Canonical Approach for Revealing Finite Baryon Density Phase

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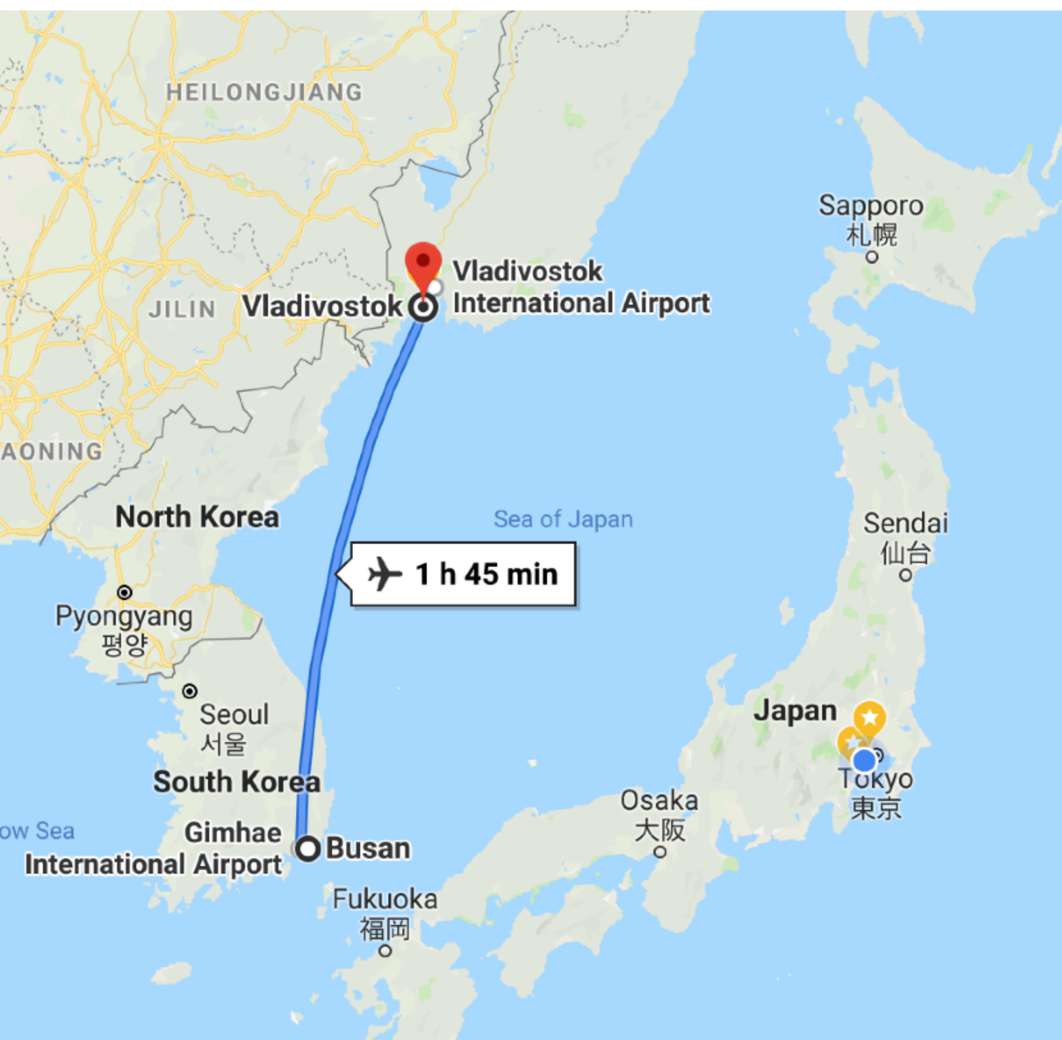

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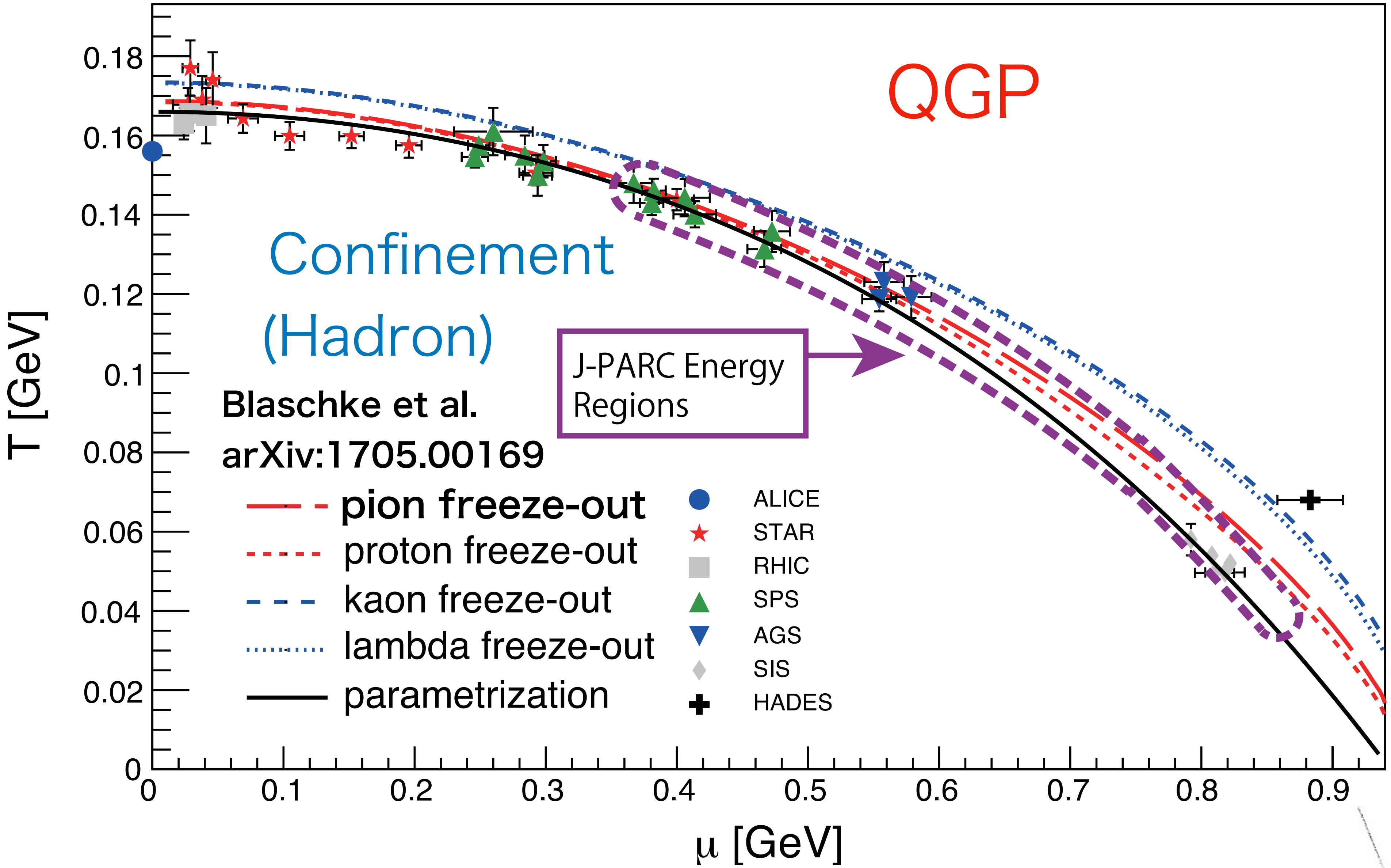
1. Introduction — Finite Density Regions
2. Sign Problem
3. Canonical Approach
4. Experimental data at RHIC — Higher Moments
5. How to find QCD phase transition line ?
6. What should we do next ?



μ :Chemical Potential

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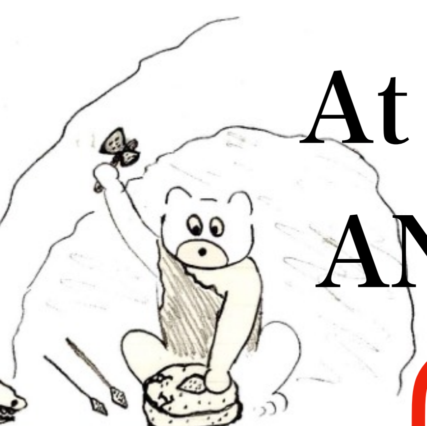
Lattice QCD + μ

$\mu N = \mu\bar{\psi}\gamma_4\psi$ **is added?**

P.Hasenfratz and F.Karsch
Physics Letters B125, (1983), 308

They found the energy density, and it becomes to diverge.



At that time, Poland was under the martial law.
AN was there, andconsidered it independently.

