Chiral kinetic theory in electromagnetic fields

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1. *Equal-time kinetic theory*
2. *Classical limit: Boltzmann equation*
3. *Quantum correction: off-shell oscillations*

Classical and quantum kinetic theory

- *Classical kinetic theory*

$$\left(p^\mu\partial_\mu + \partial^\mu m^2 \partial_\mu^p\right) f(x,p) = C \ \ [transport]$$
$$+ \ (p^2 - m^2) f(x,p) = 0 \ \ [on - shell\ constraint]$$

Many classical transport codes in HIC like AMPT, BAMPS and UrQMD

- *Quantum kinetic theory*

from $\psi(x)\psi^+(x)$ in quantum mechanics to $W(x,p)$ in quantum kinetic theory

$$W(x,p) = \int d^4y e^{ipy} \left\langle \hat{\psi}(x+\frac{y}{2}) e^{iq\int_{-\frac{1}{2}}^{\frac{1}{2}} ds \hat{A}(x+sy)y} \hat{\bar{\psi}}(x-\frac{y}{2}) \right\rangle$$

For QED: D.Vasak, M.Gyulassy and H.Elze, Ann.Phys.173, 462(1987)
For QGP: H.Elze and U.Heinz, Phys. Rept. 183, 81(1989)

Equal-time kinetic theory

- Difficulty in covariant kinetic theory

$$W(x,p)=\int d^4y e^{ipy}\left\langle \hat{\psi}(x+\frac{y}{2})e^{iq\int_{-\frac{1}{2}}^{\frac{1}{2}}ds\hat{A}(x+sy)y}\hat{\bar{\psi}}(x-\frac{y}{2})\right\rangle$$

Advantage: Covariant kinetic equations obtained from Dirac and Maxwell equations.
Disadvantage: $W(x,p)$ at a fixed time x_0 is related to $\hat{\psi}$ and $\hat{A}$ at all times ($\int dy_0$, $\int ds$), but physics distributions are defined in equal-time space $W(\vec{x},\vec{p};t)$.
Conclusion: We should go to the equal-time formalism.

- Equal-time Wigner function

$$W(\vec{x},\vec{p};t)=\int d^3\vec{y}e^{-i\vec{p}\cdot\vec{y}}\left\langle \hat{\psi}(\vec{x}+\frac{\vec{y}}{2};t)e^{-iq\int_{-\frac{1}{2}}^{\frac{1}{2}}ds\hat{A}(\vec{x}+s\vec{y};t)\cdot\vec{y}}\hat{\psi}^{+}(\vec{x}-\frac{\vec{y}}{2};t)\right\rangle$$

$$=\int dp_0 W(x,p)\gamma_0$$

For QED: I.Bialynicki-Birula, P.Gornicki and J.Rafelski (BGR), PRD44, 1825(1991)

- However, $W(\vec{x},\vec{p};t)$ is not equivalent to $W(x,p)$!

One should consider all energy moments

$$W_n(\vec{x},\vec{p};t)=\int dp_0 p_0^n W(x,p)\gamma_0 \qquad (n=0,1,2,\dots\infty)\ .$$

- Question: how to construct a full equal-time kinetic theory?

PZ and U.Heinz, Ann.Phys.245, 311(1996), PRD53, 2096(1996), PRD57, 6525(1998)
S.Ochs and U.Heinz, Ann. Phys. 266, 351(1998)

We take NJL as an example to show the 3 steps to construct a full equal-time theory

$$\mathcal{L} = \bar{\psi}\left(i\gamma^{\mu}\mathcal{D}_{\mu} - m_0\right)\psi + G(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\vec{\tau}\psi)^2, \quad \mathcal{D}_{\mu} = \partial_{\mu} + iqA_{\mu}, \quad q = diag(q_u, q_d)$$

chiral condensate: $\sigma(x) = 2G\langle\bar{\psi}\psi\rangle$

1st step: Covariant kinetic theory

Quarks in external electromagnetic fields

$$W(x,p) = \int d^4y e^{ipy} e^{iq\int_{-\frac{1}{2}}^{\frac{1}{2}} ds A(x+sy)y} \left\langle \hat{\psi}(x+\frac{y}{2})\, \hat{\bar{\psi}}(x-\frac{y}{2}) \right\rangle$$

