Transport properties of magnetized neutron stars

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QCS2019, Busan

I. Introduction

- Transport properties should be an important subject for thermal evolution of compact stars.

A.Y. Potekhin et al., Space Sci. Rev. 191 (2015) 239.
"Neutron Stars-Cooling and Transport,"

- Transport properties have been recently discussed in the context of Dirac materials in condensed-matter physics.

V. Konye and M. Ogata, PRB98, 195420 (2018).
"Magnetoresistance of a three dim Dirac gas"

- We have studied the inhomogeneous chiral phase (iCP) in quark matter.

  ⟶ H. Abuki's talk yesterday

- We have noticed that electromagnetic response of iCP leads to interesting consequences, such as spontaneous magnetization, assistance of axial anomaly for iCP, anomalous Hall effect, etc.

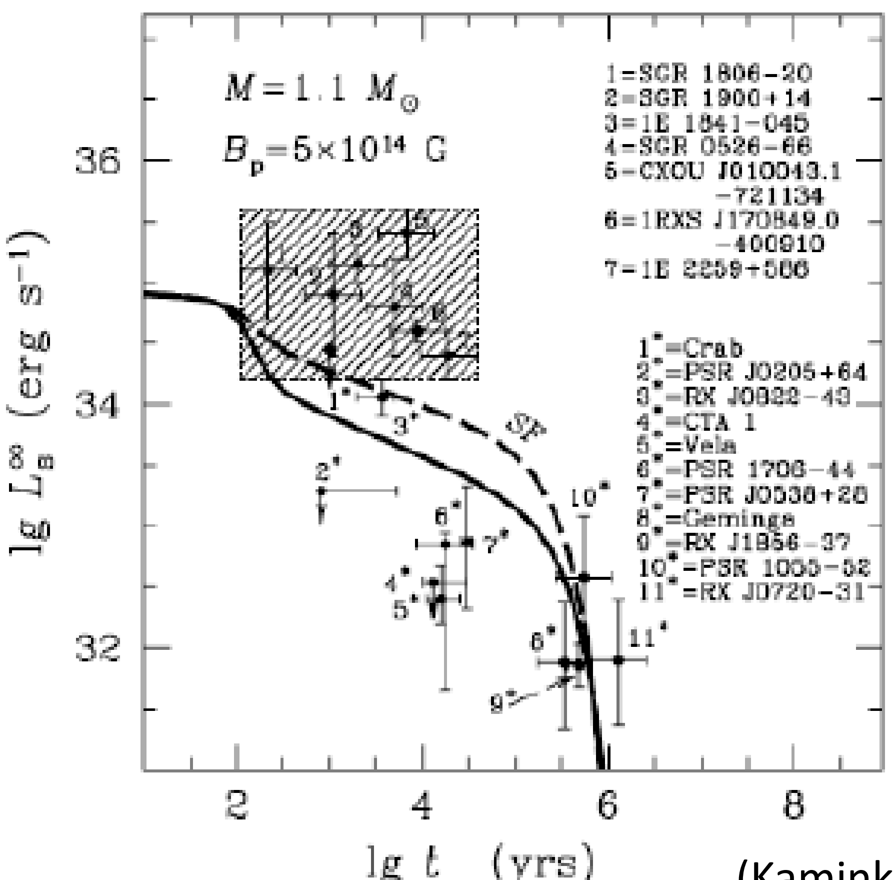
As a review paper, T.T. , JPS Conf.Proc. 20, 011008 (2018) and references therein.

- In particular, magnetic field or the Landau level plays crucial roles in these contexts.
- On these backgrounds we here consider some theoretical aspects of transport properties inside compact stars.

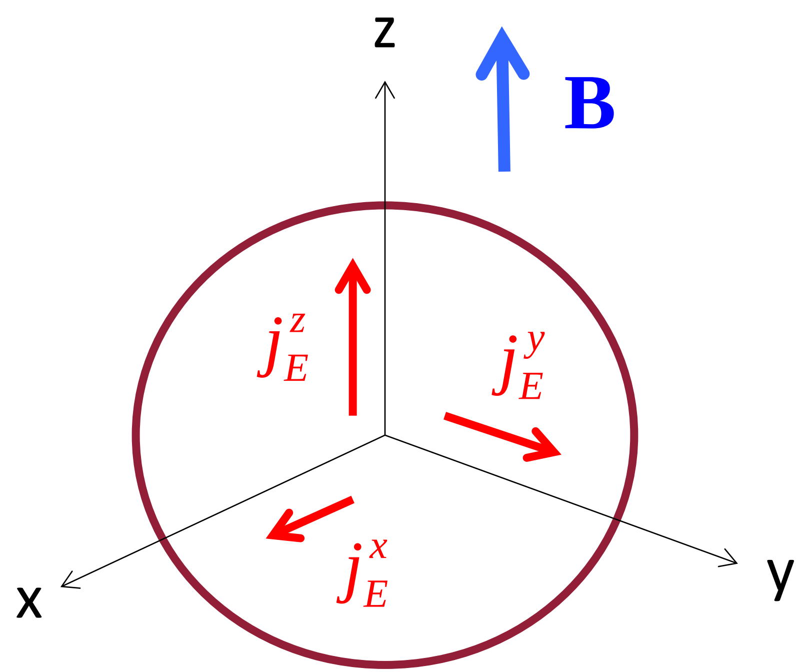
Thermal evolution of magnetars

- Neutrino luminosity
- Thermal transport

in the strong magnetic field



(Kaminkar et al., 2009)



Anisotropic heat flow

- We discuss thermal transport properties in the strong magnetic field.
- (thermal) Hall effect is highlighted in this talk, which can be decomposed into classical and quantum parts.
- Anomalous Hall effect (AHE) may be possible in iCP, which enjoys similar features to the Weyl semimetals in condensed-matter physics.
- Some topological aspects can be seen there.

