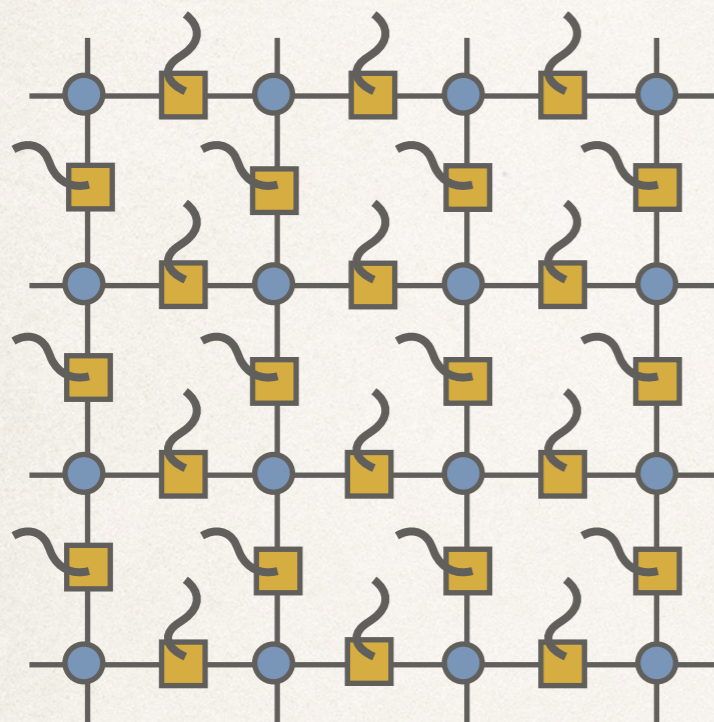
Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

▶ TPS ($d_p = d_v = 2$)

▶ Tensors (non-zero elements)

Index
0: absence
1: presence



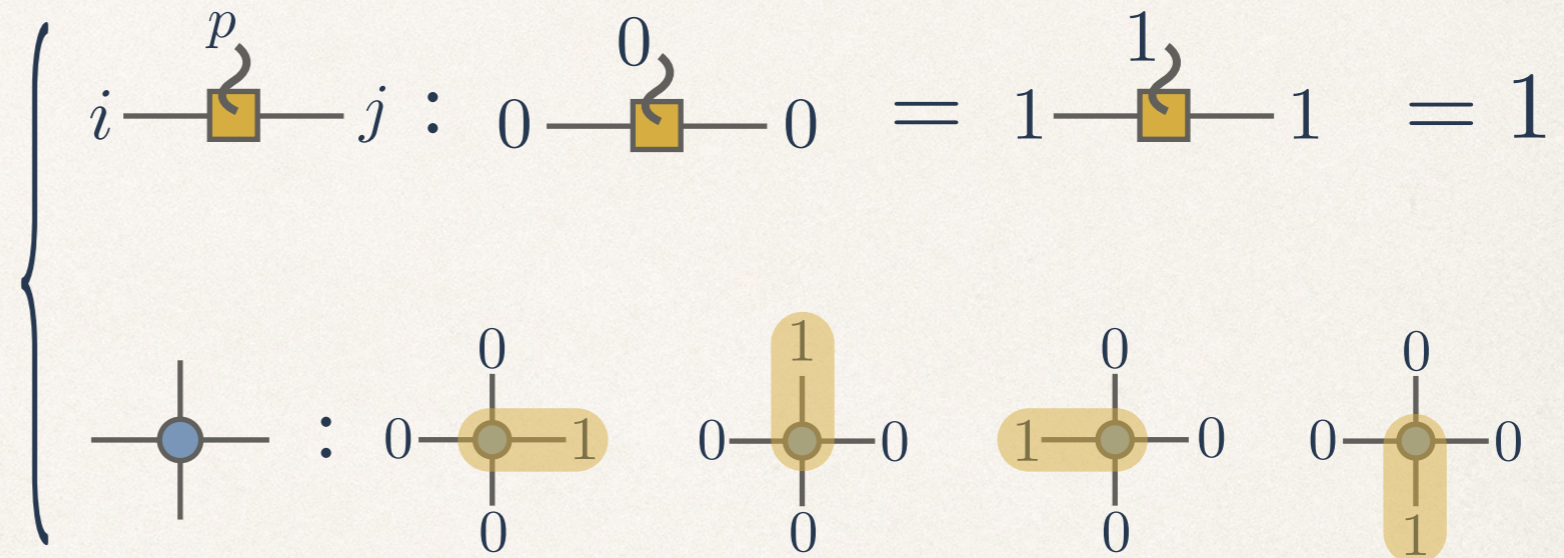
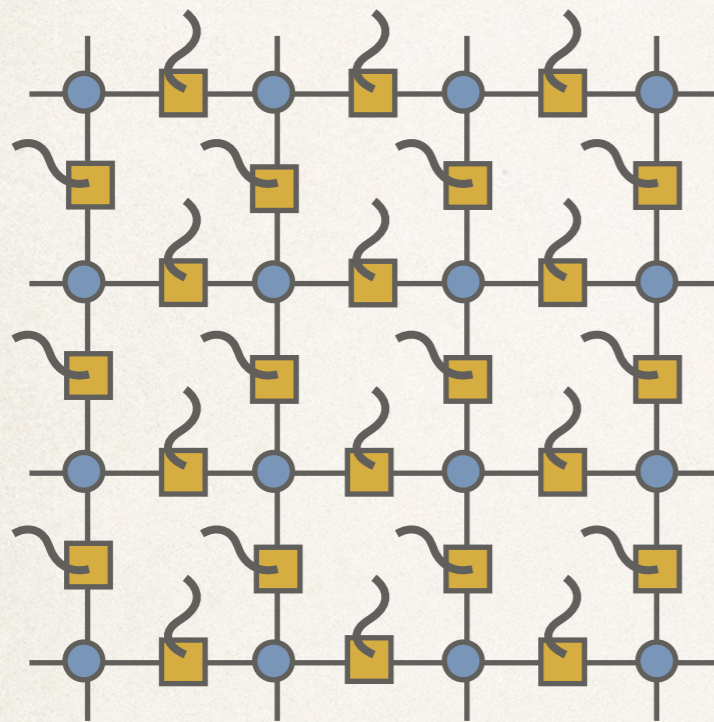
Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

▶ TPS ($d_p = d_v = 2$)

▶ Tensors (non-zero elements)

Index
0: absence
1: presence



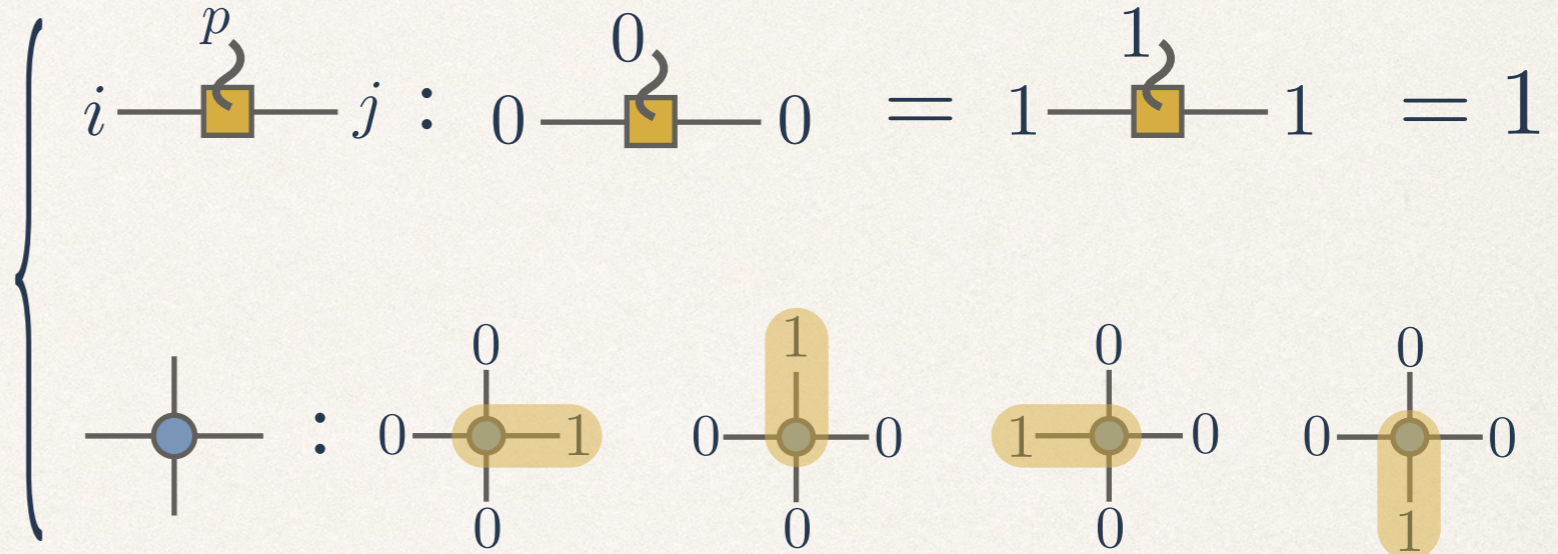
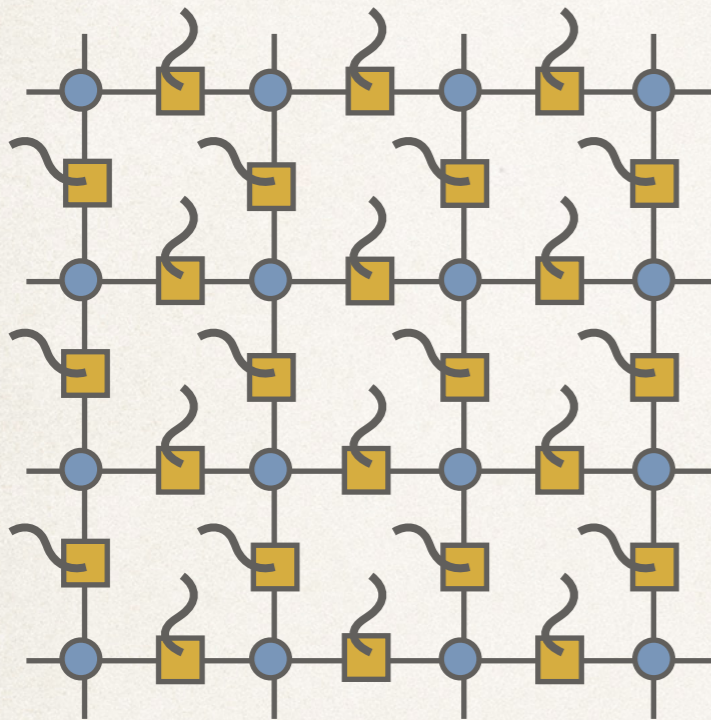
Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

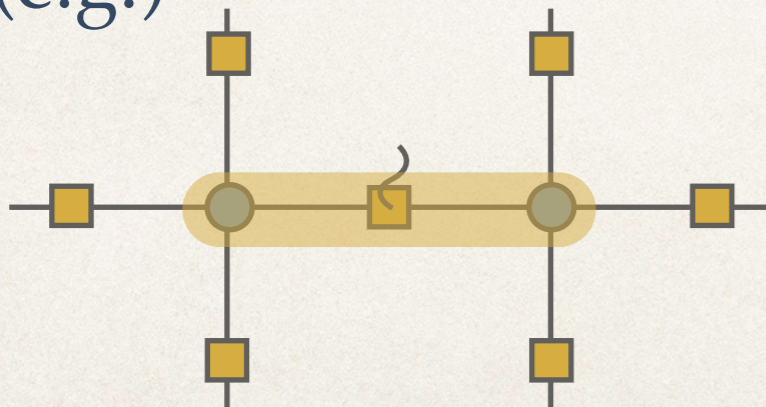
▶ TPS ($d_p = d_v = 2$)

▶ Tensors (non-zero elements)

Index
0: absence
1: presence



(e.g.)