Dirac equation at mean field level

$$\left(i\gamma^\mu \mathcal{D}_\mu - m\right)\psi = 0, \qquad m = m_0 - \sigma$$

Covariant kinetic equations

$$\left(\gamma^\mu K_\mu - M\right)W(x,p) = -\frac{i\hbar}{2}\gamma^\mu u_\mu \frac{W(x,p)-W^{th}(x,p)}{\theta} \rightarrow 2\ equations\ (real\ \&\ emaginary\ parts)$$

with electromagnetic operator K_μ and chiral operator M:

$$K_\mu = \Pi_\mu + \frac{i\hbar}{2}D_\mu, \quad \Pi_\mu = p_\mu - iq\hbar\int_{-1/2}^{1/2} dss F_{\mu\nu}(x - i\hbar s\partial_p)\partial_p^\nu, \quad D_\mu = \partial_\mu - q\int_{-1/2}^{1/2} ds F_{\mu\nu}(x - i\hbar s\partial_p)\partial_p^\nu$$

$$M = M_1 + iM_2, \ \ M_1 = m_0 - \cos(\frac{\hbar}{2}\partial_x\partial_p)\sigma(x), \quad M_2 = \sin(\frac{\hbar}{2}\partial_x\partial_p)\sigma(x),$$

Covariant transport and constraint equations

$$W^+ = \gamma_0 W \gamma_0$$

$$\begin{cases} \left[\gamma^\mu\left(K_\mu + K_\mu^+\right) - (M + M^+)\right]W(x,p) = 0, & contains\ p_\mu,\ constraint\ equation \\ \left[\gamma^\mu\left(K_\mu - K_\mu^+\right) - (M - M^+)\right]W(x,p) = i\hbar\gamma^\mu u_\mu \frac{W - W_{th}}{\theta}, & contains\ \partial_\mu,\ transport\ equation \end{cases}$$

2^{nd} step: Equal-time hierarchy

- p_0-integrating the covariant equations:

$$\begin{cases} Transport\ equations\ for\ W(\vec{x},\vec{p};t) \\ Constraint\ equation\ for\ W_1(\vec{x},\vec{p};t) \end{cases}$$

p_0-integrating $p_0 \cdot$ (covariant equations):

$$\begin{cases} Transport\ equations\ for\ W_1(\vec{x},\vec{p};t) \\ Constraint\ equation\ for\ W_2(\vec{x},\vec{p};t) \end{cases}$$

……

- Spin decomposition:

$$W(x,p) = \frac{1}{4}\left[F + i\gamma^5 P + \gamma^\mu V_\mu + \gamma^5\gamma^\mu A_\mu + \frac{1}{2}\sigma^{\mu\nu}S_{\mu\nu}\right]$$

$$W(\vec{x},\vec{p};t) = \frac{1}{4}[f_0 + \gamma^5 f_1 - i\gamma^0\gamma^5 f_2 + \gamma^0 f_3 + \gamma^5\gamma^0\vec{\gamma}\cdot\vec{g}_0 + \gamma^0\vec{\gamma}\cdot\vec{g}_1 - i\vec{\gamma}\cdot\vec{g}_2 - \gamma^5\vec{\gamma}\cdot\vec{g}_3]$$