A. Nakamura
Physics Letters B149, 1984,391
Behavior of quarks and gluons at finite temperature and density in SU(2) QCD

AN was thinking as follows:

In the continuum theories,

$$\mathcal{L} = \bar{\psi}[\partial_k \gamma_k + (\partial_4 + \mu)\gamma_4 + m]\psi$$

$$ip_\mu = \partial_\mu$$

On the lattice, (free case in the momentum space)

$$\Delta(p) = I - \kappa \sum_{\mu=1}^{4} \{(1-\gamma_\mu)e^{ip_\mu} + (1+\gamma_\mu)e^{-ip_\mu}\}$$

$$ip_4 \rightarrow ip_4 + \mu$$

$$\Rightarrow \quad I - \kappa[\sum_{\mu=1}^{3} \{(1-\gamma_\mu)e^{ip_\mu} + (1+\gamma_\mu)e^{-ip_\mu}\}]$$

$$-\kappa[(1-\gamma_4)e^{ip_4+\mu} + (1+\gamma_4)e^{-ip_4-\mu}]$$

Then we can change the hopping parameters (depending on the forward or backward)

$$\kappa e^{+\mu} \qquad \kappa e^{-\mu}$$

In the co-ordinate space with gauge field,

$$\Delta = I - \kappa \sum_{l=1}^{3} \left\{ (1-\gamma_l) U_l(x) \delta_{x', x+\hat{l}} + (1+\gamma_l) {U^{\dagger}}_l(x') \delta_{x', x-\hat{l}} \right\}$$
$$-\kappa e^{+\mu} (1-\gamma_4) U_\mu(x) \delta_{x', x+\hat{4}} - \kappa e^{-\mu} (1+\gamma_4) U^{\dagger}_\mu(x') \delta_{x', x-\hat{4}}$$

Remember $U_4 = e^{iA_4}$ **Then, we change**

$$A_4 \qquad A_4 \to A_4 + i\mu$$

- ☑ Anyway, we had got Lattice Action with μ
- ☑ Several big groups started simulation: Of course SU(3).
- ☑ Nakamura could use only a small computer, and started a simulation with SU(2).

But

Sign Problem

Lattice QCD does not work
at finite density !

Big groups failed, and only
(poor) Nakamura got results.

$$(\det \Delta(\mu))^* = \det \Delta(\mu)^\dagger = \det \Delta(-\mu^*)$$

For $\mu = 0$

$$(\det \Delta(0))^* = \det \Delta(0)$$

$\det \Delta$ ⇒ *Real*

For $\mu \neq 0$ (in general)

$\det \Delta$ ⇒ *Complex*

$$Z = \int \mathcal{D}U \prod_f \det \Delta(m_f, \mu_f) e^{-\beta S_G}$$

⇑

Complex ⇒ Sign Problem

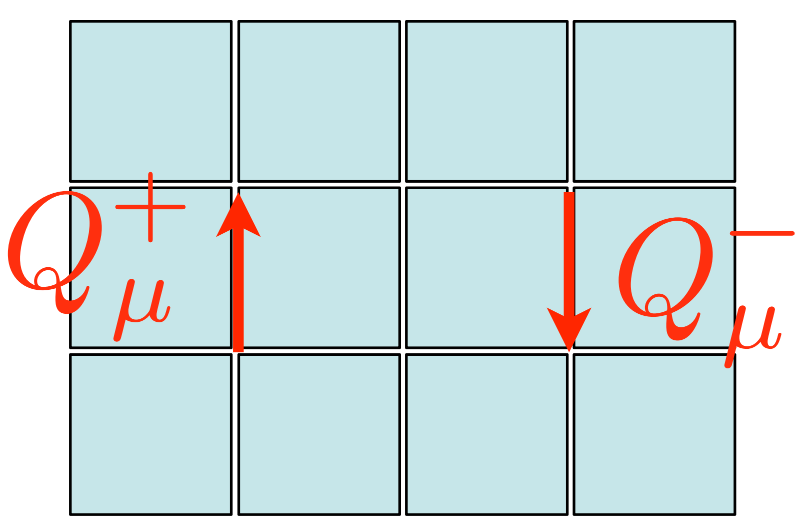
Origin of Sign Problem

Wilson Fermions $\quad \Delta = I - \kappa Q$

KS(Staggered) Fermions $\quad \Delta = m - Q'$

$$= m(I - \frac{1}{m}Q)$$

$$Q = \sum_{i=1}^{3}(Q_i^+ + Q_i^-) + (e^{+\mu}Q_4^+ + e^{-\mu}Q_4^-)$$



$$Q_\mu^+ = **U_\mu(x)\delta_{x',x+\hat{\mu}}$$

$$Q_\mu^- = **U_\mu^\dagger(x')\delta_{x',x-\hat{\mu}}$$

$$\det \Delta = e^{\mathrm{Tr}\log \Delta} = e^{\mathrm{Tr}\log(I-\kappa Q)}$$

$$= e^{-\sum_n \frac{1}{n}\kappa^n \mathrm{Tr}Q^n}$$

Hopping Parameter Exp.
or
Large Mass Expansion.

Closed loops do not vanish

Lowest μ depsnent terms

$$\kappa^{N_t} e^{\mu N_t} \mathrm{Tr}(Q^+ \cdots Q^+)$$

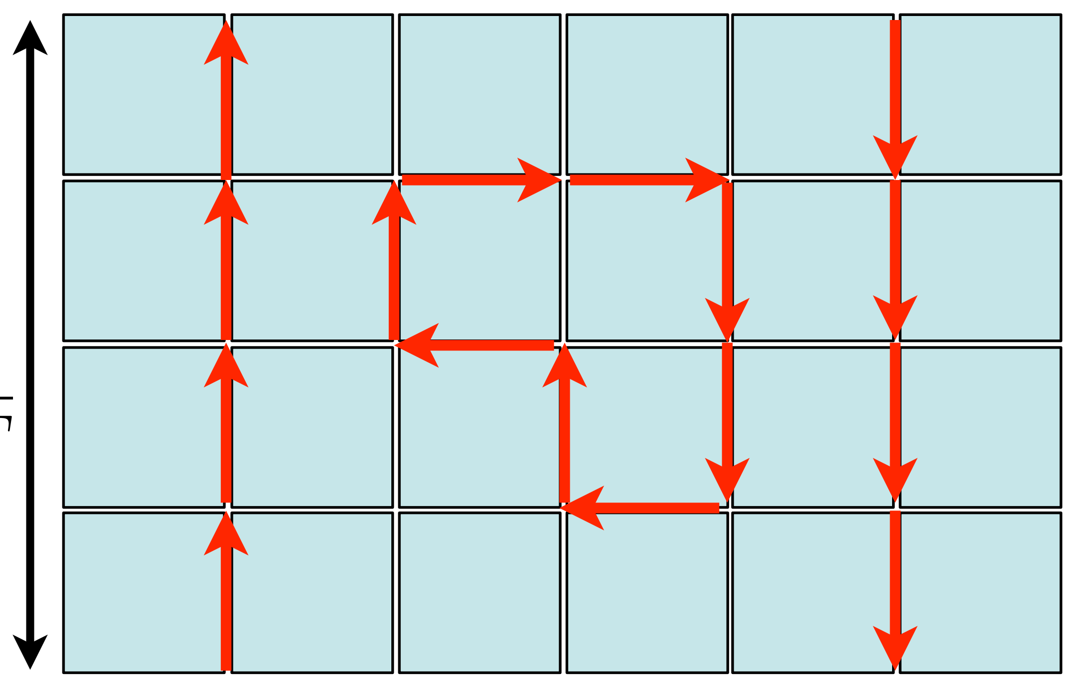
$$= **\kappa^{N_t} e^{\mu/T} \mathrm{Tr}L$$