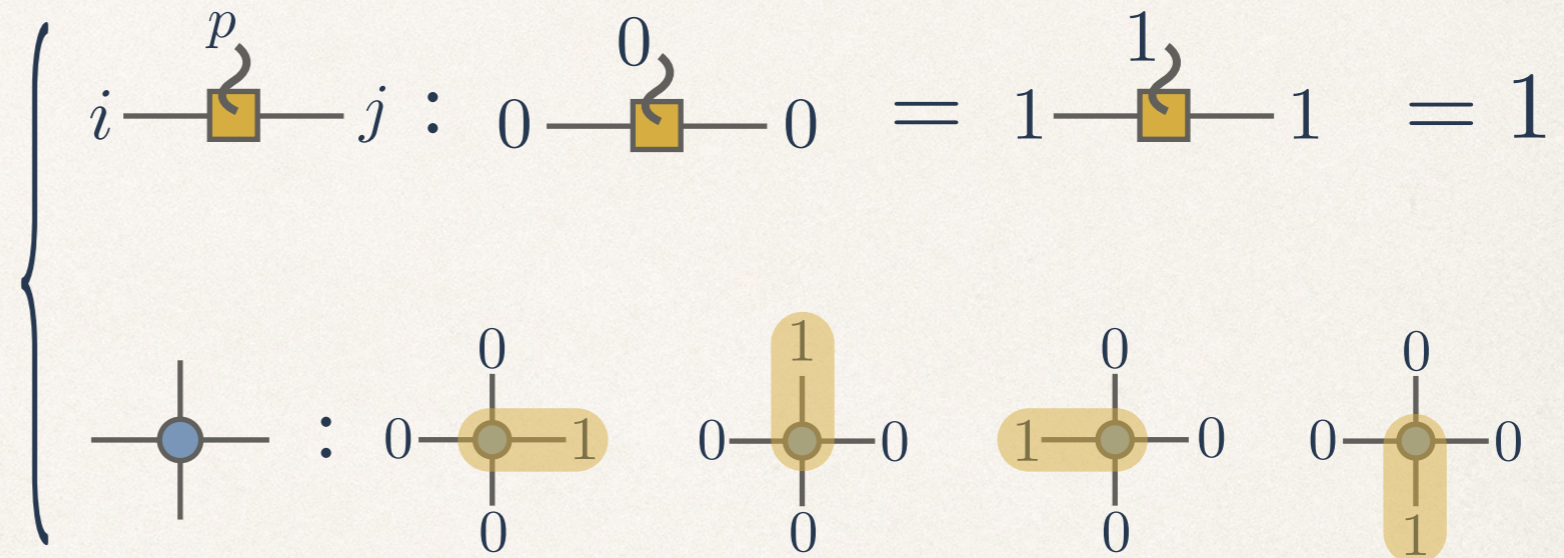
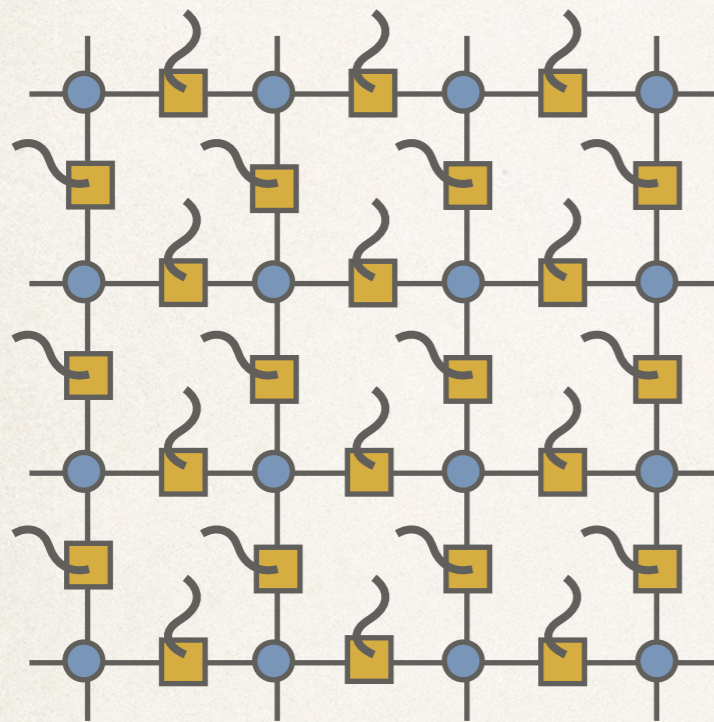
Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

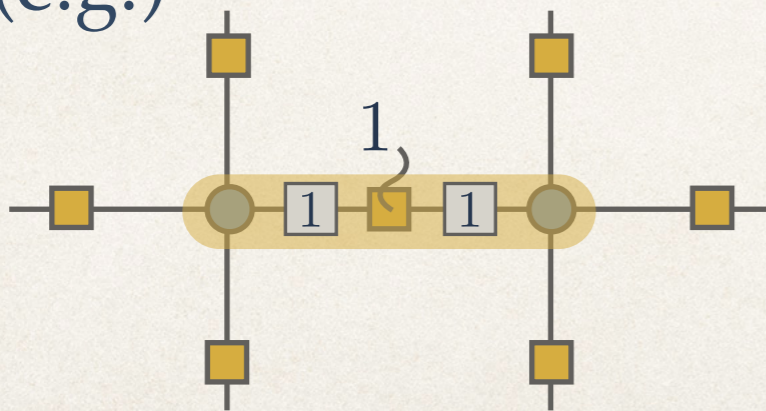
▶ TPS ($d_p = d_v = 2$)

▶ Tensors (non-zero elements)

Index
0: absence
1: presence



(e.g.)



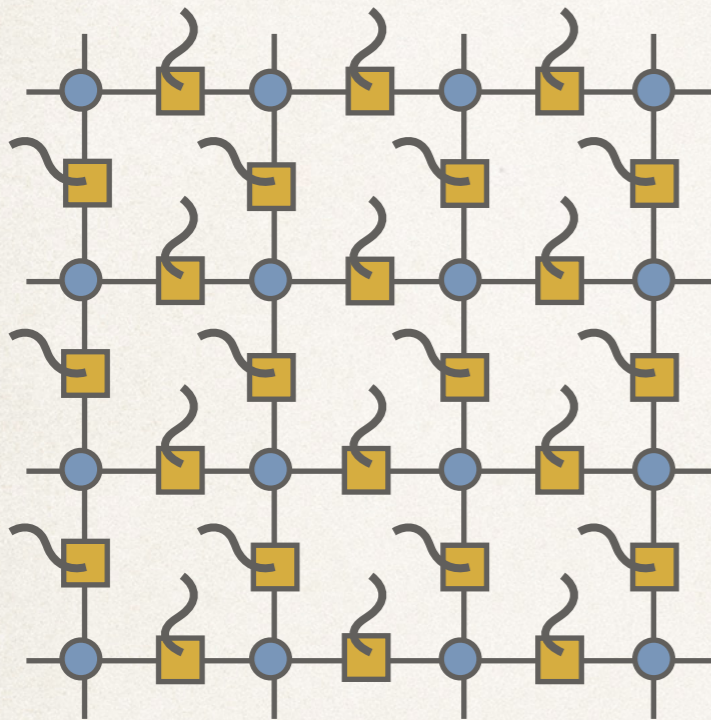
Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

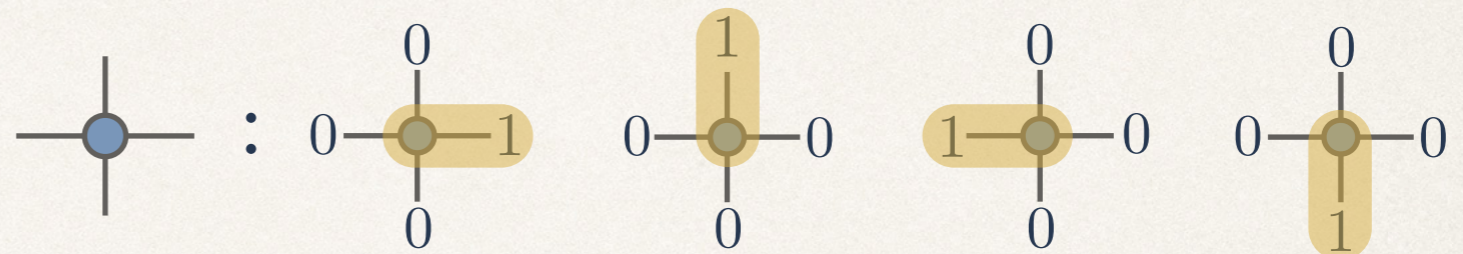
▶ TPS ($d_p = d_v = 2$)

▶ Tensors (non-zero elements)

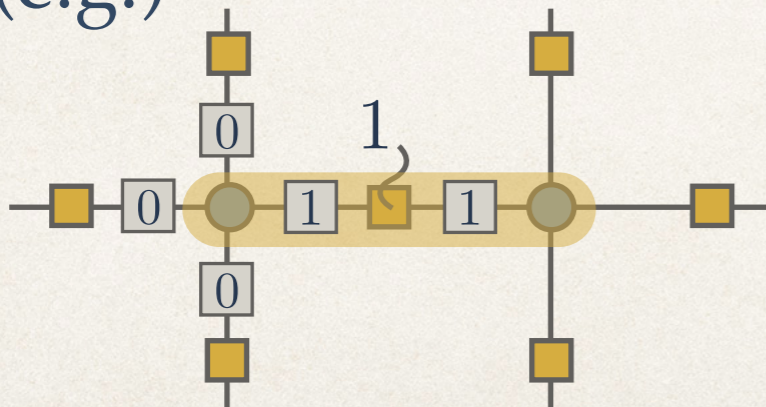
Index
0: absence
1: presence



$$i \begin{array}{c} p \\ \text{---} \end{array} \text{---} j : 0 \begin{array}{c} 0 \\ \text{---} \end{array} 0 = 1 \begin{array}{c} 1 \\ \text{---} \end{array} 1 = 1$$



(e.g.)



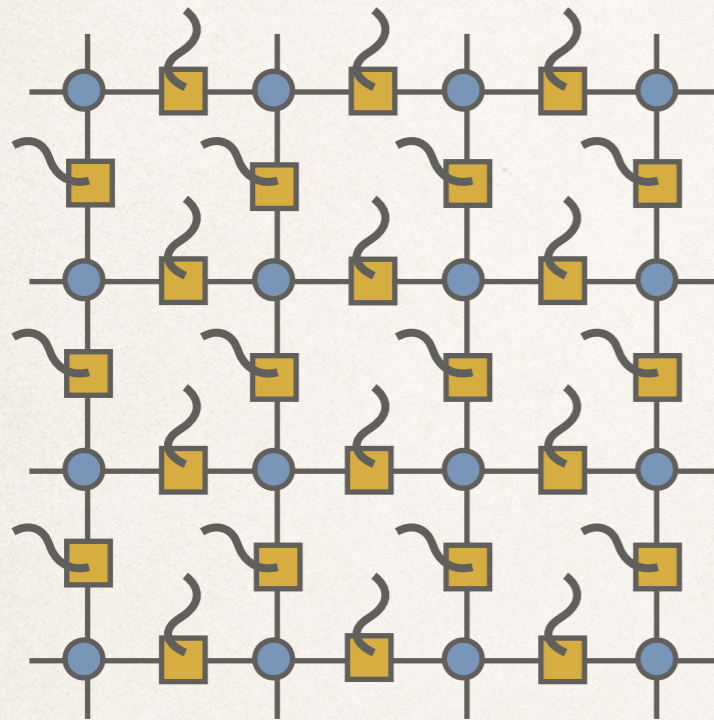
Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

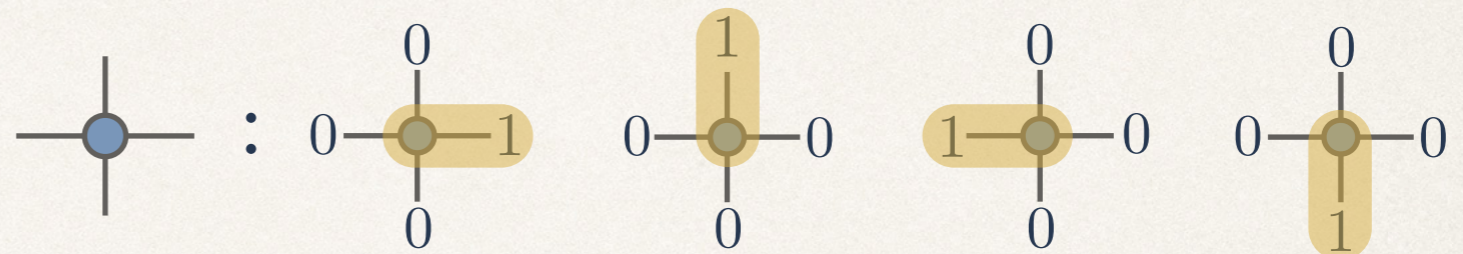
▶ TPS ($d_p = d_v = 2$)

▶ Tensors (non-zero elements)

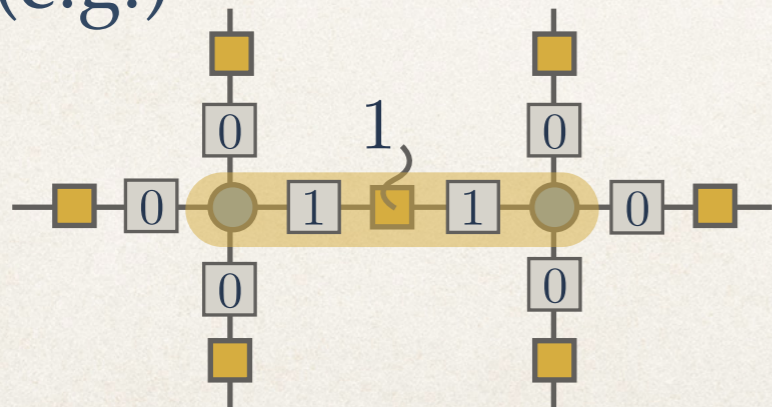
Index
0: absence
1: presence



$$i \text{---} \overset{p}{\square} \text{---} j : 0 \text{---} \overset{0}{\square} \text{---} 0 = 1 \text{---} \overset{1}{\square} \text{---} 1 = 1$$



(e.g.)



