Superconducting Qubit based Quantum Computing

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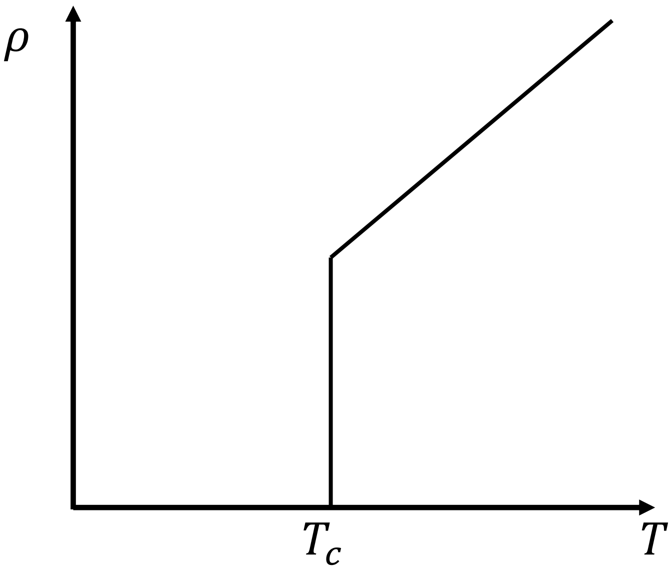
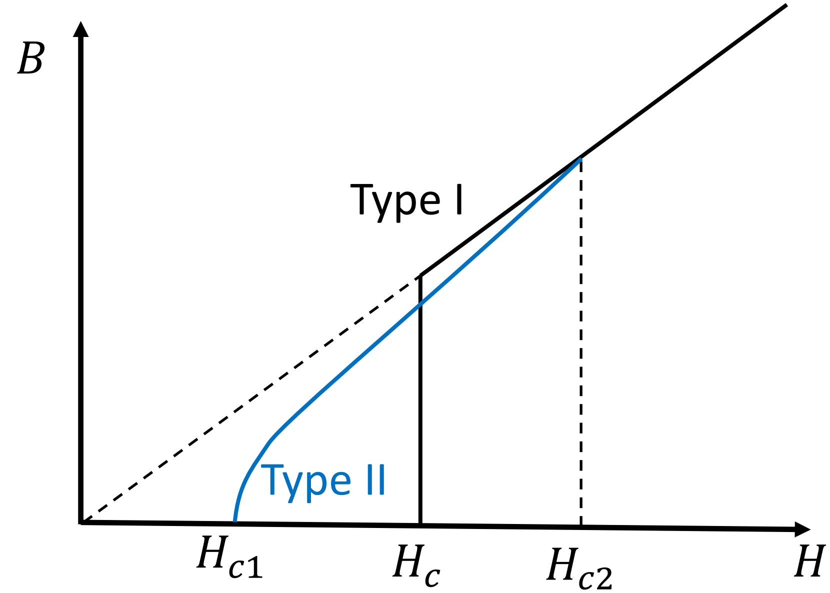
Outline

- **Basics of Superconductivity**
- **Josephson Junction**
- **Quantization of electric circuit**
- **Basic types of SC qubits**
- **Circuit QED architecture**
- **Gate operations and readout : transmon qubit**

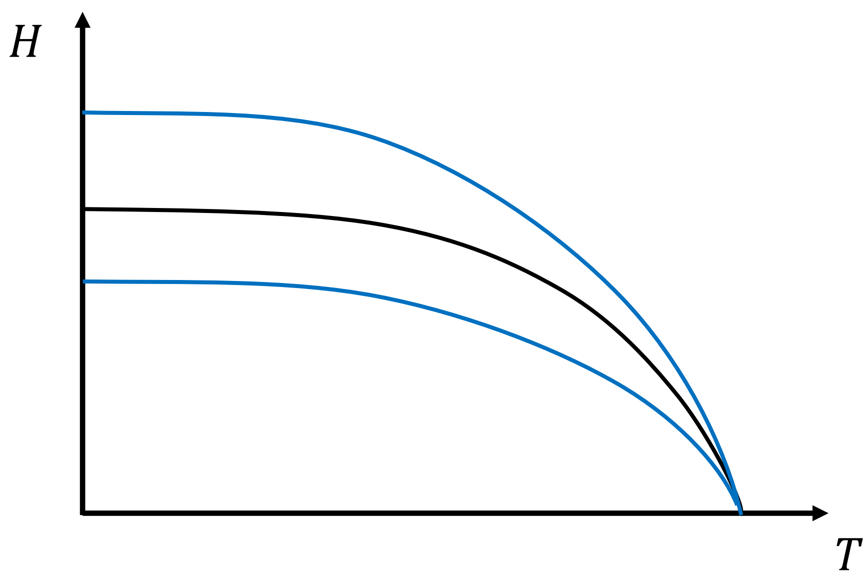
Zero DC resistivity



Meissner effect



Phase Diagram



Macroscopic wavefunction

We can derive most macroscopic SC phenomena using the macroscopic wavefunction.

Macroscopic Quantum wavefunction $$\Psi = \sqrt{n(\mathbf{r},t)}\, e^{i\theta(\mathbf{r},t)}$$

$n(\mathbf{r},t)$: Cooper pair density $= \Psi^*\Psi$

$\theta(\mathbf{r},t)$: SC phase

It satisfies the Schrodinger equation:

$$i\hbar\frac{\partial}{\partial t}\Psi = \hat{H}\,\Psi \qquad\qquad \mathbf{E} = -\boldsymbol{\nabla}\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\hat{H} = \frac{1}{2m^*}(-i\hbar\boldsymbol{\nabla} - q^*\mathbf{A})^2 + q^*\phi \qquad\qquad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$$

From the continuity equation $$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{J}_\rho = 0 \qquad (\rho = \Psi^*\Psi = n)$$

$$\mathbf{J}_\rho = \mathbb{Re}\left\{\Psi^*\left(\frac{-i\hbar}{m^*}\boldsymbol{\nabla} - \frac{q^*}{m^*}\mathbf{A}\right)\Psi\right\}$$

Supercurrent density

$$\mathbf{J}_s \equiv q^*\mathbf{J}_\rho = \boldsymbol{q}^*\mathbb{Re}\left\{\Psi^*\left(\frac{-i\hbar}{m^*}\boldsymbol{\nabla} - \frac{q^*}{m^*}\mathbf{A}\right)\Psi\right\}$$

$$= \frac{\hbar q^*}{m^*}\, n(\mathbf{r},t)\, \left\{\boldsymbol{\nabla}\theta(\mathbf{r},t) - \frac{q^*}{\hbar}\mathbf{A}(\mathbf{r},t)\right\}$$

$$\equiv \frac{\hbar q^*}{m^*}\, n(\mathbf{r},t)\, \boldsymbol{\gamma}(\mathbf{r},t) = \frac{\hbar}{q^*\Lambda}\boldsymbol{\gamma}(\mathbf{r},t) \equiv q^* n(\mathbf{r},t)\,\mathbf{v}_s(\mathbf{r},t)$$

$$\Lambda(\mathbf{r},t) = \frac{m^*}{{q^*}^2 n(\mathbf{r},t)}$$

= London coefficient

$$\Lambda(\mathbf{r},t)\mathbf{J}_s(\mathbf{r},t) = \frac{\hbar}{q^*}\boldsymbol{\gamma}(\mathbf{r},t) = \frac{\hbar}{q^*}\boldsymbol{\nabla}\theta(\mathbf{r},t) - \mathbf{A}(\mathbf{r},t)$$

$\mathbf{v}_s(\mathbf{r},t)$ = velocity

Gauge-invariant phase gradient

$$\boldsymbol{\gamma}(\mathbf{r},t) \equiv \boldsymbol{\nabla}\theta(\mathbf{r},t) - \frac{q^*}{\hbar}\mathbf{A}(\mathbf{r},t) = \frac{m^*}{\hbar}\,\mathbf{v}_s(\mathbf{r},t)$$

Gauge transformation

$$\mathbf{A} \rightarrow \mathbf{A} + \boldsymbol{\nabla}\chi$$

$$\phi \rightarrow \phi - \frac{\partial\chi}{\partial t}$$

$$\Rightarrow \quad \Psi \rightarrow \Psi\, e^{i\frac{q^*}{\hbar}\chi} \qquad \left(\theta \rightarrow \theta + \frac{q^*}{\hbar}\chi\right)$$

Dynamics of superconductor with constant density n

$$\Psi = \sqrt{n(\mathbf{r},t)}\, e^{i\theta(\mathbf{r},t)} = \sqrt{n} e^{i\theta(\mathbf{r},t)}$$

$$i\hbar \frac{\partial}{\partial t}\Psi = \hat{H}\,\Psi \qquad \text{with} \qquad \hat{H} = \frac{1}{2m^*}(-i\hbar\boldsymbol{\nabla} - q^*\mathbf{A})^2 + q^*\phi$$

We obtain two equations

$$-\hbar\frac{\partial\theta(\mathbf{r},t)}{\partial t} = \frac{1}{2m^*}\,(\hbar\boldsymbol{\nabla}\theta - q^*\mathbf{A})^2 + q^*\phi$$

$$-\hbar\,\nabla^2\theta + q^*\boldsymbol{\nabla}\cdot\mathbf{A} = 0 \quad \Rightarrow \quad \nabla^2\theta = 0 \text{ in London gauge } (\boldsymbol{\nabla}\cdot\mathbf{A} = 0)$$

First equation in terms of $\mathbf{J}_s$ or $\mathbf{v}_s$

$$-\hbar\frac{\partial\theta(\mathbf{r},t)}{\partial t} = q^*\phi + \frac{\Lambda}{2n}\mathbf{J}_s^2 = q^*\phi + \frac{m^*}{2}\mathbf{v}_s^2$$

Energy-phase relation

$$\frac{\partial\theta(\mathbf{r},t)}{\partial t} = -\frac{E}{\hbar}$$

London Equations

$$\Lambda(\mathbf{r},t)\mathbf{J}_s(\mathbf{r},t) = \frac{\hbar}{q^*}\boldsymbol{\gamma}(\mathbf{r},t) = \frac{\hbar}{q^*}\boldsymbol{\nabla}\theta(\mathbf{r},t) - \mathbf{A}(\mathbf{r},t)$$

Taking $\boldsymbol{\nabla}\times$

$$\boldsymbol{\nabla}\times\left(\Lambda(\mathbf{r},t)\mathbf{J}_s(\mathbf{r},t)\right) = -\mathbf{B}(\mathbf{r},t)$$

2nd London equation

Assuming the density $n(\mathbf{r},t)$ is constant (hence Λ is also constant)

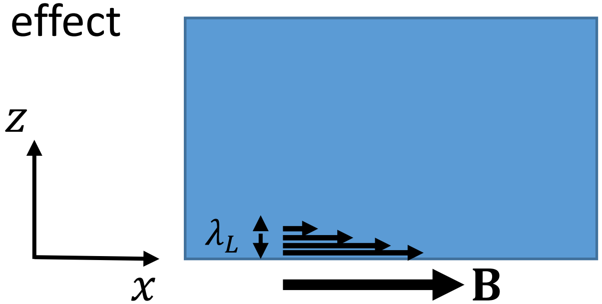
From Maxwell equation $\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\,\mathbf{J}_s$

$$\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{B} = -\nabla^2\mathbf{B} = \mu_0\boldsymbol{\nabla}\times\mathbf{J}_s = -\frac{\mu_0}{\Lambda}\mathbf{B}$$

$$\nabla^2\mathbf{B} = \frac{1}{\lambda_L^2}\mathbf{B}$$

$\lambda_L = \frac{\Lambda}{\mu_0} = \frac{m^*}{\mu_0 {q^*}^2 n}$ = London penetration depth

Meissner effect



$$B_x(z) = B_0 e^{-\frac{z}{\lambda_L}}$$

London Equations

$$\Lambda(\mathbf{r},t)\mathbf{J}_s(\mathbf{r},t) = \frac{\hbar}{q^*}\boldsymbol{\gamma}(\mathbf{r},t) = \frac{\hbar}{q^*}\nabla\theta(\mathbf{r},t) - \mathbf{A}(\mathbf{r},t)$$

Assuming the density $n(\mathbf{r},t)$ is constant (hence Λ is also constant)

$$\frac{\partial}{\partial t}(\Lambda\mathbf{J}_s) = \frac{\hbar}{q^*}\nabla\left(\frac{\partial\theta(\mathbf{r},t)}{\partial t}\right) - \frac{\partial}{\partial t}\mathbf{A}(\mathbf{r},t)$$

From the Schrodinger equation for $\Psi = \sqrt{n(\mathbf{r},t)}\, e^{i\theta(\mathbf{r},t)}$ with $n(\mathbf{r},t)$=constant,

$$-\hbar\frac{\partial\theta(\mathbf{r},t)}{\partial t} = q^*\phi + \frac{\Lambda}{2n}\mathbf{J}_s^2$$

Energy-phase relation

$$\frac{\partial}{\partial t}(\Lambda\mathbf{J}_s) = \mathbf{E} - \frac{1}{q^* n}\nabla\left(\frac{1}{2}\Lambda\mathbf{J}_s^2\right)$$

1st London equation

The 2nd term is usually negligible.

$$\frac{\partial}{\partial t}(\Lambda\mathbf{J}_s) = \mathbf{E}$$

Perfect conductivity

Fluxoid quantization

$$\Lambda(\mathbf{r},t)\mathbf{J}_s(\mathbf{r},t) = \frac{\hbar}{q^*}\boldsymbol{\gamma}(\mathbf{r},t) = \frac{\hbar}{q^*}\boldsymbol{\nabla}\theta(\mathbf{r},t) - \mathbf{A}(\mathbf{r},t)$$

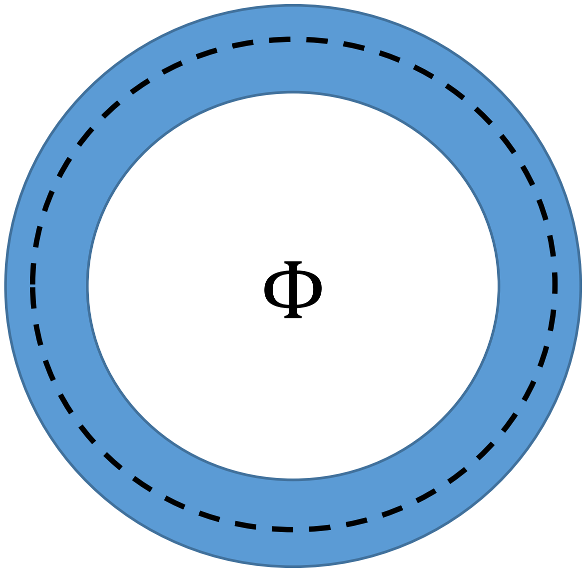
$$\frac{\hbar}{q^*}\boldsymbol{\nabla}\theta = \Lambda\mathbf{J}_s + \mathbf{A} \quad \Rightarrow \quad \frac{\hbar}{q^*}\oint \boldsymbol{\nabla}\theta \cdot d\mathbf{l} = \oint \Lambda\mathbf{J}_s \cdot d\mathbf{l} + \oint \mathbf{A}\cdot d\mathbf{l}$$

$$\frac{\hbar}{q^*}\,2n\pi = \Phi + \oint \Lambda\mathbf{J}_s \cdot d\mathbf{l} \equiv \Phi' \quad \text{(fluxoid)}$$

$$\Phi' = \Phi + \oint \Lambda\mathbf{J}_s \cdot d\mathbf{l} = n\Phi_0$$

$\Phi_0 = \frac{h}{2e}$ = superconducting flux quantum

E.g.



