Quantum Computing Basic 1

Isaac Kim(PsiQuantum)

Breakdown

15 min	Classical vs. Quantum computing
15 min	Pauli group
15 min	Clifford group
15 min	Non-Clifford gate
15 min	Single-qubit gates
30 min	Universal gate set
15 min	Quantum circuits

Classical computing

1. Starts with a fixed initial state. (ex. $000000 \ldots 000000$)
2. Apply a set of gates. (ex. NAND, OR, NOT, …)
3. Read out the outcome. (ex. 101011110)

Quantum computing

1. Starts with a fixed initial state. (ex. $|000000 \ldots 000000>$)
2. Apply a set of gates. (ex. Pauli, Clifford, T, …)
3. Read out the outcome. (ex. Measure Z_1, Z_2, …)

Comparison

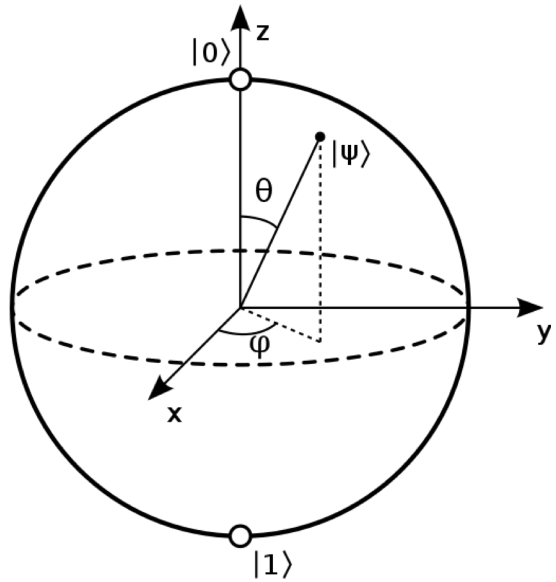
	Classical	Quantum
Gate	Not reversible(ex. OR gate)	Reversible
Readout	Does not change the state.	Changes the state.
Gate set	Discrete	Discrete
Elementary unit	Bit	Qubit

Bit

- 0 or 1
- n-bit string can represent 2^n distinct values.
- State space forms a vector space over $\{0,1\}$.
  - Basis: n-bit string

Qubit

- $\alpha|0\rangle + \beta|1\rangle$.
- n qubits can represent 2^n orthogonal states.
- State space forms a (normalized) vector space over $\mathbb{C}$.
  - Basis: $|x\rangle$, where x is an n-bit string.



$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

Classical Gates

- Input: Bits
- Output: Bits

Ex) AND gate

In 1	In 2	Out
0	0	0
0	1	0
1	0	0
1	1	1

Classical Gates

Ex) AND gate

In 1	In 2	Out
0	0	0
0	1	0
1	0	0
1	1	1

- Input: Bits
- Output: Bits

110010101

AND gate

10010101

Need to specify the input and output!

xyzw…

AND gate

AND(x,y)zw

Quantum Gates

- Input: Vector over $\mathbb{C}$.
- Output: Vector over $\mathbb{C}$.

Ex) Pauli X-gate

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

Quantum Gates

- Input: Vector over $\mathbb{C}$.
- Output: Vector over $\mathbb{C}$.

Ex) Pauli X-gate

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \beta \\ \alpha \end{pmatrix}$$

$$\alpha|0\boxed{0}1\rangle + \beta|1\boxed{0}1\rangle \rightarrow \alpha|0\boxed{1}1\rangle + \beta|1\boxed{1}1\rangle$$

Quantum Gates

- Quantum gate must be a physical process
- Except for measurement, all physical processes are unitary.

$$UU^{\dagger} = U^{\dagger}U = I$$

- This means that irreversible classical gates(ex. AND) are not allowed.

Quantum Computation generalizes classical computation

- Irreversible classical gates(ex. AND) can be implemented reversibly by adding a bit.