- We may learn some transport properties in compact stars from terrestrial experiments.

II Phenomenological transport equations

$$\mathbf{j} = L_{11}\left[\mathbf{E} - \frac{T}{e}\nabla\left(\frac{\mu}{T}\right)\right] + L_{12}\left[T\nabla\left(\frac{1}{T}\right) - \frac{1}{c^2}\nabla\phi_g\right] \; (\propto T^{i0}),$$

$$\mathbf{j}_E = L_{21}\left[\mathbf{E} - \frac{T}{e}\nabla\left(\frac{\mu}{T}\right)\right] + L_{22}\left[T\nabla\left(\frac{1}{T}\right) - \frac{1}{c^2}\nabla\phi_g\right] (\propto T^{0i}),$$

with $\mathbf{E} = -\nabla\phi.$

"gravitational" potential

(J.M. Luttinger, PR 135, A1505 (1964))

Then we can read

(Electric) conductivity:

$$\boldsymbol{\sigma} = L_{11},$$

$L_{\alpha\beta}^{ij} : \alpha, \beta = 1, 2; i, j = x, y, z$

Thermal conductivity:

$$\boldsymbol{\kappa} = T^{-1}\left(L_{22} - L_{21}L_{11}^{-1}L_{12}\right)$$

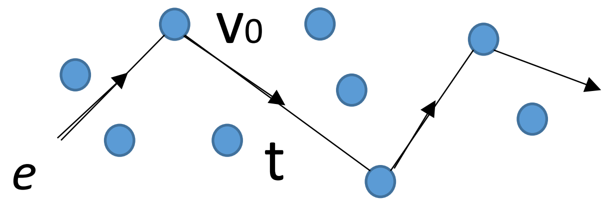
$$\simeq \frac{\pi^2}{3e^2}T\boldsymbol{\sigma} \quad \text{for low } T$$

(Wiedemann-Franz law)

Thus we need conductivity σ at T=0 to obtain thermal conductivity κ at low temperature.

Classical picture of conductivity

Drude model (Drude,1900: three years after J.J. Thomson's discovery of electrons):



$$\mathbf{v} = \mathbf{v}_0 - \frac{e\mathbf{E}t}{m^*} \rightarrow \langle \mathbf{v} \rangle = -\frac{e\mathbf{E}\tau}{m^*}$$

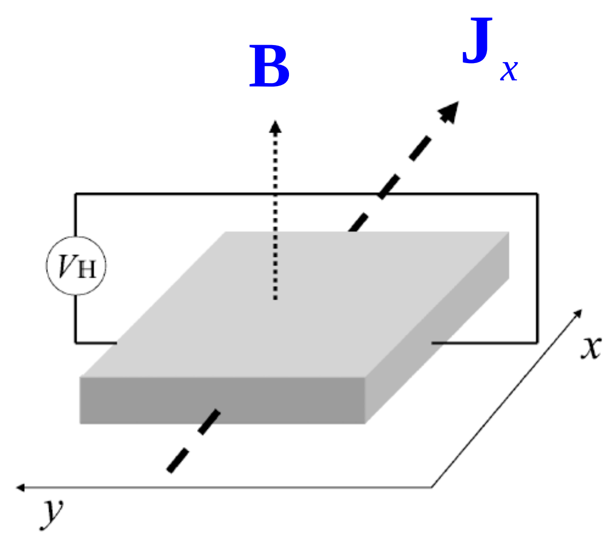
$$\langle \mathbf{j} \rangle = -ne\langle \mathbf{v} \rangle = \left(\frac{ne^2\tau}{m^*} \right) \mathbf{E},$$

n:number density

τ:relaxation time

$$\therefore \sigma_D = \frac{ne^2\tau}{m^*}$$

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} == \frac{ne^2}{m^*} \begin{pmatrix} \frac{\tau}{1+(\omega_c\tau)^2} & \frac{\omega_c\tau^2}{1+(\omega_c\tau)^2} & 0 \\ \frac{-\omega_c\tau^2}{1+(\omega_c\tau)^2} & \frac{\tau}{1+(\omega_c\tau)^2} & 0 \\ 0 & 0 & \tau \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$



$$\omega_c = eB / m^* c$$

Hall conductivity

It should be interesting to see:

(i) $B = 0(\omega_c = 0),\ \tau \to \infty$ (no dissipation) $\Rightarrow \sigma_{xy} = 0,$

$\sigma_{ii} \to \infty$ (unphysical)

(ii) $\tau \to \infty$ with B fixed $\Rightarrow \sigma_{xy} \propto 1/B$

σ_{xx} or $\sigma_{yy} \to 0, \sigma_{zz} \to \infty$ (unphysical)

Thus dissipative effects are indispensable
for the diagonal components in any case,
while it is not so effective for Hall conductivity.
This feature holds even in the fully quantum mechanical calculation.