$$\Phi' \cong \Phi = n\Phi_0$$

Microscopic picture: BCS theory

Effective attractive interaction between electrons (mediated by phonons).
Pair of electrons form a bound state → Cooper pair (boson-like).
At low temperature, the cooper pairs condense into a collective state.

$$\hat{H} = \sum_{\mathbf{k},\sigma} \varepsilon_{\mathbf{k}}\, c^{+}_{\mathbf{k}\sigma}\, c_{\mathbf{k}\sigma} + \frac{1}{2} \sum_{\substack{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4 \\ \sigma,\sigma'}} \langle \mathbf{k}_1, \mathbf{k}_2 | \hat{V} | \mathbf{k}_3, \mathbf{k}_4 \rangle c^{+}_{\mathbf{k}_1\sigma} c^{+}_{\mathbf{k}_2\sigma'} c_{\mathbf{k}_4\sigma'} c_{\mathbf{k}_3\sigma}$$

$\hat{V}$ conserves momentum --> $\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$

$$\mathbf{k}_1 = \frac{\mathbf{Q}}{2} + \mathbf{k} \quad \mathbf{k}_2 = \frac{\mathbf{Q}}{2} - \mathbf{k} \quad \mathbf{k}_3 = \frac{\mathbf{Q}}{2} + \mathbf{k}' \quad \mathbf{k}_4 = \frac{\mathbf{Q}}{2} - \mathbf{k}'$$

Zero-momentum pairing condensate assumption: $\mathbf{Q} = \mathbf{0}, \sigma' = -\sigma$

$$c^{+}_{\mathbf{k}\sigma} c^{+}_{-\mathbf{k}-\sigma} = \langle c^{+}_{\mathbf{k}\sigma} c^{+}_{-\mathbf{k}-\sigma} \rangle + \left(c^{+}_{\mathbf{k}\sigma} c^{+}_{-\mathbf{k}-\sigma} - \langle c^{+}_{\mathbf{k}\sigma} c^{+}_{-\mathbf{k}-\sigma} \rangle \right)$$

$$c_{-\mathbf{k}'-\sigma} c_{\mathbf{k}'\sigma} = \langle c_{-\mathbf{k}'-\sigma} c_{\mathbf{k}'\sigma} \rangle + \left(c_{-\mathbf{k}'-\sigma} c_{\mathbf{k}'\sigma} - \langle c_{-\mathbf{k}'-\sigma} c_{\mathbf{k}'\sigma} \rangle \right)$$

We are going to use grand canonical ensemble.

Microscopic picture: BCS theory

$$\hat{H} - \mu\hat{N} = \sum_{\mathbf{k},\sigma}(\varepsilon_{\mathbf{k}} - \mu)\, c^{+}_{\mathbf{k}\sigma}\, c_{\mathbf{k}\sigma} + \frac{1}{2}\sum_{\mathbf{k},\mathbf{k}',\sigma}\langle \mathbf{k}, -\mathbf{k}|\hat{V}|\mathbf{k}', -\mathbf{k}'\rangle c^{+}_{\mathbf{k}\sigma} c^{+}_{-\mathbf{k}-\sigma} c_{-\mathbf{k}'-\sigma} c_{\mathbf{k}'\sigma}$$

$$c^{+}_{\mathbf{k}\sigma} c^{+}_{-\mathbf{k}-\sigma} c_{-\mathbf{k}'-\sigma} c_{\mathbf{k}'\sigma} = [\langle c^{+}_{\mathbf{k}\sigma} c^{+}_{-\mathbf{k}-\sigma}\rangle + (c^{+}_{\mathbf{k}\sigma} c^{+}_{-\mathbf{k}-\sigma} - \langle c^{+}_{\mathbf{k}\sigma} c^{+}_{-\mathbf{k}-\sigma}\rangle)]$$
$$\times[\langle c_{-\mathbf{k}'-\sigma} c_{\mathbf{k}'\sigma}\rangle + (c_{-\mathbf{k}'-\sigma} c_{\mathbf{k}'\sigma} - \langle c_{-\mathbf{k}'-\sigma} c_{\mathbf{k}'\sigma}\rangle)]$$

$$\approx \langle c_{-\mathbf{k}'-\sigma} c_{\mathbf{k}'\sigma}\rangle c^{+}_{\mathbf{k}\sigma} c^{+}_{-\mathbf{k}-\sigma} + \langle c^{+}_{\mathbf{k}\sigma} c^{+}_{-\mathbf{k}-\sigma}\rangle c_{-\mathbf{k}'-\sigma} c_{\mathbf{k}'\sigma} - \langle c^{+}_{\mathbf{k}\sigma} c^{+}_{-\mathbf{k}-\sigma}\rangle\langle c_{-\mathbf{k}'-\sigma} c_{\mathbf{k}'\sigma}\rangle$$

$$\hat{H}_{BCS} = \sum_{\mathbf{k},\sigma}\xi_{\mathbf{k}} c^{+}_{\mathbf{k}\sigma}\, c_{\mathbf{k}\sigma} - \frac{1}{2}\sum_{\mathbf{k},\mathbf{k}'} V(\mathbf{k},\mathbf{k}')\,(\eta^{*}_{\mathbf{k}}\eta_{\mathbf{k}'} + \eta^{*}_{-\mathbf{k}}\eta_{-\mathbf{k}'})$$
$$+ \frac{1}{2}\sum_{\mathbf{k},\mathbf{k}'} V(\mathbf{k},\mathbf{k}')\,(\eta_{\mathbf{k}'} c^{+}_{\mathbf{k}\uparrow} c^{+}_{-\mathbf{k}\downarrow} - \eta_{-\mathbf{k}'} c^{+}_{\mathbf{k}\downarrow} c^{+}_{-\mathbf{k}\uparrow} + \eta^{*}_{\mathbf{k}} c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow} - \eta^{*}_{-\mathbf{k}} c_{-\mathbf{k}'\uparrow} c_{\mathbf{k}'\downarrow})$$

$\xi_{\mathbf{k}} = \varepsilon_{\mathbf{k}} - \mu$ $\qquad$ $V(\mathbf{k},\mathbf{k}') = \langle \mathbf{k}, -\mathbf{k}|\hat{V}|\mathbf{k}', -\mathbf{k}'\rangle$

$\eta_{\mathbf{k}} = \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}\rangle$: needs to be determined self-consistently

Bogoliubov transformation

Assuming $V(\mathbf{k},\mathbf{k}')=V(-\mathbf{k},-\mathbf{k}')$, and defining the gap function

$$\Delta_{\mathbf{k}}\equiv-\sum_{\mathbf{k}'}V(\mathbf{k},\mathbf{k}')\eta_{\mathbf{k}'}=-\sum_{\mathbf{k}'}V(\mathbf{k},\mathbf{k}')\langle c_{-\mathbf{k}'\downarrow}c_{\mathbf{k}'\uparrow}\rangle \qquad \text{: gap function}$$

$$\widehat{H}_{BCS}=\sum_{\mathbf{k},\sigma}\xi_{\mathbf{k}}c^{+}_{\mathbf{k}\sigma}c_{\mathbf{k}\sigma}+\sum_{\mathbf{k}}\left(-\Delta_{\mathbf{k}}c^{+}_{\mathbf{k}\uparrow}c^{+}_{-\mathbf{k}\downarrow}-\Delta^{*}_{\mathbf{k}}c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}\right)-\sum_{\mathbf{k},\mathbf{k}'}V(\mathbf{k},\mathbf{k}')\eta^{*}_{\mathbf{k}}\eta_{\mathbf{k}'}$$

$$=\sum_{\mathbf{k}}\xi_{\mathbf{k}}-\sum_{\mathbf{k},\mathbf{k}'}V(\mathbf{k},\mathbf{k}')\eta^{*}_{\mathbf{k}}\eta_{\mathbf{k}'}+\sum_{\mathbf{k}}\left(c^{+}_{\mathbf{k}\uparrow}\quad c_{-\mathbf{k}\downarrow}\right)\begin{pmatrix}\xi_{\mathbf{k}} & -\Delta_{\mathbf{k}}\\ -\Delta^{*}_{\mathbf{k}} & \xi_{\mathbf{k}}\end{pmatrix}\begin{pmatrix}c_{\mathbf{k}\uparrow}\\ c^{+}_{-\mathbf{k}\downarrow}\end{pmatrix}$$

Bogoliubov transformation $\begin{pmatrix}c_{\mathbf{k}\uparrow}\\ c^{+}_{-\mathbf{k}\downarrow}\end{pmatrix}=\begin{pmatrix}u^{*}_{\mathbf{k}} & v_{\mathbf{k}}\\ -v^{*}_{\mathbf{k}} & u_{\mathbf{k}}\end{pmatrix}\begin{pmatrix}\gamma_{\mathbf{k}\uparrow}\\ \gamma^{+}_{\mathbf{k}\downarrow}\end{pmatrix}$ $\qquad |u_{\mathbf{k}}|^2+|v_{\mathbf{k}}|^2=1$

Quasiparticle operators $\begin{pmatrix}\gamma_{\mathbf{k}\uparrow}\\ \gamma^{+}_{\mathbf{k}\downarrow}\end{pmatrix}=\begin{pmatrix}u_{\mathbf{k}} & -v_{\mathbf{k}}\\ v^{*}_{\mathbf{k}} & u^{*}_{\mathbf{k}}\end{pmatrix}\begin{pmatrix}c_{\mathbf{k}\uparrow}\\ c^{+}_{-\mathbf{k}\downarrow}\end{pmatrix}$ **Linear combination of electron and hole parts!**

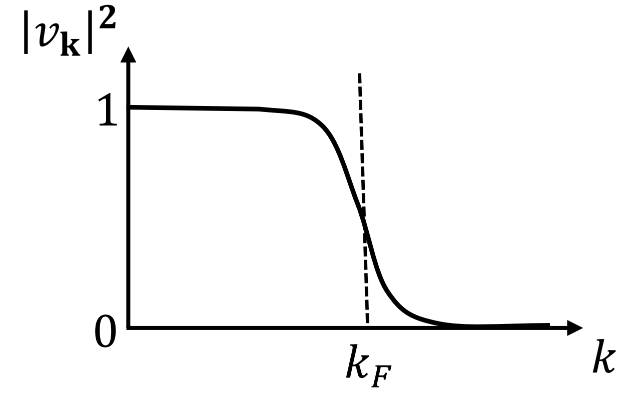
We will determine $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ that diagonalize the BCS Hamiltonian.

Self-consistency relations

To make the BCS Hamiltonian diagonal after the Bogoliubov transformation,

$$|u_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right) \qquad |v_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{k}}}{E_{\mathbf{k}}}\right)$$

$$u_{\mathbf{k}}^* v_{\mathbf{k}} = \frac{\Delta_{\mathbf{k}}}{2E_{\mathbf{k}}}$$



$$\Delta_{\mathbf{k}} = -\sum_{\mathbf{k}'} V(\mathbf{k},\mathbf{k}')\langle c_{-\mathbf{k}'\downarrow} c_{\mathbf{k}'\uparrow}\rangle = -\sum_{\mathbf{k}'} V(\mathbf{k},\mathbf{k}') u_{\mathbf{k}'}^* v_{\mathbf{k}'}\left(1 - \langle \gamma_{\mathbf{k}\uparrow}^+ \gamma_{\mathbf{k}\uparrow}\rangle - \langle \gamma_{\mathbf{k}\downarrow}^+ \gamma_{\mathbf{k}\downarrow}\rangle\right)$$

$$\Rightarrow \quad \Delta_{\mathbf{k}} = -\sum_{\mathbf{k}'} V(\mathbf{k},\mathbf{k}') \frac{\Delta_{\mathbf{k}'}}{2E_{\mathbf{k}'}}\left(1 - \langle \gamma_{\mathbf{k}\uparrow}^+ \gamma_{\mathbf{k}\uparrow}\rangle - \langle \gamma_{\mathbf{k}\downarrow}^+ \gamma_{\mathbf{k}\downarrow}\rangle\right)$$

gap equation

The presence of quasiparticles decreases the gap. (Gap decreases with T.)
The phase of the gap function is the phase difference between $v_{\mathbf{k}}$ and $u_{\mathbf{k}}$.
In the ground state (largest gap), the phase is constant.

Excitation energy

After diagonalization, $\hat{H}_{BCS} = E_0 + \sum_{\mathbf{k}} (E_{\mathbf{k}} \gamma^{+}_{\mathbf{k}\uparrow} \gamma_{\mathbf{k}\uparrow} + E_{\mathbf{k}} \gamma^{+}_{\mathbf{k}\downarrow} \gamma_{\mathbf{k}\downarrow})$

$$E_0 = \sum_{\mathbf{k}} (\xi_{\mathbf{k}} - E_{\mathbf{k}} + \Delta_{\mathbf{k}} \eta^{*}_{\mathbf{k}})$$

$E_{\mathbf{k}} = \sqrt{\xi^2_{\mathbf{k}} + |\Delta_{\mathbf{k}}|^2}$: quasiparticle excitation energy

BCS Ground state

$|\Psi_{BCS}\rangle$ should satisfy $\gamma_{\mathbf{k}\uparrow} |\Psi_{BCS}\rangle = \gamma_{\mathbf{k}\downarrow} |\Psi_{BCS}\rangle = 0$

$$|\Psi_{BCS}\rangle \propto \prod_{\mathbf{k}} (\gamma_{\mathbf{k}\uparrow} \gamma_{\mathbf{k}\downarrow})\, |0\rangle = \prod_{\mathbf{k}} (u_{\mathbf{k}} c_{\mathbf{k}\uparrow} - v_{\mathbf{k}} c^{+}_{-\mathbf{k}\downarrow})\, (v_{\mathbf{k}} c^{+}_{\mathbf{k}\uparrow} + u_{\mathbf{k}} c_{-\mathbf{k}\downarrow}) |0\rangle$$

$$= \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c^{+}_{\mathbf{k}\uparrow} c^{+}_{-\mathbf{k}\downarrow})\, |0\rangle$$ After normalization

