Lee-Yang zeros analysis of effective theories toward lattice QCD

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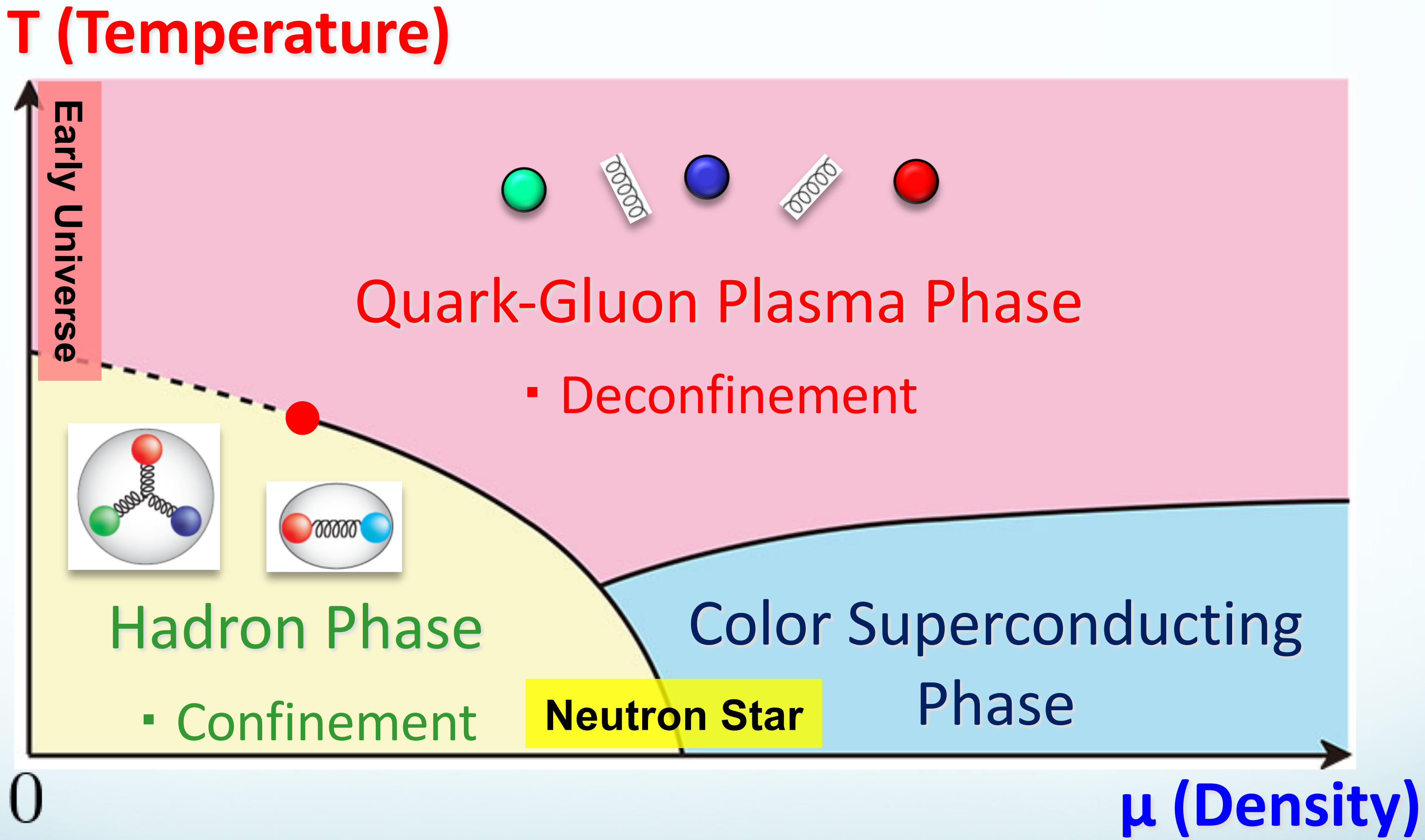
Pukyong National University

The Future of lattice-QCD studies in Korea
@ PKNU (2019.9.6)

Contents

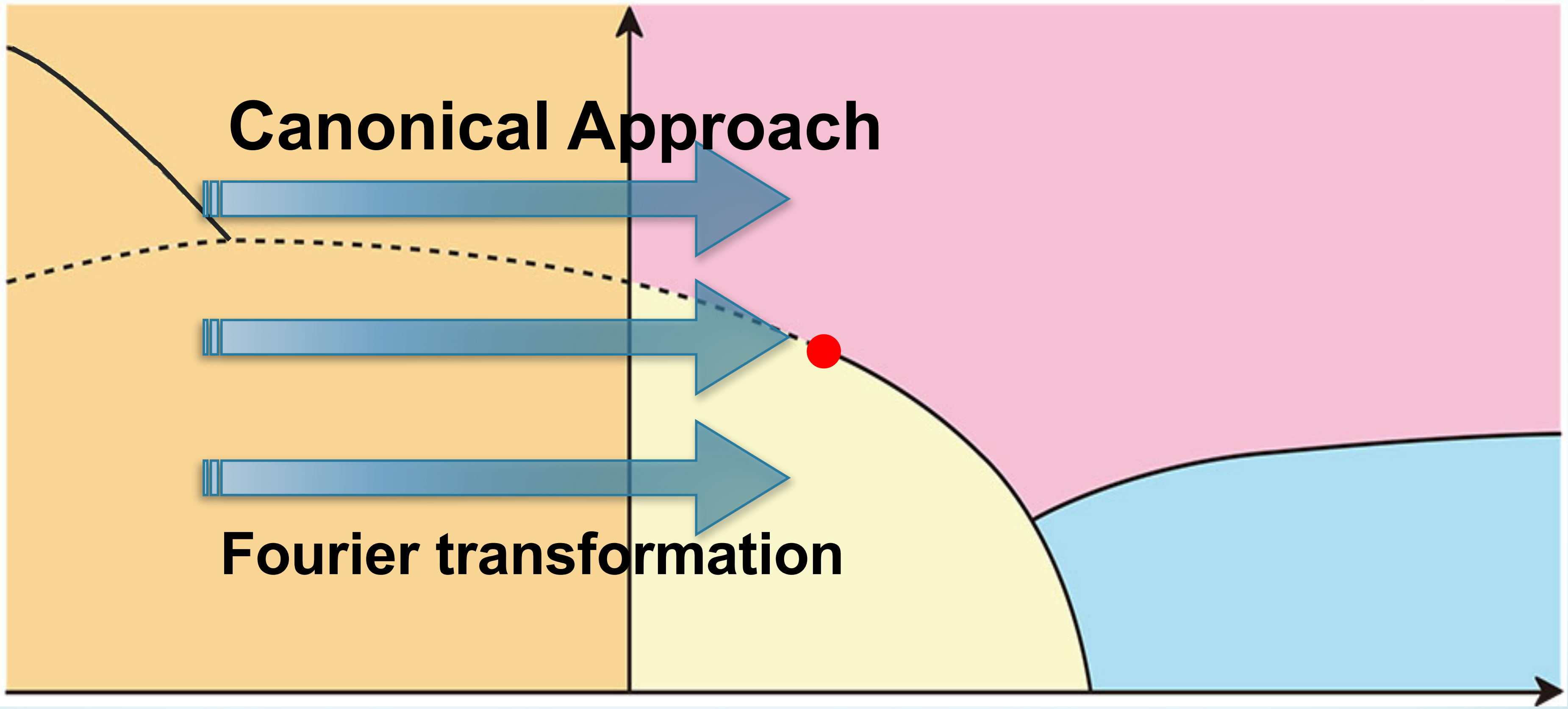
QCD Phase diagram (Prediction)



Where are the critical point and the phase transition line?
Lattice QCD at finite density: Existence of the sign problem

QCD Phase diagram (Prediction)

T (Temperature)



$[\det D(i\mu_{qI})]^* = \det D(i\mu_{qI})$ 0 $[\det D(\mu_q)]^* = \det D(-\mu_q^*)$ μ_q^2

Pure imaginary chemical potential: $\mu_q = i\mu_{qI}$

Canonical Approach

Fugacity expansion

Grand Canonical partition function

$$\begin{aligned} \underline{Z_{\mathrm{GC}}(\mu_q, T, V)} &= \mathrm{Tr}\left(e^{-(\hat{H}-\mu_q\hat{N})/T}\right) \\ &= \sum_n \langle n|\, e^{-(\hat{H}-\mu_q\hat{N})/T}\, |n\rangle \\ &= \sum_n \langle n|\, e^{-\hat{H}/T}\, |n\rangle\, e^{n\mu_q/T} \\ &= \sum_n \underline{Z(n, T, V)}\xi^n \end{aligned}$$

Fugacity: $\xi = e^{\mu_q/T}$

Canonical partition function

Canonical Approach

Fugacity expansion

Grand Canonical partition function

$$Z_{\mathrm{GC}}(\mu_q, T, V) = \sum_{n=-\infty}^{\infty} Z(n, T, V)\xi^n$$

Fugacity: $\xi = e^{\mu_q/T}$

Canonical partition function

Fourier transformation

A. Hasenfrantz & D. Toussaint, Nucl. Phys. B371 (1992)

$$Z(n, T, V) = \int_0^{2\pi} \frac{d(\mu_{qI}/T)}{2\pi} e^{-in\mu_{qI}/T} Z_{\mathrm{GC}}(\mu_q = i\mu_{qI}, T, V)$$

We can calculate Z_{GC} with Monte Carlo Method at pure imaginary μ_q.

$$[\det D(i\mu_{qI})]^* = \det D(i\mu_{qI})$$

History

Basic Idea of Canonical Approach

A. Hasenfrantz, D. Toussaint, Nucl. Phys. B371 **(1992)**

❌ **Numerical instability of (discrete) Fourier transformation**

Sign Problem ? ⇒ No, this is caused by cancelation of significant digits !