III Anomalous Hall effect (AHE)

N. Nagaosa et al., RMP 82, 1539 (2010).
Di Xiao et al., RMP 82, 1959 (2010) .

Appearance of a large spontaneous Hall current
in response to ***E*** even in the absence of magnetic field.
So the Hall conductivity consists of two parts,

Magnetic resistivity $\rho \propto \sigma^{-1}$:

$$\rho_{xy} = R_0 B_z + R_s M_z$$

(AHE)

- We can expect AHE for magnetic (ferro or anti-ferro) materials, Weyl semimetals or iCP.
- We shall see that the intrinsic contribution to AHE can be regarded as an "unquantized" version of the quantum Hall effect.

→ First, we'd like to demonstrate AHE by considering iCP in the absence of magnetic field.

Inhomogeneous chiral phase and AHE

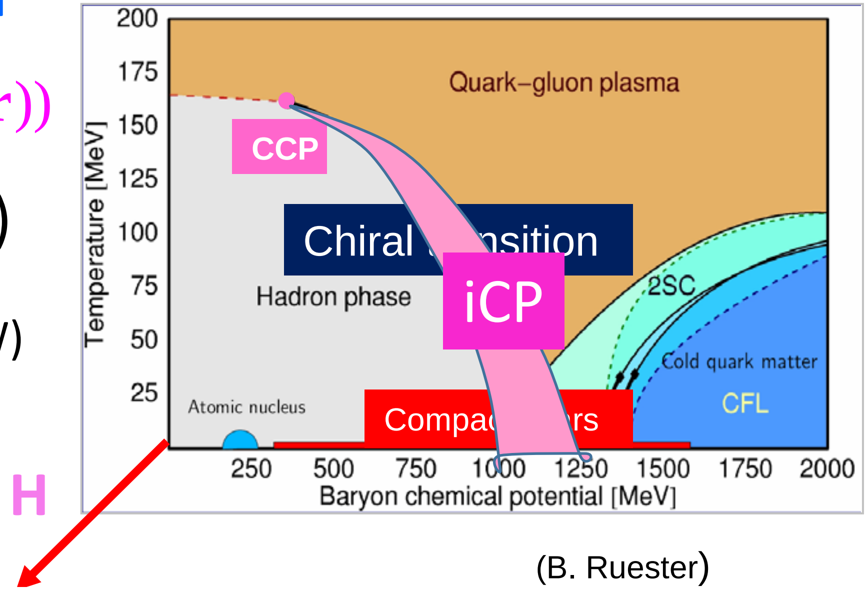
➡ **Generalized order parameter**

$$M \equiv \langle \bar{q}q \rangle + i\langle \bar{q}i\gamma_5\tau_3 q\rangle = \Delta(\mathbf{r})\exp(i\theta(\mathbf{r}))$$

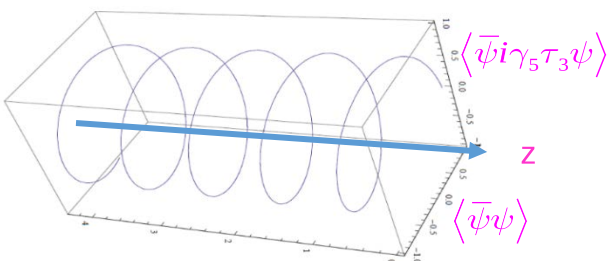
Inhomogeneous chiral phase (iCP)

ex) Dual Chiral Density Wave (DCDW)

(T. T. and E. Nakano, hep-ph/0408294.
E. Nakano and T. T., PR**D 71** (2005) 114006.)

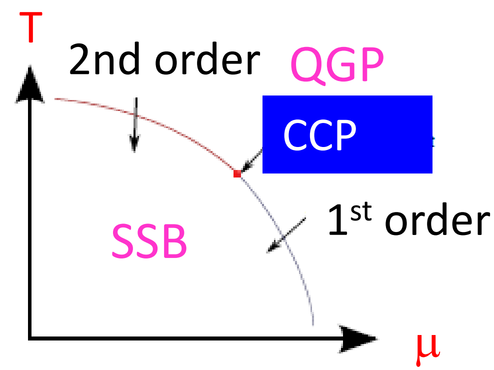


(B. Ruester)

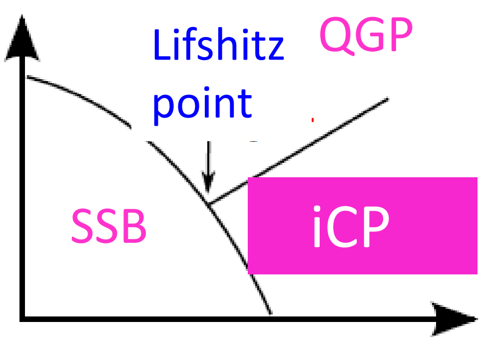


$$\langle \bar{q}q \rangle = \Delta\cos(qz)$$

$$\langle \bar{q}i\gamma_5\tau_3 q\rangle = \Delta\sin(qz)$$



Usual chiral transition



Inhomogeneous chiral transition

NJL model for simplicity.