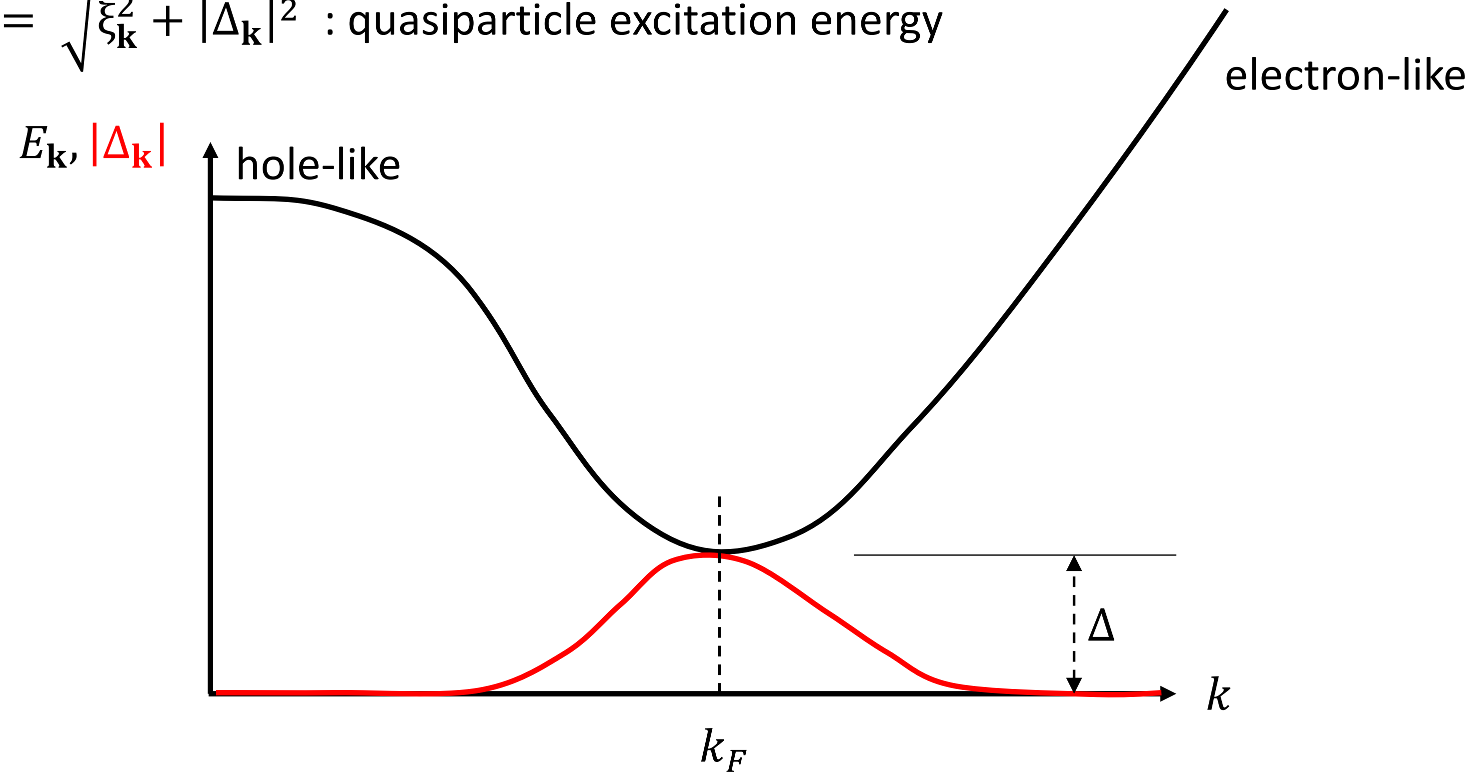
$$|\Psi_{BCS}\rangle = \prod_{\mathbf{k}} \left(|u_{\mathbf{k}}| + |v_{\mathbf{k}}| e^{i\varphi} c^{+}_{\mathbf{k}\uparrow} c^{+}_{-\mathbf{k}\downarrow}\right) |0\rangle$$

Number of electrons is not fixed.
Phase is fixed.

Excitation spectrum: quasiparticle

$$\hat{H}_{BCS} = E_0 + \sum_{\mathbf{k}} (E_{\mathbf{k}} \gamma^{+}_{\mathbf{k}\uparrow} \gamma_{\mathbf{k}\uparrow} + E_{\mathbf{k}} \gamma^{+}_{\mathbf{k}\downarrow} \gamma_{\mathbf{k}\downarrow})$$

$E_{\mathbf{k}} = \sqrt{\xi_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}$: quasiparticle excitation energy



Quasiparticle operators $\begin{pmatrix} \gamma_{\mathbf{k}\uparrow} \\ \gamma^{+}_{\mathbf{k}\downarrow} \end{pmatrix} = \begin{pmatrix} u_{\mathbf{k}} & -v_{\mathbf{k}} \\ v^{*}_{\mathbf{k}} & u^{*}_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} c_{\mathbf{k}\uparrow} \\ c^{+}_{-\mathbf{k}\downarrow} \end{pmatrix}$ **Linear combination of electron and hole parts!**

Some properties of BCS ground state

$$|\Psi_\varphi\rangle = \prod_{\mathbf{k}} \left(|u_{\mathbf{k}}| + |v_{\mathbf{k}}| e^{i\varphi} c^{+}_{\mathbf{k}\uparrow} c^{+}_{-\mathbf{k}\downarrow}\right) |0\rangle$$

Total number of electrons is not fixed. (Grand canonical ensemble).

$$\langle \hat{N}_e \rangle = \left\langle \sum_{\mathbf{k}} (c^{+}_{\mathbf{k}\uparrow} c_{\mathbf{k}\uparrow} + c^{+}_{-\mathbf{k}\downarrow} c_{-\mathbf{k}\downarrow}) \right\rangle = \sum_{\mathbf{k}} 2|v_{\mathbf{k}}|^2$$

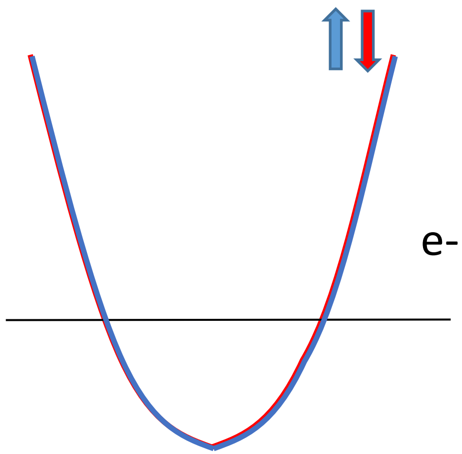
Total electron number is even. (pairing into Cooper pairs)

Cooper pair number $N = \frac{N_e}{2}$

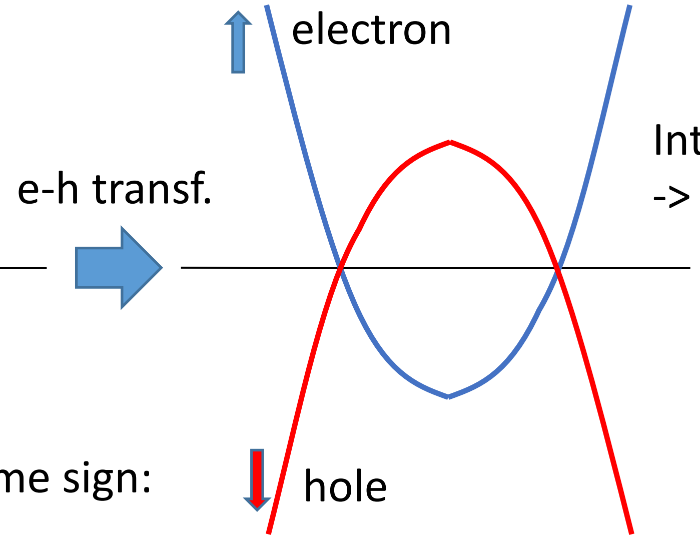
To project to a fixed N state,

$$|N\rangle = \int_0^{2\pi} d\varphi \, e^{-iN\varphi} |\Psi_\varphi\rangle \quad \Longleftrightarrow \quad |\Psi_\varphi\rangle = \frac{1}{2\pi} \sum_{N=0}^{\infty} e^{iN\varphi} |N\rangle$$

Excitation energy spectrum: Semiconductor model

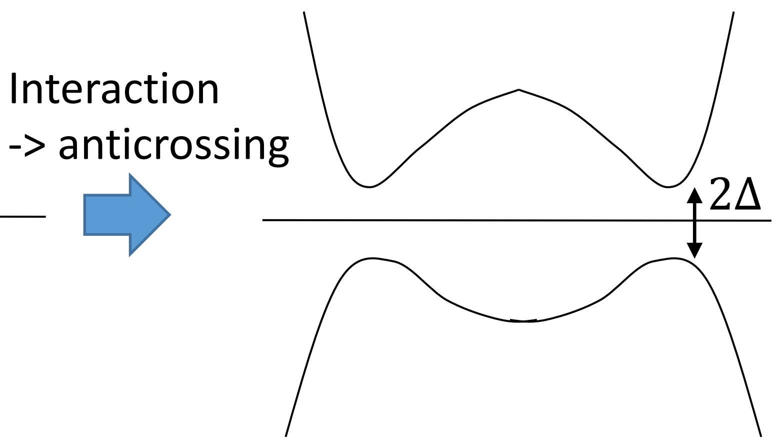


Effective masses of same sign:
attractive interaction



Effective masses of opposite sign:
repulsive interaction
(e.g. exciton)

Semiconductor model

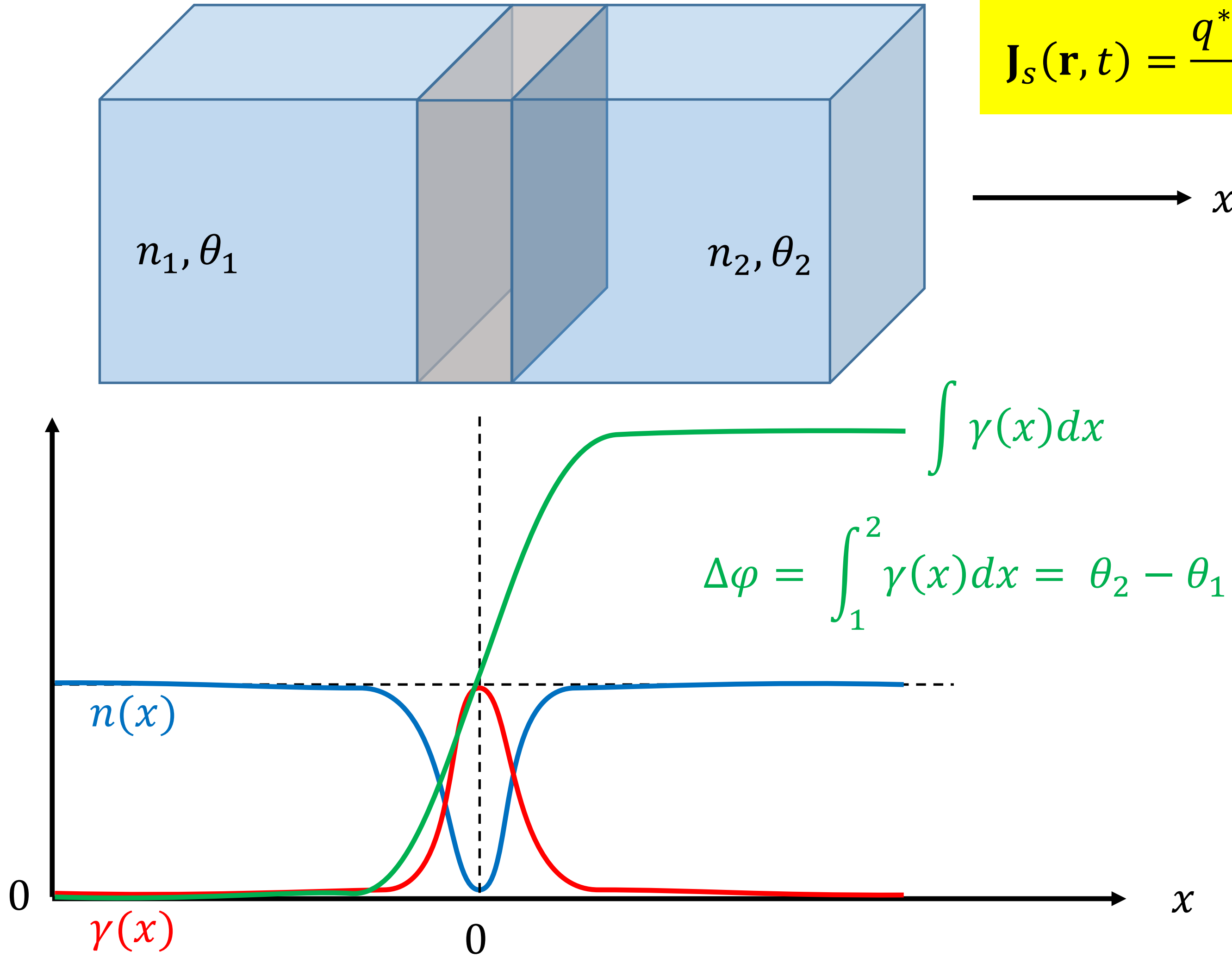


$$\hat{H}_{BCS} = E_0 + \sum_{\mathbf{k}} (E_{\mathbf{k}} \gamma^+_{\mathbf{k}\uparrow} \gamma_{\mathbf{k}\uparrow} + E_{\mathbf{k}} \gamma^+_{\mathbf{k}\downarrow} \gamma_{\mathbf{k}\downarrow}) = E_0 + \sum_{\mathbf{k}} (E_{\mathbf{k}} \gamma^+_{\mathbf{k}\uparrow} \gamma_{\mathbf{k}\uparrow} - E_{\mathbf{k}} \tilde{\gamma}^+_{\mathbf{k}\downarrow} \tilde{\gamma}_{\mathbf{k}\downarrow})$$

$\tilde{\gamma}^+_{\mathbf{k}\downarrow} \equiv \gamma_{\mathbf{k}\downarrow}$ (e-h like transformation)

Josephson Junction: THE most important component in SC qubit

$$\mathbf{J}_s(\mathbf{r},t) = \frac{q^*\hbar n(\mathbf{r},t)}{m^*}\boldsymbol{\gamma}(\mathbf{r},t)$$



$$\Delta\varphi = \int_1^2 \gamma(x)dx = \theta_2 - \theta_1 + \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A}\cdot d\mathbf{x}$$

$$J_S = J_S(\Delta\varphi)$$

DC Josephson effect

$J_S = J_S(\Delta\varphi)$: current is a function of gauge-invariant phase difference $\Delta\varphi$

It should satisfy

Periodicity of 2π: $J_S(\Delta\varphi + 2\pi) = J_S(\Delta\varphi)$

Time-reversal symmetry ($\Delta\varphi$ → - $\Delta\varphi$, J_S → $-J_S$) : $J_S(-\Delta\varphi) = -J_S(\Delta\varphi)$

$$\Rightarrow \quad J_S(\Delta\varphi) = J_c \sin \Delta\varphi + \sum_{m=2}^{\infty} J_m \sin(m\Delta\varphi)$$

For many cases (e.g. tunnel junction)

$$J_S(\Delta\varphi) = J_c \sin \Delta\varphi$$

DC Josephson equation

AC Josephson effect

$$\Delta\varphi = \theta_2 - \theta_1 + \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A}\cdot d\mathbf{x} \longrightarrow \frac{\partial\Delta\varphi}{\partial t} = \frac{\partial\theta_2}{\partial t} - \frac{\partial\theta_1}{\partial t} + \frac{2\pi}{\Phi_0}\int_1^2 \frac{\partial\mathbf{A}}{\partial t}\cdot d\mathbf{x}$$