Toffoli gate

In 1	In 2	In 3	Out 1	Out 2	Out 3
0	0	0	0	0	0
0	1	0	0	1	0
1	0	0	1	0	0
1	1	0	1	1	1
0	0	1	0	0	1
0	1	1	0	1	1
1	0	1	1	0	1
1	1	1	1	1	0

Compute AND(1,2) and add to XOR to the third bit.

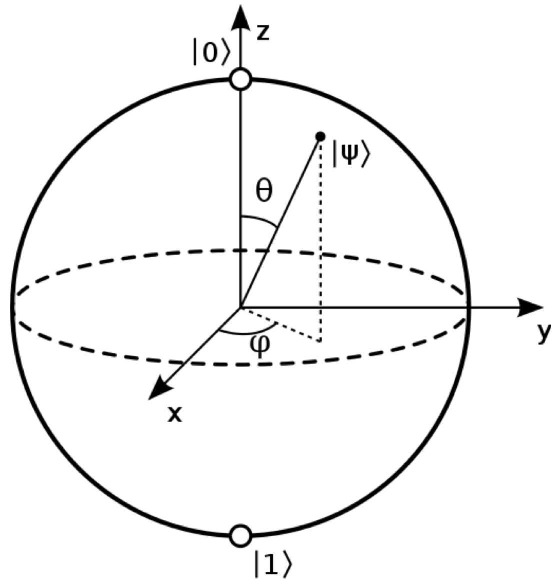
- Therefore, any classical computation using N logic operations can be implemented on a quantum computer with N extra qubits.

Quantum Gate

- Clifford hierarchy : This is a very useful way to organize quantum gates.
  - First level: Pauli
  - Second level: Clifford
  - Third level: non-Clifford
- The second level can be simulated efficiently on a classical computer. [Gottesman-Knill theorem]

Pauli gates

- $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
- $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- $X^2 = Y^2 = Z^2 = I$.
- They all anticommute.
- They are all Hermitian.
- They are all unitary.



$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

Up to a global phase, the gate set forms a finite group of order 4.

Pauli string

- $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$
- $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- Ex) $X_1 Z_2 Y_3 Y_4$
    - Interpretation: Apply X on qubit 1, Z on qubit 2, Y on qubit 3, and Y on qubit 4.

Clifford group

- Clifford group maps a Pauli string to a Pauli string.
- Any C such that $C^{-1}PC$ is a Pauli string for any Pauli string P.
- Generators
  - Hadamard: $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
  - Phase: S= $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$
  - CNOT: $CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$
- Still a finite group.
- Board: Let’s check that these gates are in the Clifford group.
- Quick question: Up to a phase, what is the order of the Clifford group for single qubit?

Non-Clifford

- Clifford + Pauli gate generates a finite group -> Cannot be universal.
- Adding almost any gate completes the universal gate set.
- Popular examples are in the third level of Clifford hierarchy.
  - C is in the third level if $C^{-1}PC$ is a Clifford for any Pauli P.
- Examples
  - Controlled-Phase gate: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$
  - T-gate: $\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{i} \end{pmatrix}$

Toffoli
-> Not in the Clifford hierarchy

In 1	In 2	In 3	Out 1	Out 2	Out 3
0	0	0	0	0	0
0	1	0	0	1	0
1	0	0	1	0	0
1	1	0	1	1	1
0	0	1	0	0	1
0	1	1	0	1	1
1	0	1	1	0	1
1	1	1	1	1	0

Why the third level?

- In quantum error correction, we can often perform nearly perfect Pauli + Clifford.
- However, non-Clifford gates are generally more difficult.
- The standard approach is to distill low-noise non-Clifford gates from noisy non-Clifford gates.
- These gates can be distilled.
    - Controlled-Phase gate: $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}$
    - T-gate: $\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{i} \end{pmatrix}$
    - Toffoli
- However, not every gate can be distilled.