R.Fukuda, A.Nakamura, S.Oka, PRD93 **(2016)**

1.234567890123456 – 1.234567890123455 = 0.000000000000001
(16 significant digits) (1 significant digits)

↓ **Multiple-precision arithmetic**

1.234567890123456666666
– 1.234567890123455555555 = 0.000000000000001111111111
(24 significant digits) (10 significant digits)

History

Basic Idea of Canonical Approach

A. Hasenfrantz, D. Toussaint, Nucl. Phys. B371 **(1992)**

✕ **Numerical instability of (discrete) Fourier transformation**

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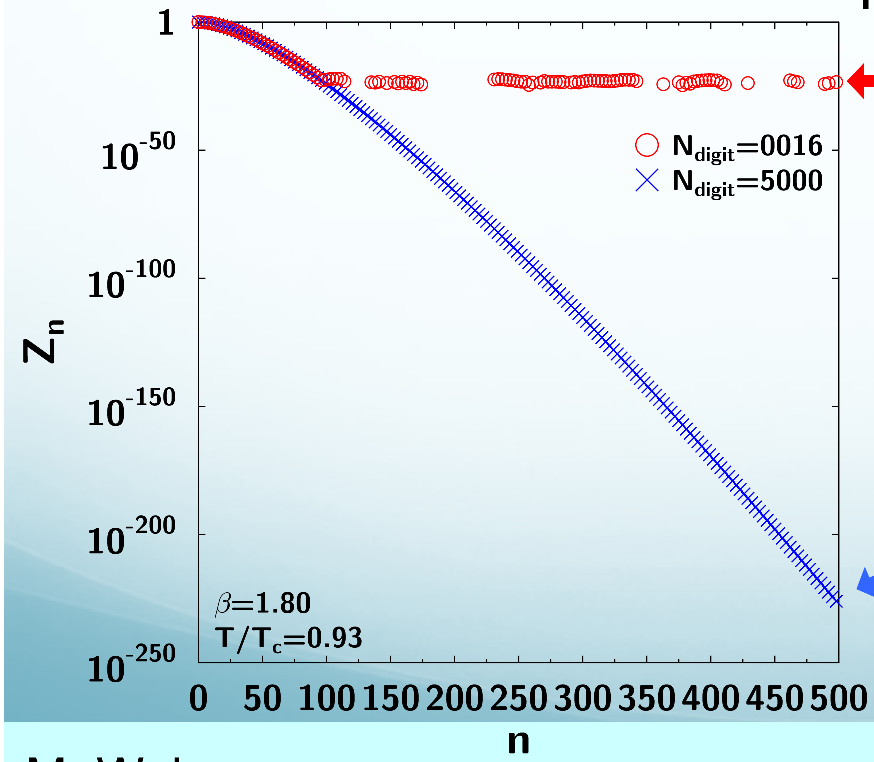


In double-precision arithmetic, cancelation of significant digits occurs at high n region.

In multiple-precision arithmetic, we can evaluate Zn up to high n region with accuracy.

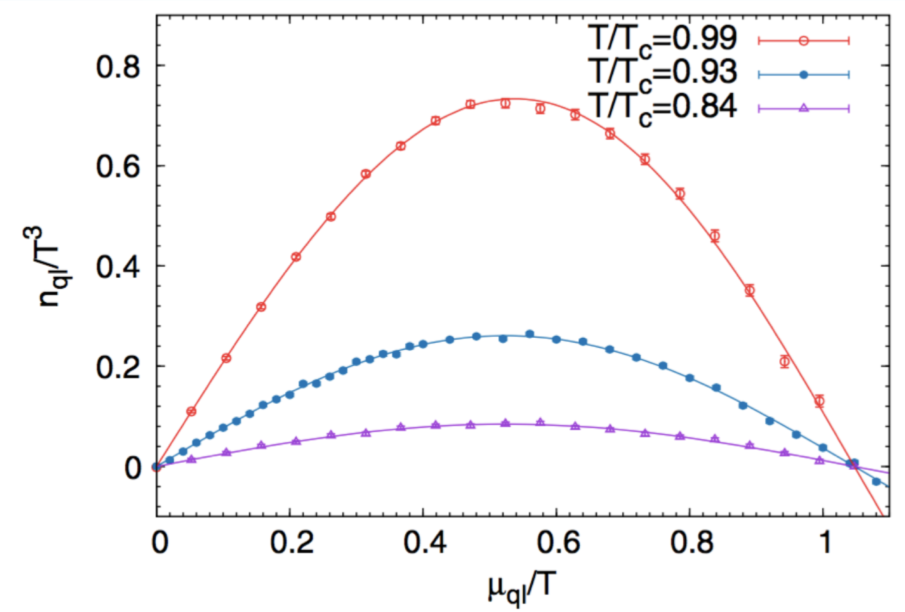
Number density Integration method

How to calculate $Z_{\mathrm{GC}}(\mu_q = i\mu_{qI}, T, V)$

V.G. Bornyakov et al., PRD95, 094506 (2017)

Quark number density

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q}\ln Z_{\mathrm{GC}}$$
$$= \frac{1}{VT^3}\frac{1}{Z_{\mathrm{GC}}}\int \mathcal{D}U\ \det D(\mu_q)\, e^{-S_G}\,\mathrm{Tr}\left[D^{-1}\frac{\partial D}{\partial(\mu_q/T)}\right]$$



$$n_q = i n_{qI} \qquad \theta = \frac{\mu_{qI}}{T}$$

Approximated by a Fourier series.

$$\frac{n_{qI}}{T^3}(\theta) \sim \sum_{k=1}^{N_{\sin}} f_k \sin(k\theta)$$

Outline

Lattice QCD

$$n_q(\mu_q = i\mu_{qI}, T, V)$$

Integration method
V. Bornyakov et al., PRD95(2017)

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q} \ln Z_{\rm GC}$$

$$Z_{\rm GC}(\mu_q = i\mu_{qI}, T, V)$$

Fourier transformation

$$Z(n, T, V)$$

$$Z_{\rm GC}(\mu_q, T, V) = \sum_{n=-\infty}^{\infty} Z(n, T, V)\xi^n \qquad \xi = e^{\mu_q/T}$$

If we get Z_n for all n, we can search at ANY density!

Outline

Lattice QCD

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Integration method
V. Bornyakov et al., PRD95(2017)

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$$Z_{\rm GC}(\mu_q, T, V) = \sum_{n=-\mathbf{N_{max}}}^{\mathbf{N_{max}}} Z(n, T, V)\xi^n \qquad \xi = e^{\mu_q/T}$$

In numerical calculations, n is finite.

Contents

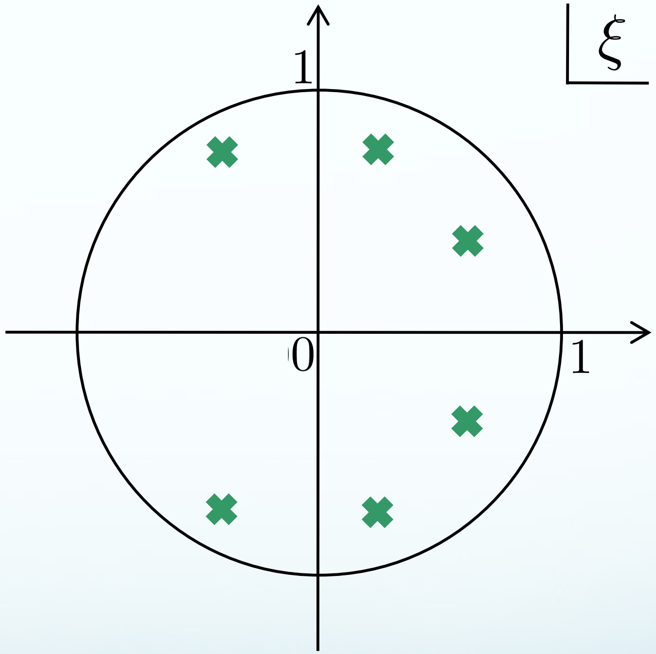
Lee-Yang Zeros

Zeros of Z_{GC} called Lee-Yang Zeros contain a valuable information on the phase transitions of a system.

T.D. Lee & C.N. Yang, Phys. Rev. 87, 404&410 (1952)

$$Z_{\rm GC}(\mu_q, T, V) = \sum_{n=-\mathbf{N_{max}}}^{\mathbf{N_{max}}} Z(n, T, V)\xi^n = 0$$

There are $2N_{max}$ LYZs in the complex $\xi = e^{\mu_q/T}$ plane.



N_{max} ~ small

Lee-Yang Zeros

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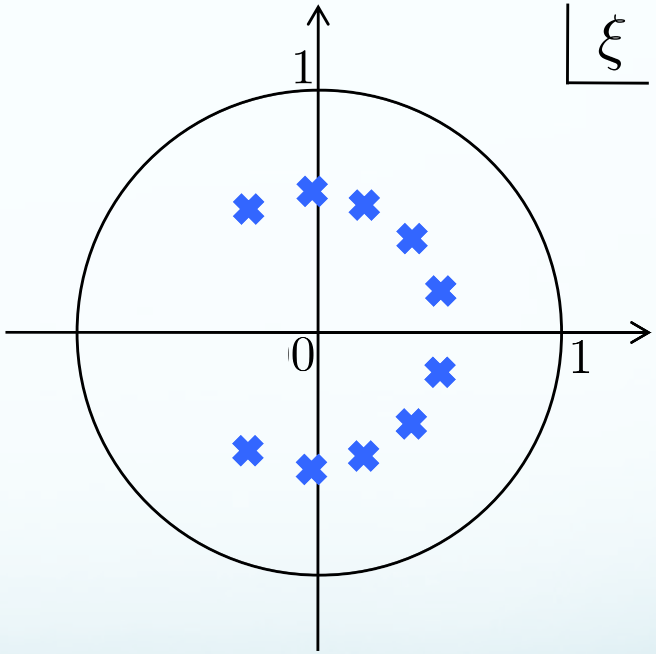