→ 16 transport and 16 constraint equations

- Chiral condensate is coupled to Wigner function

$$\sigma(x) = G\int d^3\vec{p} f_3(\vec{x},\vec{p};t)$$

3^{rd} step: Truncating the hierarchy

- Semi-classical expansion

$$W(x,p) = W^{(0)}(x,p) + \hbar W^{(1)}(x,p) + \cdots$$
$$W(\vec{x},\vec{p};t) = W^{(0)}(\vec{x},\vec{p};t) + \hbar W^{(1)}(\vec{x},\vec{p};t) + \cdots$$
$$K_\mu = K_\mu^{(0)} + \hbar K_\mu^{(1)} + \cdots$$
$$M = M^{(0)} + \hbar M^{(1)} + \cdots$$

- Constraint equations to the zeroth order in $\hbar$ (classical limit)

1) on-shell condition: $E_p^2 = m^2 + \vec{p}^2$

2) only 2 independent components: $f_0^{(0)}$ and $\vec{g}_0^{(0)}$

$$f_1^{(0)} = \frac{\vec{p}}{E_p}\cdot\vec{g}_0^{(0)}, \qquad f_2^{(0)} = 0, \qquad f_3^{(0)} = \frac{m}{E_p} f_0^{(0)}$$
$$\vec{g}_1^{(0)} = \frac{\vec{p}}{E_p} f_0^{(0)}, \qquad \vec{g}_2^{(0)} = \frac{\vec{p}\times\vec{g}_0^{(0)}}{m}, \qquad \vec{g}_3^{(0)} = \frac{E_p^2\vec{g}_0^{(0)} - \left(\vec{p}\cdot\vec{g}_0^{(0)}\right)\vec{p}}{E_p m}$$

3) gap equation:

$$(m_0 - \sigma(x))\left(1 + 2G\int\frac{d^3\vec{p}}{(2\pi)^3}\frac{f_0^{(0)}(\vec{x},\vec{p};t)}{E_p}\right) - m_0 = 0$$

- Physics of the components

$f_0^{(0)}$: *number density*, $\vec{g}_0^{(0)}$: *spin density*

$f_1^{(0)}$: *helicity density*, $f_2^{(0)}$: *topologic charge density*, $f_3^{(0)}$: *mass density*

$\vec{g}_1^{(0)}$: *number current*, $\vec{g}_3^{(0)}$: *magnetic moment*

Classical transport equations

- Transport equations to the first order in $\hbar$

Vlasov equation for number density $f_0^{(0)}$:

$$\left(D_t+\frac{\vec{p}}{E_p}\cdot\vec{D}\right)f_0^{(0)}-\frac{1}{2E_p}\left(\vec{\nabla}m^2\cdot\vec{\nabla}_p\right)f_0^{(0)}=0$$

Bargmann-Michel-Telegdi equation for spin density $\vec{g}_0^{(0)}$

V.Bargmann, L.Michel and V.Telegdi, Phys. Rev. Lett. **2**, 435(1959).

$$\left(D_t+\frac{\vec{p}}{E_p}\cdot\vec{D}\right)\vec{g}_0^{(0)}-\frac{1}{2E_p}\left(\vec{\nabla}m^2\cdot\vec{\nabla}_p\right)\vec{g}_0^{(0)}=\frac{q}{E_p^2}\left[\vec{p}\times\left(\vec{E}\times\vec{g}_0^{(0)}\right)-E_p\vec{B}\times\vec{g}_0^{(0)}\right]$$

$$D_t=\partial_t+q\vec{E}\cdot\vec{\nabla}_p,\qquad \vec{D}=\vec{\nabla}+q\vec{B}\times\vec{\nabla}_p$$

the effective collision terms on the right hand side is due to the quark spin interaction with the electromagnetic fields.