Using the energy-phase relation $-\hbar\frac{\partial\theta(\mathbf{r},t)}{\partial t} = q^*\phi + \frac{\Lambda}{2n}\mathbf{J}_s^2$ and $J_1 = J_2$

$$\frac{\partial\Delta\varphi}{\partial t} = -\frac{q^*}{\hbar}(\phi_2 - \phi_1) + \frac{2\pi}{\Phi_0}\int_1^2 \frac{\partial\mathbf{A}}{\partial t}\cdot d\mathbf{x}$$

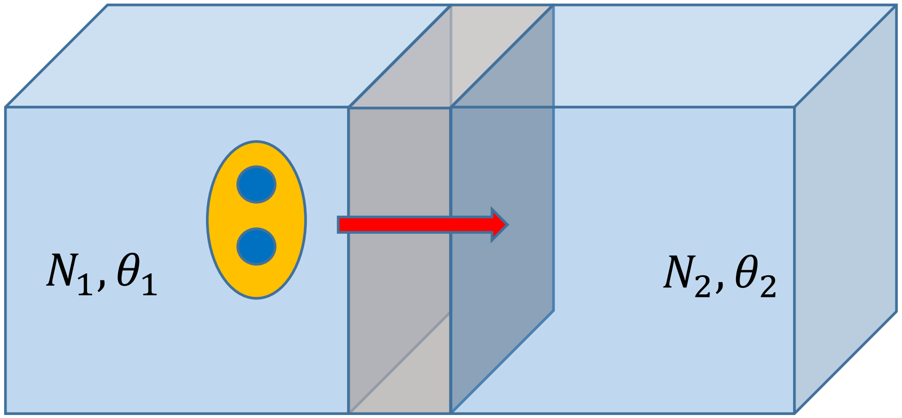
$$= -\frac{2\pi}{\Phi_0}\int_1^2 \left(-\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}\right)\cdot d\mathbf{x} = -\frac{2\pi}{\Phi_0}\int_1^2 \mathbf{E}\cdot d\mathbf{x} = \frac{2\pi}{\Phi_0}(V_2 - V_1)$$

$$\frac{\partial\Delta\varphi}{\partial t} = \frac{2\pi}{\Phi_0}V$$

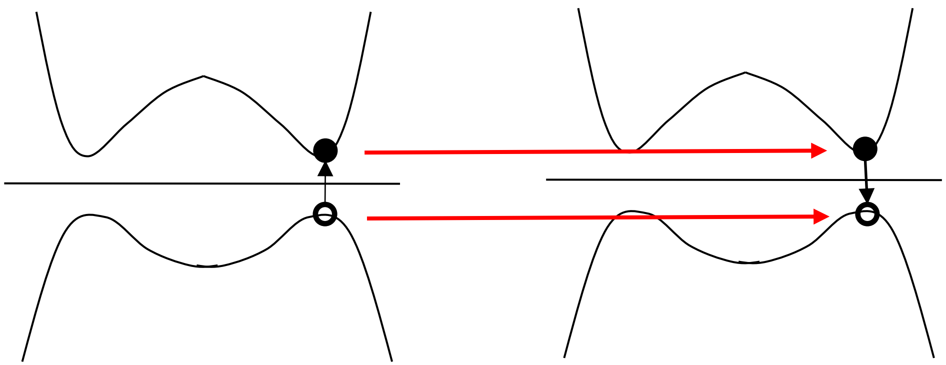
$V = V_2 - V_1$

AC Josephson equation

Josephson Junction : Cooper pair number representation



- Tunneling is possible when the barrier is thin enough.
- Single particle tunneling is suppressed due to the energy gap.
- Tunneling of a Cooper pair does not change energy!



Tunneling of a Cooper pair

$|N_1, N_2\rangle \rightarrow |N_1 - 1, N_2 + 1\rangle$ (1 -> 2)

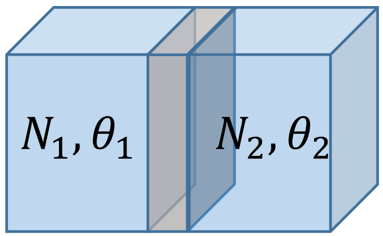
$|N_1, N_2\rangle \rightarrow |N_1 + 1, N_2 - 1\rangle$ (2 -> 1)

$$\langle N_1, N_2 | \hat{H}_T | N_1 - 1, N_2 + 1 \rangle = \langle N_1, N_2 | \hat{H}_T | N_1 + 1, N_2 - 1 \rangle = t$$

$\hat{H}_T$: tunnel-coupling Hamiltonian

$$\hat{H}_T = t \sum_{N_1, N_2} [|N_1, N_2\rangle\langle N_1 - 1, N_2 + 1| + |N_1 - 1, N_2 + 1\rangle\langle N_1, N_2|]$$

Josephson Junction : Cooper pair number representation



BCS GS: $|\Psi_\varphi\rangle = \frac{1}{2\pi} \sum_{N=0}^{\infty} e^{iN\varphi} |N\rangle$

$$|\Psi_{12}\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle = \frac{1}{4\pi^2} \sum_{N_1,N_2} e^{iN_1\theta_1} e^{iN_2\theta_2} |N_1, N_2\rangle$$

$$\widehat{H}_T |\Psi_{12}\rangle = \frac{t}{4\pi^2} \sum_{N_1',N_2'} [|N_1', N_2'\rangle\langle N_1' - 1, N_2' + 1| + |N_1' - 1, N_2' + 1\rangle\langle N_1', N_2'|] \sum_{N_1,N_2} e^{iN_1\theta_1} e^{iN_2\theta_2} |N_1, N_2\rangle$$

$$= \frac{t}{4\pi^2} \sum_{N_1,N_2} e^{iN_1\theta_1} e^{iN_2\theta_2} (|N_1 + 1, N_2 - 1\rangle + |N_1 - 1, N_2 + 1\rangle)$$

$$= \frac{t}{4\pi^2} \sum_{N_1,N_2} \left(e^{i(N_1-1)\theta_1} e^{i(N_2+1)\theta_2} + e^{i(N_1+1)\theta_1} e^{i(N_2-1)\theta_2}\right) |N_1, N_2\rangle$$

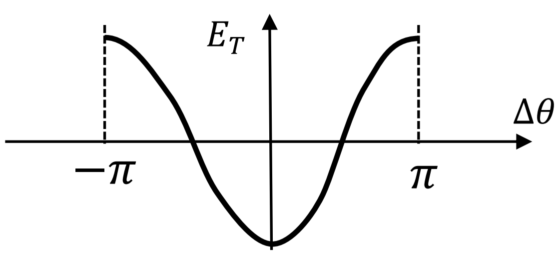
$$= \frac{t}{4\pi^2} \left(e^{i(\theta_2-\theta_1)} + e^{-i(\theta_2-\theta_1)}\right) \sum_{N_1,N_2} e^{iN_1\theta_1} e^{iN_2\theta_2} |N_1, N_2\rangle = 2t\cos(\theta_2 - \theta_1)\, |\Psi_{12}\rangle$$

$E_T = 2t\cos\Delta\theta \qquad \Delta\theta = \theta_2 - \theta_1$

$t < 0$ to make $\Delta\theta = 0$ is the ground state (for $|t| \to 0$).

$$E_T = -E_J \cos\Delta\theta$$

E_J : Josephson energy



Relation between number and phase

$|\Psi_n\rangle$ $\equiv$ state after n Cooper pairs have tunneled from 1 to 2

$$|\Psi_n\rangle = \hat{T}_n|\Psi_{12}\rangle \qquad\qquad \hat{T}_n|N_1, N_2\rangle = |N_1 - n, N_2 + n\rangle$$

$$= \frac{1}{4\pi^2} \sum_{N_1,N_2} e^{iN_1\theta_1} e^{iN_2\theta_2} |N_1 - n, N_2 + n\rangle$$

$$= \frac{1}{4\pi^2} \sum_{N_1,N_2} e^{i(N_1+n)\theta_1} e^{i(N_2-n)\theta_2} |N_1, N_2\rangle$$

$$= \frac{1}{4\pi^2} e^{in\theta_1} e^{-in\theta_2} \sum_{N_1,N_2} e^{iN_1\theta_1} e^{iN_2\theta_2} |N_1, N_2\rangle$$

$$= e^{-in\Delta\theta}|\Psi_{12}\rangle$$

$$i\frac{\partial}{\partial\Delta\theta}|\Psi_n\rangle = n|\Psi_n\rangle \quad \Longrightarrow \quad n = i\frac{\partial}{\partial\Delta\theta}$$

Josephson equations: DC Josephson equation

Time-evolution of $|\Psi_{12}\rangle$

$$\hat{H}_T|\Psi_{12}\rangle = -E_J \cos\Delta\theta\, |\Psi_{12}\rangle \quad \Rightarrow \quad |\Psi(t)\rangle = e^{i\frac{t}{\hbar}E_J\cos\Delta\theta}|\Psi_{12}\rangle$$

Number of Cooper pairs tunneled at time t

$$n(t) = i\frac{\partial}{\partial\Delta\theta}\left(i\frac{t}{\hbar}E_J\cos\Delta\theta\right) = \frac{t}{\hbar}E_J\sin\Delta\theta$$

Current across the junction

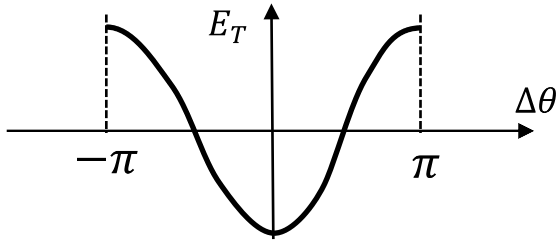
$$I = (-2e)\,\frac{dn}{dt} = \frac{-2eE_J}{\hbar}\sin\Delta\theta$$

$$I = -I_c\sin\Delta\theta$$ **DC Josephson effect**

$$I_c \equiv \frac{2eE_J}{\hbar}$$ **Josephson critical current**

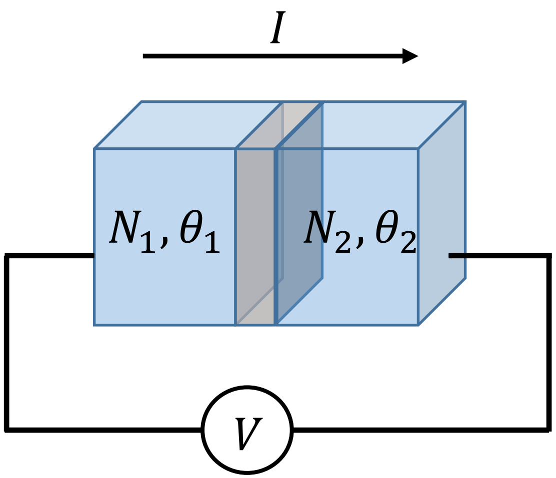
In most literature, $\Delta\theta$ is defined as $\theta_1 - \theta_2$ (or the direction of the current is defined to be opposite (2->1)) and DC Josephson equation is written as $I = I_c\sin\Delta\theta$

Josephson equations: AC Josephson equation



$$E_T = -E_J \cos \Delta\theta$$

The state with a finite phase difference is not the ground state.
Changing the phase difference away from 0 costs energy.



Power supplied by an external voltage source = $\frac{dE_{JJ}}{dt}$

$$-I(V_2 - V_1) = \frac{d}{dt}\left[-E_J \cos \Delta\theta\right]$$

$$I_c V \sin \Delta\theta = E_J \sin \Delta\theta \, \frac{d}{dt}\Delta\theta$$

$$\frac{d}{dt}\Delta\theta = \frac{2eV}{\hbar}$$

AC Josephson effect

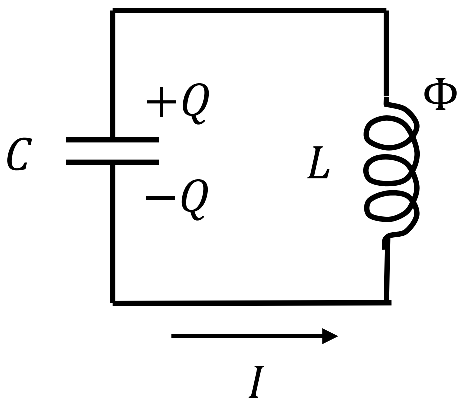
Fully microscopic theory (Ambegaokar-Baratoff): $E_J = \frac{\pi\hbar}{4e^2}\frac{\Delta(T)}{R_n}\tanh\left[\frac{\Delta(T)}{2k_B T}\right]$

Quantization procedure for electrical circuits

1. Define independent variables (i.e. degrees of freedom). : node flux
2. Find equations of motion for these variables. : Kirchhoff's law
3. Find Lagrangian : capacitive energy – inductive energy
4. Find Hamiltonian : conjugate momenta, Legendre transformation
5. Conjugate variables become quantum operators with commutation relations.

Quantum Fluctuations in Electrical Circuits, Michel Devoret (1997)

Classical LC oscillator



$$V = \frac{Q}{C} \qquad \Phi = LI$$

Equation of motion for Φ: $-\frac{d\Phi}{dt} = V = \frac{Q}{C} \qquad \frac{dQ}{dt} = I = \frac{\Phi}{L}$

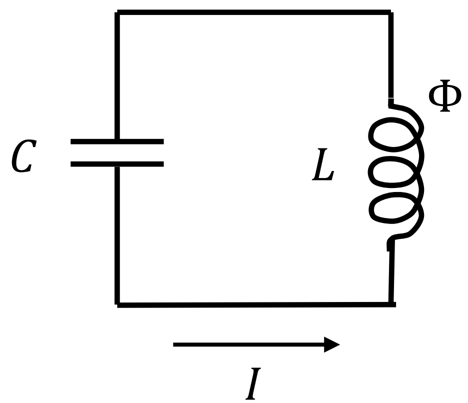
$$\frac{d^2\Phi}{dt^2} = -\frac{1}{C}\frac{dQ}{dt} = -\frac{\Phi}{LC} \quad \Rightarrow \quad \frac{d^2\Phi}{dt^2} + \omega_0^2\Phi = 0$$

Oscillation frequency : $\omega_0 = \frac{1}{\sqrt{LC}}$

$E = \frac{Q^2}{2C} + \frac{\Phi}{2L}$ (=Classical Hamiltonian H)

Dissipative element (i.e. resistance) makes the oscillation to decay.