$$\mathcal{L}_{NJL} = \bar{\psi}(i\partial\!\!\!/ - m_0)\psi + G\left[(\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\boldsymbol{\tau}\psi)^2\right], m_0 = 0.$$

$$\rightarrow \mathcal{L}_{MF} = \bar{\psi}\left(i\partial\!\!\!/ - \frac{1+\gamma_5\tau_3}{2}M(z) - \frac{1-\gamma_5\tau_3}{2}M^*(z)\right)\psi - \frac{M(z)^2}{4G},$$

$$M(z) = -2G\Delta e^{iqz} (\equiv Me^{iqz})$$

After the Weinberg transformation,

$$\psi_W = \exp[i\gamma_5\tau_3\mathbf{q}\cdot\mathbf{r}/2]\psi,$$

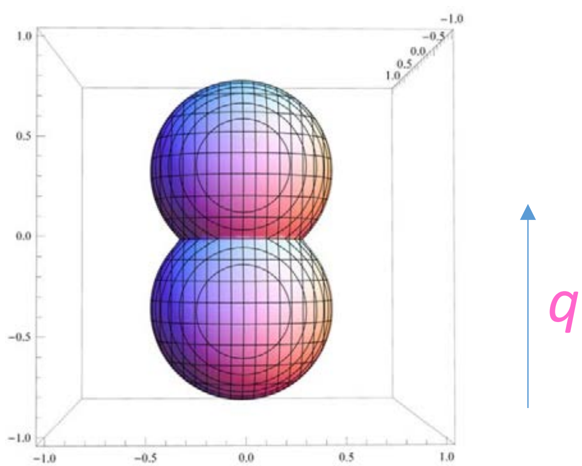
$$\mathcal{L}_{MF} = \bar{\psi}_W\left[i\partial\!\!\!/ - \underline{M} - 1/2\underline{\gamma_5\tau_3 q\!\!\!/}\right]\psi_W - G\Delta^2$$

Axial-vector MF

with $M = -2G\Delta$ and $q^\mu = (0, \mathbf{q})$.

single-particle energy:

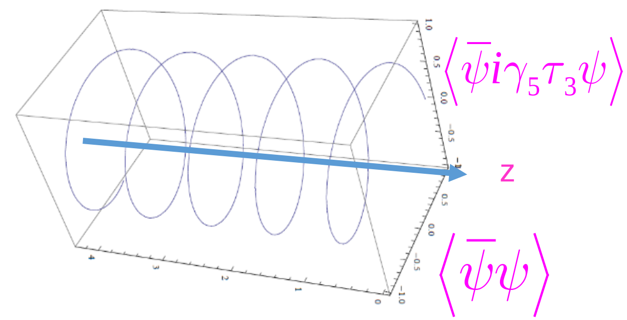
$$E_{s=\pm1}(p) = \sqrt{E_p^2 + |\mathbf{q}|^2/4 + s\sqrt{(\mathbf{p}\cdot\mathbf{q})^2 + M^2|\mathbf{q}|^2}}$$



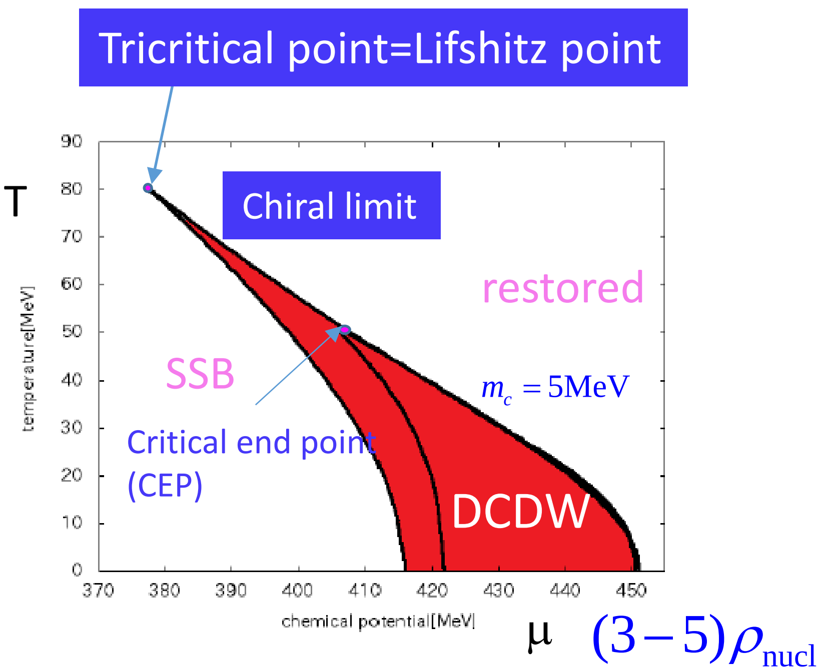
Deformation of the Fermi surface

Dual Chiral Density Wave (DCDW)

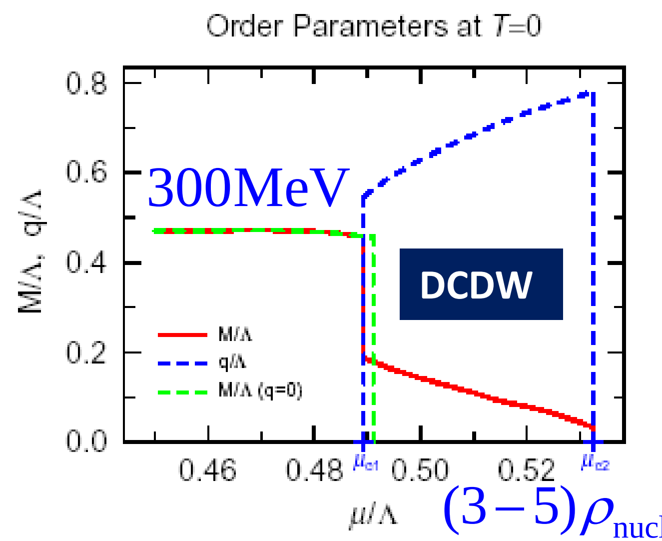
$$\langle \bar{q}q \rangle = \Delta \cos qz,$$
$$\langle \bar{q} i \gamma_5 \tau_3 q \rangle = \Delta \sin qz$$



ref. T.T. and E. Nakano, hep-ph/0408294
PRD71(2005)114006.