- A homogeneous solution

$$f_0^{(0)}=\frac{\left(\vec{B}\times\vec{p}\right)^2}{m^2B^2}$$

$$\vec{g}_0^{(0)}=\frac{q}{m^2}\left(\vec{B}+\frac{\vec{p}}{E_p}\times\vec{E}\right)$$

Quantum off-shell effect

- Constraint equations to the first order in $\hbar$

$$\int dp_0 p_0 F^{(1)} - \frac{1}{2}\vec{D}\cdot\vec{g}_2^{(0)} = m f_0^{(1)} \qquad \int dp_0 p_0 P^{(1)} - \frac{1}{2}\vec{D}\cdot\vec{g}_3^{(0)} = 0$$

$$\int dp_0 p_0 V_0^{(1)} - \vec{p}\cdot\vec{g}_1^{(1)} = m f_3^{(1)} \qquad \int dp_0 p_0 \vec{V}^{(1)} - \frac{1}{2}\vec{D}\times\vec{g}_0^{(0)} - \vec{p} f_0^{(1)} = 0$$

$$\int dp_0 p_0 A_0^{(1)} - \vec{p}\cdot\vec{g}_0^{(1)} = 0 \qquad \int dp_0 p_0 \vec{A}^{(1)} - \frac{1}{2}\vec{D}\times\vec{g}_1^{(0)} - \vec{p} f_1^{(1)} = m\vec{g}_3^{(1)}$$

$$\int dp_0 p_0 S_{0i}^{(1)}\vec{e}_i + \frac{1}{2}\vec{D} f_3^{(0)} + \vec{p}\times\vec{g}_3^{(1)} = 0 \qquad \int dp_0 p_0 S_{jk}^{(1)}\varepsilon^{jki}\vec{e}_i - \vec{p}\times\vec{g}_2^{(1)} = m\vec{g}_0^{(1)}$$

- Shell shift from $\delta(p_0 - E_p)$ to $\delta(p_0 - (E_p + \hbar\delta E_p))$?

There is no solution of a definite shell shift δE_p !

- From on-shell to off-shell

$$\delta(p_0 - E_p) \;\rightarrow\; \delta(p_0 - E_p) - \hbar\mathcal{A}(p)$$

$\mathcal{A}(p)$: *a continuoys function of* p_0

The off-shell terms $\Delta E_p = \int dp_0 p_0 W^{(0)}(x,p)\mathcal{A}(p)$ is determined by the constraint equations:

$$\Delta E_p(f_0) = -\frac{\vec{B}\cdot\vec{g}_0^{(0)}}{2E_p}, \qquad \Delta E_p(f_1) = -\frac{\vec{B}\cdot\vec{p}}{2E_p^2} f_0^{(0)}, \qquad \Delta E_p(f_2) = -\frac{\vec{E}\cdot\vec{g}_0^{(0)}}{2m}$$

$$\Delta E_p(f_3) = \frac{\vec{p}\cdot\left(\vec{E}\times\vec{g}_0^{(0)}\right)}{2mE_p} - \frac{\vec{B}\cdot\vec{g}_0^{(0)}}{2m} + \frac{(\vec{B}\cdot\vec{p})\left(\vec{p}\cdot\vec{g}_0^{(0)}\right)}{2mE_p^2}, \qquad \Delta E_p(\vec{g}_0) = -\left(\frac{\vec{E}\times\vec{p}}{2E_p^2} - m\frac{\vec{B}}{2E_p}\right) f_0^{(0)}$$

$$\Delta E_p(\vec{g}_1) = -\frac{\vec{E}\times\vec{g}_0^{(0)}}{2E_p} - \frac{\vec{B}\left(\vec{p}\cdot\vec{g}_0^{(0)}\right)}{2E_p^2}, \qquad \Delta E_p(\vec{g}_2) = \frac{m\vec{E}}{2E_p^2} f_0^{(0)}, \qquad \Delta E_p(\vec{g}_3) = \frac{m\vec{B}}{2E_p^2} f_0^{(0)}$$