$$\kappa^{N_t} e^{-\mu N_t} \mathrm{Tr}(Q^- \cdots Q^-)$$

$$= **\kappa^{N_t} e^{-\mu/T} \mathrm{Tr}L^\dagger$$

$\mathrm{Tr}L$: Polyakov Loop

$N_t = \frac{1}{T}$



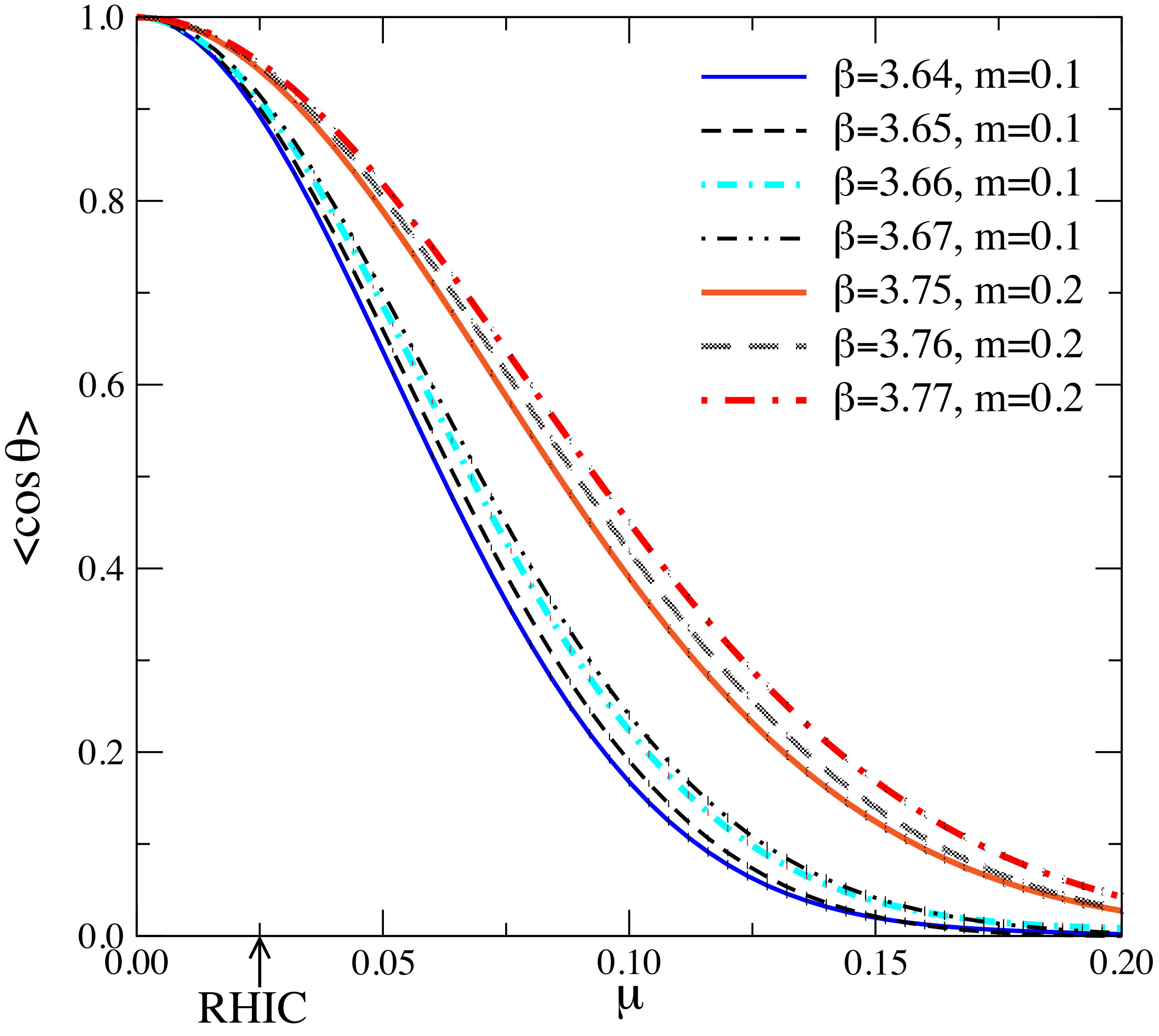
Combine both

$$**\kappa^{N_t}\left(\cosh\frac{\mu}{T}\Re\mathrm{Tr}L + i\sinh\frac{\mu}{T}\Im\mathrm{Tr}L\right)$$

Sign Problem is sever
when μ is large
when T is low

Allton et al., Phys.Rev.D.66. 074507 (arXiv:hep-lat/0204010)

$$\det D = |\det D| e^{i\theta}$$



No Sign problem cases

1. Pure imaginary chemical potential

$$(\det \Delta(\mu))^* = \det \Delta(-\mu^*)$$

$$\mu = i\mu_I \quad \Rightarrow \quad (\det \Delta(\mu_I))^* = \det \Delta(\mu_I)$$

2. Color SU(2)

$$U_\mu^* = \sigma_2 U_\mu \sigma_2$$

$$\det \Delta(U, \gamma_\mu)^* = \det \Delta(U^*, \gamma_\mu^*) = \det \sigma_2 \Delta(U, \gamma_\mu^*) \sigma_2$$

$$= \det \Delta(U, \gamma_\mu)$$

3. Iso vector (finite iso-spin)

$$\mu_d = -\mu_u$$

$$\det \Delta(\mu_u) \det \Delta(\mu_d) = \det \Delta(\mu_u) \det \Delta(-\mu_u)$$

$$= \det \Delta(\mu_u) \det \Delta(\mu_u)^* = |\det \Delta(\mu_u)|^2$$

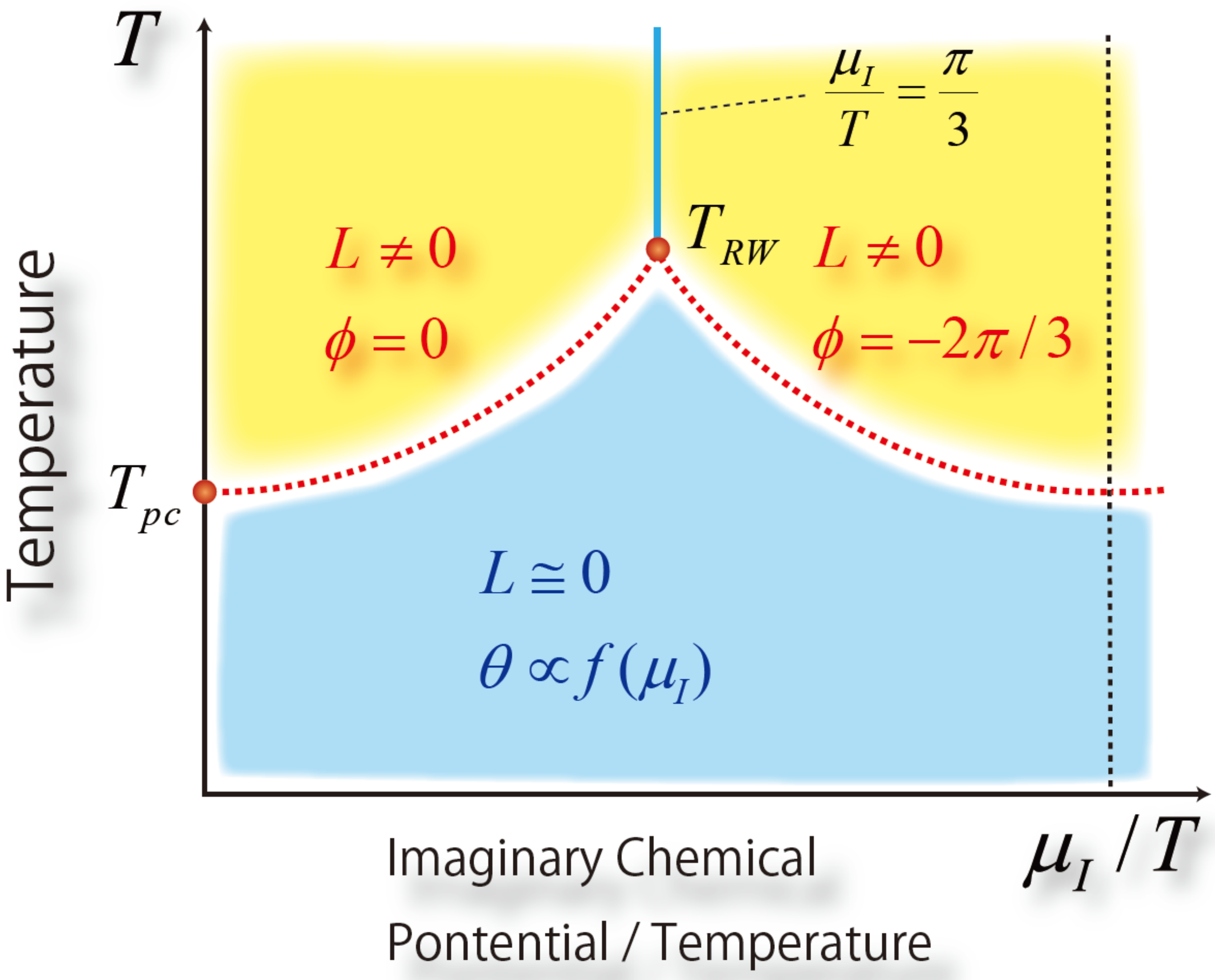
(Phase Quench)