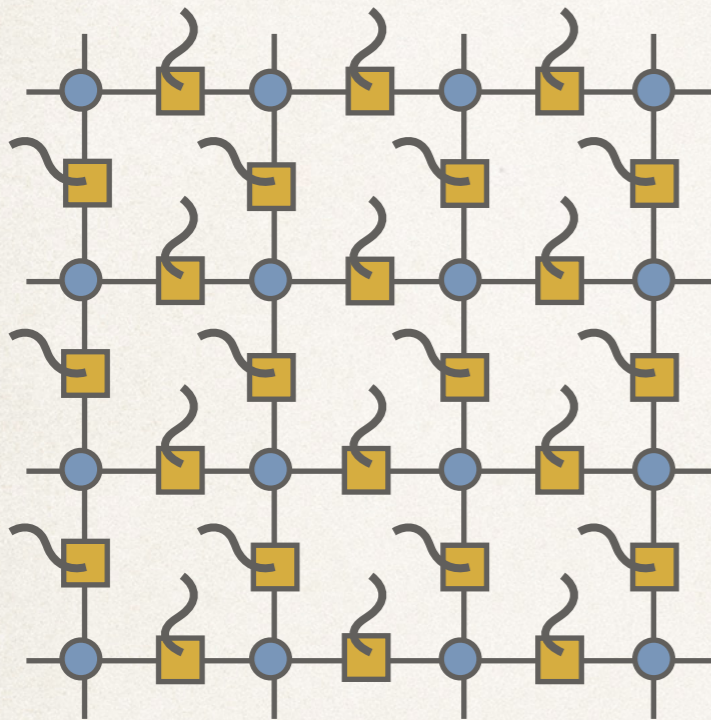
Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

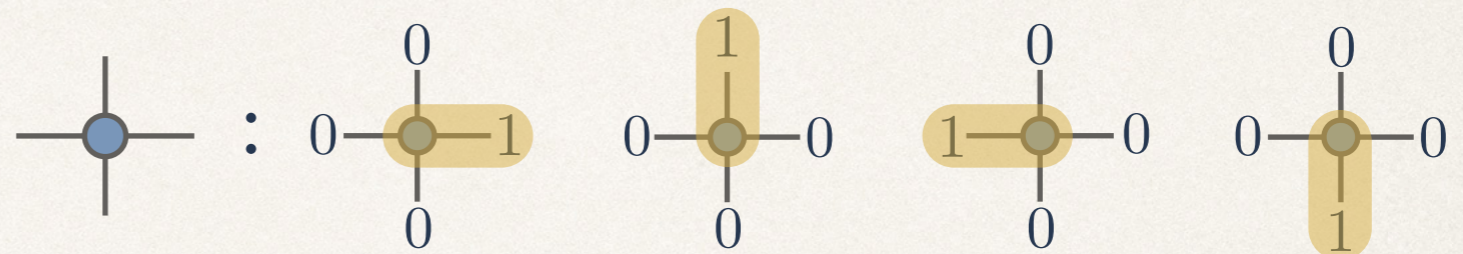
▶ TPS ($d_p = d_v = 2$)

▶ Tensors (non-zero elements)

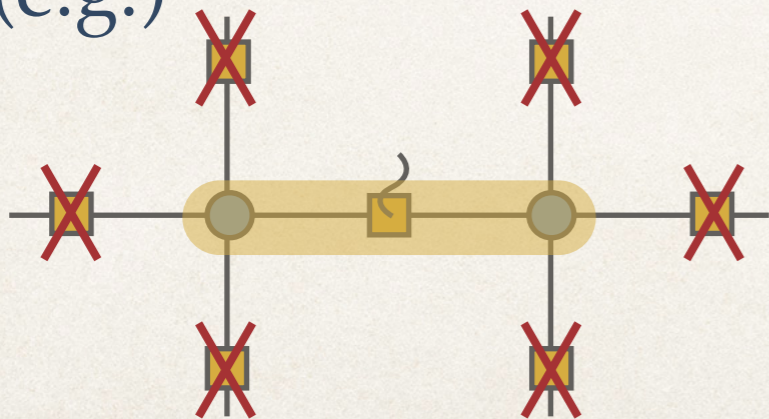
Index
0: absence
1: presence



$$i \text{---} \overset{p}{\square} \text{---} j : 0 \text{---} \overset{0}{\square} \text{---} 0 = 1 \text{---} \overset{1}{\square} \text{---} 1 = 1$$



(e.g.)



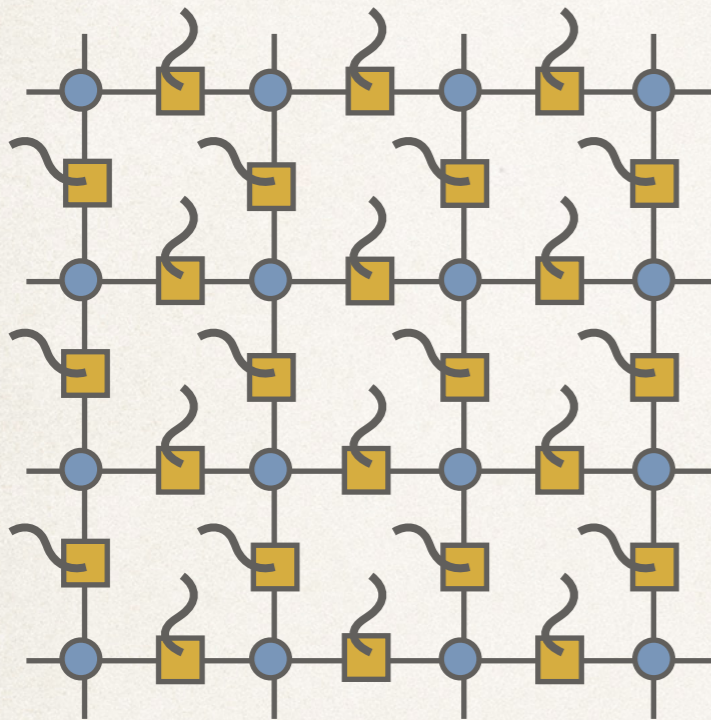
Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

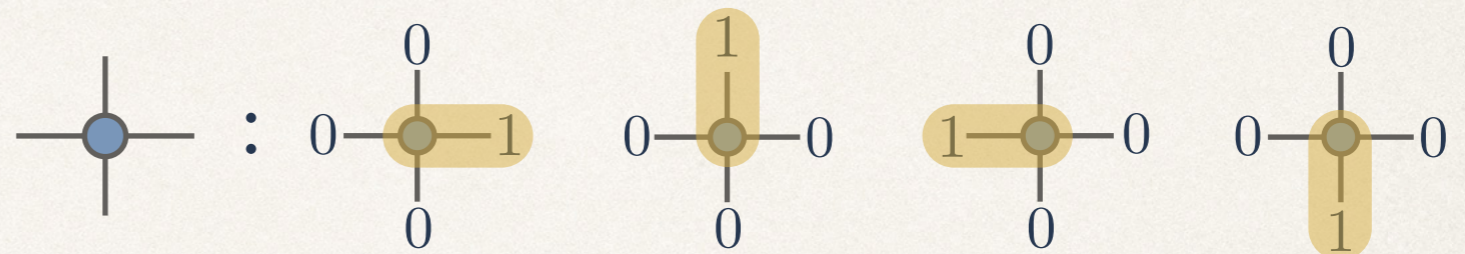
▶ TPS ($d_p = d_v = 2$)

▶ Tensors (non-zero elements)

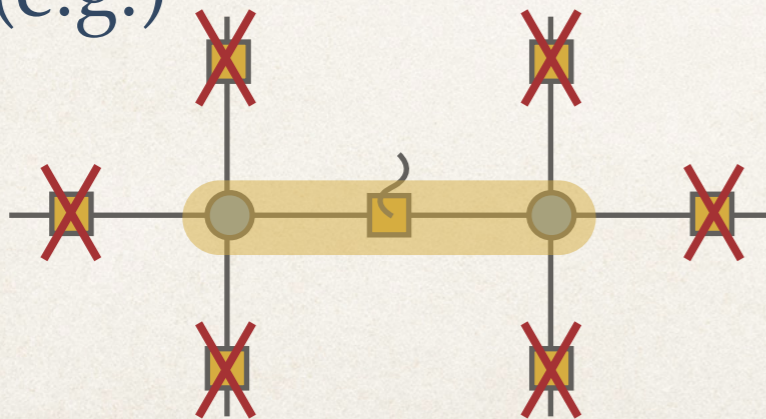
Index
0: absence
1: presence



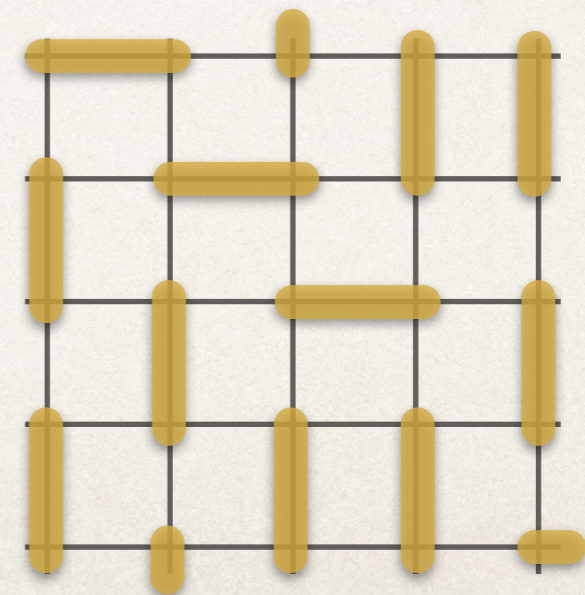
$$i \begin{array}{c} p \\ \text{---} \end{array} \text{---} j : 0 \begin{array}{c} 0 \\ \text{---} \end{array} \text{---} 0 = 1 \begin{array}{c} 1 \\ \text{---} \end{array} \text{---} 1 = 1$$



(e.g.)