Quantum transport equations

- *Transport equations to the second order in $\hbar$:*

Vlasov equation for number density $f_0^{(1)}$:

$$\left(D_t + \frac{\vec{p}}{E_p}\cdot\vec{D}\right) f_0^{(1)} = \frac{\vec{E}}{2E_p^2}\cdot\left(\vec{D}\times\vec{g}_0^{(0)}\right) - \frac{\vec{B}}{2E_p^3}\cdot(\vec{p}\cdot\vec{D})\vec{g}_0^{(0)} + \frac{(\vec{B}\times\vec{p})\left(\vec{E}\times\vec{g}_0^{(0)}\right)}{E_p^4}$$

Bargmann-Michel-Telegdi equation for spin density $\vec{g}_0^{(1)}$:

$$\left(D_t + \frac{\vec{p}}{E_p}\cdot\vec{D}\right)\vec{g}_0^{(1)} = \frac{q}{E_p^2}\left[\vec{p}\times\left(\vec{E}\times\vec{g}_0^{(1)}\right) - E_p\vec{B}\times\vec{g}_0^{(1)}\right] - \left(\frac{\vec{B}}{2E_p^3} + \frac{\vec{E}\times\vec{p}}{2E_p^4}\right)\vec{p}\cdot\vec{D}f_0^{(0)}$$
$$-\left(\frac{(\vec{p}\cdot\vec{E})(\vec{E}\times\vec{p})}{E_p^5} + \frac{\vec{p}\times(\vec{B}\times\vec{E})}{2E_p^4}\right) f_0^{(0)}$$

Application 1： Chiral kinetic equation (in chiral limit)

To the zeroth order in $\hbar$ (classical limit):

classical constraint equations $\begin{cases} p^\mu J_\mu^{\chi(0)} = 0 \\ p_\mu J_\nu^{\chi(0)} - p_\nu J_\mu^{\chi(0)} = 0 \end{cases}$

classical transport equation (Vlasov equation): $p^\mu \left(\partial_\mu - QF_{\mu\nu}\partial_p^\nu\right) f_\chi^{(0)} = 0$

To the first order in $\hbar$:

constraint equations $\begin{cases} p^\mu J_\mu^{\chi(1)} = 0 \\ \varepsilon^{\mu\nu\rho\sigma}\left(\partial_\rho - QF_{\rho\theta}\partial_p^\theta\right) J_\sigma^{\chi(0)} = -2\chi\left(p^\mu J_{\chi(1)}^\nu - p^\nu J_{\chi(1)}^\mu\right) \end{cases}$

general transport equation for $f_\chi = f_\chi^{(0)} + \hbar f_\chi^{(1)}$:

$$
\begin{aligned}
0 &= \delta\left(p^2 - \hbar\frac{\chi Q}{p\cdot n} p_\lambda \widetilde{F}^{\lambda\nu} n_\nu\right) \\
&\times \left\{ p\cdot\nabla + \hbar\frac{\chi}{2\,(p\cdot n)^2}\left[(\partial_\mu n_\sigma)p^\sigma - QF_{\mu\alpha}n^\alpha\right]\epsilon^{\mu\nu\lambda\rho} n_\nu p_\lambda \nabla_\rho \right. \\
&\left. -\hbar\frac{\chi}{2p\cdot n}\epsilon^{\mu\nu\lambda\rho}\left(\partial_\mu n_\nu\right) p_\lambda \nabla_\rho + \hbar\frac{\chi Q}{2p\cdot n} p_\lambda \left(\partial_\sigma \widetilde{F}^{\lambda\nu}\right) n_\nu \partial_p^\sigma \right\} f_\chi
\end{aligned}
$$

1) Complete solution to $\hbar^1$
2) General non-equilibrium distribution f_χ
3) General electromagnetic fields
4) Frame dependence n_μ
5) on-shell but shifted shell

A.Huang, S.Shi, Y.Jiang, J.Liao and PZ, PRD98, 036010(2018)