Phase Structure in pure imaginary

$$(\det \Delta(\mu))^* = \det \Delta(\mu)^\dagger = \det \Delta(-\mu^*)$$

$$\mu = i\mu_I \Rightarrow \det \Delta\text{: Real !}$$

Phase diagram in μI region



Polyakov loop

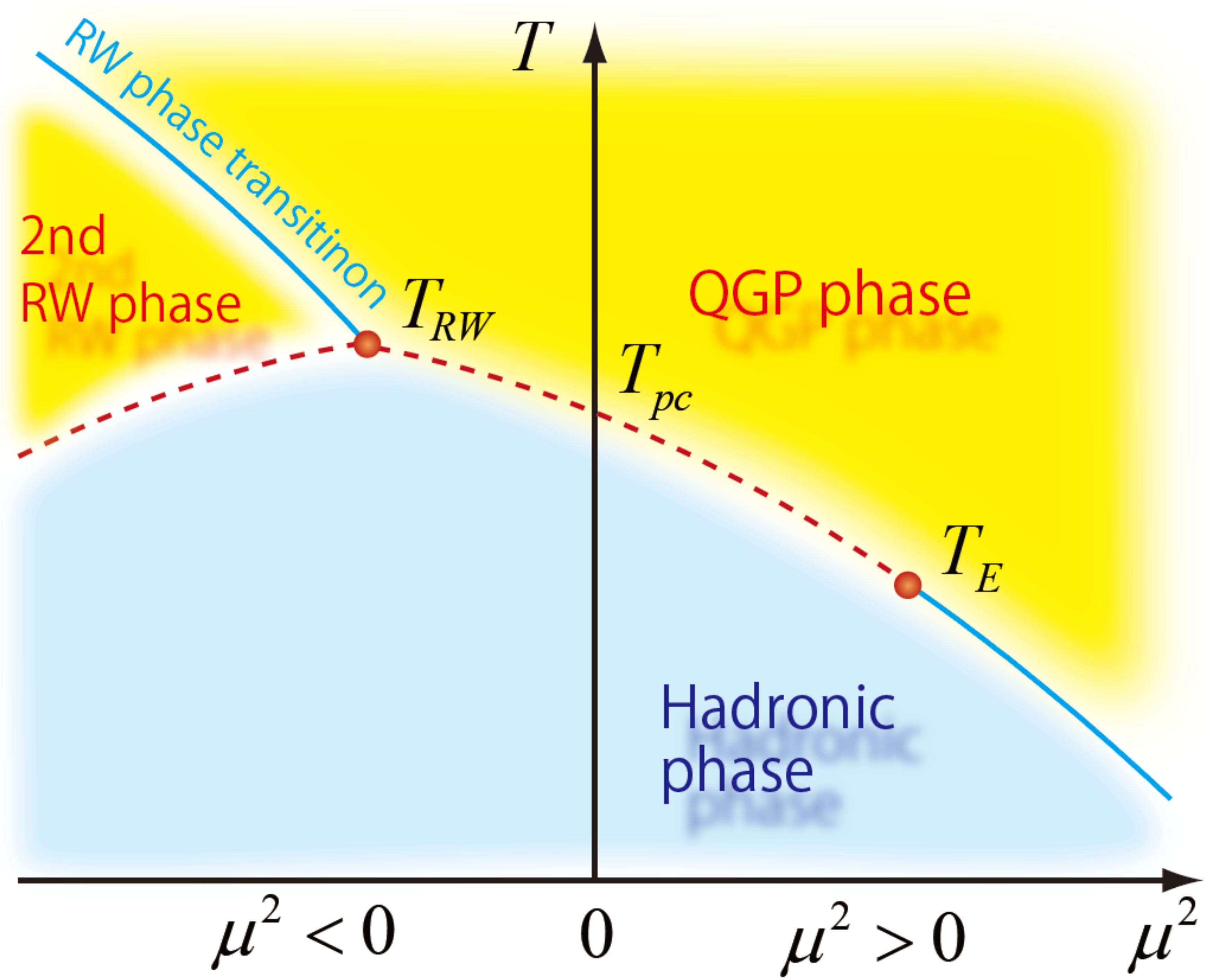
$$P = L_P \exp(i\phi_P)$$

If μ is pure imaginary

there is no sign problem.

$$(\det \Delta(\mu))^* = \det \Delta(-\mu^*)$$

Imaginary to real chemical potential



Many Approaches to Sign Problem

- Taylor Expansion
- Canonical Approach
- Density of State
- Complex Langevin

Content

Canonical Approach

proposed by
A.Hasenfratz and Toussaint in 1992
to solve the sign problem.
But it did not work.
We traced the cause and solve it with
multiple precision numerical calculations

Canonical Approach

$$Z(\mu, T) \Leftrightarrow Z_n(T)$$

Grand Canonical — Canonical

$$Z(\mu, T) = \mathrm{Tr}\, e^{-(H-\mu\hat{N})/T}$$

$$= \sum_n \langle n | e^{-(H-\mu\hat{N})/T} | n \rangle$$

If $[H, \hat{N}] = 0$

$$= \sum_n \langle n | e^{-H/T} | n \rangle \, e^{\mu n/T}$$

$$= \sum_n Z_n(T) \xi^n \qquad \left(\xi \equiv e^{\mu/T}\right)$$

Fugacity

Personal History about Sign Problem

We were looking for

A Reduction Formula for Wilson Fermions

$$\det \Delta = \det Q$$

Matrix Δ is smaller than Q

- ☆Keitaro Nagata and Atsushi Nakamura
  Phys. Rev. D82,094027 (arXiv:1009.2149)
- ☆A. Alexandru and U. Wenger
  Phys.Rev.D83:034502,2011 (arXiv:1009.2197)
- ☆ One more group

For KS Fermions, the reduction formula was known.

- Gibbs Formula(*)
  - P.E.Gibbs, Phys.Lett. B172 (1986) 53-61

$$\det \Delta = z^{-N} \begin{vmatrix} -B(-V) - z & 1 \\ -V^2 & -z \end{vmatrix}$$

$$= \left| \begin{pmatrix} BV & 1 \\ -V^2 & 0 \end{pmatrix} - zI \right|$$

$$= \det (P - zI)$$

$$= \prod (\lambda_i - z)$$

P

$$z \equiv e^{-\mu}$$

- P is $(2 \times N_c \times N_x \times N_y \times N_z)^2$ (Matrix Reduction)
- Determinant for any value of μ

*) A similar formula was developped by Neuberger (1997) for a chiral fermion and applied by Kikukawa(1998).

The same matrix transformation like KS case cannot be employed, due to the fact that

$r \pm \gamma_4$ have no inverse, if the Wilson term $r = 1$.

Gibbs started to multiply V to the fermion matrix Δ.

Instead, we multiply $P = (c_a r_- + c_b r_+ V z^{-1})$

Here,

$$V = \left(\begin{array}{c|c|ccc|c} 0 & U_4(t=1) & 0 & \cdots & & 0 \\ \hline 0 & 0 & U_4(t=2) & \cdots & & 0 \\ \hline 0 & 0 & 0 & \cdots & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & \cdots & U_4(t=N_t-2) & 0 \\ \hline 0 & 0 & & \cdots & 0 & U_4(t=N_t-1) \\ \hline -U_4(t=N_t) & 0 & & \cdots & 0 & 0 \end{array} \right)$$

c_a and c_b are arbitary non-zero numbers.

$$\det P = (c_a c_b z^{-1})^{N/2}$$

if we take the following trick, Borici (2004)

$$r_+ r_- = \frac{r^2 - 1}{4} = \epsilon \to 0$$

where $r_\pm \equiv \frac{r \pm \gamma_4}{2}$

After very long calculation (See Nagata-Nakamura arXiv:1009.2149), we get

$$\det \Delta(\mu) = (c_a c_b)^{-N/2} z^{-N/2}$$

$$\times \left(\prod_{i=1}^{N_t} \det(\alpha_i) \right) \det \left(z^{N_t} + Q \right)$$

$$\frac{\det \Delta(\mu)}{\det \Delta(0)} = \frac{\det(\xi + Q)}{\det(1+Q)}$$

$$\xi \equiv e^{-\mu/T}$$
(fugacity)

Q is $(4N_cN_xN_yN_z) \times (4N_cN_xN_yN_z)$ matrix.

No N_t !

In case of KS matrix, the corresponding matrix is $(2N_cN_xN_yN_z) \times (2N_cN_xN_yN_z)$

Diagonalize Q,

$$Q \rightarrow \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \cdots & \\ & & & \lambda_{N_{red}} \end{pmatrix}$$

$$\det(\xi + Q) = \prod(\xi + \lambda_n)$$

λ_n does not depend on μ.

Once we calculate λ_n,
we can evaluate $\det \Delta(\mu)$ for any μ.

$$\det(\xi + Q) = \prod(\xi + \lambda_k) = \sum C_n \xi^n$$

$$Z = \int \mathcal{D}U \det \Delta e^{-\beta S_G}$$

$$Z = \sum_n \left(\int \mathcal{D}U C_n e^{-\beta S_G} \right) \xi^n$$
$$= \sum_n z_n \, \xi^n$$

$$\xi \equiv e^{\mu/T}$$

Fugacity Expansion

$$Z(\mu, T) = \sum_n z_n(T)(e^{\mu/T})^n$$

$Z(\mu, T)$: Grand Canonical Partition Function

$z_n(T)$: Canonical Partition Function

Inverse transformation:

$$z_n = \int \frac{d\theta}{2\pi} e^{i\theta n} Z_{GC}(\theta \equiv \frac{\mathrm{Im}\mu}{T}, T)$$

A.Hasenfratz and Toussaint (1992)

z_n can be determined in imaginary μ regions.

This is Canonical approach by

A.Hasenfratz and Toussaint (1992)

$$Z_n = \int \frac{d\theta}{2\pi} e^{in\theta} Z(\theta = \frac{\mu_I}{T})$$

In pure Imaginary μ, there is no sign problem.

It was known that this method does not work.

Why ???

Check by an analytic method (Winding Number Expansion)

$$Z_n = \int \frac{d\theta}{2\pi} e^{in\theta} Z(\theta = \frac{\mu_I}{T})$$

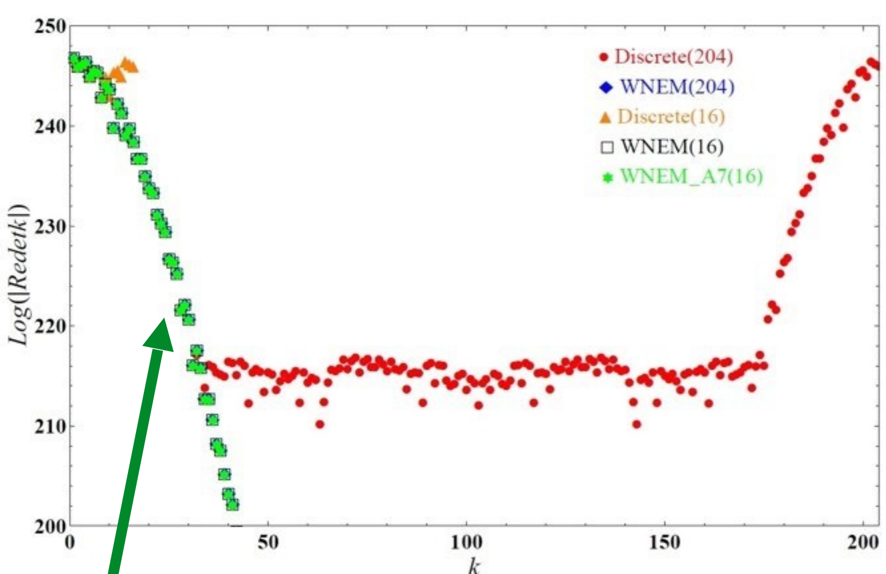
A. Hasenfratz and D. Toussaint

$$Z(\mu) = \int DU \det \Delta(\mu) e^{-S_G}$$

Kentucky: Winding Number Expansion

Meng et al., 2008

The original method does not work due to numerical errors.