guarantees
1-dimer at vertex
→
and
all configurations



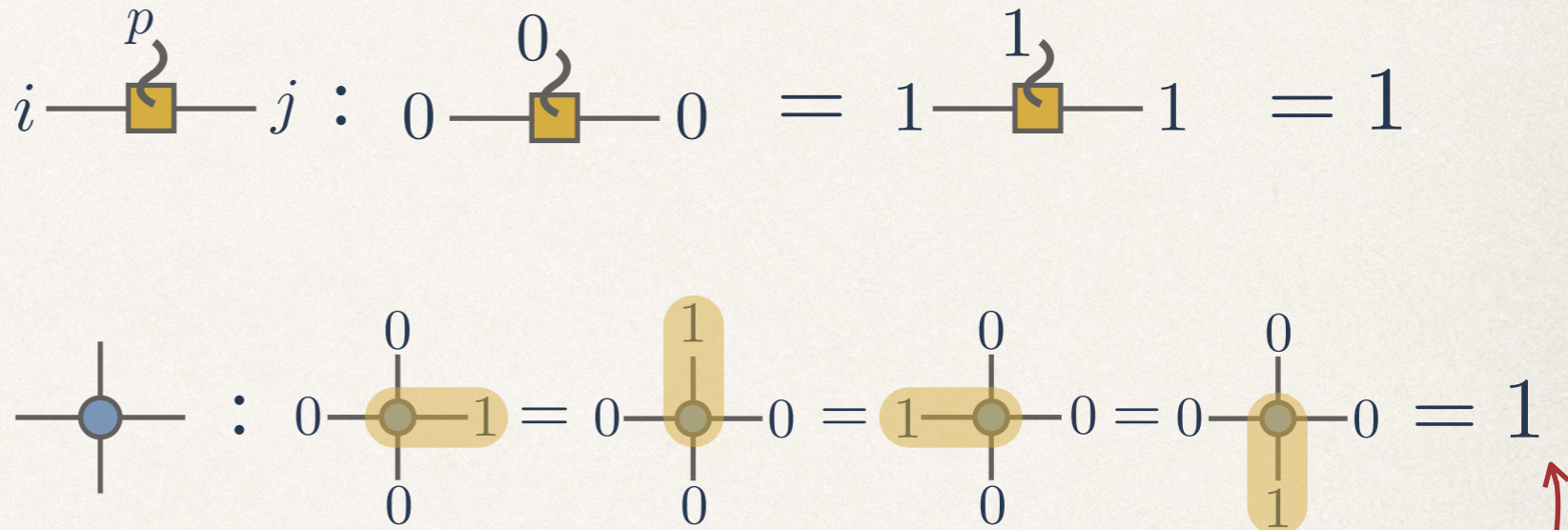
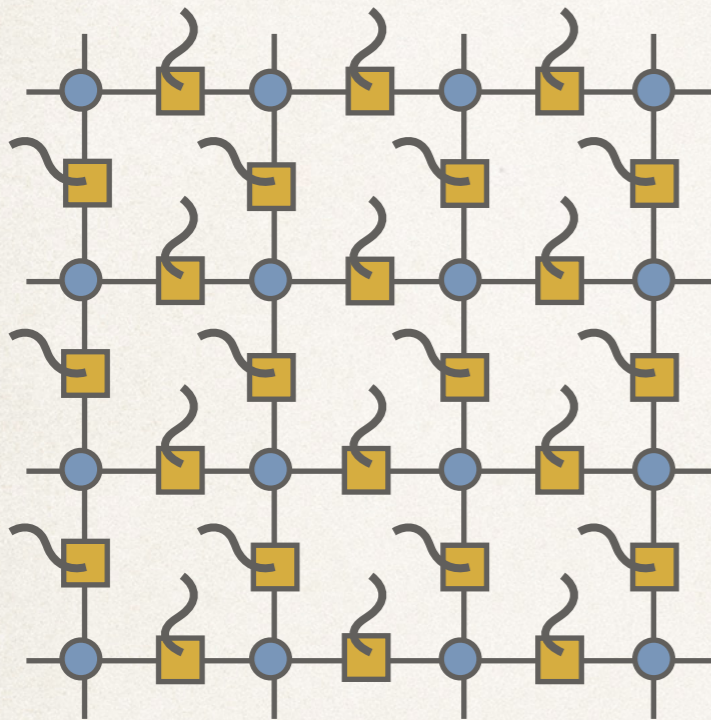
Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

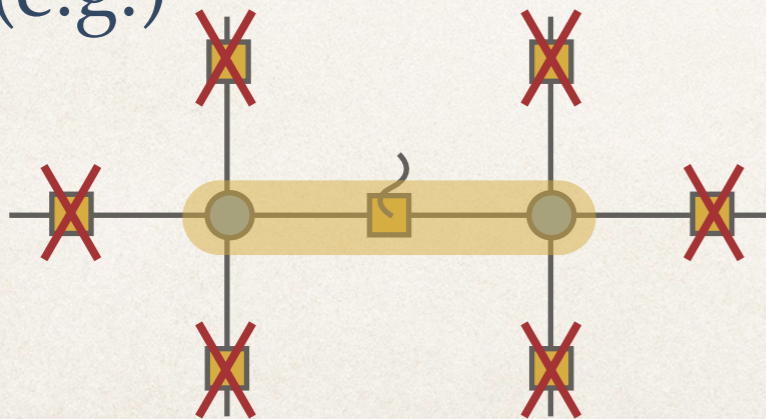
▶ TPS ($d_p = d_v = 2$)

▶ Tensors (non-zero elements)

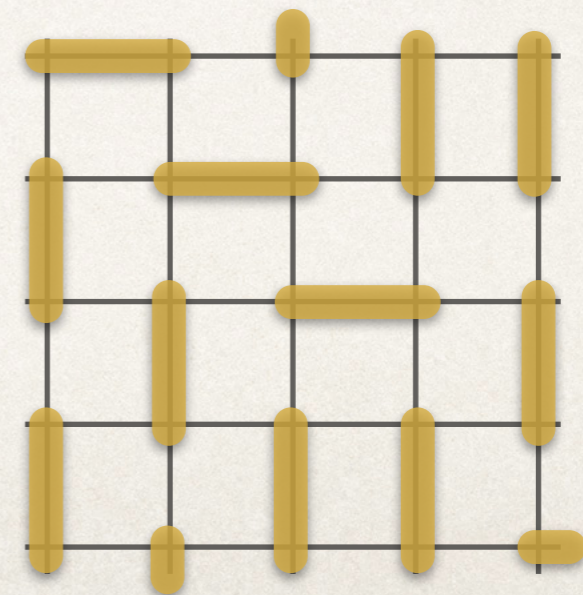
Index
0: absence
1: presence



(e.g.)



guarantees
1-dimer at vertex
→
and
all configurations



equal weight

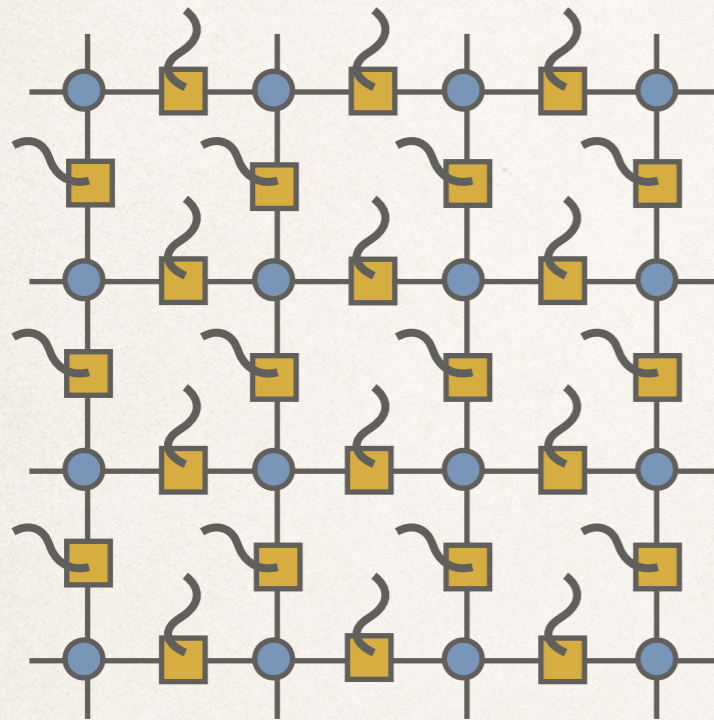
Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

▶ TPS ($d_p = d_v = 2$)

▶ Tensors (non-zero elements)

Index
0: absence
1: presence



$$\left. \begin{aligned}
 & i \text{---} \overset{p}{\square} \text{---} j : 0 \text{---} \overset{0}{\square} \text{---} 0 = 1 \text{---} \overset{1}{\square} \text{---} 1 = 1 \\
 & \text{---} \circ \text{---} : 0 \begin{array}{c} 0 \\ | \\ \circ \\ | \\ 0 \end{array} 1 = 0 \begin{array}{c} 1 \\ | \\ \circ \\ | \\ 0 \end{array} 0 = 1 \begin{array}{c} 0 \\ | \\ \circ \\ | \\ 0 \end{array} 0 = 0 \begin{array}{c} 0 \\ | \\ \circ \\ | \\ 1 \end{array} 0 = 1
 \end{aligned} \right\}$$

$$\text{▶ } |RK\rangle = \left| \begin{array}{cccc} \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \end{array} \right\rangle + \left| \begin{array}{cccc} \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \end{array} \right\rangle + \left| \begin{array}{cccc} \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \end{array} \right\rangle + \dots$$

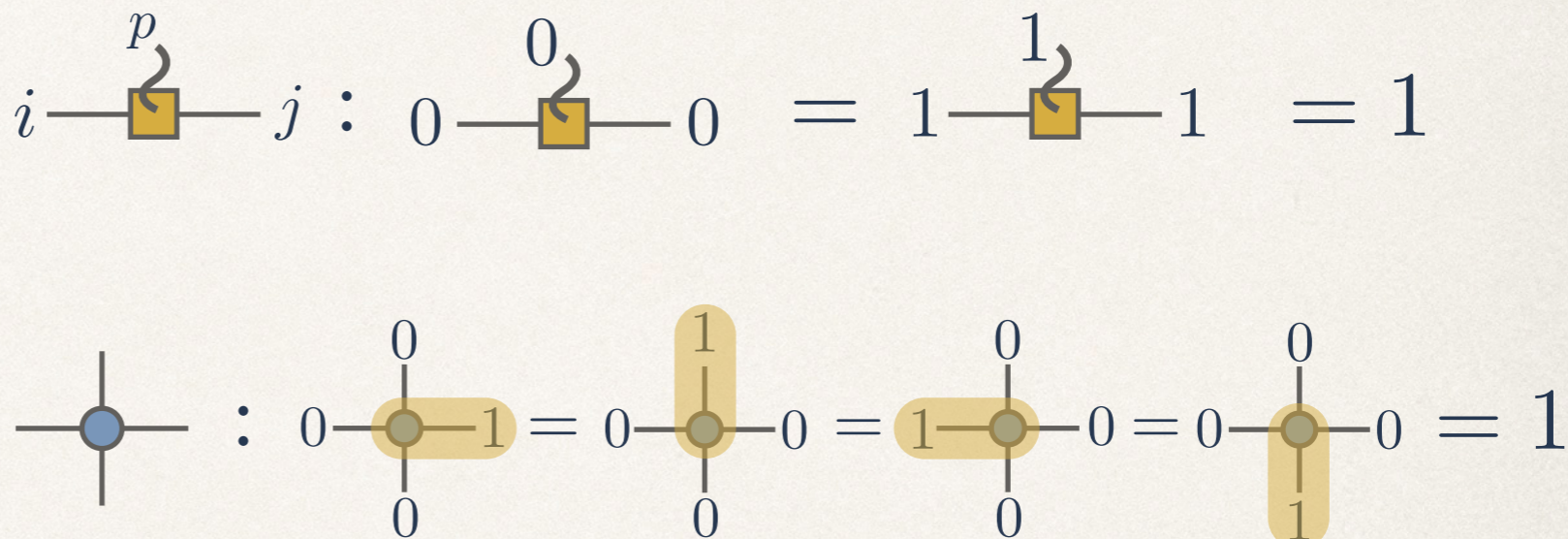
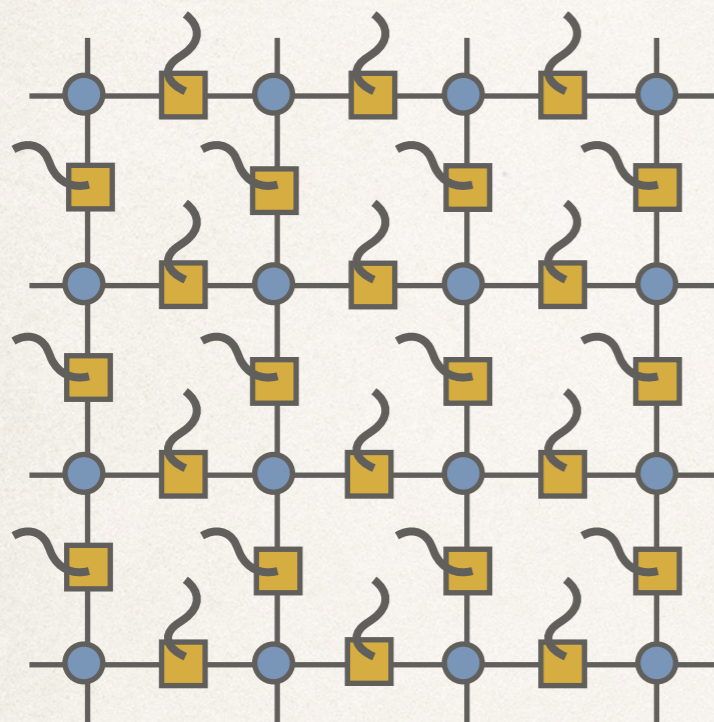
Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

▶ TPS ($d_p = d_v = 2$)

▶ Tensors (non-zero elements)

Index
0: absence
1: presence



▶ $|RK\rangle = \left| \begin{array}{cccc} \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \end{array} \right\rangle + \left| \begin{array}{cccc} \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \end{array} \right\rangle + \left| \begin{array}{cccc} \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \end{array} \right\rangle + \dots$