N_{max} ~ large

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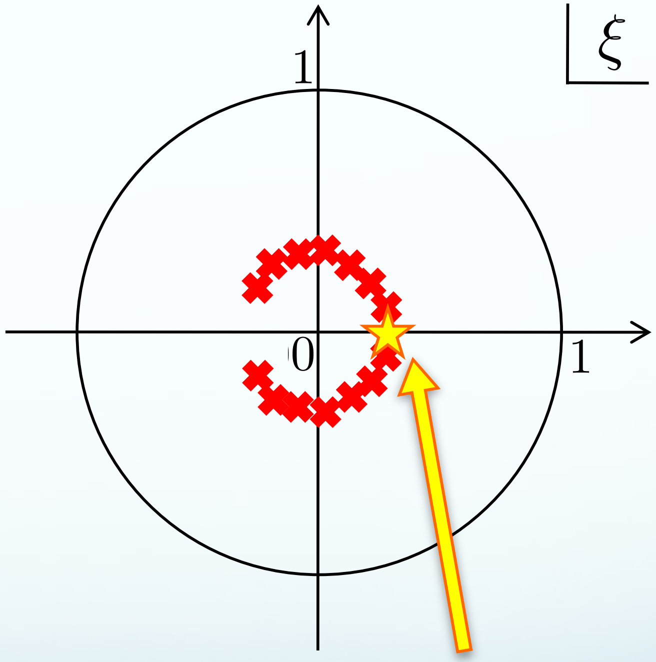

Phase Transition
N_{max} ~ infinity
(V ~ infinity)

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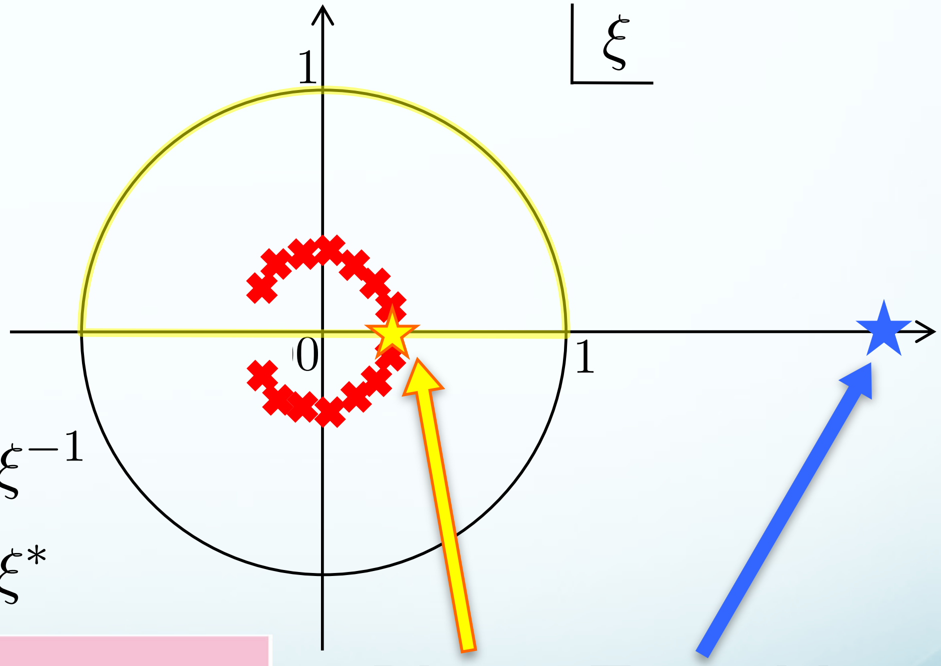


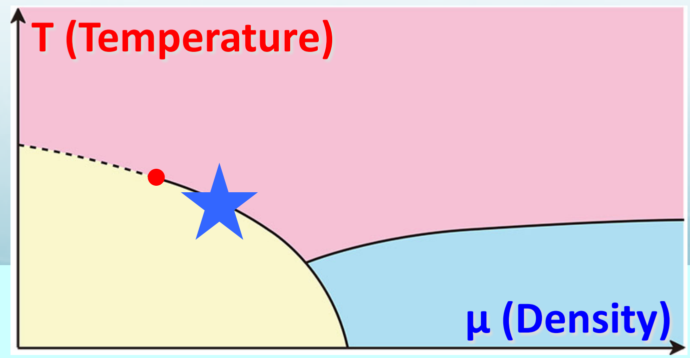
Z(n) properties

$Z(n,T,V) = Z(-n,T,V)$ ⇒ $\xi \leftrightarrow \xi^{-1}$

$Z(n,T,V)$: **Real values** ⇒ $\xi \leftrightarrow \xi^{*}$

Phase Transition
N_{max} ~ infinity
(V ~ infinity)

$\xi = e^{\mu_q/T} =$ ★



Outline

Lattice QCD

$n_q(\mu_q = i\mu_{qI}, T, V)$

Integration method
V. Bornyakov et al., PRD95(2017)

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q}\ln Z_{\rm GC}$$

$Z_{\rm GC}(\mu_q = i\mu_{qI}, T, V)$

Fourier transformation

$Z(n, T, V)$

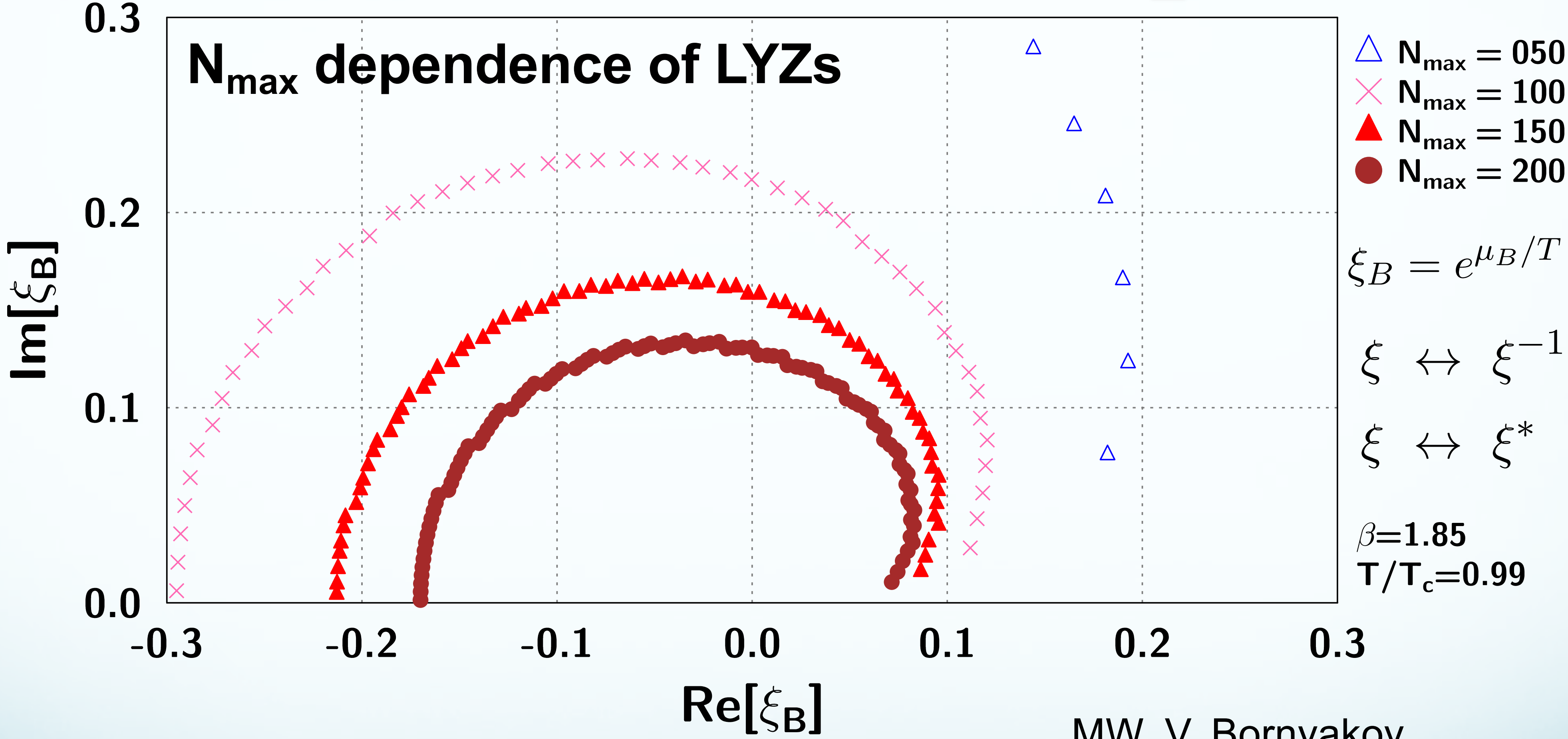
$$Z_{\rm GC}(\mu_q, T, V) = \sum_{n=-\mathbf{N_{max}}}^{\mathbf{N_{max}}} Z(n, T, V)\xi^n \qquad \xi = e^{\mu_q/T}$$