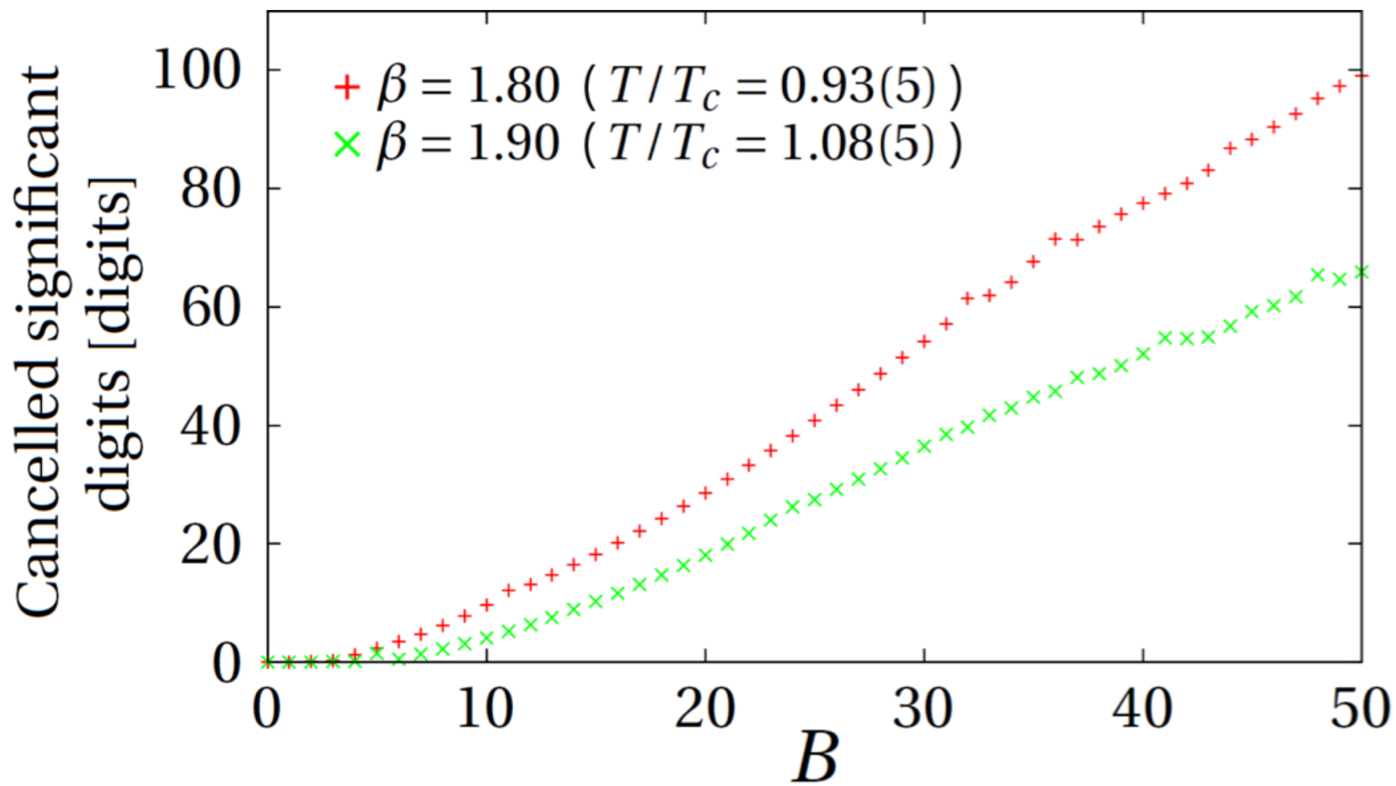
$$\det \Delta = \exp(Tr \log \Delta)$$

$$\log(I - \kappa Q) = -\sum_n \frac{\kappa^n Q^n}{n}$$

$$\det \Delta = \exp(\sum_n (W_n \xi^n + W_{-n} \xi^{-n}))$$

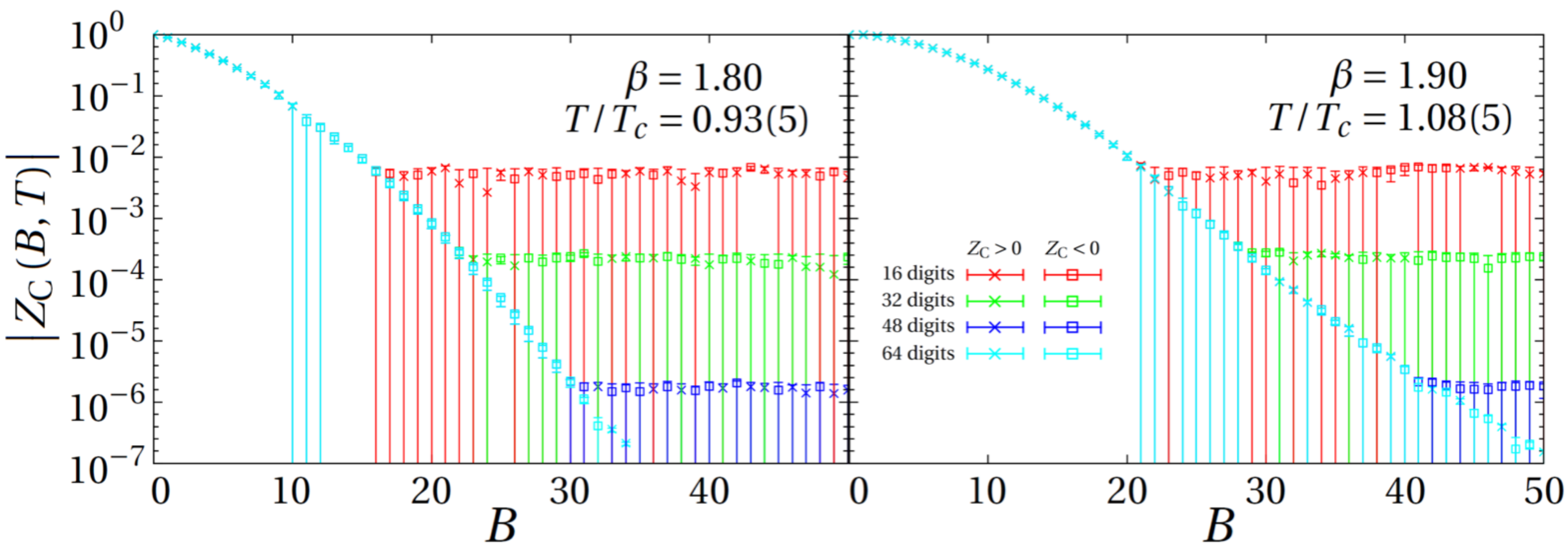
Take W_n for $|n| \leq 6$ and do the Fourier Trans. analytically.

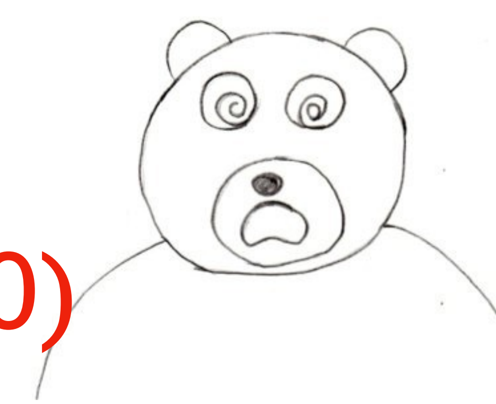
Big Cancellation in FFT !



S.Oka, arXiv:1511.04711
Talk at LATTICE 2015

Fukuda, Nakamura, Oka,
arXiv:1511.04711
Phys.RevD93, 094508 (2016)





θintegration → **Multi-Precision (50 - 100)**

$$z_n = \int \frac{d\theta}{2\pi} e^{i\theta n} Z_{GC}(\theta \equiv \frac{\mathrm{Im}\mu}{T}, T)$$

Using Multiple-precision, we have beaten Sign Problem.