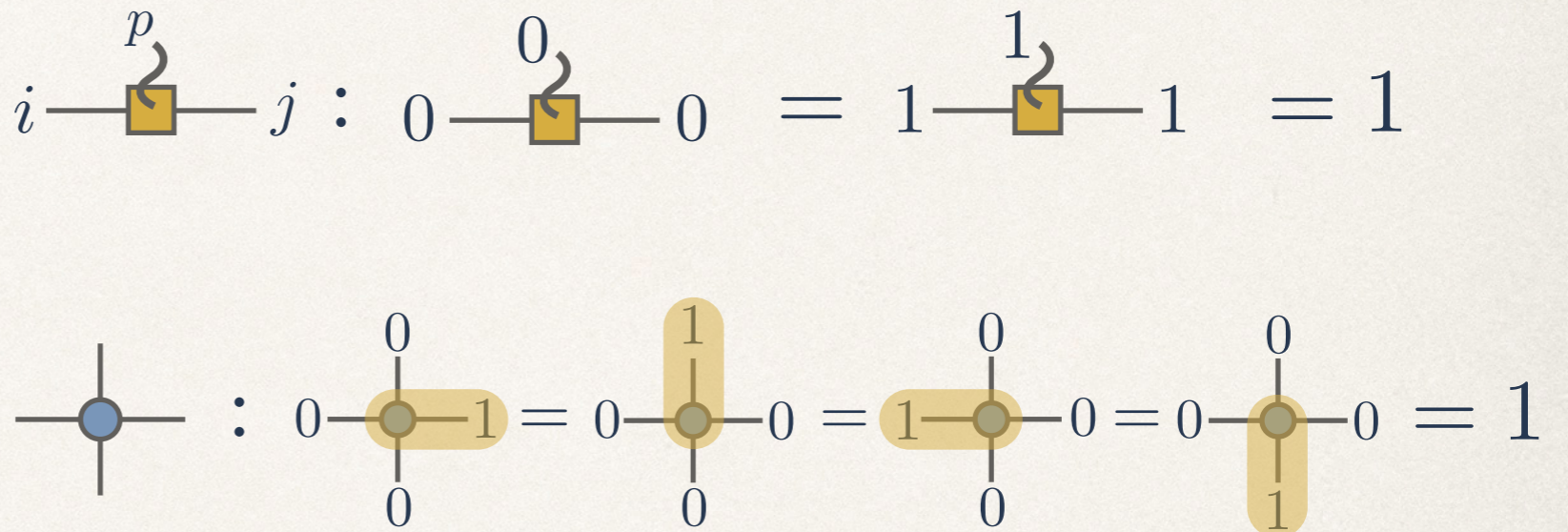
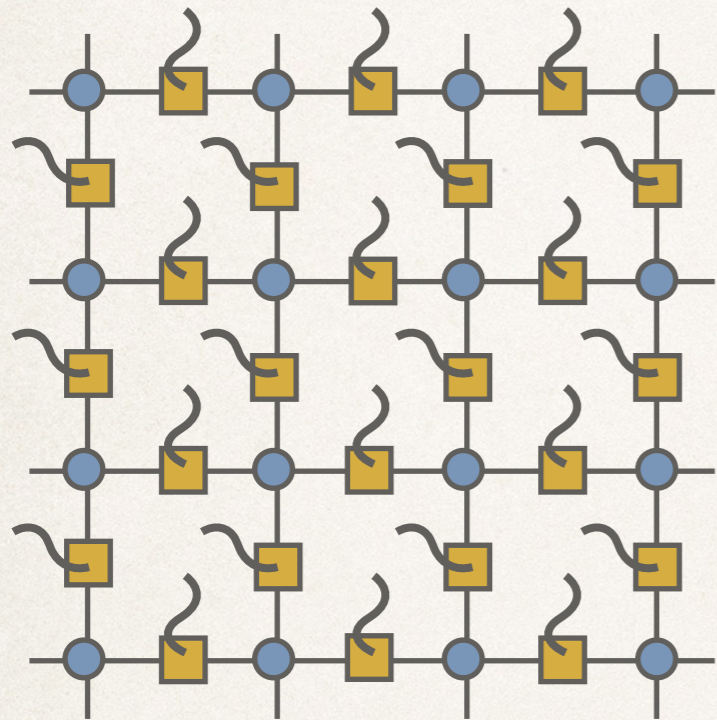
Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

▶ TPS ($d_p = d_v = 2$)

▶ Tensors (non-zero elements)

Index
0: absence
1: presence



▶ $|\text{RK}\rangle = \left| \begin{array}{ccc} \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \end{array} \right\rangle + \left| \begin{array}{ccc} \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \end{array} \right\rangle + \left| \begin{array}{ccc} \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \end{array} \right\rangle + \dots$

Exact RK wavefunction in $D=2$ TPS representation!

Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

► Generalization

$$\left\{ \begin{array}{l}
 i \text{---} \boxed{p} \text{---} j : 0 \text{---} \boxed{0} \text{---} 0 = 1 \text{---} \boxed{1} \text{---} 1 = 1 \\
 \text{---} \textcircled{p} \text{---} : 0 \text{---} \textcircled{0} \text{---} 1 = 0 \text{---} \textcircled{1} \text{---} 0 = 1 \text{---} \textcircled{0} \text{---} 0 = 0 \text{---} \textcircled{0} \text{---} 1 = 1
 \end{array} \right.$$

Optimization - Exact Construction

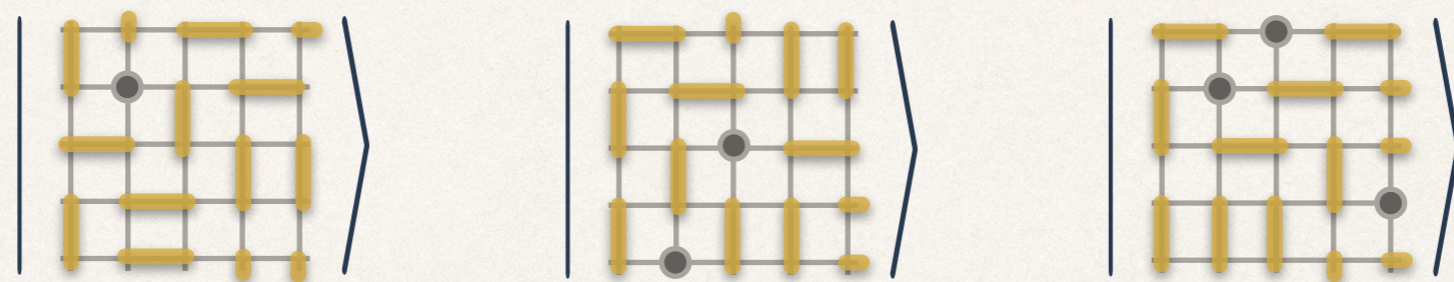
❖ Example 1: Quantum Dimer Liquid

► Generalization

$$\left\{ \begin{array}{l}
 i \text{---} \boxed{p} \text{---} j : 0 \text{---} \boxed{0} \text{---} 0 = 1 \text{---} \boxed{1} \text{---} 1 = 1 \\
 \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} : 0 \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} = 0 \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} = 1 \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} = 0 \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} = 1
 \end{array} \right.$$

Add

$$\begin{array}{c}
 0 \\
 | \\
 0 \text{---} \text{---} 0 = c \\
 | \\
 0
 \end{array}$$



Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

► Generalization

$$\left\{ \begin{array}{l}
 i \text{---} \overset{p}{\square} \text{---} j : 0 \text{---} \overset{0}{\square} \text{---} 0 = 1 \text{---} \overset{1}{\square} \text{---} 1 = 1 \\
 \text{---} \circ \text{---} : 0 \text{---} \overset{0}{\circ} \text{---} 1 = 0 \text{---} \overset{1}{\circ} \text{---} 0 = 1 \text{---} \overset{0}{\circ} \text{---} 0 = 0 \text{---} \overset{0}{\circ} \text{---} 1 = 1
 \end{array} \right.$$

Add

$$\begin{array}{c}
 0 \\
 | \\
 0 \text{---} \circ \text{---} 0 \\
 | \\
 0
 \end{array} = c$$

$$\text{► } |\text{MD}\rangle = c \left| \begin{array}{ccc} \text{---} \circ \text{---} & & \\ \text{---} \circ \text{---} & & \\ \text{---} \circ \text{---} & & \end{array} \right\rangle + c^2 \left| \begin{array}{ccc} \text{---} \circ \text{---} & & \\ \text{---} \circ \text{---} & & \\ \text{---} \circ \text{---} & & \end{array} \right\rangle + c^3 \left| \begin{array}{ccc} \text{---} \circ \text{---} & & \\ \text{---} \circ \text{---} & & \\ \text{---} \circ \text{---} & & \end{array} \right\rangle + \dots$$

Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

► Generalization

$$\left\{ \begin{array}{l}
 i \text{---} \boxed{p} \text{---} j : 0 \text{---} \boxed{0} \text{---} 0 = 1 \text{---} \boxed{1} \text{---} 1 = 1 \\
 \text{---} \text{---} \text{---} \text{---} : 0 \text{---} \text{---} \text{---} \text{---} = 0 \text{---} \text{---} \text{---} \text{---} = 1 \text{---} \text{---} \text{---} \text{---} = 0 \text{---} \text{---} \text{---} \text{---} = 1
 \end{array} \right.$$

Add

$$\begin{array}{c}
 0 \\
 | \\
 0 \text{---} \text{---} 0 = c \\
 | \\
 0 \\
 \\
 0 \\
 | \\
 0 \text{---} \text{---} 1 = \tilde{c} \\
 | \\
 1
 \end{array}$$

$$\text{► } |\text{MD}\rangle = c \left| \begin{array}{ccc} \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \end{array} \right\rangle + c^2 \left| \begin{array}{ccc} \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \end{array} \right\rangle + c^3 \left| \begin{array}{ccc} \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \end{array} \right\rangle + \dots$$

Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

► Generalization

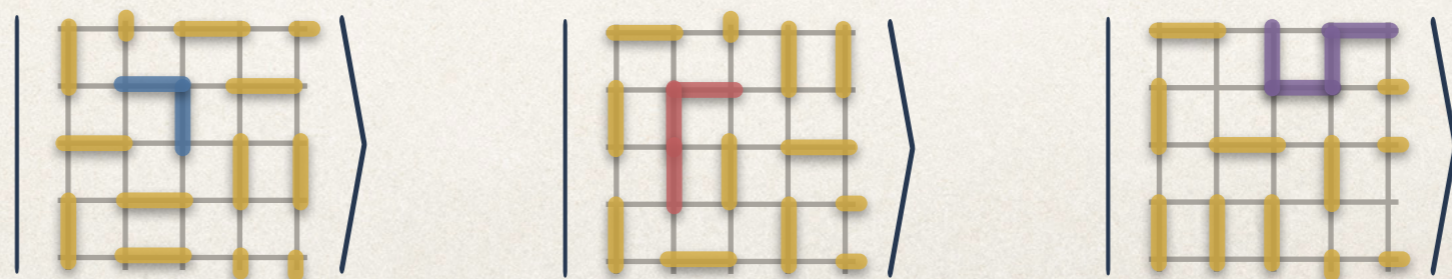
$$\left\{ \begin{array}{l}
 i \text{---} \boxed{p} \text{---} j : 0 \text{---} \boxed{0} \text{---} 0 = 1 \text{---} \boxed{1} \text{---} 1 = 1 \\
 \text{---} \text{---} \text{---} \text{---} : 0 \text{---} \text{---} \text{---} \text{---} = 0 \text{---} \text{---} \text{---} \text{---} = 1 \text{---} \text{---} \text{---} \text{---} = 0 \text{---} \text{---} \text{---} \text{---} = 1
 \end{array} \right.$$

Add

$$\begin{array}{c}
 0 \\
 | \\
 0 \text{---} \text{---} 0 = c \\
 | \\
 0
 \end{array}$$

$$\begin{array}{c}
 0 \\
 | \\
 0 \text{---} \text{---} 1 = \tilde{c} \\
 | \\
 1
 \end{array}$$

$$\text{► } |\text{MD}\rangle = c \left| \begin{array}{ccc} \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \end{array} \right\rangle + c^2 \left| \begin{array}{ccc} \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \end{array} \right\rangle + c^3 \left| \begin{array}{ccc} \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \end{array} \right\rangle + \dots$$



Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

► Generalization

$$\left\{ \begin{array}{l}
 i \text{---} \boxed{p} \text{---} j : 0 \text{---} \boxed{0} \text{---} 0 = 1 \text{---} \boxed{1} \text{---} 1 = 1 \\
 \text{---} \bullet \text{---} : 0 \text{---} \boxed{0} \text{---} 1 = 0 \text{---} \boxed{1} \text{---} 0 = 1 \text{---} \boxed{0} \text{---} 0 = 0 \text{---} \boxed{0} \text{---} 1 = 1
 \end{array} \right.$$

Add

$$\begin{array}{c}
 0 \\
 | \\
 0 \text{---} \bullet \text{---} 0 = c \\
 | \\
 0
 \end{array}$$

$$\begin{array}{c}
 0 \\
 | \\
 0 \text{---} \bullet \text{---} 1 = \tilde{c} \\
 | \\
 1
 \end{array}$$