Lee-Yang zeros

Phase transition point

Contents

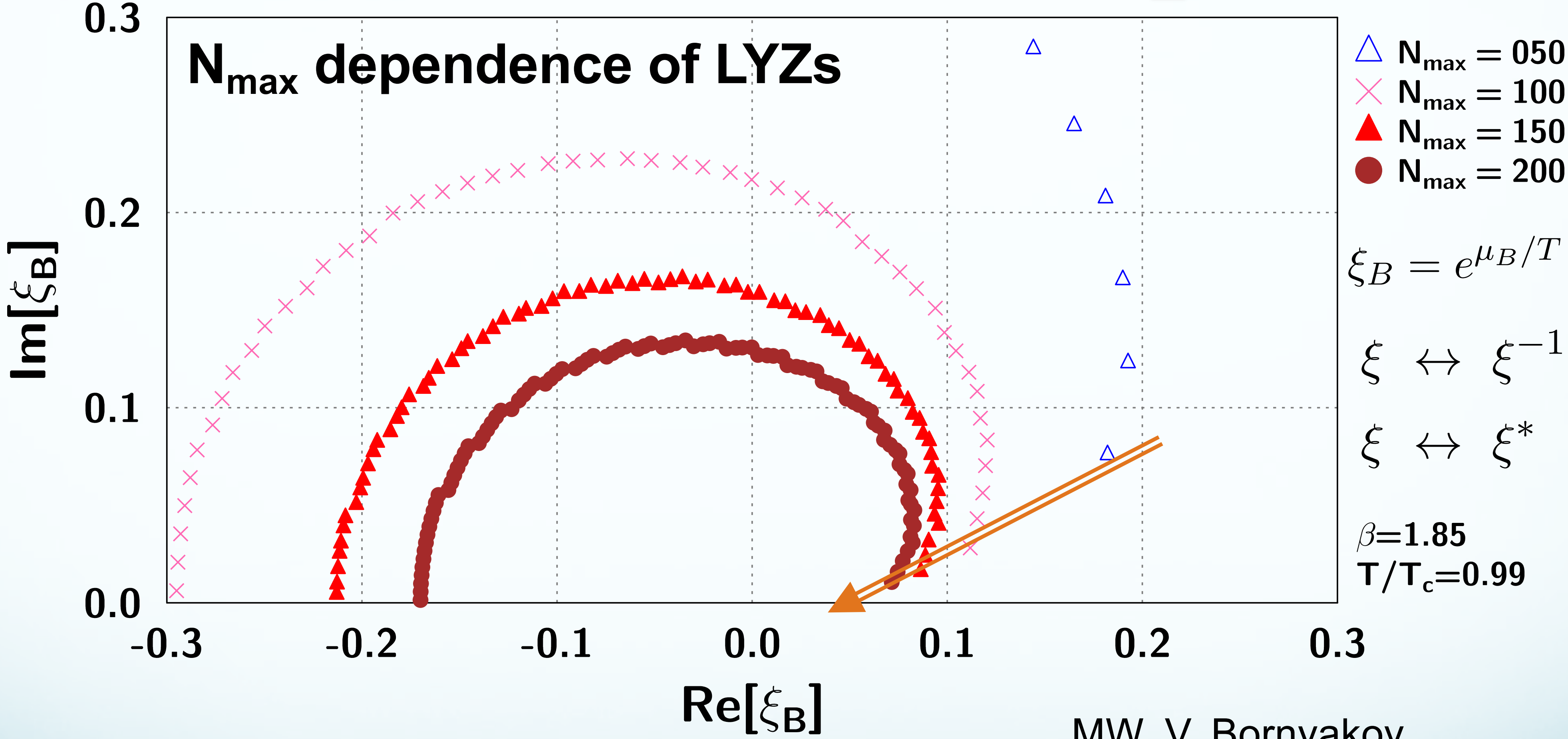
Lattice QCD Results (T/T_c=0.99)



As N_{max} increases, right edges of LYZs approach to the real positive axis.
Phase transition point: $\mu_B/T \sim 3\text{-}3.5$?

MW, V. Bornyakov, D. Boyda, V. Goy, H. Iida, A. Molochkov, A. Nakamura, V. Zakharov, PLB793, 227 (2019)

Lattice QCD Results (T/T_c=0.99)



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Outline

NJL model

$n_q(\mu_q = i\mu_{qI}, T)$

Lattice QCD

$n_q(\mu_q = i\mu_{qI}, T, V)$

Integration method
V. Bornyakov et al., PRD95(2017)

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q}\ln Z_{\rm GC}$$

$Z_{\rm GC}(\mu_q = i\mu_{qI}, T, V)$

Fourier transformation

$Z(n, T, V)$

We can check whether the canonical approach works well or not from the NJL model.

$$Z_{\rm GC}(\mu_q, T, V) = \sum_{n=-\mathbf{N_{max}}}^{\mathbf{N_{max}}} Z(n, T, V)\xi^n \qquad \xi = e^{\mu_q/T}$$

NJL model

$n_q(\mu_q, T)$

Lee-Yang zeros

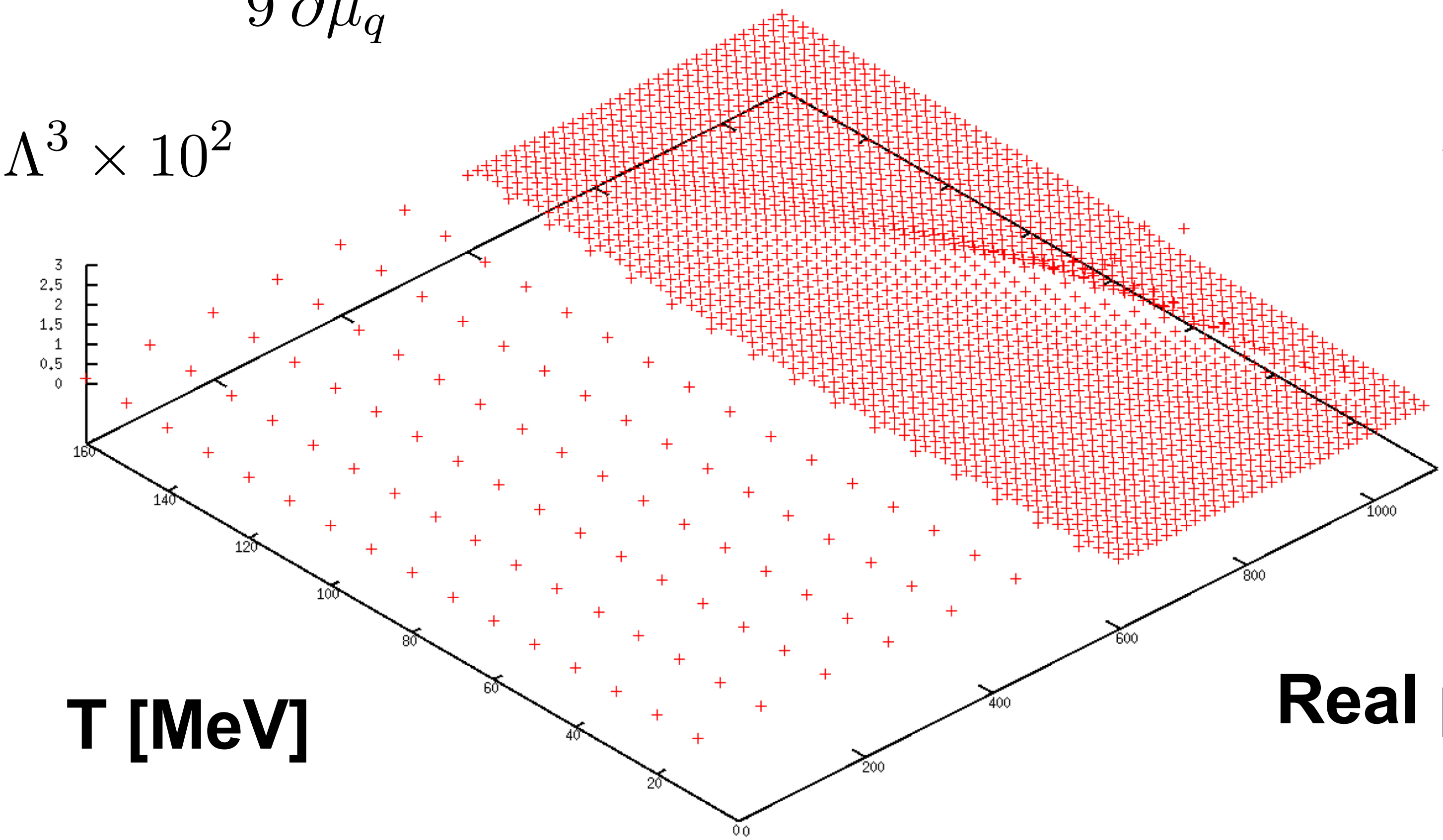
Phase transition point

Contents

Susceptibility in the NJL model

$$\chi_B = \frac{1}{9}\frac{\partial n_q}{\partial \mu_q}$$

$\chi_B T/\Lambda^3 \times 10^2$



Two-flavor NJL
G=5.5GeV^{-2}
m_q=5.5MeV
Λ=631MeV

T [MeV]

Real μ_B [MeV]

Critical Point: (T,μ_B) ~ (49,981) [MeV]