S. Karasawa and T.T., PRD92 (2015) 116004.



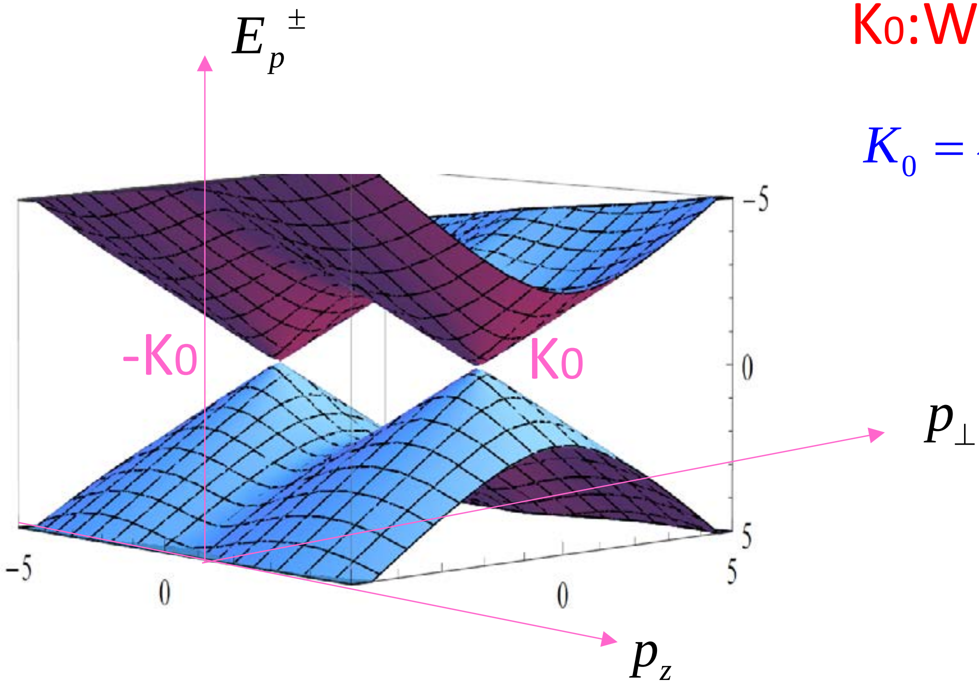
Resemblance to Weyl semimetals

(T.T., R. Yoshiike, K. Kashiwa, PLB 785(2018) 46.)

Dual Chiral Density Wave (DCDW)

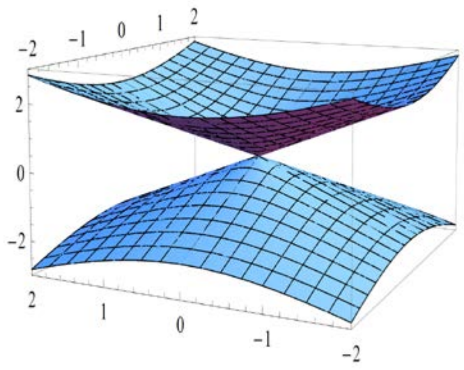
single-particle energy:

$$E_{s=\pm 1}(p) = \sqrt{E_p^2 + |\mathbf{q}|^2/4 + s\sqrt{(\mathbf{p}\cdot\mathbf{q})^2 + M^2|\mathbf{q}|^2}}$$



K_0:Weyl nodes (points) $(0,0,\pm K_0)$

$$K_0 = \sqrt{(q/2)^2 - M^2} \simeq q/2$$



Weyl cone

Weyl semimetal

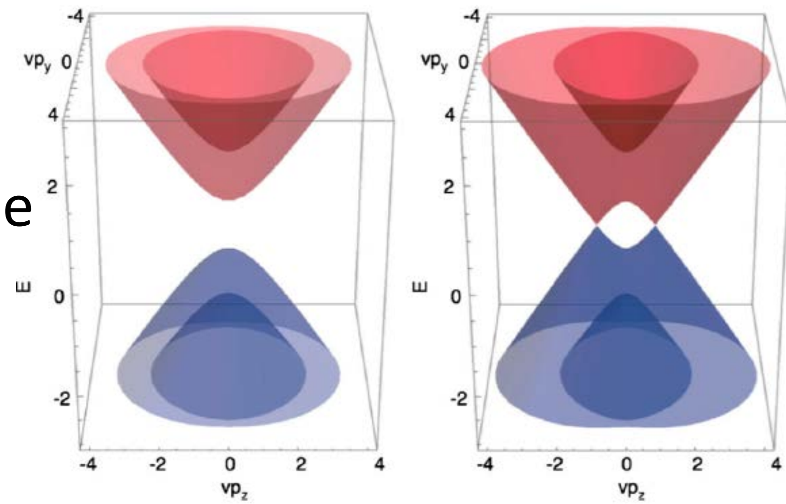
(E. Fradkin, *Field Theories of Condensed Matter Physics*, Chap. 16, Cambridge U. Press, 2013.
N.P. Armitage, E.J. Mele, A. Vishwanath, Rev.Mod.Phys. **90**, 015001 (2018).)

$$\mathscr{H}_{\rm TI}(\mathbf{k}) = \sum_{\mu=1}^{4} R_\mu(\mathbf{k})\alpha_\mu,$$

Doping of magnetic impurities may induce CPT odd term to evade the Kramer's theorem,

$$b\Sigma_3 = b\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix}\left(=\gamma^5\gamma^0\gamma^3\right)$$

$|b| < M\,(=R_4)$ $\qquad$ $|b| > M$



Weyl nodes

To summarize,
we know there have been many common features between nuclear physics and condensed-matter physics such as superfluidity or magnetism in compact stars.
This is a new encounter of nuclear physics with condensed-matter physics.