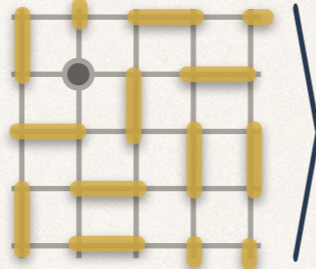
$$\text{► } |\text{MD}\rangle = c \left| \begin{array}{ccc} \bullet & & \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \end{array} \right\rangle + c^2 \left| \begin{array}{ccc} & \bullet & \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \bullet & & \end{array} \right\rangle + c^3 \left| \begin{array}{ccc} & & \bullet \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ & \bullet & \end{array} \right\rangle + \dots$$

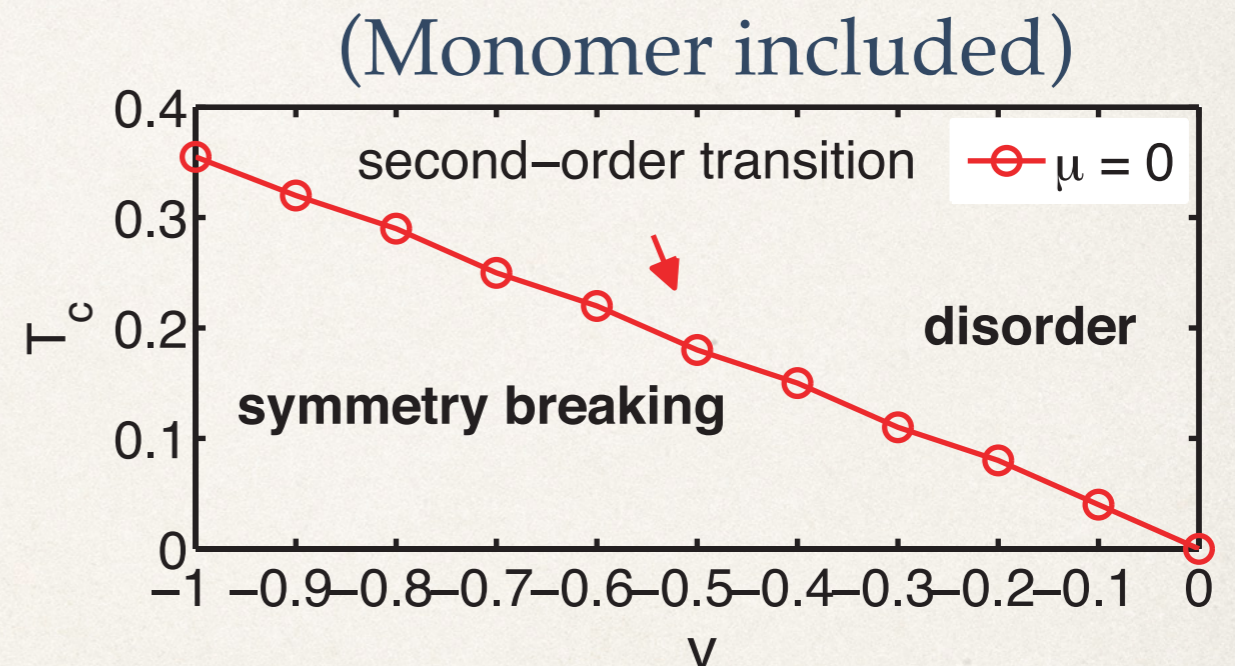
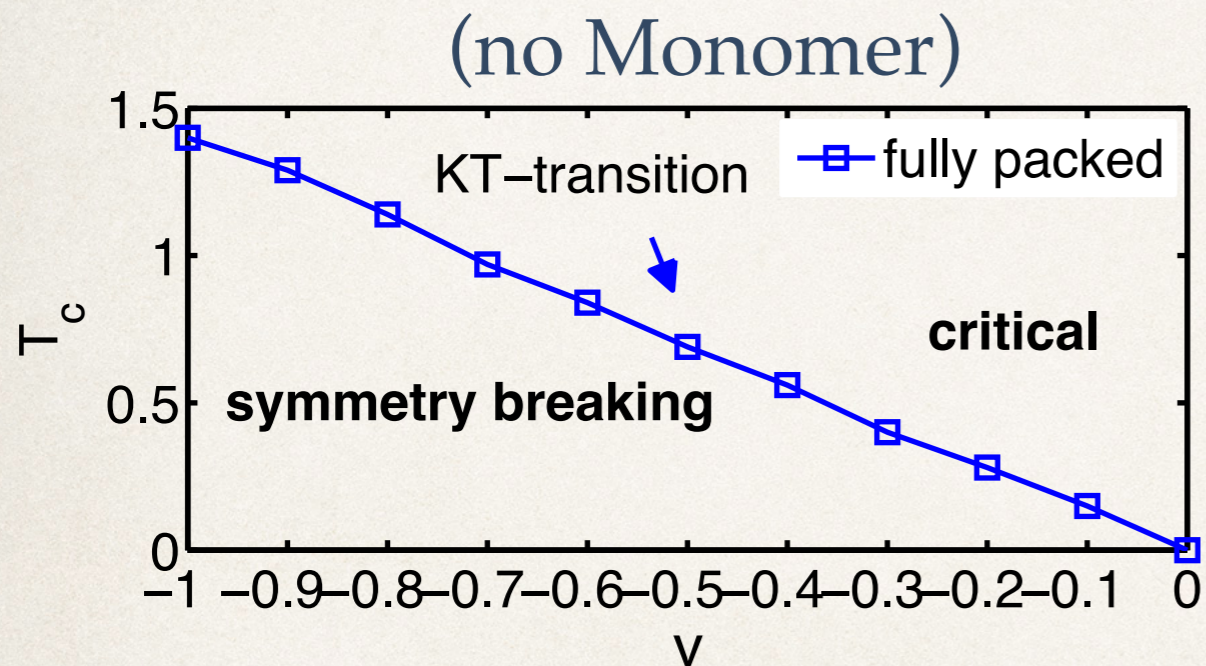
$$\text{► } |\text{Qmer}\rangle = \tilde{c} \left| \begin{array}{ccc} & \text{---} & \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ & \text{---} & \end{array} \right\rangle + \tilde{c}^2 \left| \begin{array}{ccc} & & \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ & \text{---} & \end{array} \right\rangle + \tilde{c}^3 \left| \begin{array}{ccc} & & \text{---} \\ \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} \\ & \text{---} & \end{array} \right\rangle + \dots$$

Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

- ▶ ex) Monomer-Dimer model [Sazi, Wei, Ziyu (2015)]

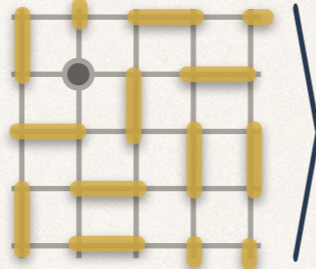
$$\bullet \quad |\text{MD}\rangle = \sum_{\text{config}} e^{-\frac{v}{T}} \left| \begin{array}{c} \text{grid with dimers} \end{array} \right\rangle$$


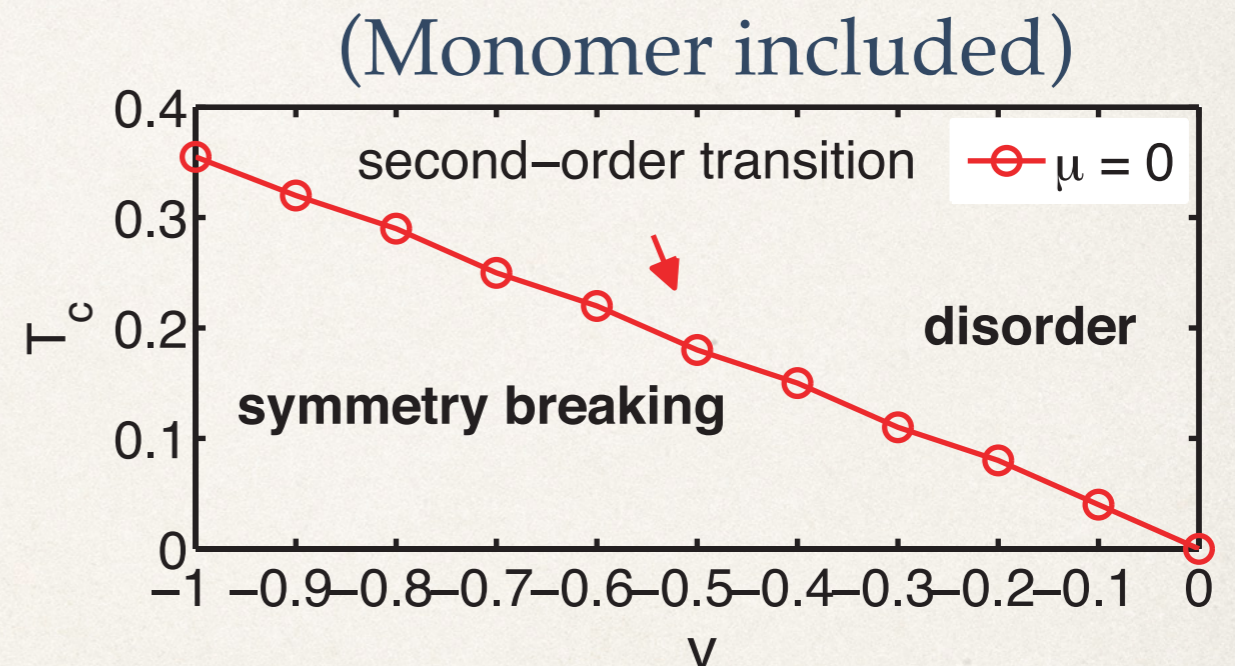
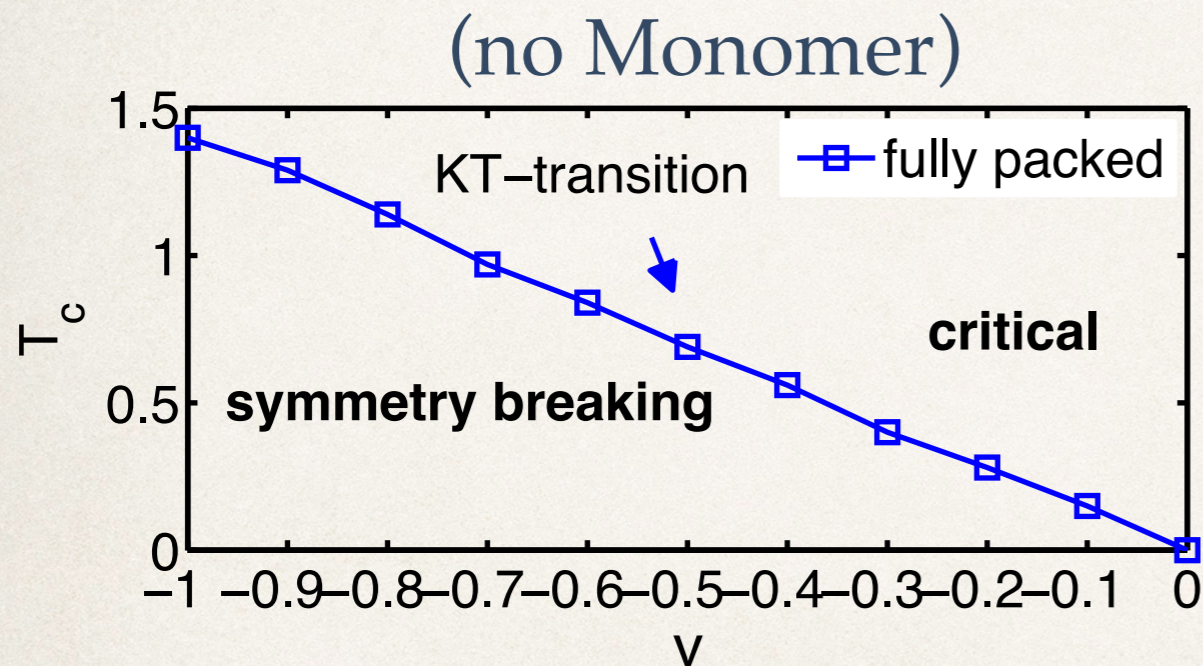


Optimization - Exact Construction

❖ Example1: Quantum Dimer Liquid

- ▶ ex) Monomer-Dimer model [Sazi, Wei, Ziyu (2015)]

$$\bullet \quad |\text{MD}\rangle = \sum_{\text{config}} e^{-\frac{v}{T}} \left| \begin{array}{c} \text{grid with dimers} \end{array} \right\rangle$$