Quantization of an LC oscillator



From the equation of motion, we can find Lagrangian

$$C\frac{d^2\Phi}{dt^2} = -\frac{\Phi}{L}$$

Lagrangian = capacitive energy – inductive energy

$$\mathcal{L} = \frac{C}{2}\dot{\Phi}^2 - \frac{\Phi^2}{2L}$$

$$Q = \frac{\partial\mathcal{L}}{\partial\dot{\Phi}} = C\dot{\Phi}$$

Conjugate variable to the flux

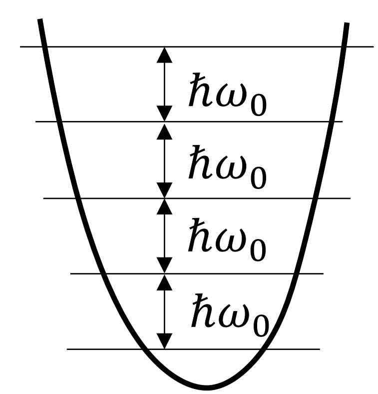
$$\mathcal{H} = \frac{Q^2}{2C} + \frac{\Phi}{2L}$$

Classical Hamiltonian $Q \leftrightarrow p, \Phi \leftrightarrow x$

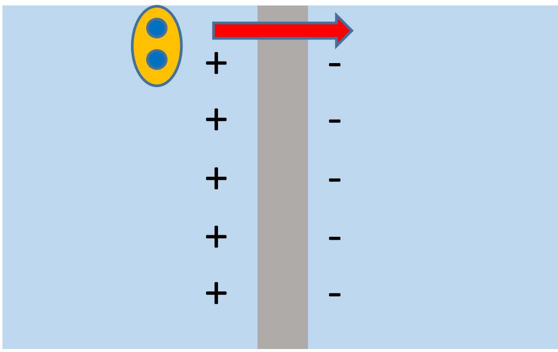
Superconducting LC oscillator

- For Large C and L → Dissipationless, "classical" ideal LC oscillator
- For small C and L → Quantum harmonic oscillator (linear)

$$\hat{H} = \frac{\hat{Q}^2}{2C} + \frac{\hat{\Phi}^2}{2L} \qquad [\hat{Q}, \hat{\Phi}] = i\hbar \quad \Longleftrightarrow \quad \hat{H} = \frac{\hat{p}^2}{2m} + \frac{k\hat{x}^2}{2}$$



Nonlinear element: Josephson junction

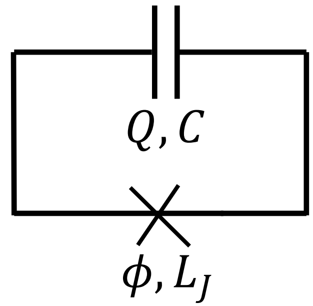


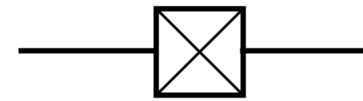
$+Q \qquad -Q$

$$Q = -CV \qquad I = -\frac{dQ}{dt}$$

For small JJs, we also need to consider the charging energy.

$$E = \frac{Q^2}{2C} = \frac{(-2e)^2}{2C} n^2 = E_C n^2 \qquad E_C \equiv \frac{(2e)^2}{2C}$$





Total energy: Charging energy + Josephson energy

$$H = \frac{Q^2}{2C} - E_J \cos\varphi \qquad (\varphi = \Delta\theta)$$

Define a branch flux $\Phi = \frac{\hbar}{2e}\varphi = \frac{\Phi_0}{2\pi}\varphi$

$$H = \frac{Q^2}{2C} - E_J \cos\left(2\pi \frac{\Phi}{\Phi_0}\right)$$

Josephson equations can be considered the classical Hamiltonian equations of motion.

$$I = -I_C \sin\varphi \quad \Leftrightarrow \quad \frac{dQ}{dt} = \frac{\partial H}{\partial \Phi}$$

$$\frac{d\varphi}{dt} = \frac{2e}{\hbar} V \quad \Leftrightarrow \quad \frac{d\Phi}{dt} = -\frac{\partial H}{\partial Q}$$

$$Q \leftrightarrow x, \Phi \leftrightarrow p$$

Nonlinear potential due to the cos term.

$$\frac{1}{L_J} \equiv \frac{\partial^2 E}{\partial \Phi^2} \Rightarrow L_J = \frac{\Phi_0}{2\pi I_C \cos\left(2\pi \frac{\Phi}{\Phi_0}\right)}$$

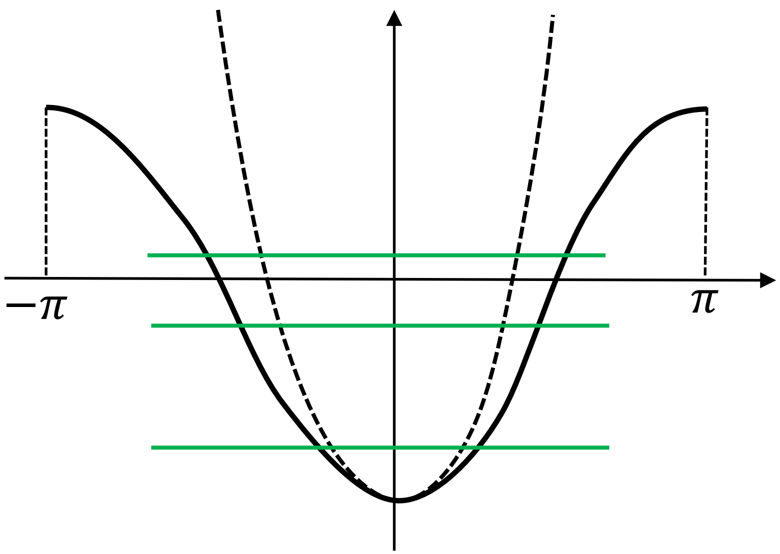
Quantization of the Josephson junction Hamiltonian

$$H = \frac{Q^2}{2C} - E_J \cos\left(2\pi \frac{\Phi}{\Phi_0}\right) \Rightarrow \hat{H} = \frac{\hat{Q}^2}{2C} - E_J \cos\left(2\pi \frac{\hat{\Phi}}{\Phi_0}\right) \qquad [\hat{Q}, \hat{\Phi}] = i\hbar$$

In terms of the number and phase operators:

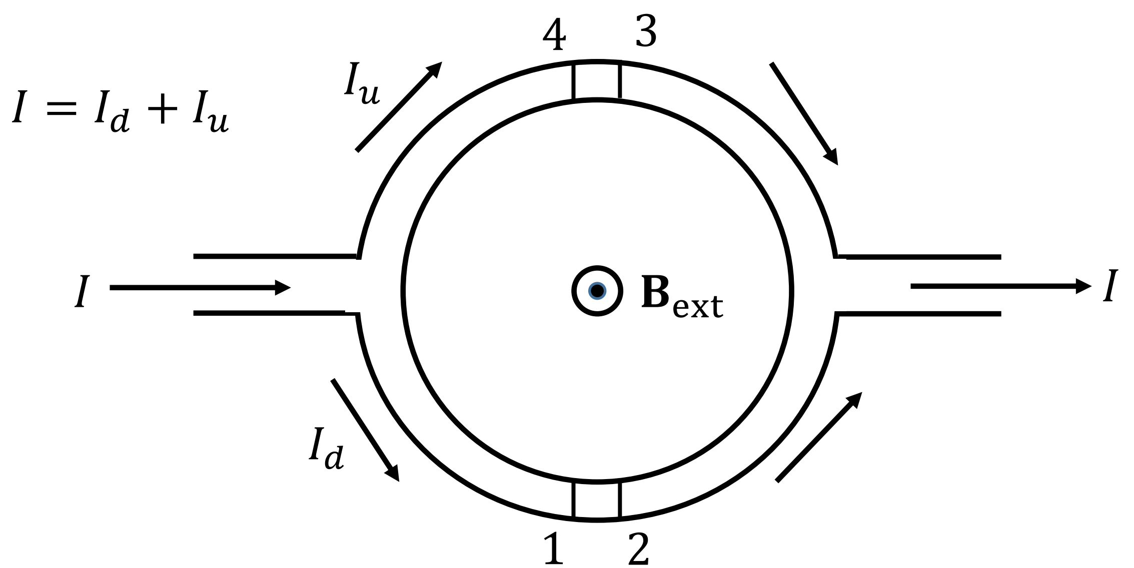
$$\hat{n} = \frac{\hat{Q}}{-2e} \qquad \hat{\varphi} = 2\pi \frac{\hat{\Phi}}{\Phi_0} \qquad \hat{H} = E_C \hat{n}^2 - E_J \cos\hat{\varphi} \qquad [\hat{n}, \hat{\varphi}] = -i$$

The commutation relation indicates that $\hat{n} = i\frac{\partial}{\partial\varphi}$ in the φ representation, which we saw earlier.



Josephson junction is the only dissipationless nonlinear element in the superconducting qubit circuits.

Tunable Josephson junction

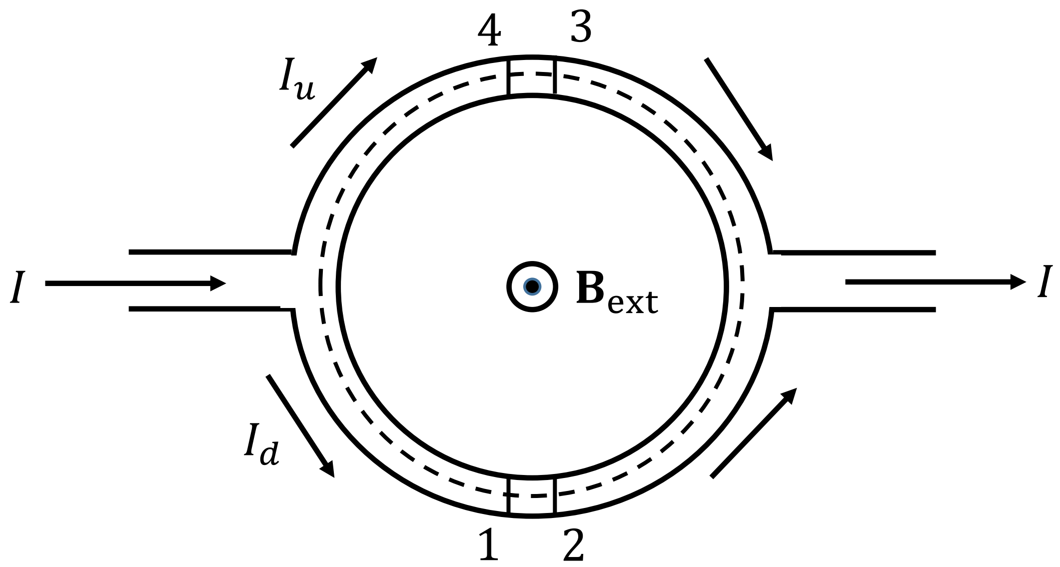


$$\Lambda(\mathbf{r},t)\mathbf{J}_s(\mathbf{r},t) = \frac{\hbar}{q^*}\boldsymbol{\gamma}(\mathbf{r},t) = \frac{\hbar}{q^*}\nabla\theta(\mathbf{r},t) - \mathbf{A}(\mathbf{r},t)$$

$$\varphi_d = \int_1^2 \boldsymbol{\gamma}\cdot d\mathbf{l} = \theta_2 - \theta_1 + \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A}\cdot d\mathbf{l} \qquad I_d = -I_c \sin\varphi_d$$

$$\varphi_u = \int_4^3 \boldsymbol{\gamma}\cdot d\mathbf{l} = \theta_4 - \theta_3 + \frac{2\pi}{\Phi_0}\int_4^3 \mathbf{A}\cdot d\mathbf{l} \qquad I_u = -I_c \sin\varphi_u$$

Tunable Josephson junction



Phase change around the dashed loop (1 -> 2 -> 3 -> 4 -> 1)

$$\theta_2 - \theta_1 = \varphi_d - \frac{2\pi}{\Phi_0}\int_1^2 \mathbf{A} \cdot d\mathbf{l} \qquad \theta_3 - \theta_2 = -\frac{2\pi}{\Phi_0}\int_2^3 (\mathbf{A} + \Lambda \mathbf{J}_s) \cdot d\mathbf{l}$$

$$\theta_4 - \theta_3 = -\varphi_u - \frac{2\pi}{\Phi_0}\int_3^4 \mathbf{A} \cdot d\mathbf{l} \qquad \theta_1 - \theta_4 = -\frac{2\pi}{\Phi_0}\int_4^1 (\mathbf{A} + \Lambda \mathbf{J}_s) \cdot d\mathbf{l}$$

$$2n\pi = \varphi_d - \varphi_u - \frac{2\pi}{\Phi_0}\oint \mathbf{A} \cdot d\mathbf{l} = \varphi_d - \varphi_u - \frac{2\pi}{\Phi_0}\Phi$$

Tunable Josephson junction

$$\varphi_d - \varphi_u = \frac{2\pi}{\Phi_0}\Phi + 2n\pi$$

$$I = I_d + I_u = -I_c \sin\varphi_d - I_c \sin\varphi_u = -2I_c \sin\left(\frac{\varphi_d + \varphi_u}{2}\right)\cos\left(\frac{\varphi_d - \varphi_u}{2}\right)$$

$$= -2I_c \cos\left(\pi\frac{\Phi}{\Phi_0} + n\pi\right)\sin\left(\varphi_d - \pi\frac{\Phi}{\Phi_0} - n\pi\right)$$

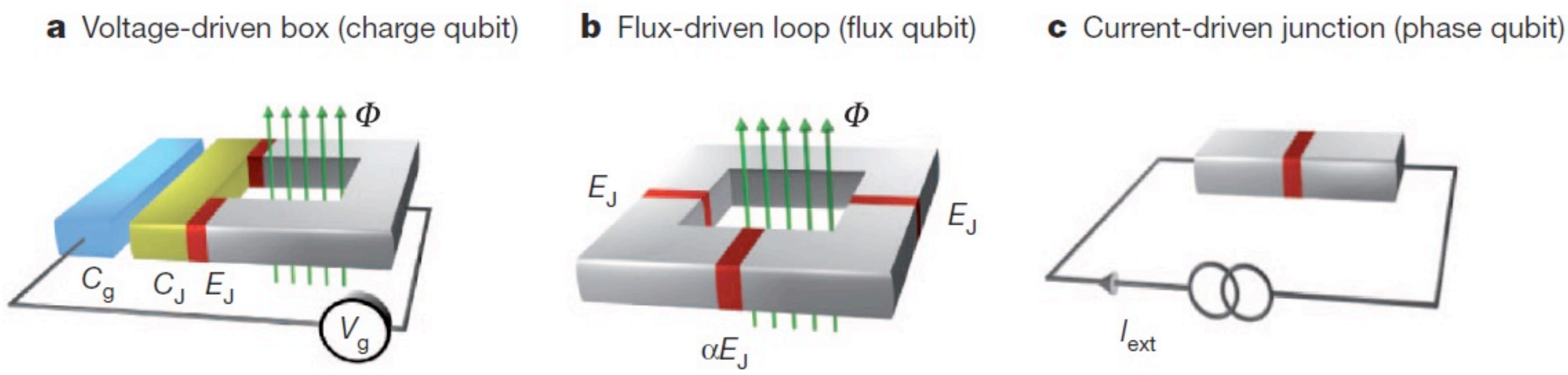
$$= -2I_c \cos\left(\pi\frac{\Phi}{\Phi_0}\right)\sin\left(\varphi_d - \pi\frac{\Phi}{\Phi_0}\right)$$

For small induced flux (i.e. for small loop inductance) compared with the external flux, it can be considered a tunable Josephson junction with the paramters controlled by the external flux.