Correspondence:

DCDW	Weyl semimetal
Dynamical mass M	Spin-orbit coupling (SOC) strength M
Momentum k	Bloch momentum k
Wave vector q	Spin splitting due to magnetic impurities or magnetic field b

IV What can we learn from Weyl semimetals (WSM)?

- AHE is one of the main subjects to specify WSM experimentally.

N.P. Armitage, E.J. Mele, A. Vishwanath, Rev.Mod.Phys.**90**, 015001 (2018).

- It may affect the transport properties through modification of the Maxwell equation.

- Phenomenological implications should be also interesting for compact stars.

In the following we demonstrate AHE in DCDW phase

We can evaluate the Hall coefficient by way of the Kubo formula, considering a linear response to an electric field.

For translational invariant systems,

Hall conductivity σ_{xy} :

$$\sigma_{xy} = \frac{e^2}{\hbar}\int \frac{d^3k}{(2\pi)^3} B_z(\mathbf{k}) f(E_k)$$

$$= \int \frac{dk_z}{2\pi}\left[\frac{e^2}{\hbar}\int \frac{d^2k}{(2\pi)^2} B_z(\mathbf{k}) f(E_k)\right],$$

In terms of the Berry curvature, ***B(k)*** in the momentum space, which is a generalization of the TKNN formula for 2D quantum Hall systems.

D.J. Thouless, M. Kohmoto, M.P. Nightingale, M. den Nijs, PRL 49, 405 (1982)

Berry's curvature for DCDW:

(T.T., R. Yoshiike, K. Kashiwa, PLB 785(2018) 46.)

$$B_{s,z} = \frac{-1}{2E_s^3}\left(s\varepsilon_0 + \frac{q}{2}\right)\left[\simeq \frac{p'_z}{2|\mathbf{p}'|^3} \text{ for } s = \text{-1 around the Weyl node}\right],$$

$$p' = (p_x, p_y, p_z - K_0), \quad \varepsilon_0 = \left(p_z^2 + m^2\right)^{1/2}$$

$$\sigma_{xy}^{\text{Dirac}} = \sum_{s=\pm 1}\int \frac{dp_z}{2\pi}\int \frac{d^2\mathbf{p}_\perp}{(2\pi)^2}\frac{1}{2E_s^3}\left(s\varepsilon_0 + q/2\right)$$

$$= \lim_{\Lambda_z,\Lambda_\perp\to\infty}\frac{-1}{2\pi^2}\sum_{s=\pm 1}\int_0^{\Lambda_z} dp_z\left\{\frac{s\varepsilon_0 + q/2}{\left[\Lambda_\perp^2 + \left(s\varepsilon_0 + q/2\right)^2\right]^{1/2}} - \text{sign}\left(s\varepsilon_0 + q/2\right)\right\}$$

- First term diverges and proper regularization is needed.

P. Goswami and S. Tewari, PRB88, 245107 (2013)

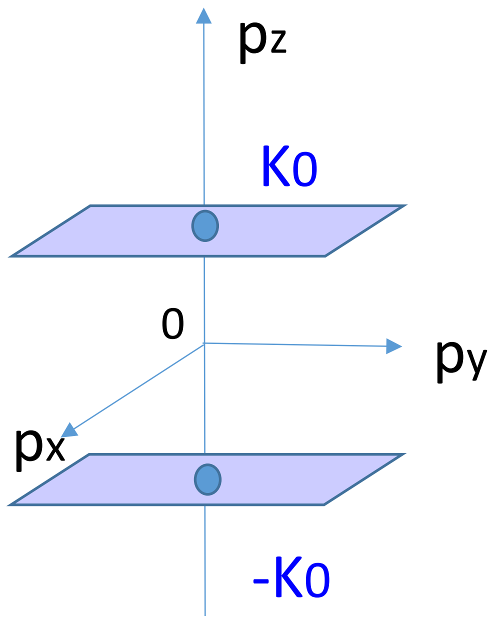
- Second term gives a topological number.

$$\sigma_{xy}^{\text{Dirac}} = \lim_{\Lambda_z \to \infty} \frac{e^2}{4\pi^2} \sum_s \int_0^{\Lambda_z} dp_z \,\text{sign}\left(s\sqrt{m^2 + p_z^2} + q/2 \right)$$

$$= \frac{e^2}{4\pi^2}(2K_0) = \left(\int \frac{dp_z}{2\pi} = \frac{2K_0}{2\pi} \right)(\nu = 1)\frac{e^2}{2\pi}$$

Chern number

"A stack of 2D quantum Hall systems"



	S=1	S=-1
	1	-1
	1	1
	1	-1

$$K_0 = \sqrt{(q/2)^2 - M^2}$$

Axion electrodynamics

(F.Wilczek, PRL 58, 1799 (1987).
E.J. Ferrer, V. de la Incera, PLB 769 (2017) 208.)