General problems are defined by adding simple configurations

Optimization - Exact Construction

❖ Example2: SU(2) Resonating Valence Bond

$$\text{---} = |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$$

$$\blacktriangleright |\text{RVB}\rangle = \left| \begin{array}{cccc} \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \end{array} \right\rangle + \left| \begin{array}{cccc} \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \end{array} \right\rangle + \left| \begin{array}{cccc} \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \\ \text{---} & \text{---} & \text{---} & \text{---} \end{array} \right\rangle + \dots$$

Index	
0:	0 $S = 0$
1:	\uparrow $\left. \vphantom{\begin{matrix} 1: \\ 2: \end{matrix}} \right\} S = \frac{1}{2}$
2:	\downarrow

$$\left\{ \begin{array}{l} \text{---} \circ \text{---} : \begin{array}{c} 0 \\ \uparrow \\ 0 \end{array} = \begin{array}{c} 0 \\ \downarrow \\ 0 \end{array} = 1 \quad (\text{cyclic}) \\ = T_{\text{short}} \\ \text{---} \square \text{---} : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \end{array} \right. \rightarrow \text{generates singlet!}$$

Optimization - Exact Construction

❖ Example2: SU(2) Resonating Valence Bond

$$\text{—} = |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$$

$$\blacktriangleright |\text{RVB}\rangle = \left| \begin{array}{cccc} \text{—} & \text{—} & \text{—} & \text{—} \\ \text{—} & \text{—} & \text{—} & \text{—} \\ \text{—} & \text{—} & \text{—} & \text{—} \\ \text{—} & \text{—} & \text{—} & \text{—} \end{array} \right\rangle + \left| \begin{array}{cccc} \text{—} & \text{—} & \text{—} & \text{—} \\ \text{—} & \text{—} & \text{—} & \text{—} \\ \text{—} & \text{—} & \text{—} & \text{—} \\ \text{—} & \text{—} & \text{—} & \text{—} \end{array} \right\rangle + \left| \begin{array}{cccc} \text{—} & \text{—} & \text{—} & \text{—} \\ \text{—} & \text{—} & \text{—} & \text{—} \\ \text{—} & \text{—} & \text{—} & \text{—} \\ \text{—} & \text{—} & \text{—} & \text{—} \end{array} \right\rangle + \dots$$

- \blacktriangleright Constraints:
 - a. SU(2) symmetry
 - b. equal weight superposition

Index	
0:	0 $S = 0$
1:	\uparrow $\left. \vphantom{\begin{matrix} 1: \\ 2: \end{matrix}} \right\} S = \frac{1}{2}$
2:	\downarrow

$$\left\{ \begin{array}{l} \text{—} \circ \text{—} : \begin{array}{c} 0 \\ \uparrow \\ 0 \end{array} = \text{—} \circ \text{—} \begin{array}{c} 0 \\ \downarrow \\ 0 \end{array} = 1 \quad (\text{cyclic}) \\ = T_{\text{short}} \\ \text{—} \square \text{—} : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \end{array} \right. \rightarrow \text{generates singlet!}$$

Optimization - Exact Construction

❖ Example2: SU(2) Resonating Valence Bond

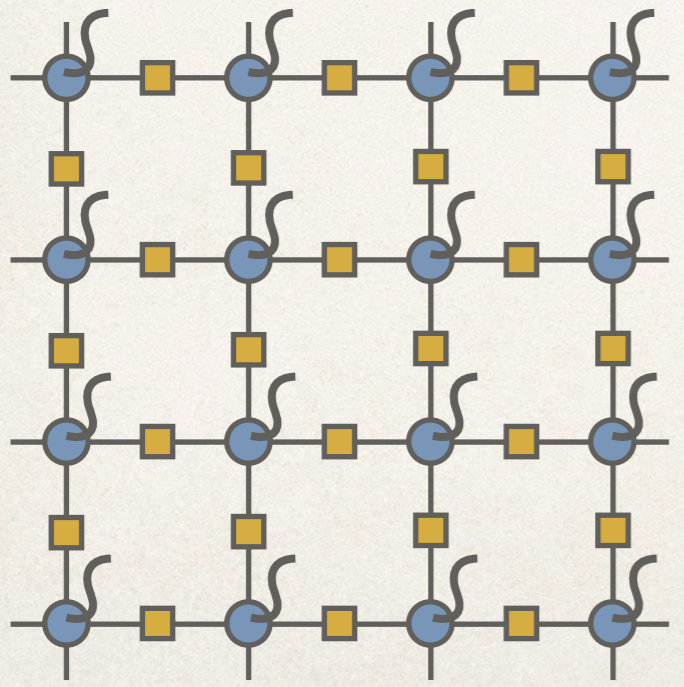
$$\text{---} = |\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$$

$$\text{▶ } |\text{RVB}\rangle = \left| \begin{array}{cccc} \text{---} & & & \\ & \text{---} & & \\ & & \text{---} & \\ & & & \text{---} \end{array} \right\rangle + \left| \begin{array}{cccc} \text{---} & & & \\ & \text{---} & & \\ & & \text{---} & \\ & & & \text{---} \end{array} \right\rangle + \left| \begin{array}{cccc} & \text{---} & & \\ & & \text{---} & \\ & & & \text{---} \\ & \text{---} & & \end{array} \right\rangle + \dots$$

- ▶ Constraints: $\left\{ \begin{array}{l} \text{a. SU(2) symmetry} \\ \text{b. equal weight superposition} \end{array} \right.$

Index		
0:	0	$S = 0$
1:	\uparrow	$S = \frac{1}{2}$
2:	\downarrow	

▶ TPS ($d_v = 3$)

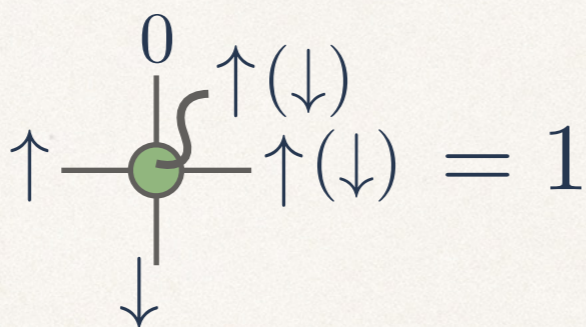
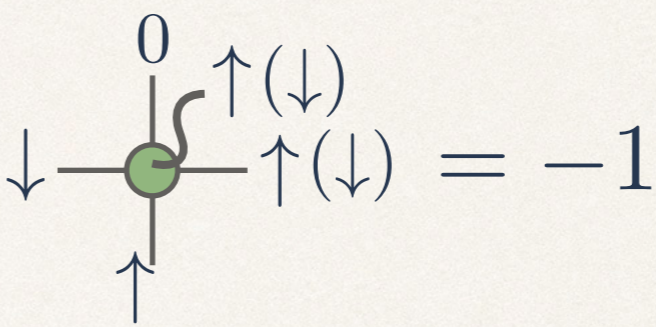


$$\left\{ \begin{array}{l} \text{---} : \begin{array}{c} 0 \\ \uparrow \\ 0 \end{array} = \begin{array}{c} 0 \\ \downarrow \\ 0 \end{array} = 1 \quad (\text{cyclic}) \\ = T_{\text{short}} \\ \text{---} : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \end{array} \right. \rightarrow \text{generates singlet!}$$

Optimization - Exact Construction

❖ Example1: Quantum Dimer Liquid

► Generalization

Add:  = 1  = -1 (cyclic)

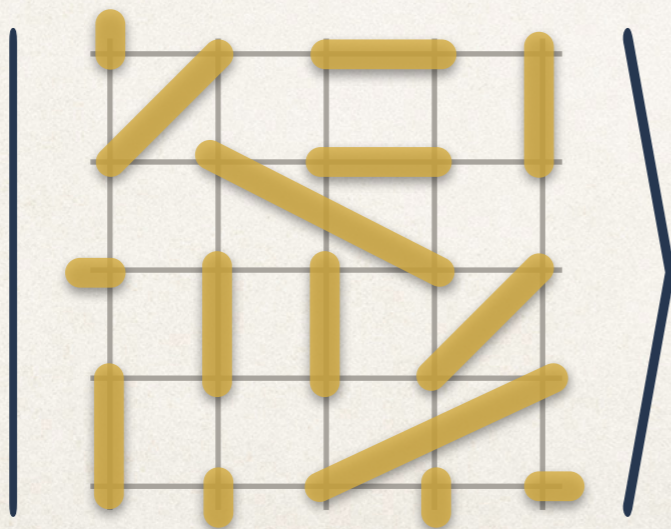
Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

▶ Generalization

Add: $\begin{array}{c} 0 \\ \uparrow \\ \text{---} \circ \text{---} \\ \downarrow \\ \uparrow(\downarrow) \end{array} = 1$ $\begin{array}{c} 0 \\ \downarrow \\ \text{---} \circ \text{---} \\ \uparrow \\ \uparrow(\downarrow) \end{array} = -1$ (cyclic)

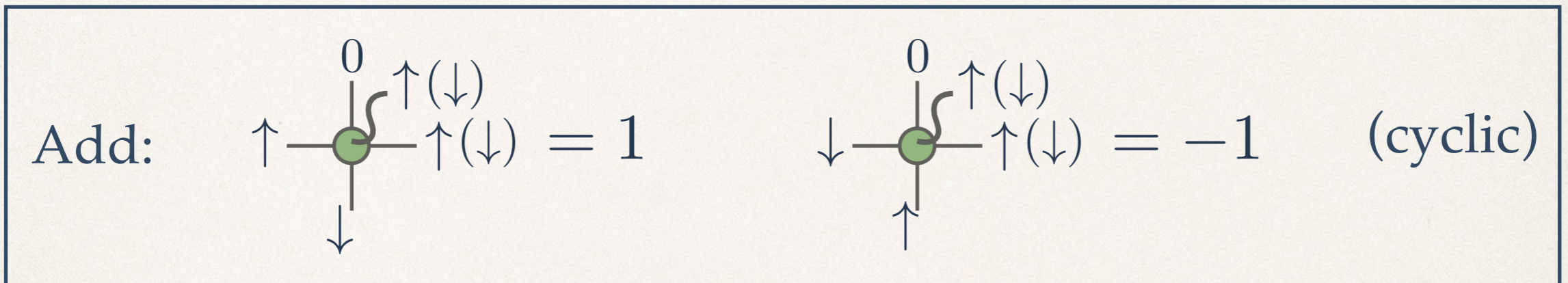
▶ Long-range RVB appears!



Optimization - Exact Construction

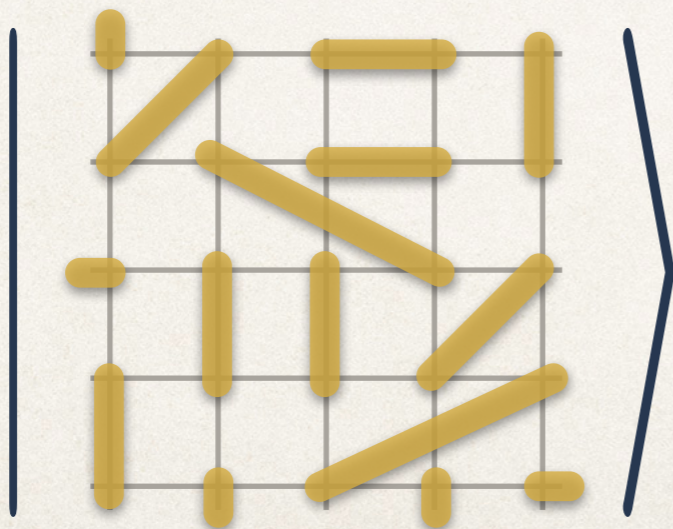
❖ Example 1: Quantum Dimer Liquid

▶ Generalization



$$= T_{\text{long}}$$

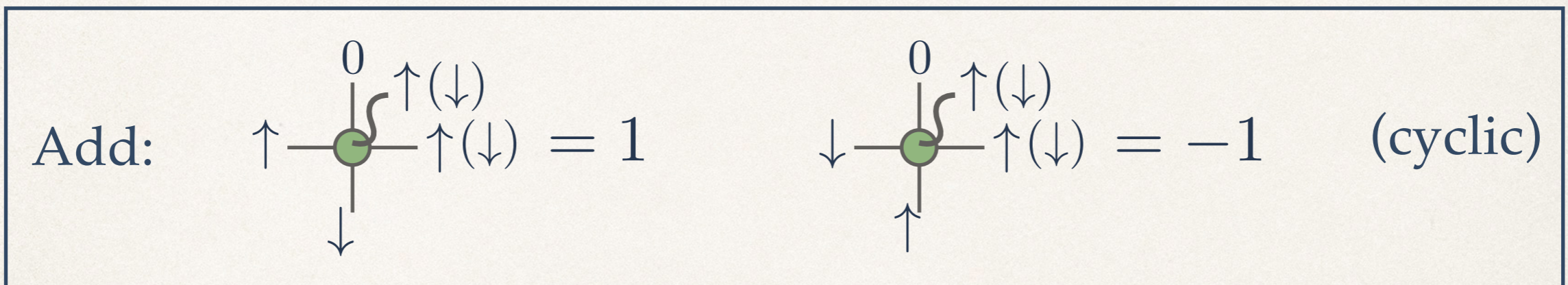
▶ Long-range RVB appears!