But to make Canonical Approach workable,

we had to solve 2 problems:

1. Z_{GC} is not a direct observable in lattice QCD

2. We should perform simulations at many imaginary μ points.

Integration Method

Not Z_G but n_B in imaginary μ → z_n

$$n_B = \frac{1}{3V} T \frac{\partial}{\partial \mu} \log Z_G$$

$$= \frac{N_f}{3N_s^3 N_t} \int \mathcal{D}U e^{-S_G} \mathrm{Tr}\Delta^{-1} \frac{\partial \Delta}{\partial \mu} \det \Delta$$

(For pure imaginary μ, n_B is also imaginary)

Then, for fixed T

$$Z(\theta \equiv \frac{\mu}{T}) = \exp(V \int_0^{\theta} n_B d\theta')$$

$$z_k = \frac{3}{2\pi} \int_{-\pi/3}^{+\pi/3} d\theta \exp\left(i\, k\theta + \int_0^{\theta} n_B d\theta' \right)$$

- Multi-precision calculation
- Integration Method

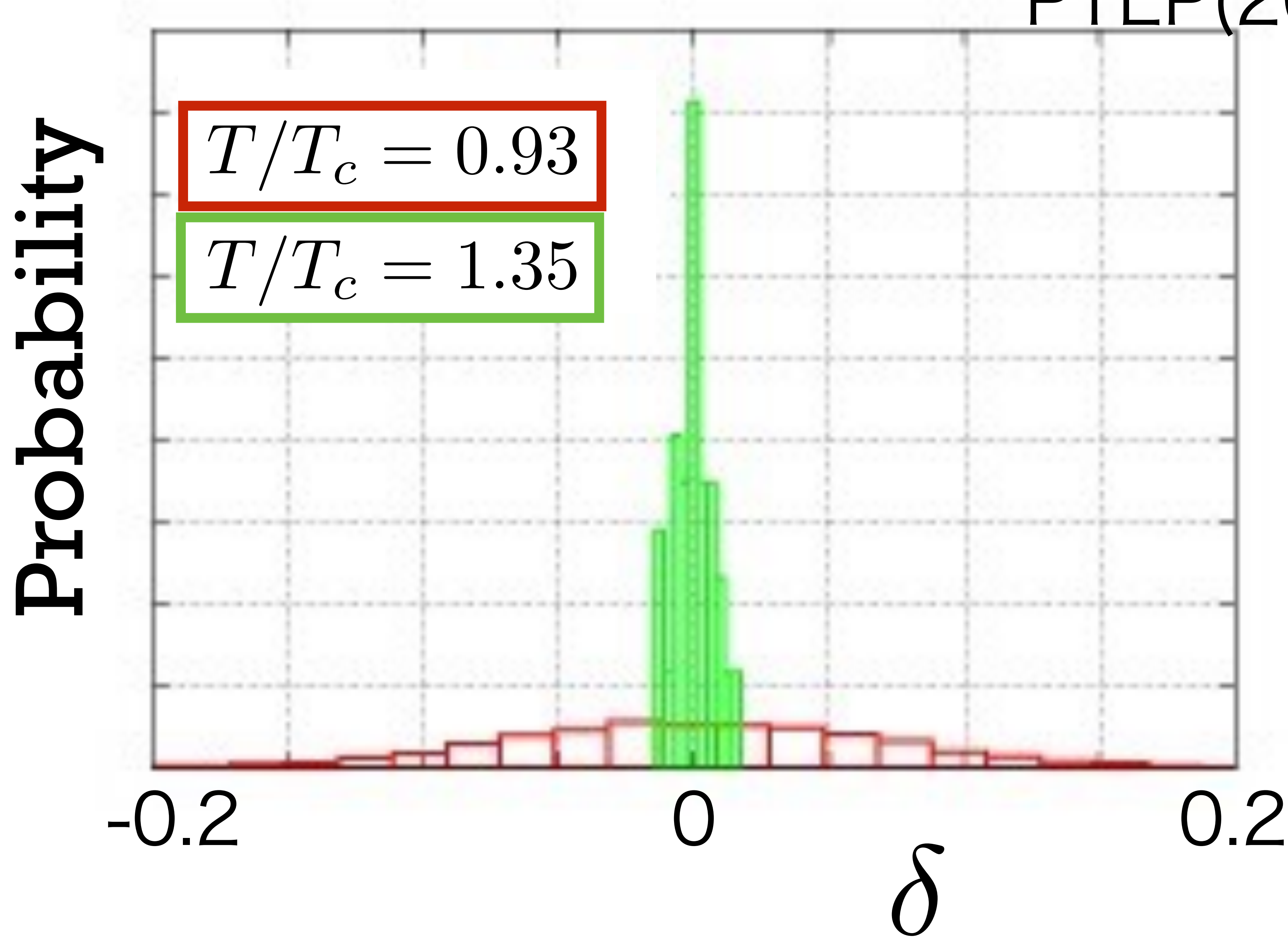
I thought we have beaten
Sign Problem.

But !

Hidden Sign Problem ?

Z_n have phase on each configuration !

V.Goy et al.,
PTEP(2017) 031D01



$z_n \simeq |z_n| e^{in\delta}$

$Z_n = \langle z_n \rangle$ are real positive.

References

A.Li et al.(Kentucky), Phys.Rev.D82:054502,2010, arXiv:1005.4158

A.Suzuki et al.(Zn Collaboration), Lattice 2016 Proceedings,

V.Goy et al.(Vladivostok), Prog Theor Exp Phys (2017) (3): 031D01,arXiv:1611.08093

Where comes the phase of z_n ?

A.Li et al.(Kentucky), Phys.Rev.D82:054502,2010,
arXiv:1005.4158

$$Z = \int \mathcal{D}U \left(\det \Delta(\mu)\right)^{N_f} e^{-S_G} \quad = e^{\log(1-\kappa Q)}$$

$$\begin{aligned}\det \Delta(\mu) &= \det(1-\kappa Q(\mu)) \\ &= \exp\left(A_0 + \sum_{n>0}[e^{in\phi}W_n + e^{-in\phi}W_n^\dagger]\right) \\ &= \exp\left(A_0 + \sum_{n} A_n \cos(n\phi + \delta_n)\right)\end{aligned}$$

$$\begin{aligned}A_n &\equiv 2|W_n| \\ \delta_n &\equiv arg(W_n)\end{aligned}$$

We use $W_{-n} = W_n$

Then,

$$z_n \propto \int_0^{2\pi} \frac{d\phi}{2\pi} e^{-ik\phi} e^{A_0 + A_1\cos(\phi+\delta_1) + A_1 \cos(2\phi+\delta_2)\cdots}$$

In the lowest order,

$$\int_0^{2\pi} \frac{d\phi}{2\pi} e^{-ik\phi} e^{A_0 + A_1 \cos(\phi+\delta_1)} = e^{A_0} \int_{\delta_1}^{2\pi+\delta_1} \frac{d\phi'}{2\pi} e^{-ik(\phi'-\delta_1)} e^{A_1 \cos\phi'}$$

$$= e^{A_0+ik\delta_1} \int_{\delta_1}^{2\pi+\delta_1} \frac{d\phi'}{2\pi} e^{-ik\phi'} e^{A_1 \cos\phi'}$$

$$= e^{A_0+ik\delta_1} \int_0^{2\pi} \frac{d\phi'}{2\pi} e^{-ik\phi'} e^{A_1 \cos\phi'}$$