Phase transition of the NJL model

$$\chi_B = \frac{1}{9}\frac{\partial n_q}{\partial \mu_q}$$

$$\chi_B T/\Lambda^3 \times 10^2$$

Two-flavor NJL
G=5.5GeV^{-2}
m_q=5.5MeV
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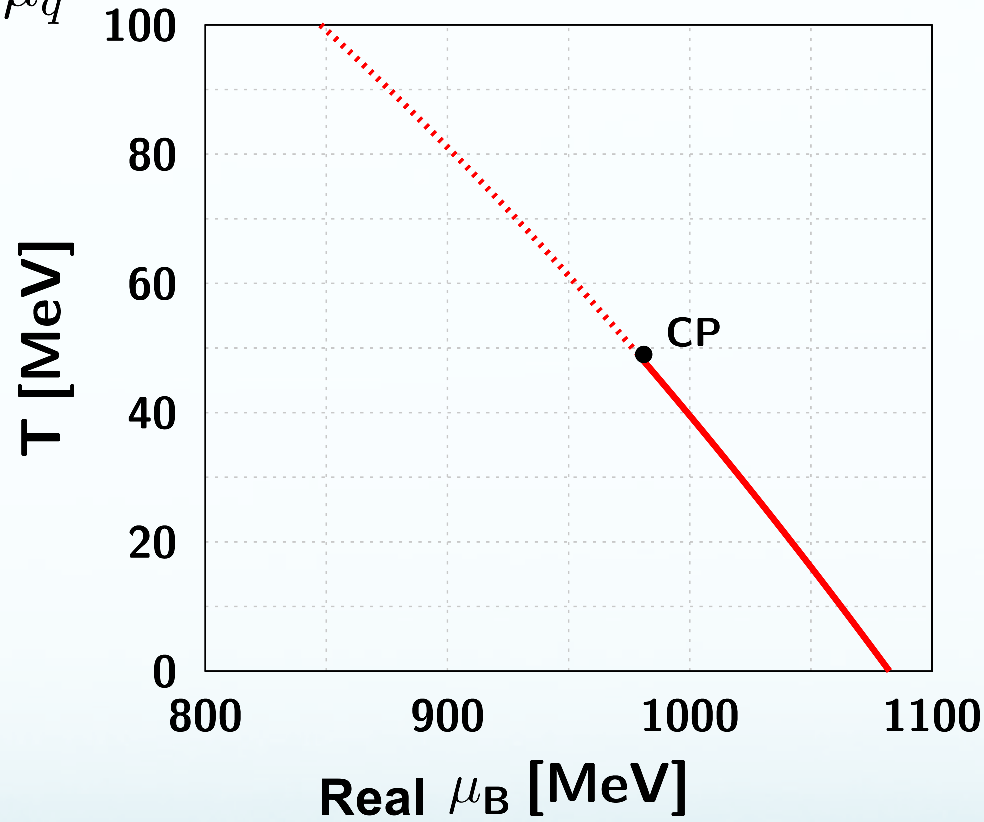
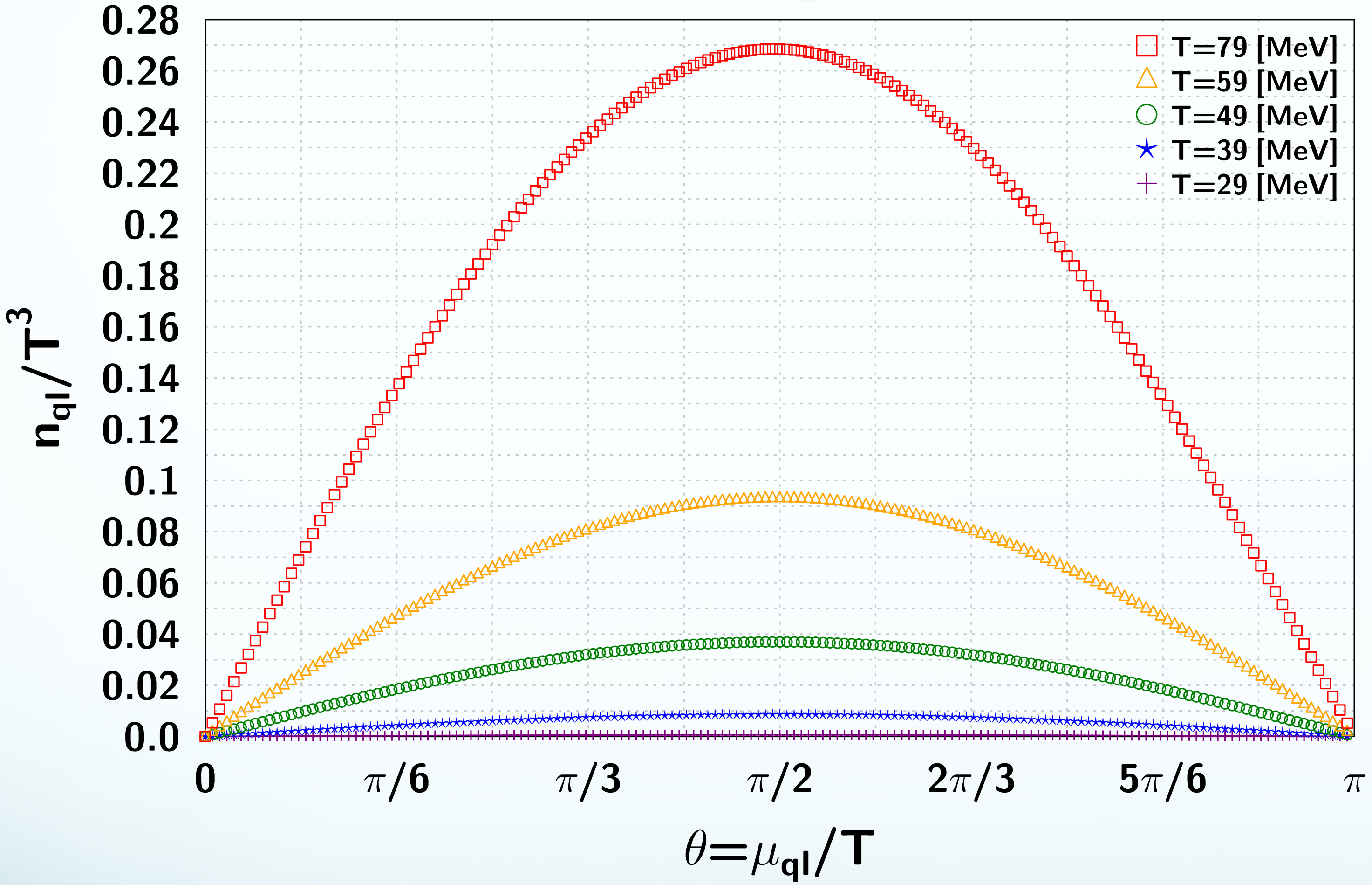


Critical Point: $(T,\mu_B) \sim (49,981)$ [MeV]

Number density in the NJL model



Number density

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q}\ln Z_{\rm GC}$$

$$n_q = i n_{qI}$$
$$\mu_q = i \mu_{qI}$$

Number density integration method (V. Bornyakov et al., PRD95(2017))

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q}\ln Z_{\rm GC} \sim \sum_{k=1}^{N_{\rm sin}} f_k \sin(k\theta)$$

We fit the number density as it was done in the lattice simulations.

Outline

NJL model

$n_q(\mu_q = i\mu_{qI}, T)$

Lattice QCD

$n_q(\mu_q = i\mu_{qI}, T, V)$

Integration method
V. Bornyakov et al., PRD95(2017)

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q}\ln Z_{\rm GC}$$

$Z_{\rm GC}(\mu_q = i\mu_{qI}, T, V)$

Fourier transformation

$Z(n, T, V)$

We can check whether the canonical approach works well or not from the NJL model.

$$Z_{\rm GC}(\mu_q, T, V) = \sum_{n=-\mathbf{N_{max}}}^{\mathbf{N_{max}}} Z(n, T, V)\xi^n \qquad \xi = e^{\mu_q/T}$$