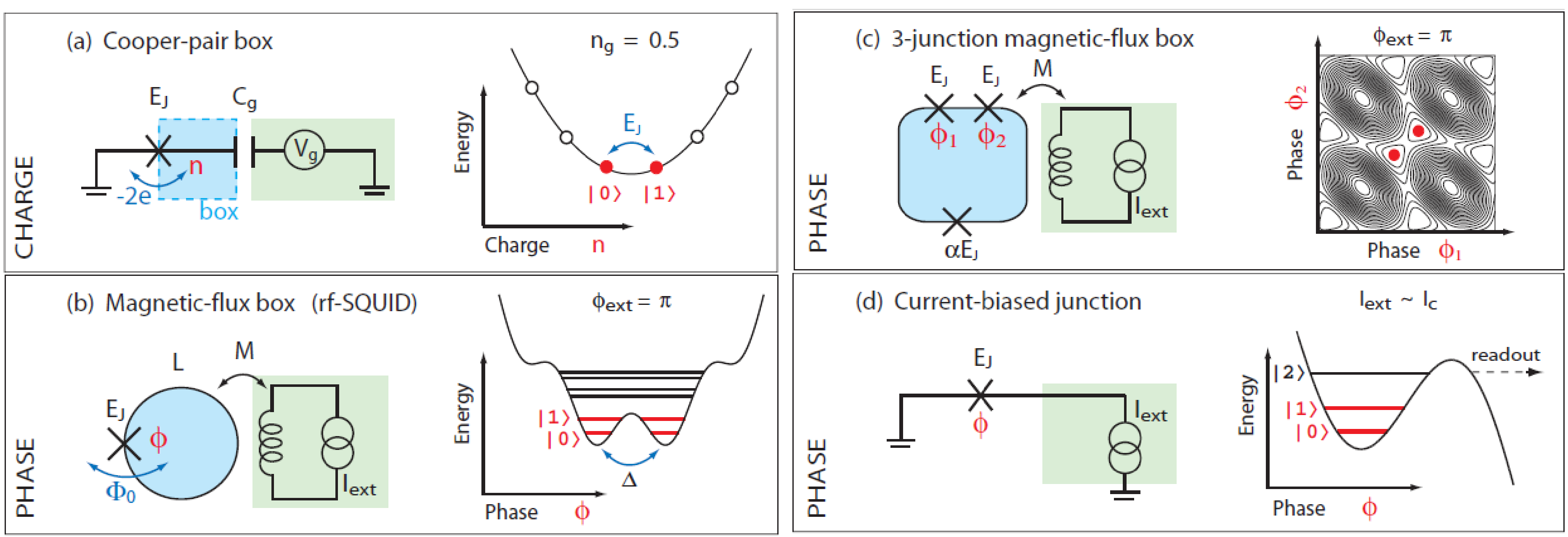
$$I = -\bar{I}_c \sin\bar{\varphi}$$

$$\bar{I}_c = 2I_c \cos\left(\pi\frac{\Phi_{ext}}{\Phi_0}\right) \qquad \bar{\varphi} = \varphi_d - \pi\frac{\Phi_{ext}}{\Phi_0} = \varphi_u + \pi\frac{\Phi_{ext}}{\Phi_0} = \frac{\varphi_d + \varphi_u}{2}$$

Basic types of Superconducting qubits

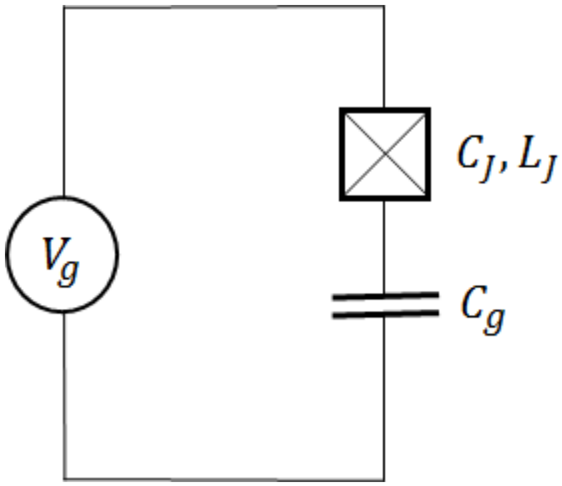


You and Nori, Nature **474**, 589 (2011)



You and Nori, Phys. Today **58**, 42 (2005)

Charge qubit : Cooper pair box



Josephson junction is capacitively connected to external voltage.

Superconducting island between JJ and the capacitor -> Cooper pair box

$$\hat{H} = 4E_C\left(\hat{N} - n_g\right)^2 - E_J \cos\hat{\varphi}$$

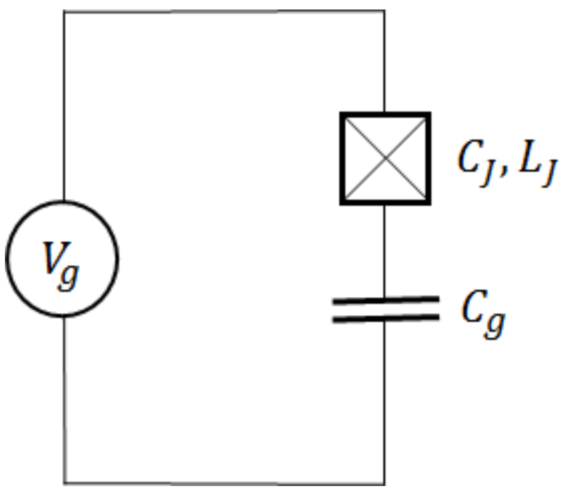
$$E_C = \frac{e^2}{2\left(C_J + C_g\right)} \qquad n_g = \frac{C_g V_g}{2e}$$

$\hat{N}$ = Number of Cooper pair tunneled into the SC island
$\hat{\varphi}$ = Phase different across JJ

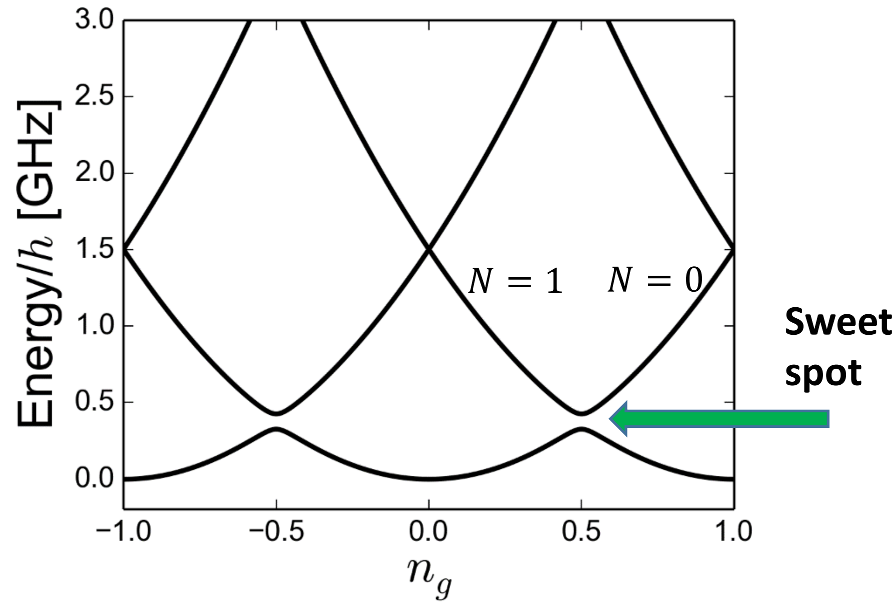
$$[\hat{N}, \hat{\varphi}] = i$$

$C_g V_g = 2en_g$ is called gate charge or offset charge, which can be controlled by the voltage.

Charge qubit



$E_J \ll E_C$



$$\hat{H} = 4E_C\left(\hat{N} - n_g\right)^2 - E_J \cos\hat{\phi}$$

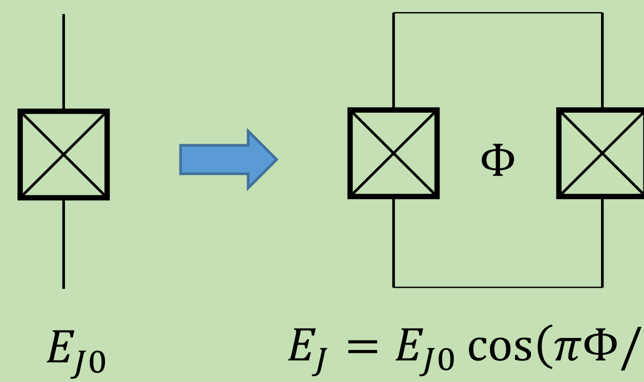
$$E_C = \frac{e^2}{2(C_J + C_g)} \qquad n_g = \frac{C_g V_g}{2e}$$

$\hat{N}$ = Number of Cooper pair in SC island

$\hat{\phi}$ = Phase different across JJ

$$[\hat{N}, \hat{\phi}] = i$$

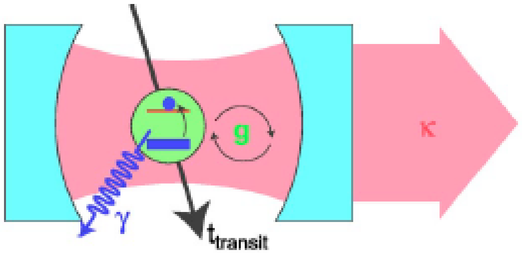
Tunable JJ

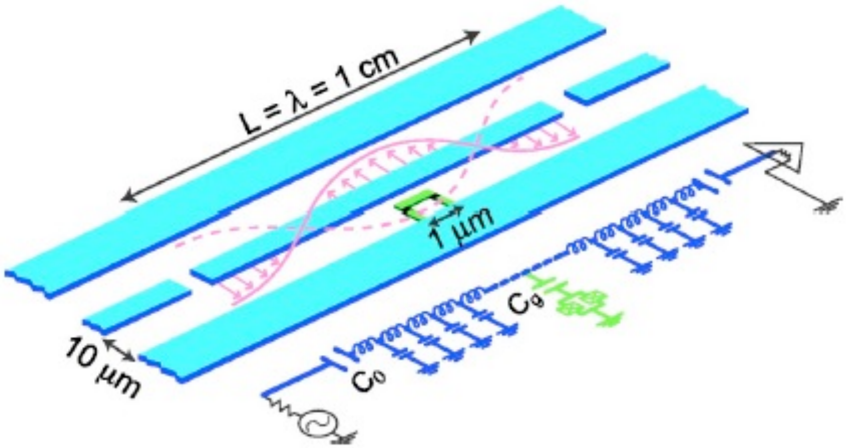


$$E_J = E_{J0}\cos(\pi\Phi/\Phi_0)$$

Circuit QED

A. Blais et al, PRA **69**, 062320 (2004)





Cavity QED
Atom in an optical cavity

Circuit QED
Artificial atom (qubit) in a superconducting microwave cavity

Jaynes-Cummings Hamiltonian

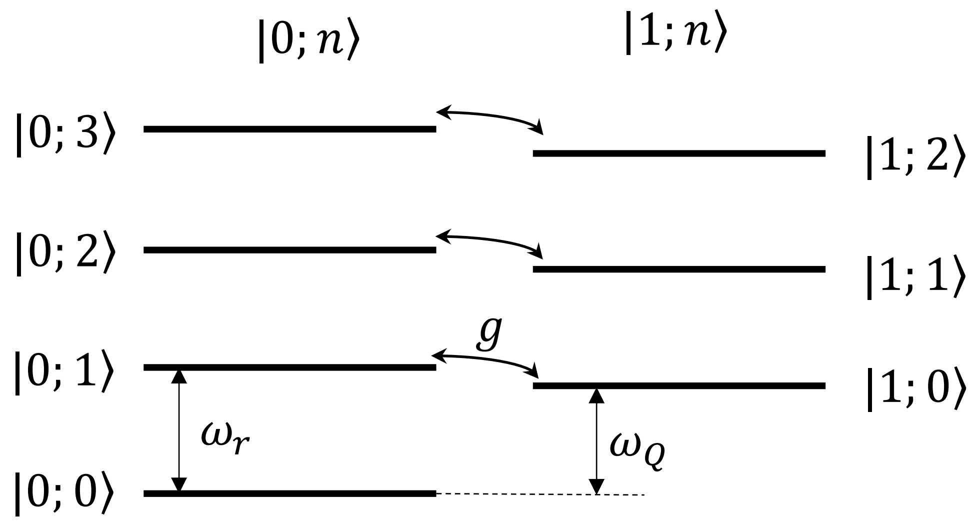
$$\hat{H}_{JC} = -\frac{\hbar\omega_Q}{2}\,\hat{\sigma}_z + \hbar\omega_r\,\hat{a}^+\hat{a} + g\,(\hat{\sigma}^+ + \hat{\sigma}^-)(\hat{a}^+ + \hat{a})$$

Qubit   Cavity mode   Coupling $\propto \mathbf{d}\cdot\mathbf{E}$

$$= -\frac{\hbar\omega_Q}{2}\,\hat{\sigma}_z + \hbar\omega_r\,\hat{a}^+\hat{a} + g\,(\hat{\sigma}^+\hat{a} + \hat{\sigma}^-\hat{a}^+)$$