$\rightarrow$ Maxwell equation

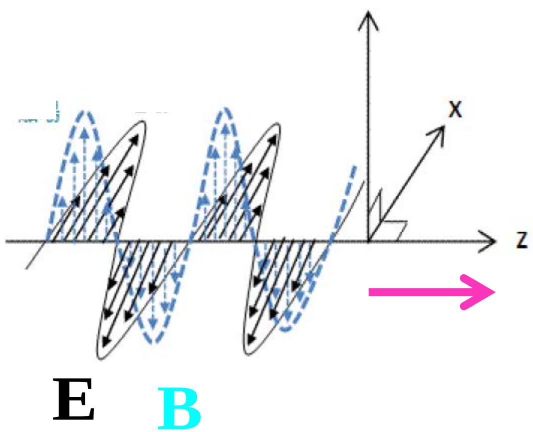
$$\nabla \mathbf{E} = j^0 + j^0_{\text{anom}},$$

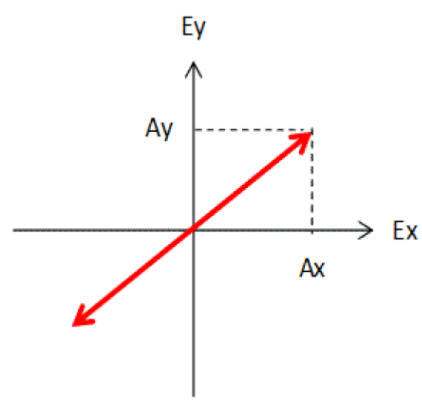
$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \mathbf{j} + \mathbf{j}_{\text{AHE}} + (\mathbf{j}_{\text{CME}} \parallel \mathbf{B}),$$

$$\nabla \mathbf{B} = 0, \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

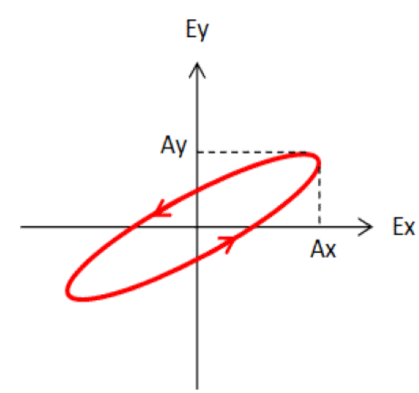
Some phenomenological implications may be possible, such as modification of transport properties inside compact stars.

(Example in Weyl semimetal) (A. Grushin, PRD 86, 045001 (2012).)





Linearly polarized



Elliptically polarized

V Hall conductivity in the presence of magnetic field

Streda formula

(P. Streda, J.Phys. C15(1972) L717).

$$\sigma_{ij} = \sigma_{ij}^{\mathrm{I}} + \sigma_{ij}^{\mathrm{II}},$$

- Classical Drude-Zener result (Dissipative),

$$\sigma_{xy}^{\mathrm{I}} = -\omega_c \tau \sigma_{xx},$$

(Effect of the Fermi surface)

where relaxation time $\tau = \hbar/2\Gamma\left(\Gamma = -\mathrm{Im}\,\Sigma\left(E_F\right)\right)$.

- Quantum effect (Non-dissipative),

$$\sigma_{xy}^{\mathrm{II}} = -\sigma_{yx}^{\mathrm{II}} = ec \left.\frac{\partial N(E)}{\partial B}\right|_{E=E_F},$$

where $N(E)$ is the number of states below the energy E defined by

$$N(E) = \int_{-\infty}^{E} \mathrm{Tr}\,\delta(\eta - H)\,d\eta = N(0) + \underline{N_F(E)}$$

Fermi sea

AHE(Dirac sea)

Three remarks about Hall conductivity:

(i) Quantised Hall effect in 2D Hall systems,

(P. Streda, J.Phys. C15(1972) L717).

$$\sigma_{xy}^{\mathrm{I}} = 0,\ \sigma_{xy}^{\mathrm{II}} = -\frac{e^2}{2\pi} n,\ n:$$ number of bands below the Fermi energy

(ii) If we can neglect the effect of impurities,

(V. Konye and M. Ogata, PRB98, 195420 (2018).)

$$\sigma_{xy}^{\mathrm{I}} = \frac{en_e}{B}, \sigma_{xy}^{\mathrm{II}} = 0.$$

(iii) AHE is given by $N(0)$ in $\sigma_{xy}^{\mathrm{II}}$ by way of spectral asymmetry.

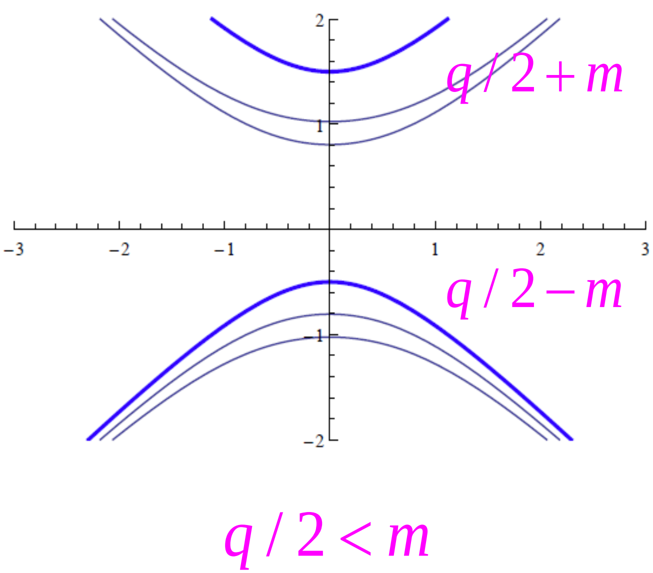
(T.T., H. Abuki, in preparation)