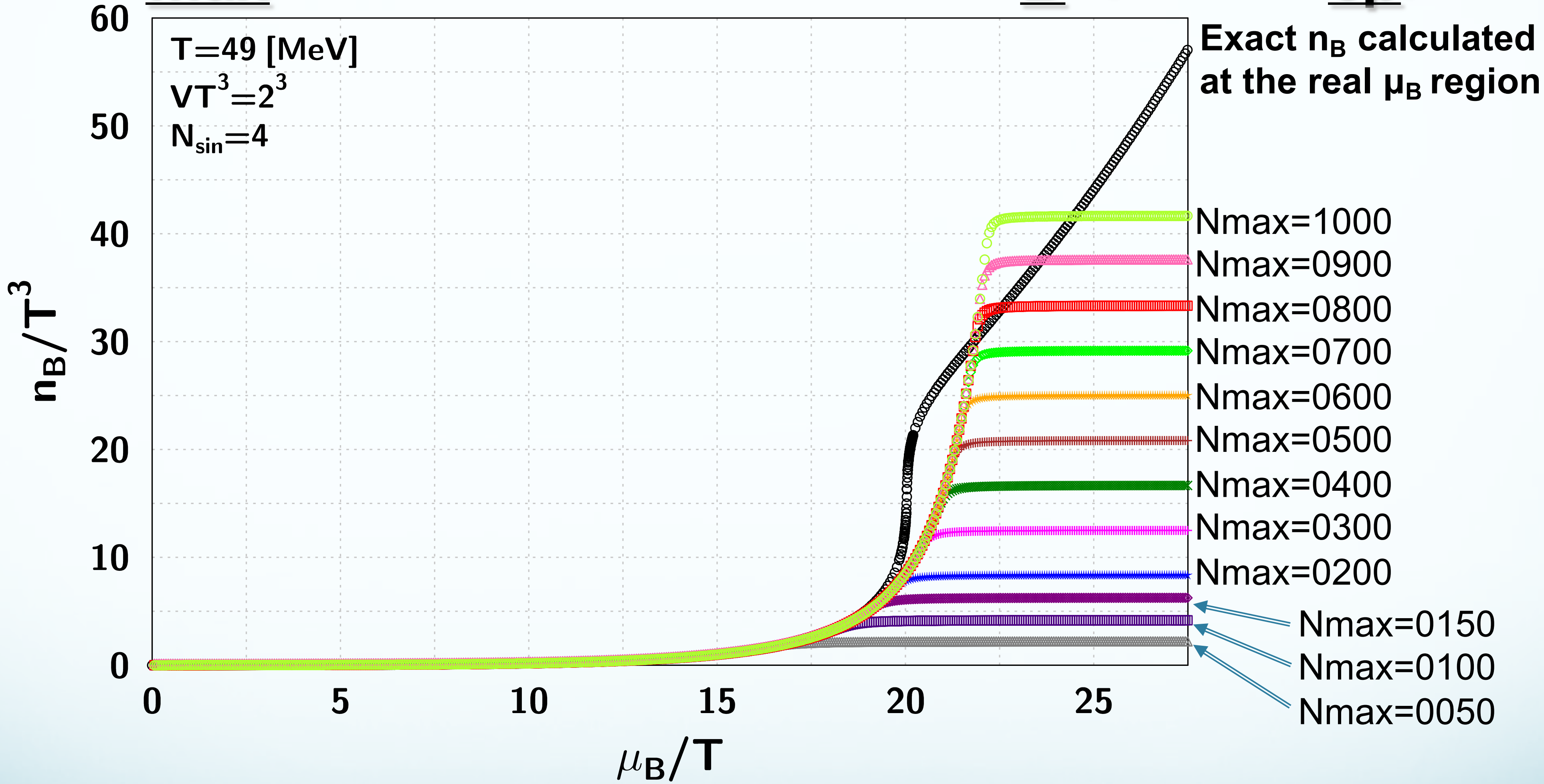
NJL model

$n_q(\mu_q, T)$

Lee-Yang zeros

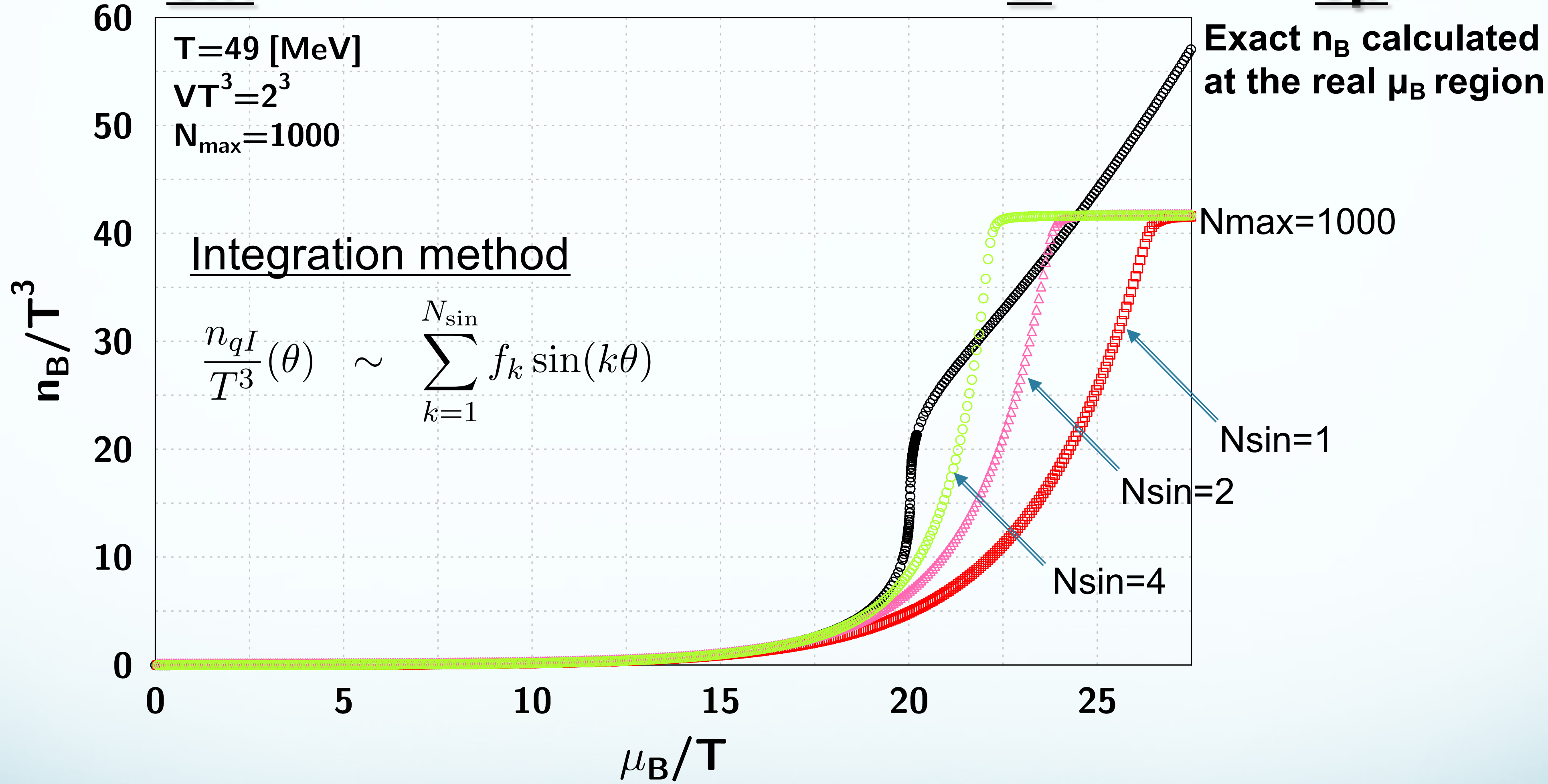
Phase transition point

N_{max} dependence of n_B ($T = T_{cp}$)



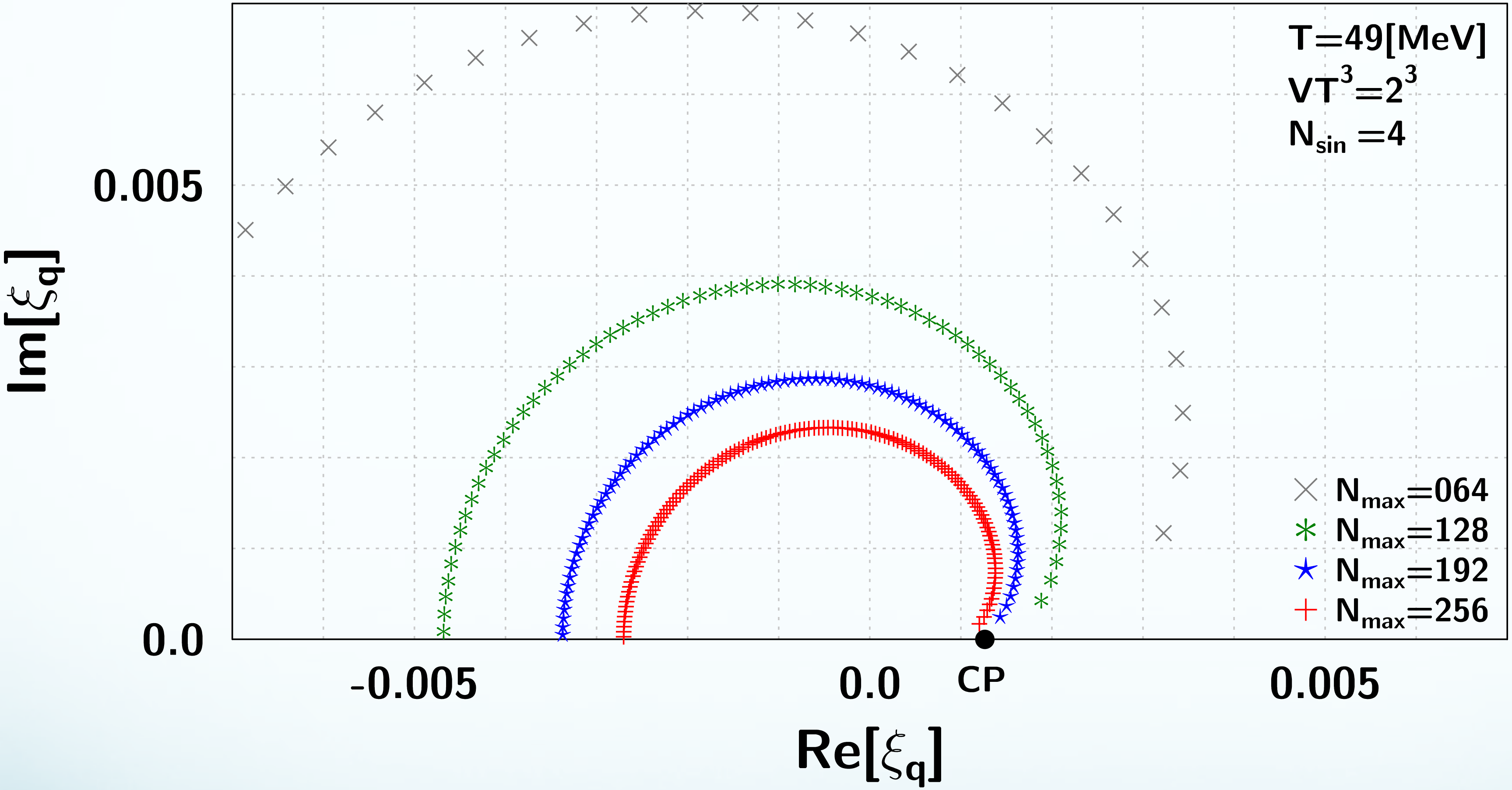
Under the phase transition density, exact n_B calculated at the real μ_B region can be reconstructed from the results of the canonical approach of $N_{max} \geqq 200$.

N_{sin} dependence of n_B ($T = T_{cp}$)



As N_{sin} increases, the difference around the phase transition point becomes small.

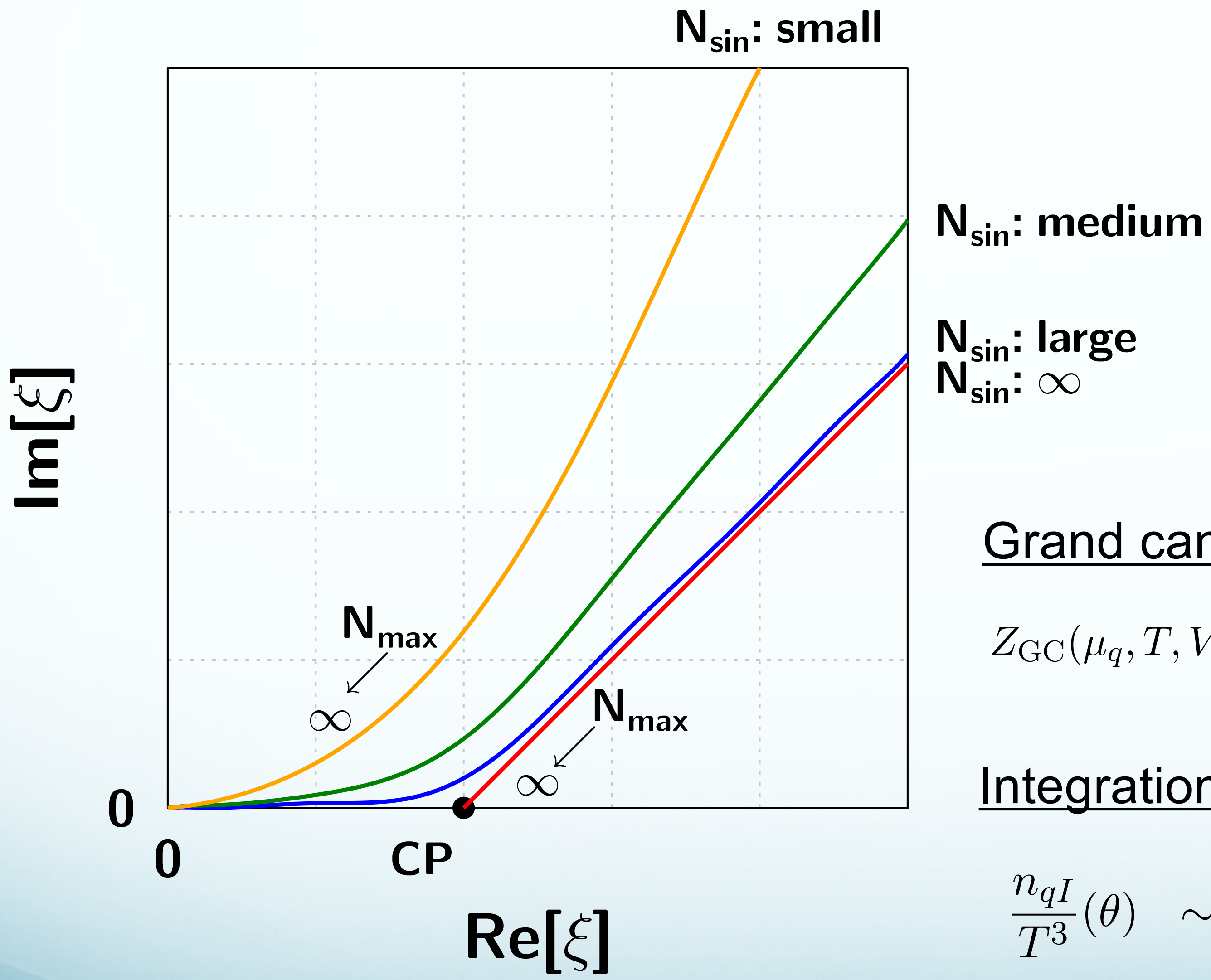
N_{max} dependence ($T = T_{cp}$)



$$\xi = e^{\mu_q/T}$$

As N_{max} increases, edges of LYZs approach to the real axis. But we find that for the finite Nsin, edges of LYZs pass over the expected CP.