M.Stephanov and Y.Yin, PRL109, 162001 (2012);
D.Son and N.Yamamoto. PRD, 87, 85016(2013);
J.Chen, S.Pu, Q.Wang and X.Wang, PRL110, 262301(2013);
Y.Hidaka, S.Pu and D.Yang. PRD95, 091901(2017);
Wu, Hou, Ren, PRD 96 (2017)096015; and ……

Application 2: Mass correction to CKE

Introduce chiral components

$$f_\chi = f_0 + \chi f_1, \qquad \chi = \pm$$

For small quark mass, constant magnetic field, and to the first order in $\hbar$,

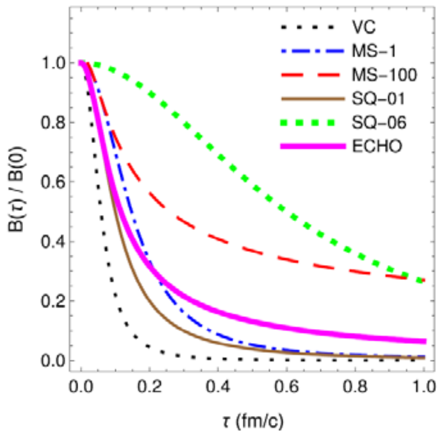
$$\partial_t f_\chi + \dot{\vec{x}} \cdot \vec{\nabla} f_\chi + \dot{\vec{p}} \cdot \vec{\nabla}_p f_\chi = \frac{\hbar m}{\sqrt{G}} A \left[\vec{g}_0^{(0)}\right]$$

Conclusion:
mass correction to CME is linear in $\hbar$ and therefore should be small.

Z.Wang and PZ, PRD100, 014015(2019)

Application 3: Chiral Magnetic effect in HIC

Strong electromagnetic fields before thermalization in heavy ion collisions
→ possible non-equilibrium CME

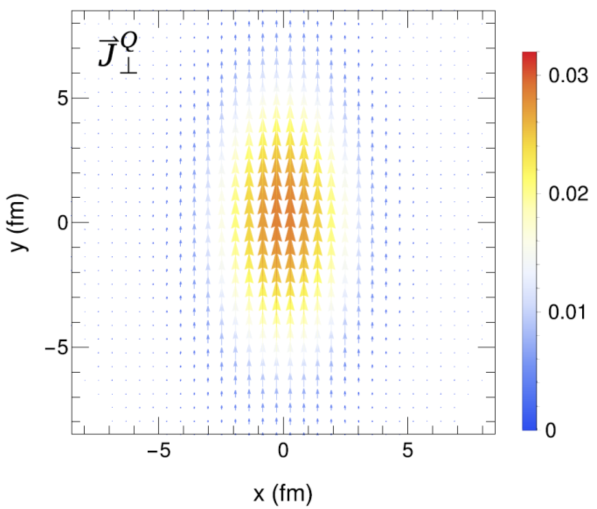


Suppose $\vec{E}=0$ and $\vec{B}=B(t)\vec{e}_y$,

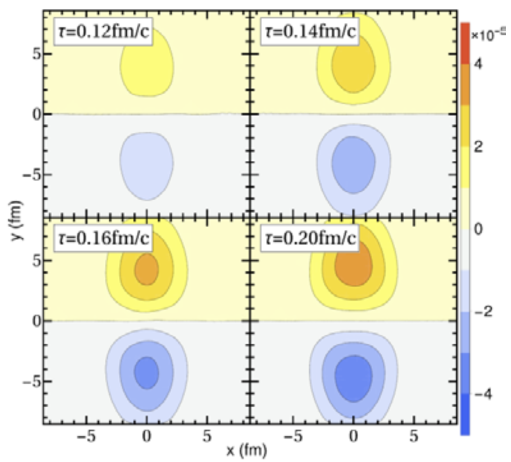
$$\begin{cases} \left(\partial_t+\dot{\vec{x}}\cdot\vec{\nabla}+\dot{\vec{p}}\cdot\vec{\nabla}_p\right)f_\chi=-\frac{f_\chi-f_{\chi\,th}}{\tau_R} \\ \dot{\vec{x}}=\frac{1}{1+Q\vec{B}\cdot\vec{b}_\chi}\frac{\vec{p}}{|\vec{p}|}\left(1+2Q\vec{B}\cdot\vec{b}_\chi\right) \\ \dot{\vec{p}}=\frac{1}{1+Q\vec{B}\cdot\vec{b}_\chi}Q\frac{\vec{p}}{|\vec{p}|}\times\vec{B} \end{cases}$$