$$= e^{A_0+ik\delta_1} I_k(A_1)$$

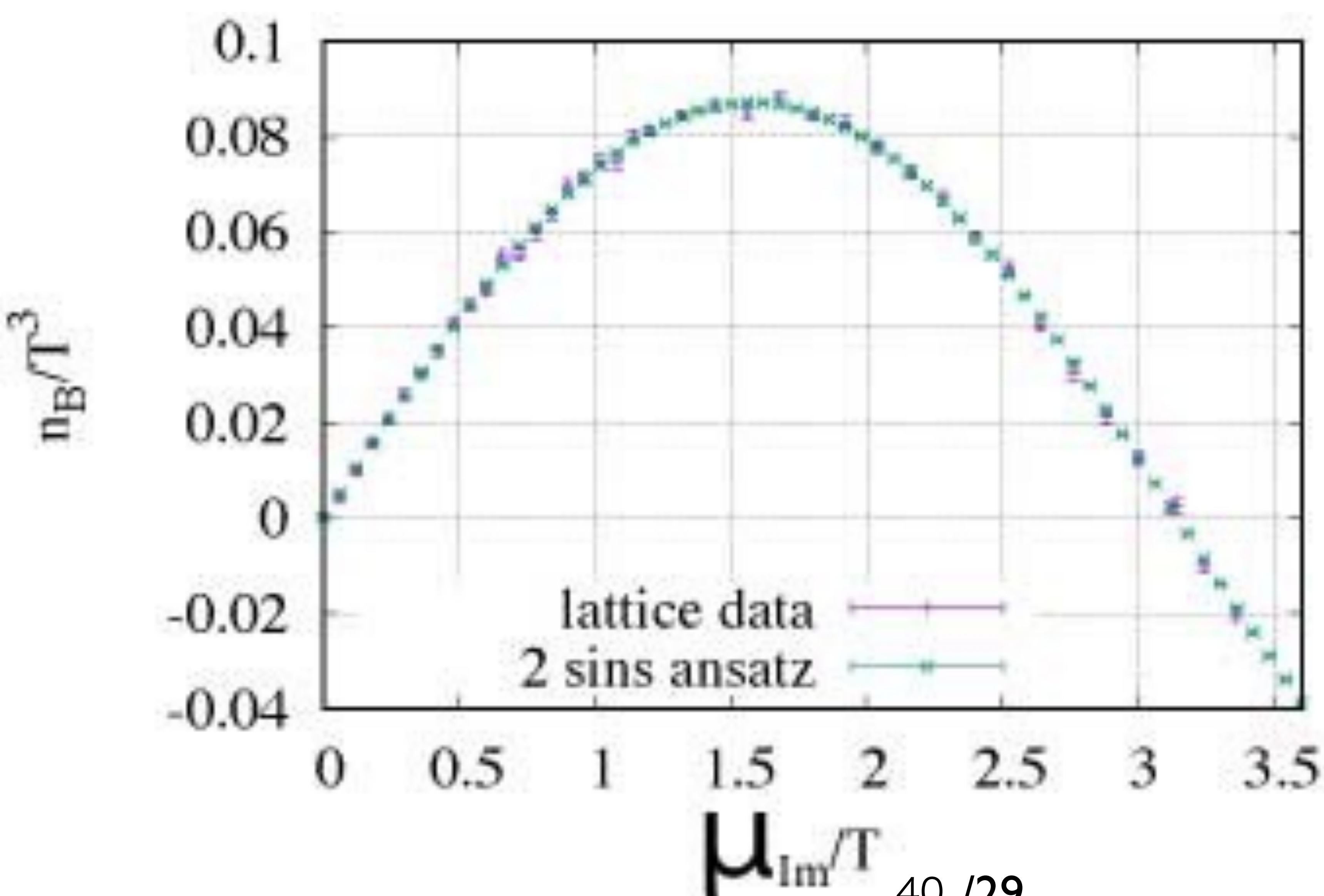
$$\propto z_k$$

where we use

$$I_n(z) = \frac{(-1)^n}{2\pi} \int_0^{2\pi} e^{z\cos t} e^{-int} dt$$

A Remark of Function Form of $n_B(\mu_I)$

Preliminary



$n_B(\mu_I)$ is well approx-imated by sine function at *T*<*Tc*.

Takahashi et al. Phy. Rev. D 91 (1) (2015) 014501.
Bornyakov et al., Phys.Rev. D95, 094506 (2017)

Number density in Imaginary μ

We expand the number density as

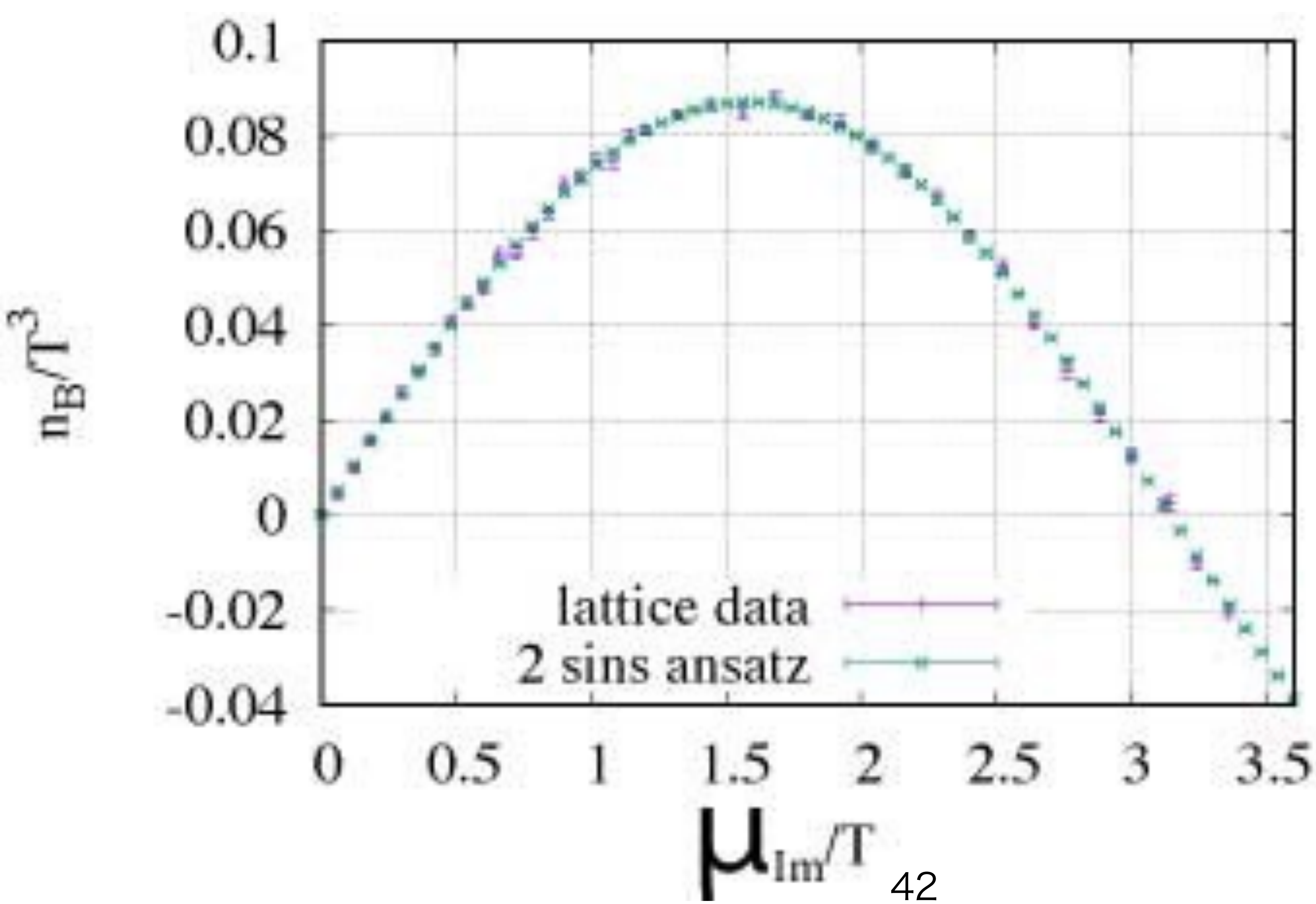
$$n_B/T^3 = \sum_{k=1}^{k_{max}} f_{3k}\sin(k\theta)$$ **Confinement phase** $T < T_c$

$$n_B/T^3 = \sum_{k=1}^{k_{max}} a_{2k-1}\theta^{2k-1}$$ **DeConfinement phase** $T > T_c$

Fittine functions are much more robust against the hidden sign problem, because a fitting curve include many points.

$$\theta \equiv \frac{\mu}{T}$$

A Remark of Function Form of $n_B(\mu_I)$



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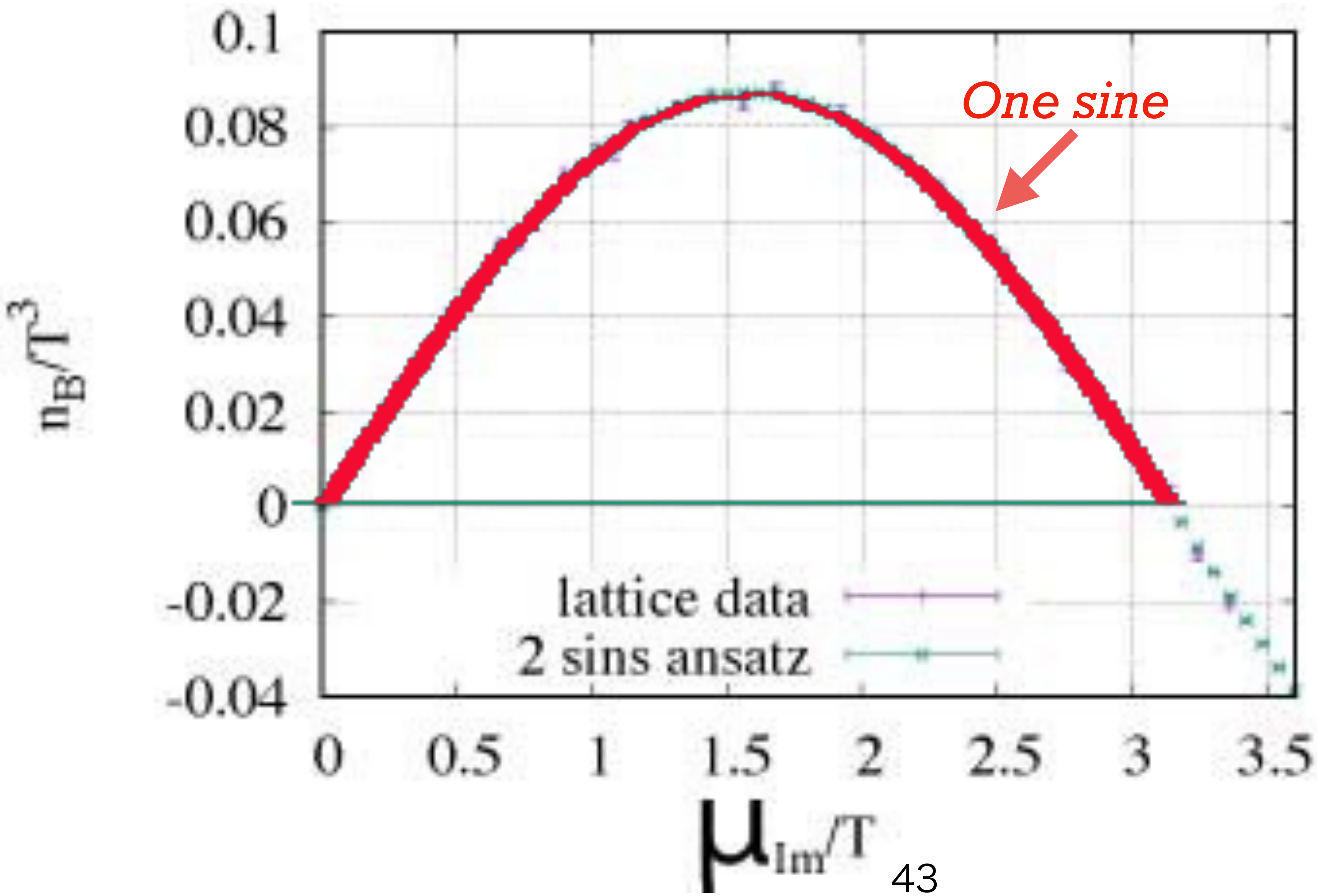
A Remark of Function Form of $n_B(\mu_I)$