Schematic flows of the edges of LYZs



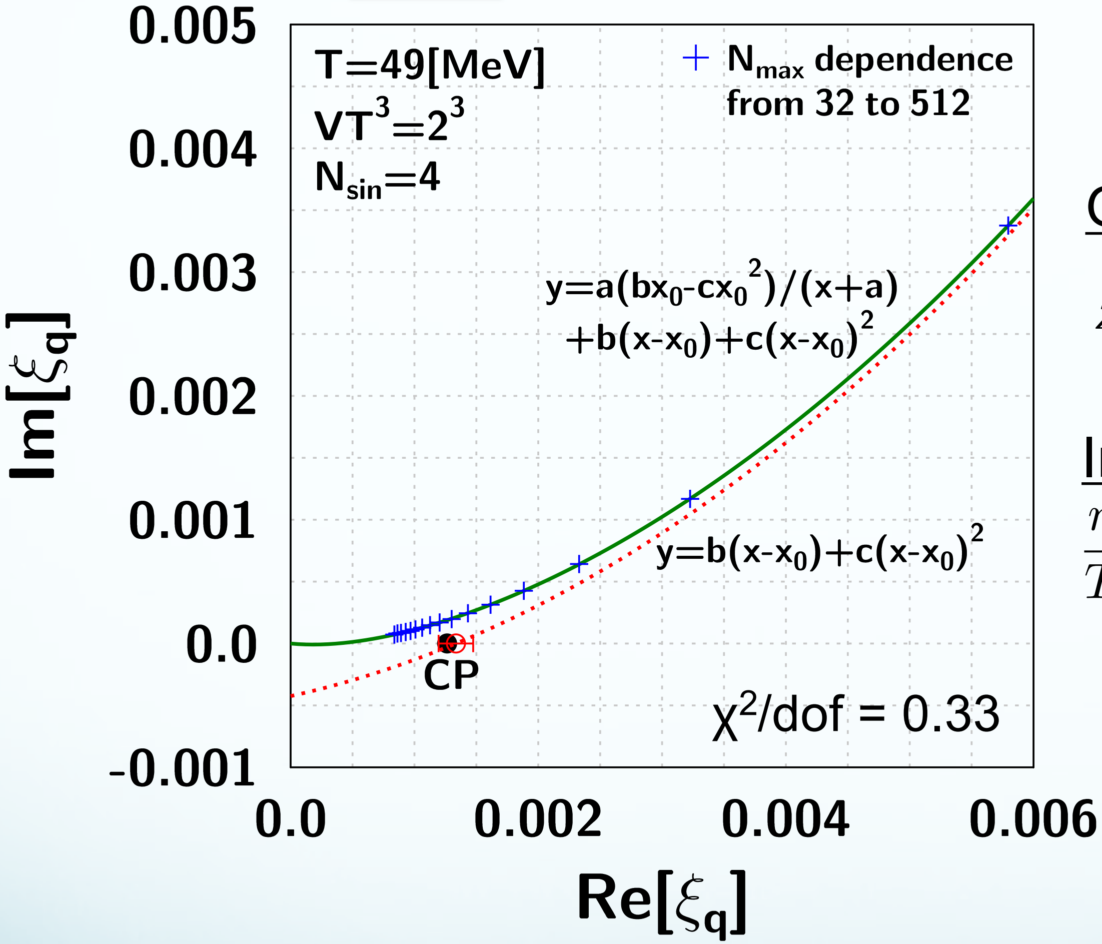
Grand canonical partition function

$$Z_{\rm GC}(\mu_q, T, V) = \sum_{n=-\mathbf{N_{max}}}^{\mathbf{N_{max}}} Z(n, T, V)\xi^n$$

Integration method

$$\frac{n_{qI}}{T^3}(\theta) \sim \sum_{k=1}^{N_{\rm sin}} f_k \sin(k\theta)$$

N_{max} dependence ($T = T_{cp}$)



$$\xi = e^{\mu_q/T}$$

Grand canonical partition function

$$Z_{\rm GC}(\mu_q, T, V) = \sum_{n=-\mathbf{N_{max}}}^{\mathbf{N_{max}}} Z(n, T, V)\xi^n$$

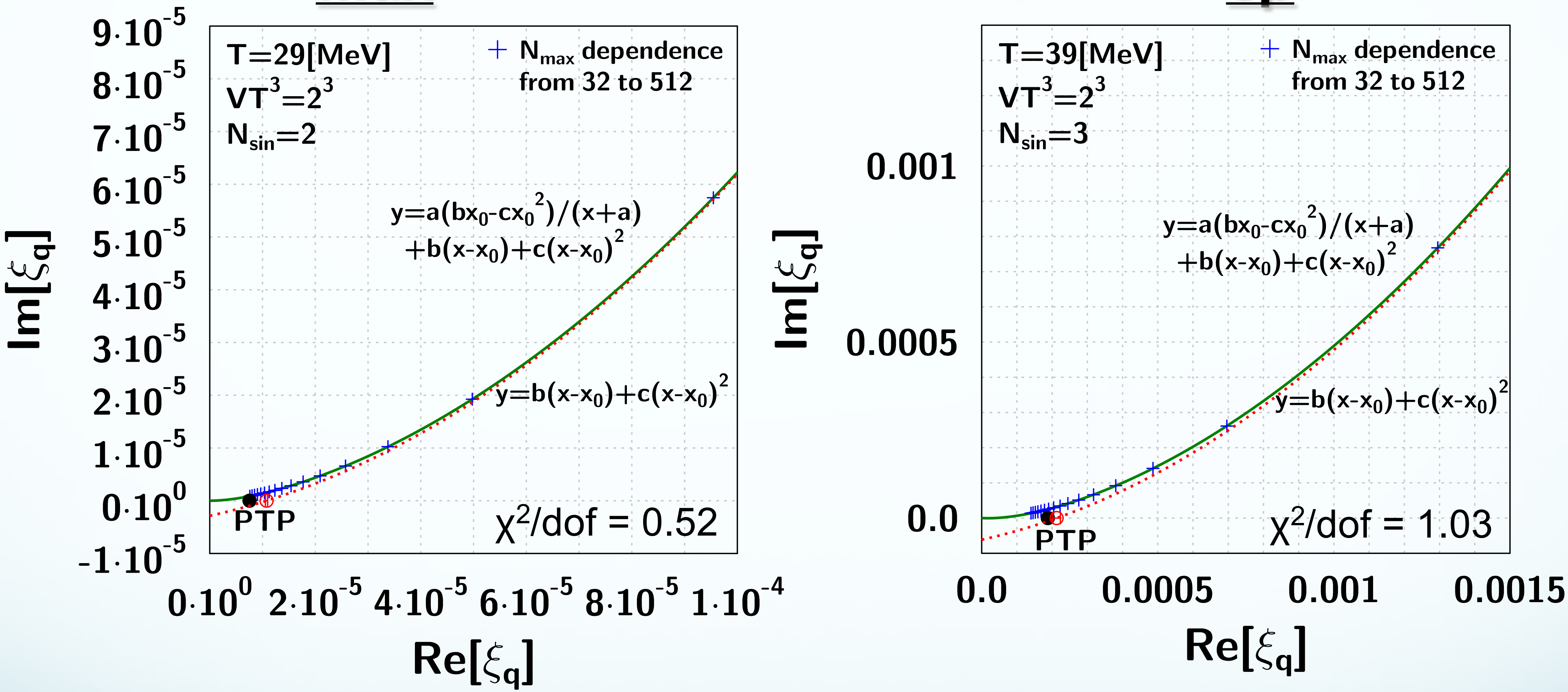
Integration method

$$\frac{n_q}{T^3} = \frac{1}{VT^2}\frac{\partial}{\partial \mu_q}\ln Z_{\rm GC} \sim \sum_{k=1}^{N_{\rm sin}} f_k \sin(k\theta)$$

M.W., A. Hosaka, PLB975, 548 (2019)