Optimization - Exact Construction

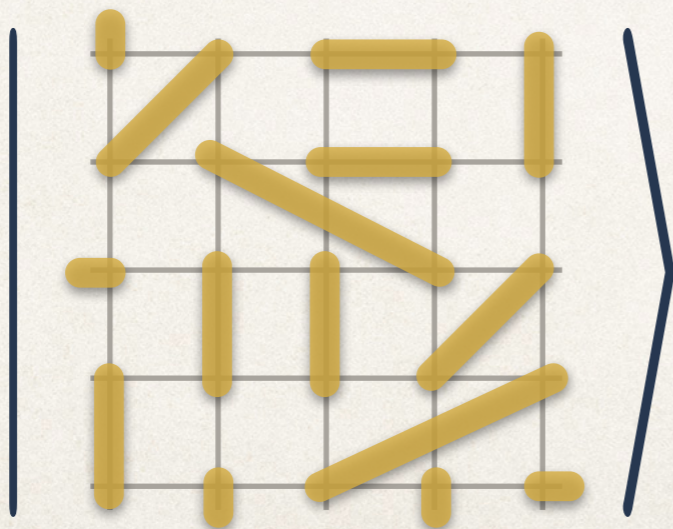
❖ Example 1: Quantum Dimer Liquid

▶ Generalization



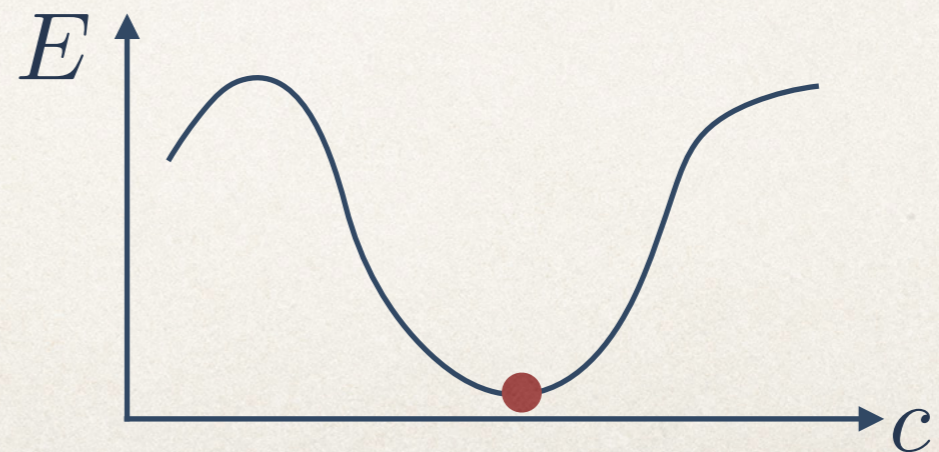
$$= T_{\text{long}}$$

▶ Long-range RVB appears!



▶ Variational ansatz

$$T(c) = T_{\text{short}} + c T_{\text{long}}$$



Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

► Generalization

Add:

$$\uparrow \text{---} \text{---} \uparrow(\downarrow) = 1 \qquad \downarrow \text{---} \text{---} \uparrow(\downarrow) = -1 \quad (\text{cyclic})$$

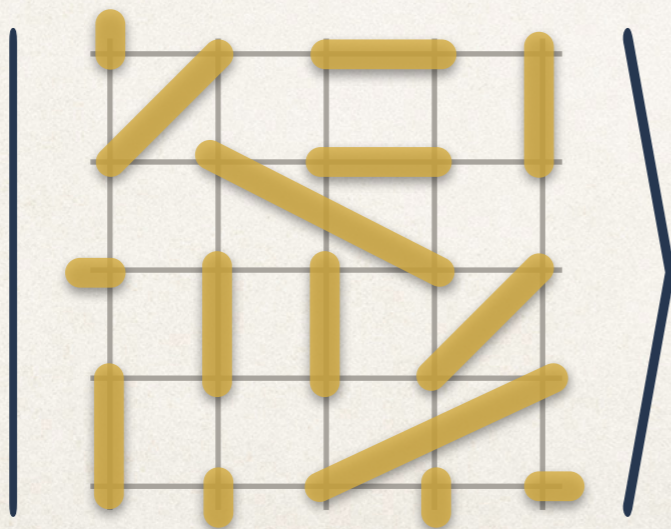
Optimization - Exact Construction

❖ Example 1: Quantum Dimer Liquid

▶ Generalization

Add: $\begin{array}{c} 0 \\ \uparrow \\ \text{---} \circ \text{---} \\ \downarrow \\ \uparrow(\downarrow) \end{array} = 1$ $\begin{array}{c} 0 \\ \downarrow \\ \text{---} \circ \text{---} \\ \uparrow \\ \uparrow(\downarrow) \end{array} = -1$ (cyclic)

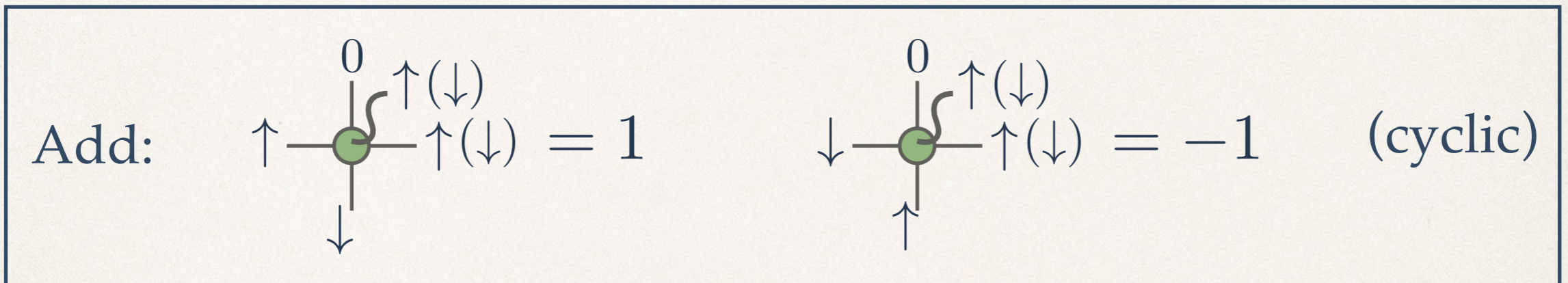
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Optimization - Exact Construction

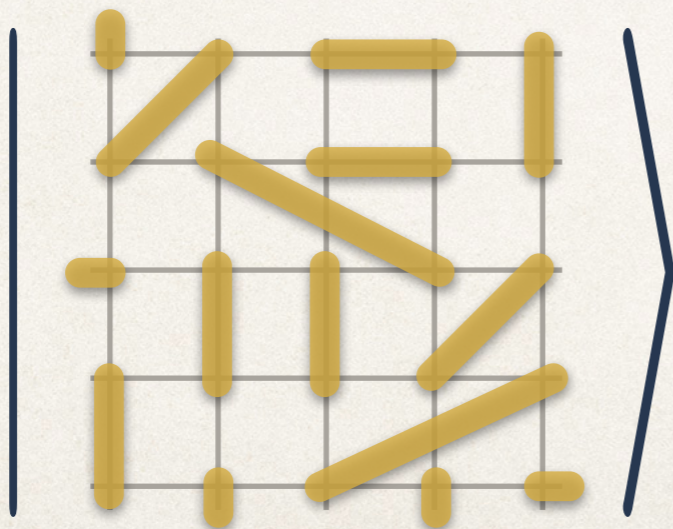
❖ Example 1: Quantum Dimer Liquid

▶ Generalization



$$= T_{\text{long}}$$

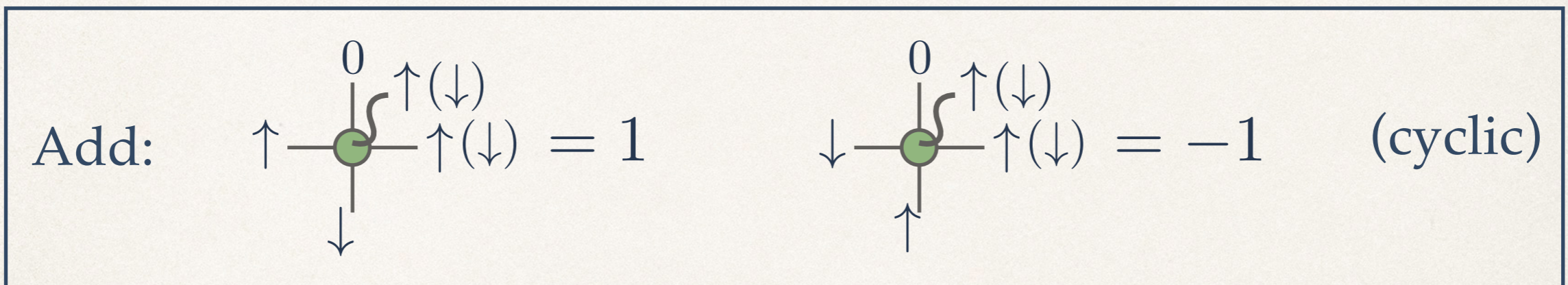
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Optimization - Exact Construction

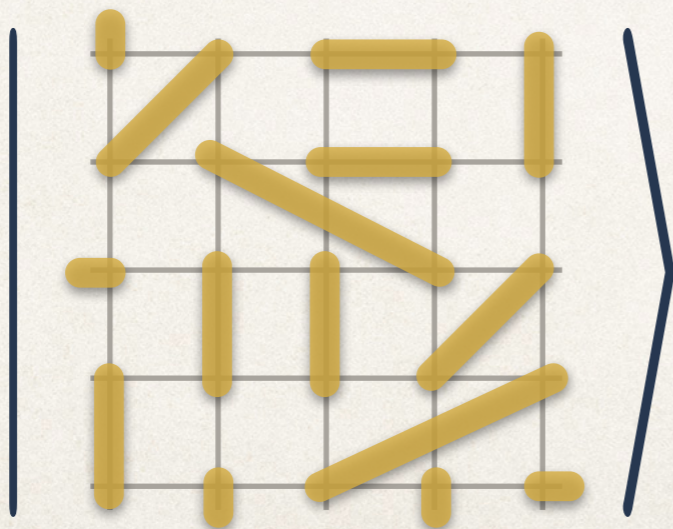
❖ Example 1: Quantum Dimer Liquid

▶ Generalization



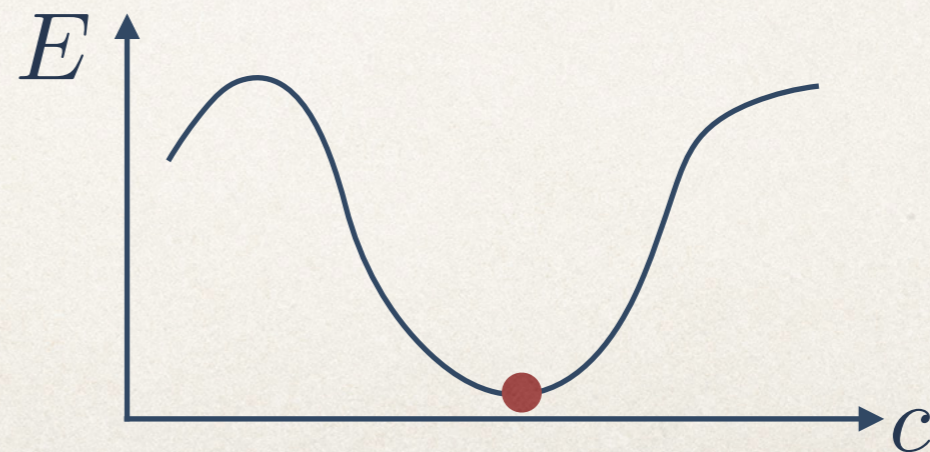
$$= T_{\text{long}}$$

▶ Long-range RVB appears!



▶ Variational ansatz

$$T(c) = T_{\text{short}} + c T_{\text{long}}$$



Outline

— Introduction

- Overview on Tensor Network Applications
- Frustrated Quantum Spin Systems

— Algorithms for Optimization

- Exact Constructions
- Numerical Optimizations

— Algorithms for Measurement

- Corner Transfer Matrix Renormalization Group

Numerical Optimization

- ❖ Imaginary Time Evolution
 - ▶ Generalization