+ initial distribution + initial imbalance ($1\pm\lambda$ for $\chi=\pm$) + B(t)

Transverse charge current



Net charge density

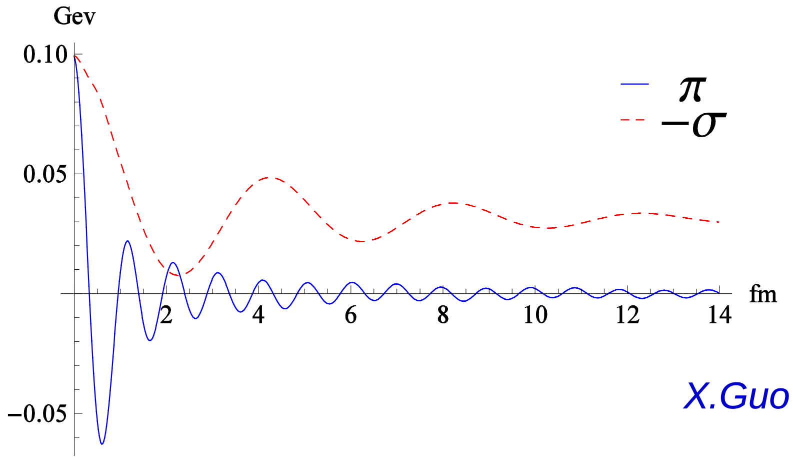


A.Huang, Y.Jiang, S.Shi, J.Liao and PZ, PLB777, 177(2018)

Application 4: Off-shell oscillations

*The solution of the **full** transport equations for a homogeneous system:*

$$\partial_t f_0 = 0$$
$$\partial_t f_1 + 2m f_2 - 2\pi f_3 = 0$$
$$\partial_t f_2 + 2\vec{p}\cdot\vec{g}_3 - 2m f_1 = 0$$
$$\partial_t f_3 - 2\vec{p}\cdot\vec{g}_2 + 2\pi f_1 = 0$$
$$\partial_t \vec{g}_0 = 0$$
$$\partial_t \vec{g}_1 + 2m\vec{g}_2 - 2\pi\vec{g}_3 = 0$$
$$\partial_t \vec{g}_2 + 2\vec{p} f_3 - 2m\vec{g}_1 = 0$$
$$\partial_t \vec{g}_3 - 2\vec{p} f_2 + 2\pi\vec{g}_1 = 0$$



X.Guo and PZ, PRD98, 016007(2018)

Conclusion: Off-shell effect → multi-component coupling → memory effect → strong oscillations.

Summary

1) *To have physics distributions, we should go from covariant to equal-time formalism. The question is how to set up a complete equal-time theory. In general case, the full set* $\left\{W_n(\vec{x},\vec{p};t) = \int dp_0 p_0^n W(x,p),\ \ n = 0,1,2,\ldots\infty\right\}$ *is equivalent to* $W(x,p)$*. Only in on-shell case,* $W(\vec{x},\vec{p};t)$ *itself is equivalent to* $W(x,p)$*.*

2) *In a strong electromagnetic field, spin is the first order quantum correction to the classical Boltzmann transport, it leads to CME, CVE and other chiral anomaly effects.*

3) *For off-shell particles, the multi-component coupling leads to a strong memory effect and in turn a strong oscillation.*