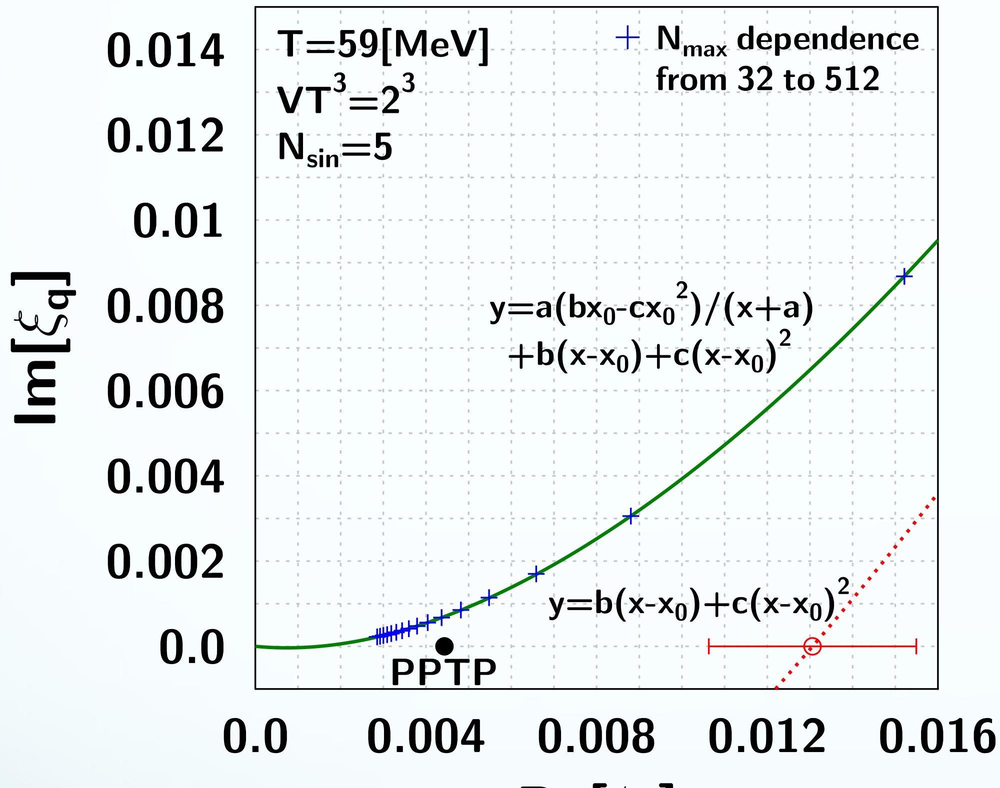
We have succeeded in subtracting the first term associated with the finite N_{sin} effect from the fitted function. The resulting curve represented the dotted curve nicely reproduces the expected CP in the NJL model.

N_{max} dependence ($T < T_{cp}$)

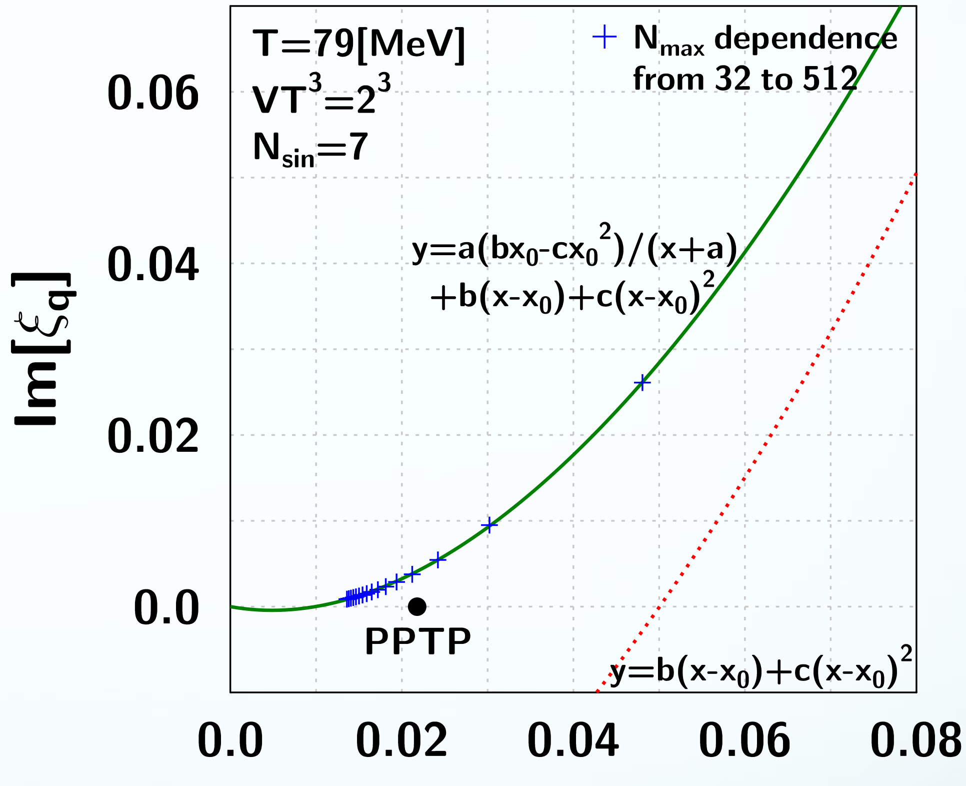


This extrapolation procedure works well to obtain the expected phase transition points (PTP).

N_{max} dependence ($T > T_{cp}$)



χ^2/dof = 29



χ^2/dof = 1586

Our results are different from the pseudo phase transition points (PPTP), which is consistent with the disappearance of PTP.

Contents

1. **Canonical Approach**
2. **Lee-Yang zeros (LYZs)**
3. **Lattice QCD Results**

   MW, V.G. Bornyakov, D.L. Boyda, V.A. Goy, H. Iida, A.V. Molochkov, A. Nakamura, V.I. Zakharov, PLB793, 227 (2019)

4. **Study of the NJL model**

   MW, A. Hosaka, PLB795, 548 (2019)

5. **Study of the PNJL model and Progress of Recent Lattice QCD Calculations**
6. **Summary & Future works**

N_{max} dependence ($T < T_{cp}$) in the PNJL model

Preliminaries

$$y = \frac{b\left(cx_0 - dx_0^2\right)}{x+b} + c\,(x - x_0) + d\,(x - x_0)^2$$

$$y = c\,(x - x_0) + d\,(x - x_0)^2$$

Zoom

The result and the expected PTP differ by about O(10).
There is room for improvement on the fitted function.

Progress of recent LQCD calculations

Previous LQCD study

FEFU group, PLB793, 227 (2019)

- Clover fermion action
- Iwasaki gauge action
- Lattice size: 4x16x16x16
- m_π/m_ρ ~ 0.80 (m_π ~ 0.7GeV)
- T/T_c = 0.84(4), 0.93(5), 0.99(5), 1.08(5), 1.20(6), 1.35(7)
- # of μ_I: 20-40 points
- 1800-3800 conf.

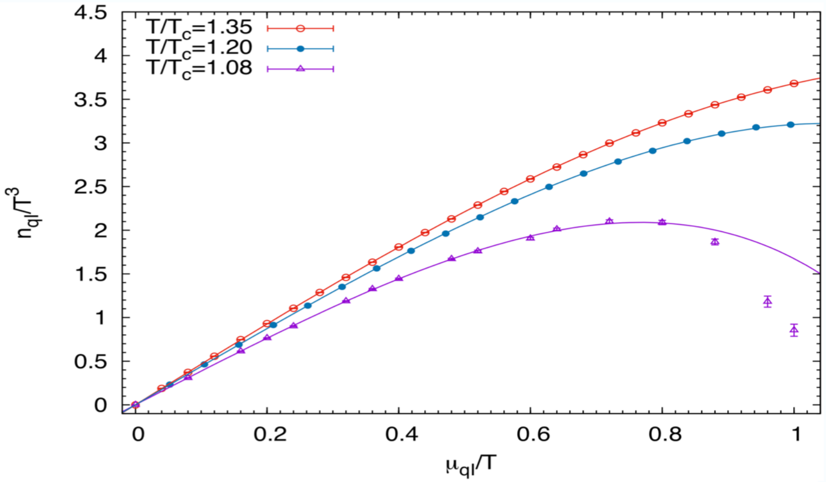
More Realistic LQCD study

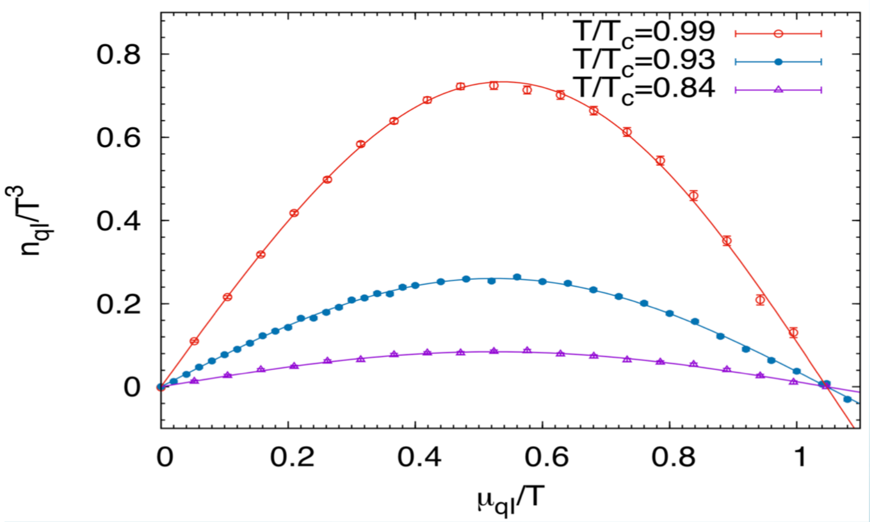
- Clover fermion action
- Iwasaki gauge action
- Lattice size: 4x**24x24x24**
- m_π/m_ρ ~ **0.42** (m_π ~ 0.33GeV)
- T/T_c ~ 0.95, 1.1 (T_c = 168(10)MeV)
- # of μ_I: 20 points
- 200, 700 conf. (preliminaries)

The search of lattice parameters (β,κ) so as to be m_π/m_ρ ~ 0.42 is done with a 16x16x16x16 lattice, which we regard as zero temperature.

Progress of recent LQCD calculations

Previous LQCD study





More Realistic LQCD study

T/T_c~1.1

Preliminaries

T/T_c~0.95

Summary

- We studied Lee-Yang zeros for Z_n obtained from the canonical approach in lattice QCD and the NJL model.
- The phase transition points can be roughly estimated from lattice QCD.
- We found the reasonable extrapolation procedure of the edge of LYZs in the NJL model.

Future works

- We search an extrapolation procedure for the PNJL model.
- Realistic lattice QCD calculations and determine the QCD phase transitions