RWA

Jaynes-Cummings Hamiltonian



In the subspace $\langle\{|0;n+1\rangle, |1;n\rangle\}\rangle$

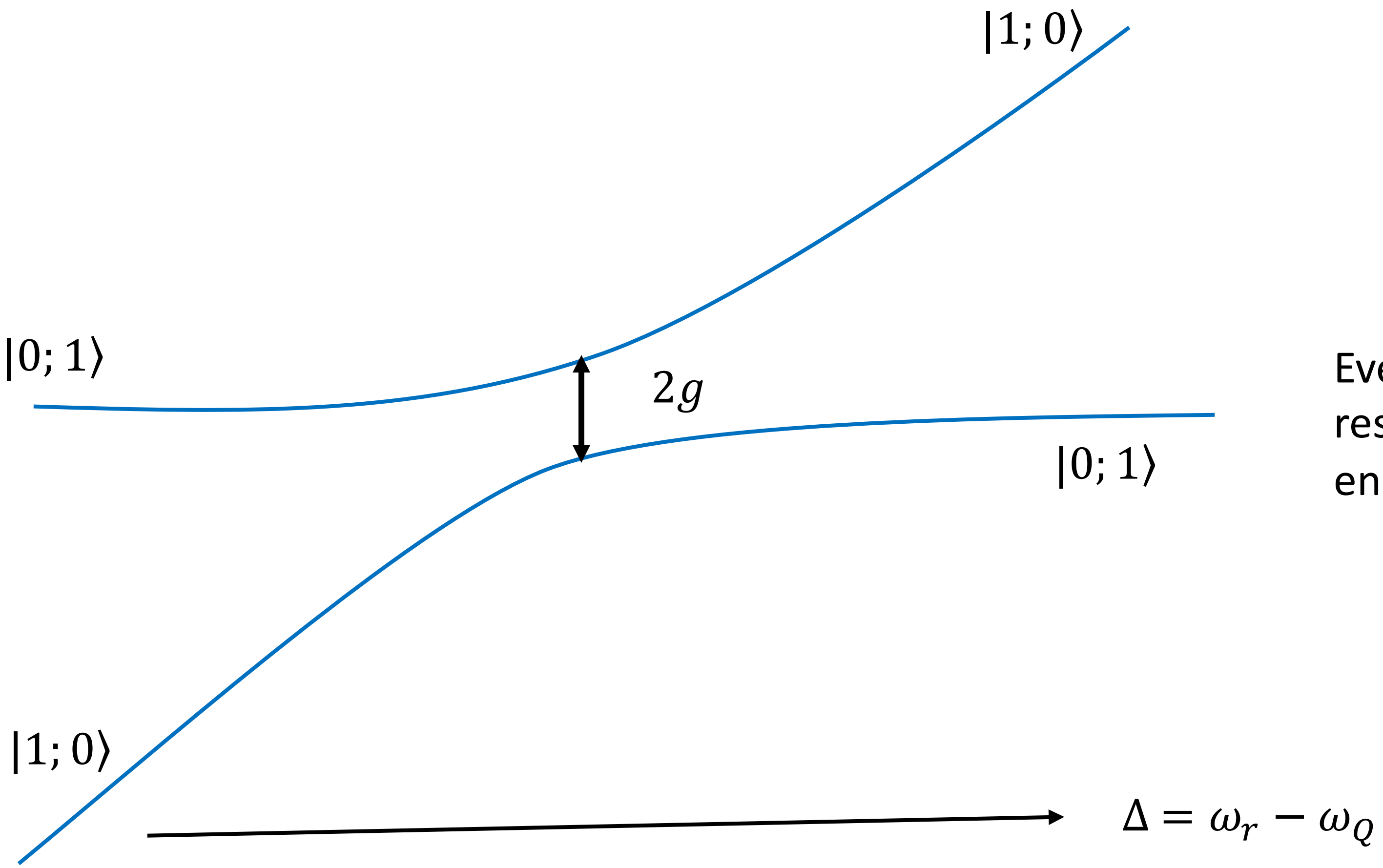
$$\hat{H} = \hbar\omega_r\left(n+\frac{1}{2}\right) + \begin{pmatrix} \frac{\Delta}{2} & g\sqrt{n+1} \\ g\sqrt{n+1} & -\frac{\Delta}{2} \end{pmatrix} \qquad \Delta = \hbar(\omega_r - \omega_Q)$$

Vacuum Rabi splitting

For n=0 (vacuum)

$$\hat{H} = \frac{1}{2}\hbar\omega_r + \begin{pmatrix} \frac{\hbar\Delta}{2} & g \\ g & -\frac{\hbar\Delta}{2} \end{pmatrix}$$

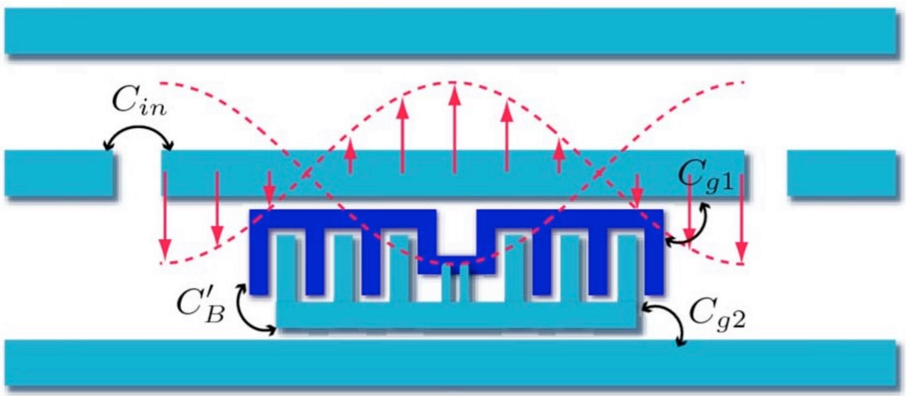
$$E = \frac{1}{2}\hbar\omega_r \pm \frac{1}{2}\sqrt{\Delta^2 + 4g^2}$$

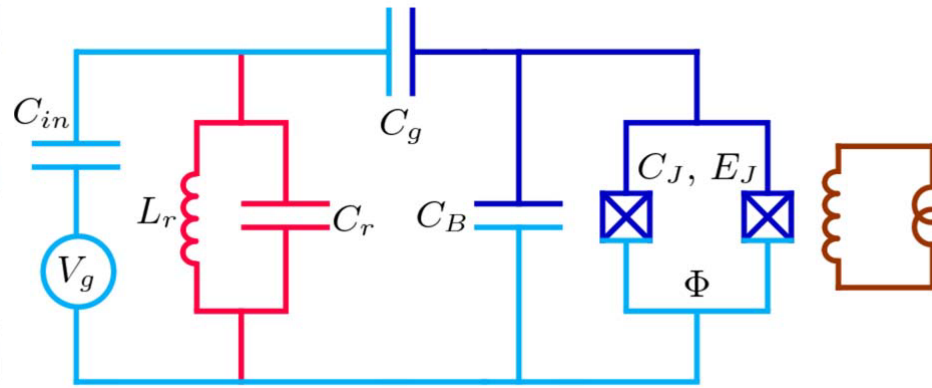


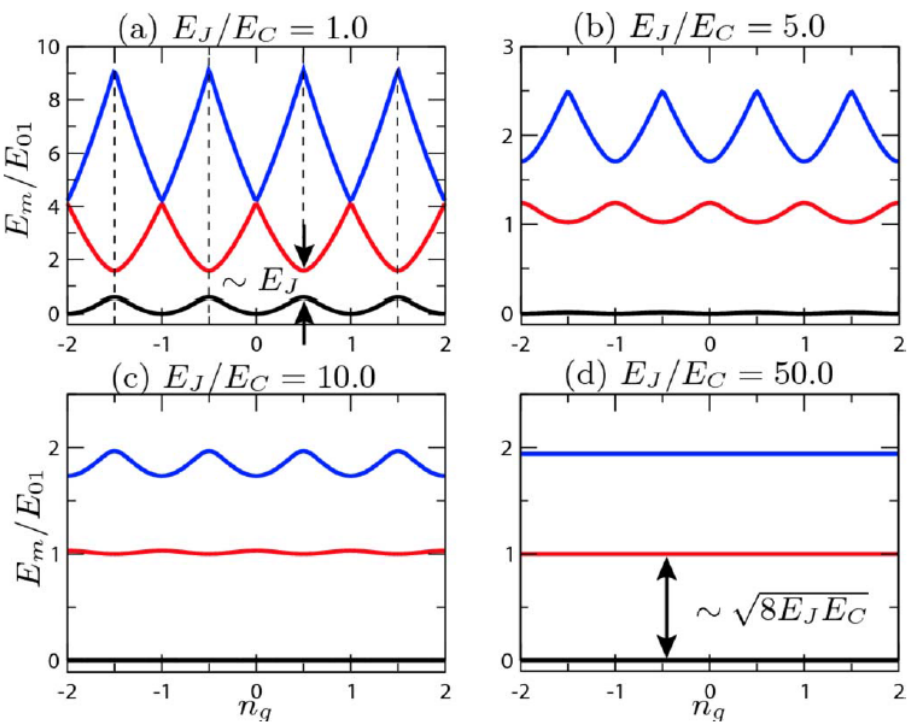
Even in the vacuum resonator, there is a energy spliting!

Transmon Qubit

Koch *et al.*, PRA **76**, 042319 (2007)

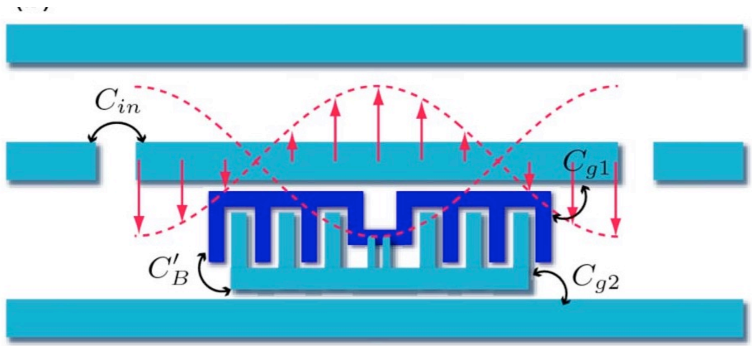






- Large shunt capacitance C_B ($E_J \gg E_C$)
- Qubit states are insensitive to charge noise (**everywhere is sweet spot**)
- Anharmonicity is reduced, but still large enough for qubit operations
- Needs microwaves for gate operations

Single qubit gate operations: microwave drive



Apply oscillating electric field (i.e. microwave) through a drive line capacitively coupled to the qubit.

$$\widehat{H}_d = -\hat{\mathbf{d}} \cdot \mathbf{E}$$

$\langle 0|\hat{\mathbf{d}}|1\rangle = \mathbf{d_{01}}$ = transition dipole matrix element

$$\mathbf{E} = \mathbf{E}_0 \cos(\omega_d t + \delta) = \frac{1}{2}\left(\mathbf{E}_0\, e^{-i(\omega_d t+\delta)} + \mathbf{E}_0^*\, e^{i(\omega_d t+\delta)}\right)$$

$$\widehat{H}_d = -\left(Ae^{i\omega_d t} + Be^{-i\omega_d t}\right)|0\rangle\langle 1| + h.c.$$

$$A = \mathbf{d_{01}} \cdot \mathbf{E}_0\, e^{i\delta} \qquad B = \mathbf{d_{01}} \cdot \mathbf{E}_0\, e^{-i\delta}$$

Single qubit

$$\widehat{H}_Q = -\frac{\hbar\omega_Q}{2}\hat{\sigma}_z + \widehat{H}_d = \begin{pmatrix} -\dfrac{\hbar\omega_Q}{2} & Ae^{i\omega_d t} + Be^{-i\omega_d t} \\ A^* e^{-i\omega_d t} + B^* e^{-i\omega_d t} & \dfrac{\hbar\omega_Q}{2} \end{pmatrix}$$

Rotating wave approximation (RWA)

In the interaction picture with $\widehat{H}_Q = \widehat{H}_0 + \widehat{H}_d = -\frac{\hbar\omega_Q}{2}\,\hat{\sigma}_z + \widehat{H}_d$

$$\widehat{H}_{dI} = e^{\frac{i\widehat{H}_0 t}{\hbar}}\widehat{H}_d e^{-\frac{i\widehat{H}_0 t}{\hbar}} = -\left(Ae^{i(\omega_d-\omega_Q)t} + Be^{-i(\omega_d+\omega_Q)t}\right)|0\rangle\langle 1| + h.c.$$

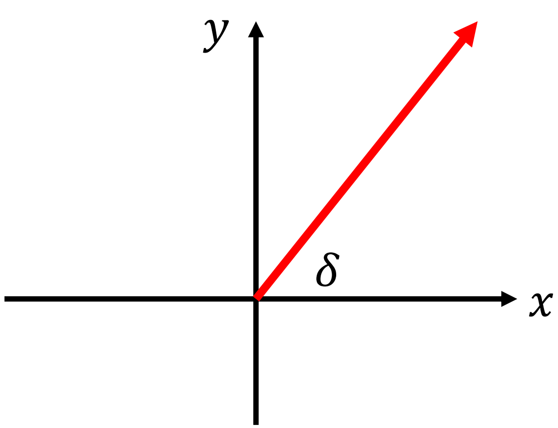
Fast oscillating term → drop

$$\approx -Ae^{i(\omega_d-\omega_Q)t}|0\rangle\langle 1| - A^* e^{-i(\omega_d-\omega_Q)t}|1\rangle\langle 0|$$

If we drive resonant microwave $\omega_d - \omega_Q = 0,\qquad \mathbf{E} = \mathbf{E}_0\cos(\omega_Q t + \delta)$

$$\widehat{H}_{dI} \approx -|A|e^{i\delta}|0\rangle\langle 1| - |A|e^{-i\delta}|1\rangle\langle 0| = -|A|\left(\cos\delta\ \hat{\sigma}_x - \sin\delta\,\hat{\sigma}_y\right)$$

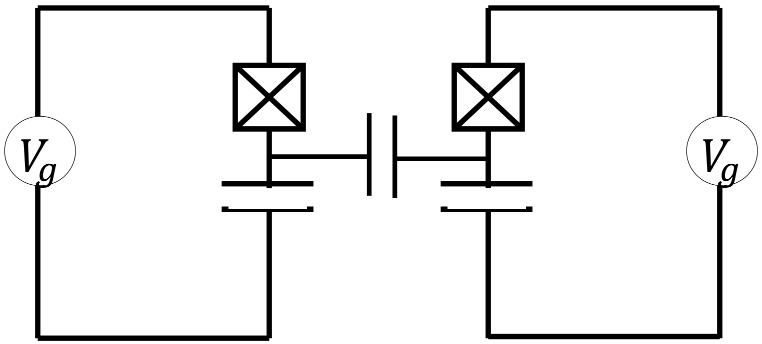
$$i\hbar\frac{\partial}{\partial t}|\psi_I(t)\rangle = \widehat{H}_{dI}|\psi_I(t)\rangle \quad \Rightarrow \quad |\psi_I(t)\rangle = e^{-\frac{i\widehat{H}_{dI}t}{\hbar}}|\psi_I(t)\rangle$$

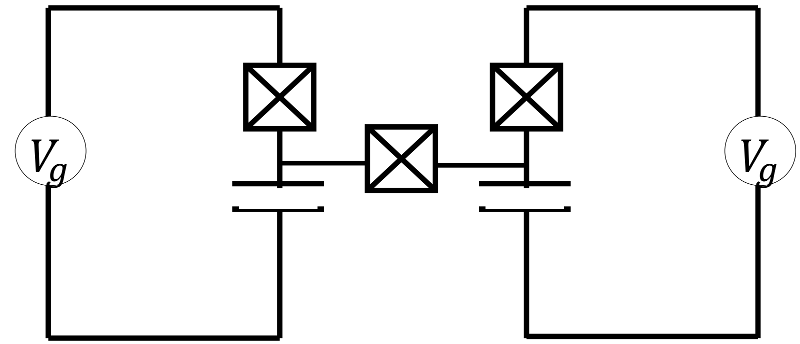


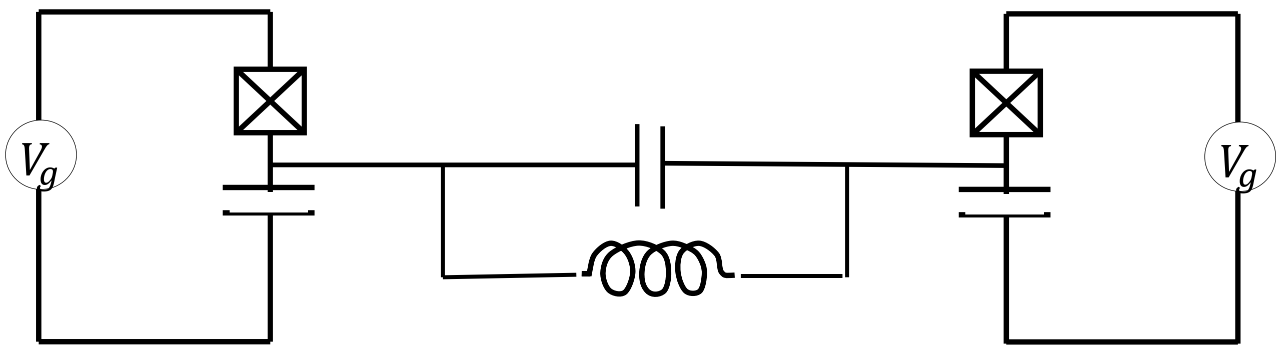
- In the rotation frame that rotates with ω_Q
- Rotation around an axis in the xy plane.
- We can realize x and y rotations by choosing the phase of the microwave.
- All single qubit gates can be created using this.