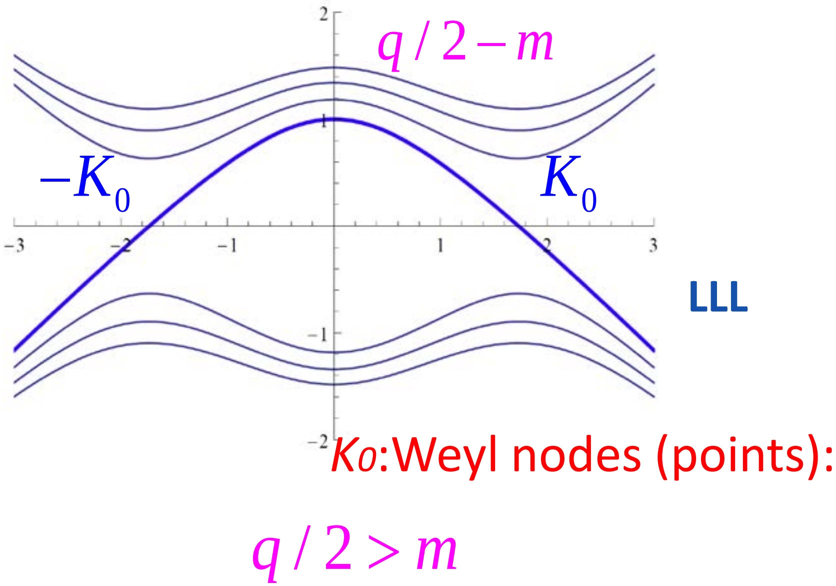
We'd like to demonstrate the property (iii) by considering DCDW in the presence of magnetic field.

VI Spectral asymmetry and AHE

$$E_{n,\zeta,\varepsilon}(p)=\varepsilon\sqrt{\left(\zeta\sqrt{m^2+p^2}+q/2\right)^2+2eBn},\quad n=1,2,3,\ldots$$

$$E_{n=0,\varepsilon}(p)=\varepsilon\sqrt{m^2+p^2}+q/2,\quad \text{(lowest Landau level(LLL))}$$





Spectral asymmetry (SA) for LLL

→ Number of “the vacuum” *N(0)*

c.f. Cheshire Cat in Prof. Young-Liang Ma ‘s talk yesterday

Anomalous number

Atiyah –Patodi-Singer, (1975,1976)
A.J. Niemi and G.W. Semenoff, Phys. Rept. 135, 99 (1986)

$$N(0) = -\frac{1}{2}\eta_H$$

$$= -\frac{1}{2}\int_{-\infty}^{+\infty} \mathrm{sign}(\lambda)\mathrm{Tr}\delta\left(\lambda - H\right)d\lambda,$$

where η_H is a topological quantity and called the APS η invariant measuring spectral asymmetry w.r.t zero energy.

$$\eta_H = \lim_{s\to 0+}\frac{eB}{2\pi}N_c\int\frac{dp}{2\pi}\sum_{\varepsilon}\mathrm{sign}\left(E_{\varepsilon,p}^{LLL}\right)\left|E_{\varepsilon,p}^{LLL}\right|^{-s}$$

$$= \begin{cases} -N_c\dfrac{eBq}{2\pi^2}, & m > \dfrac{q}{2} \\ -N_c\dfrac{eBq}{2\pi^2} + \dfrac{eB}{\pi^2}N_c\sqrt{\left(\dfrac{q}{2}\right)^2 - m^2}, & m < \dfrac{q}{2} \end{cases}.$$

T.T., K. Nishiyama and S. Karasawa, PLB 743, 66 (2015).

The anomalous Hall conductivity is *B*-independent to give through the Streda formula,

$$\sigma_{xy}^{\text{AHE}} = e\frac{\partial N(0)}{\partial B} = -\frac{e}{2}\frac{\partial \eta_H}{\partial B}$$

$$= \begin{cases} \dfrac{e^2 q}{4\pi^2} & m > \dfrac{q}{2} \\ -\dfrac{e^2}{2\pi^2}\sqrt{\left(\dfrac{q}{2}\right)^2 - m^2} + \dfrac{e^2 q}{4\pi^2} & m < \dfrac{q}{2} \quad (\to 0 \text{ as } m \to 0) \end{cases},$$

for DCDW. which is the same result to the previous one.

(T.T., R. Yoshiike, K. Kashiwa, PLB 785(2018) 46.)

- However, note the different manifestation of topological effect in two cals: one is by Berry's curvature(monopole) and the other by SA.
- Similar result can be obtained for WSM.

VII. Summary and concluding remarks

- We have discussed some theoretical aspects of transport properties of compact star matter from the viewpoint of topological material (Dirac material).
- Some common features are indicated with condensed-matter physics.
- In the presence of the magnetic field, we must carefully evaluate the Hall conductivity by way of the Kubo formula; it consists of "classical" and "quantum" components.

・ AHE is prescribed by the quantum component, and some topological features are concealed.

・ Many subjects are left for further considerations.

ex) We can easily generalize the CPT odd term,

$$\gamma_5 \boldsymbol{\gamma} \cdot \mathbf{b} \to \gamma_5 \gamma^\mu b_\mu .$$

Then we can incorporate the chiral magnetic effect, due to the effective μ_5 given by b_0,

$$\mathbf{j}_{\mathrm{CME}} = \frac{e^2}{2\pi^2} b_0 \mathbf{B}$$

(K.Fukushima et al., PRD 78, 074033 (2008).)

Even in such case, AHE is not modified by the presence of b_0 .