Two qubit gate operations

Interaction between qubits can be used to implement two qubit gates.





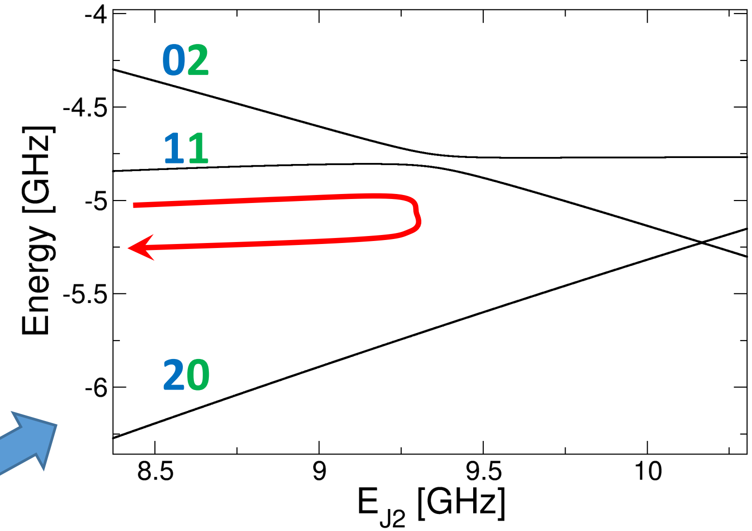


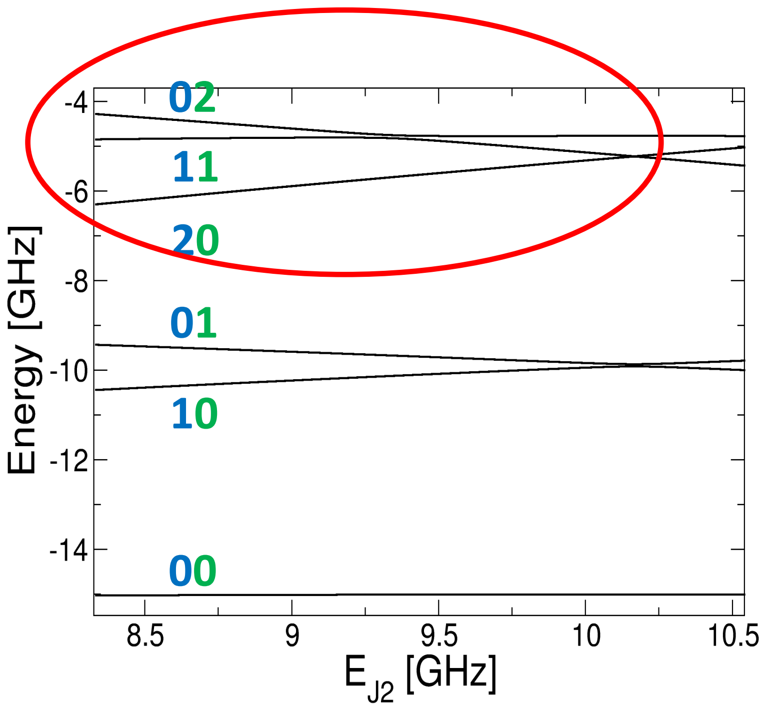
Two qubit gate : Adiabatic CPHASE gate

Two capacitively coupled transmons

OFF: detuning $(\omega_1 - \omega_2)$ >> coupling
Change the detuning such that 11 and 02 states are close.
Come back to OFF state.

Only 11 state accumulated nontrivial phase!





00 01 10 11

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{i\varphi_R} & 0 & 0 \\ 0 & 0 & e^{i\varphi_L} & 0 \\ 0 & 0 & 0 & e^{i(\varphi_L+\varphi_R+\delta\varphi)} \end{pmatrix}$$

$$= \left(\begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi_L} \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi_R} \end{pmatrix} \right) \times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\delta\varphi} \end{pmatrix}$$

Dispersive Readout

$$\hat{H}_{JC} = -\frac{\hbar\omega_Q}{2}\hat{\sigma}_z + \hbar\omega_r\,\hat{a}^+\hat{a} + g\,(\hat{\sigma}^+\hat{a} + \hat{\sigma}^-\hat{a}^+)$$

At large detuning $\Delta \gg g$, $\Delta = \hbar\,(\omega_Q - \omega_r)$

After a unitary transformation by $\hat{U} = \exp\left(\frac{g}{\Delta}(\hat{a}\hat{\sigma}^+ - \hat{a}^+\hat{\sigma}^-)\right)$ and up to the 2nd order in g,

$$\hat{H}_{\text{eff}} = \hbar\omega_r\hat{a}^+\hat{a} - \frac{\hbar}{2}\left(\omega_Q - \frac{g^2}{\Delta}(2\hat{a}^+\hat{a} + 1)\right)\hat{\sigma}_z$$

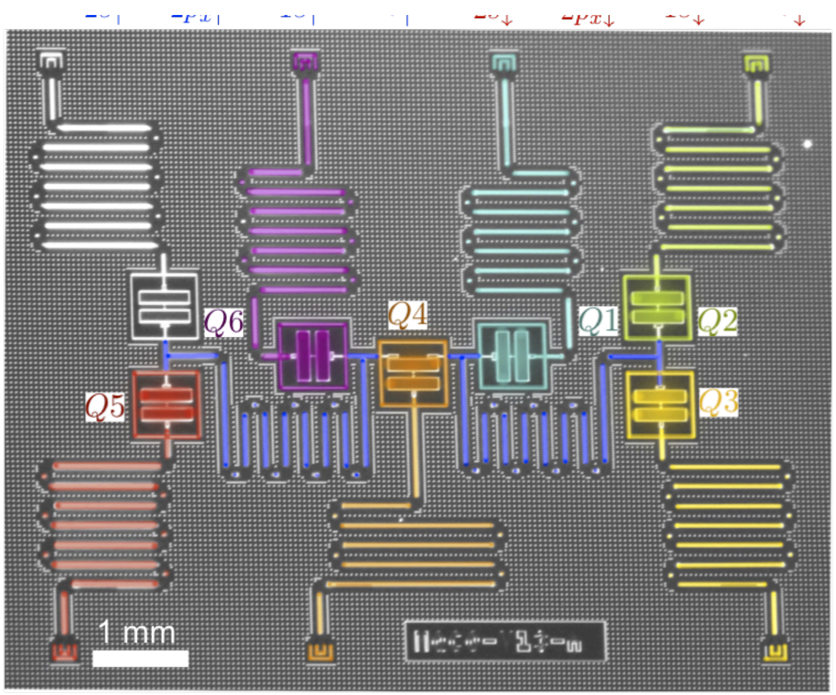
Qubit frequency is shifted depending on the # of photons in the cavity
→ ac-Stark shift

$$\hat{H}_{\text{eff}} = \hbar\left(\omega_r + \frac{g^2}{\Delta}\hat{\sigma}_z\right)\hat{a}^+\hat{a} - \frac{\hbar}{2}\left(\omega_Q - \frac{g^2}{\Delta}\right)\hat{\sigma}_z$$

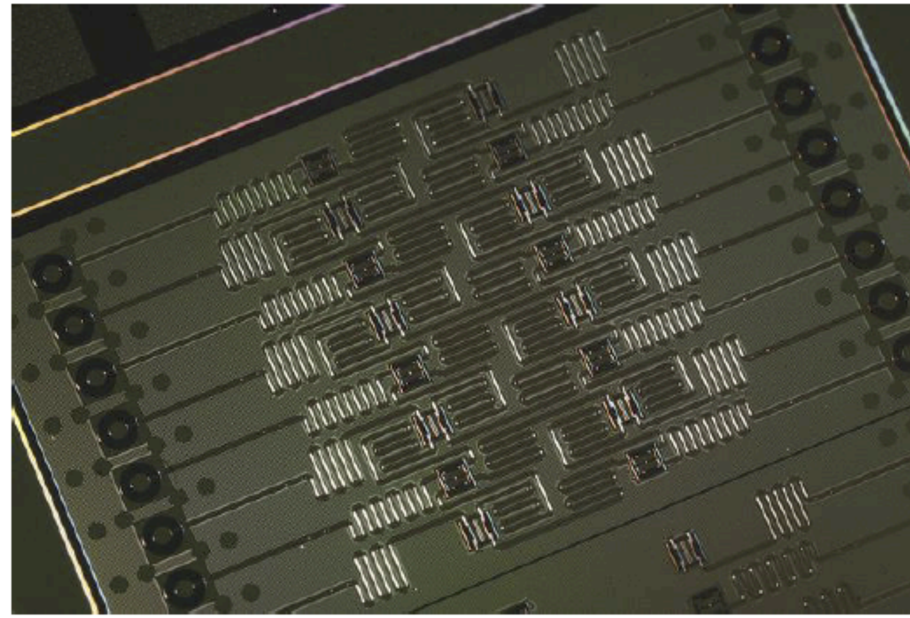
Alternatively, we can consider it as a shift in the cavity mode frequency depending on the qubit state → Dispersive readout

IBM

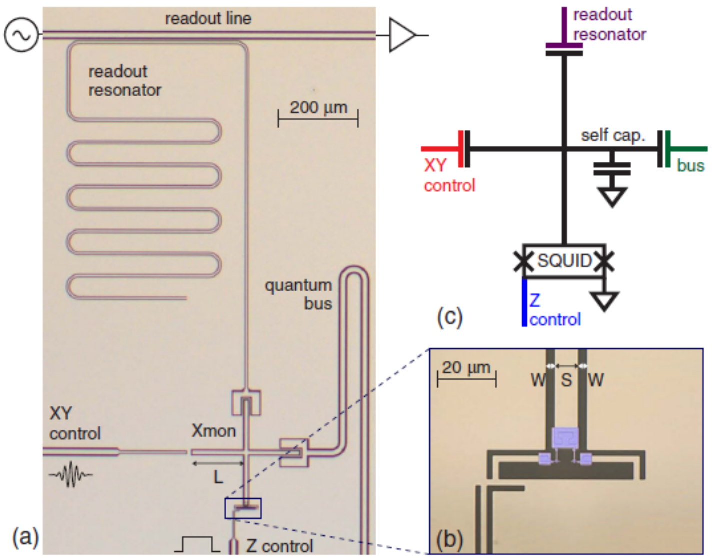
b



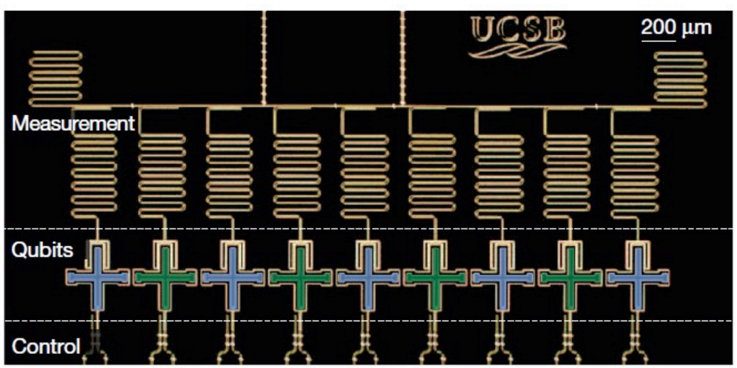
IBM 16-qubit processor



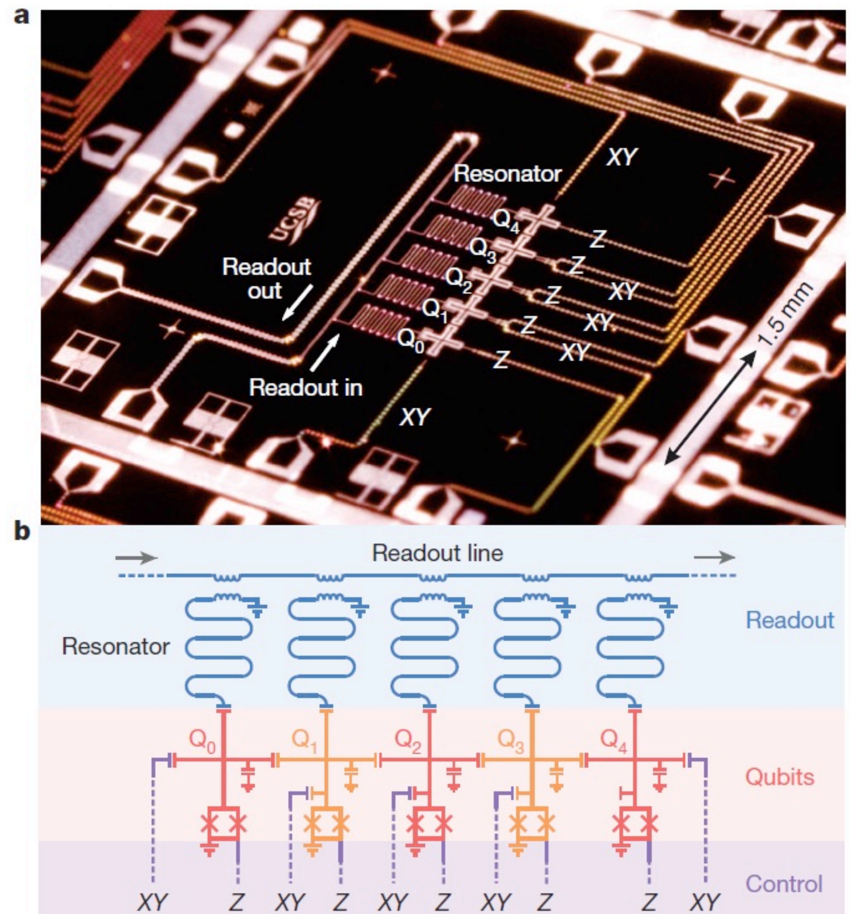
Xmons (Martinis, Google)



*Barends et al, PRL **111**, 080502 (2013)*



*Kelly et al, Nature **519**, 66 (2015)*



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