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Takahashi et al. Phy. Rev. D 91 (1) (2015) 014501.
Bornyakov et al., Phys.Rev. D95, 094506 (2017)

- Now we can say we have beaten Sign Problem for T>0 by Canonical Approach.

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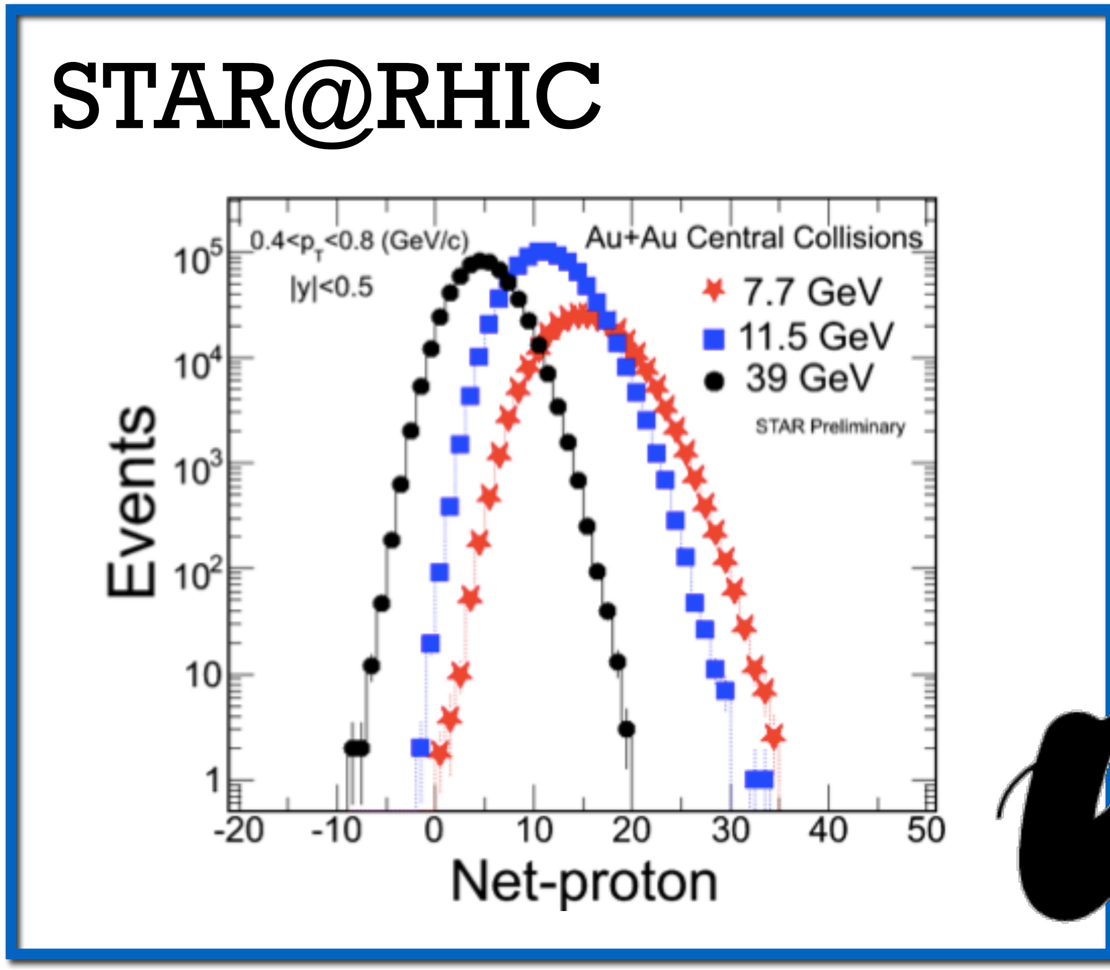
6. What should we do next ?

In 2012, at Wuhan



Prof.Nu Xu

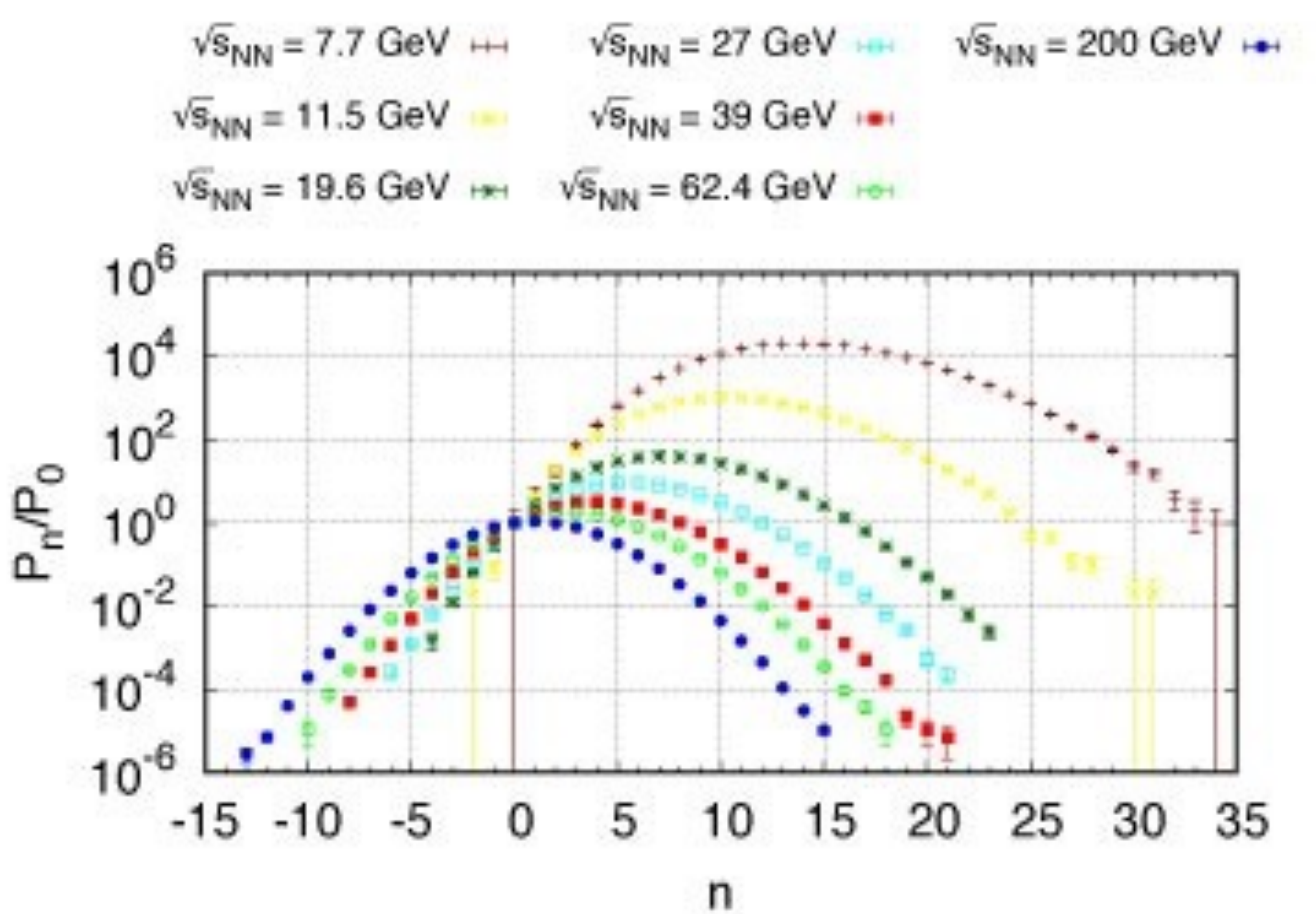
This is Canonical !

We thank Prof.Nu Xu
and Prof.Luo!

$$Z(\mu, T) = \sum_n Z_n(T)(e^{\mu/T})^n$$

Experimental data and Fugacity Expansion

$$Z(\mu, T) = \sum_n Z_n(T) z_n(T)^n$$



Talk by Denis

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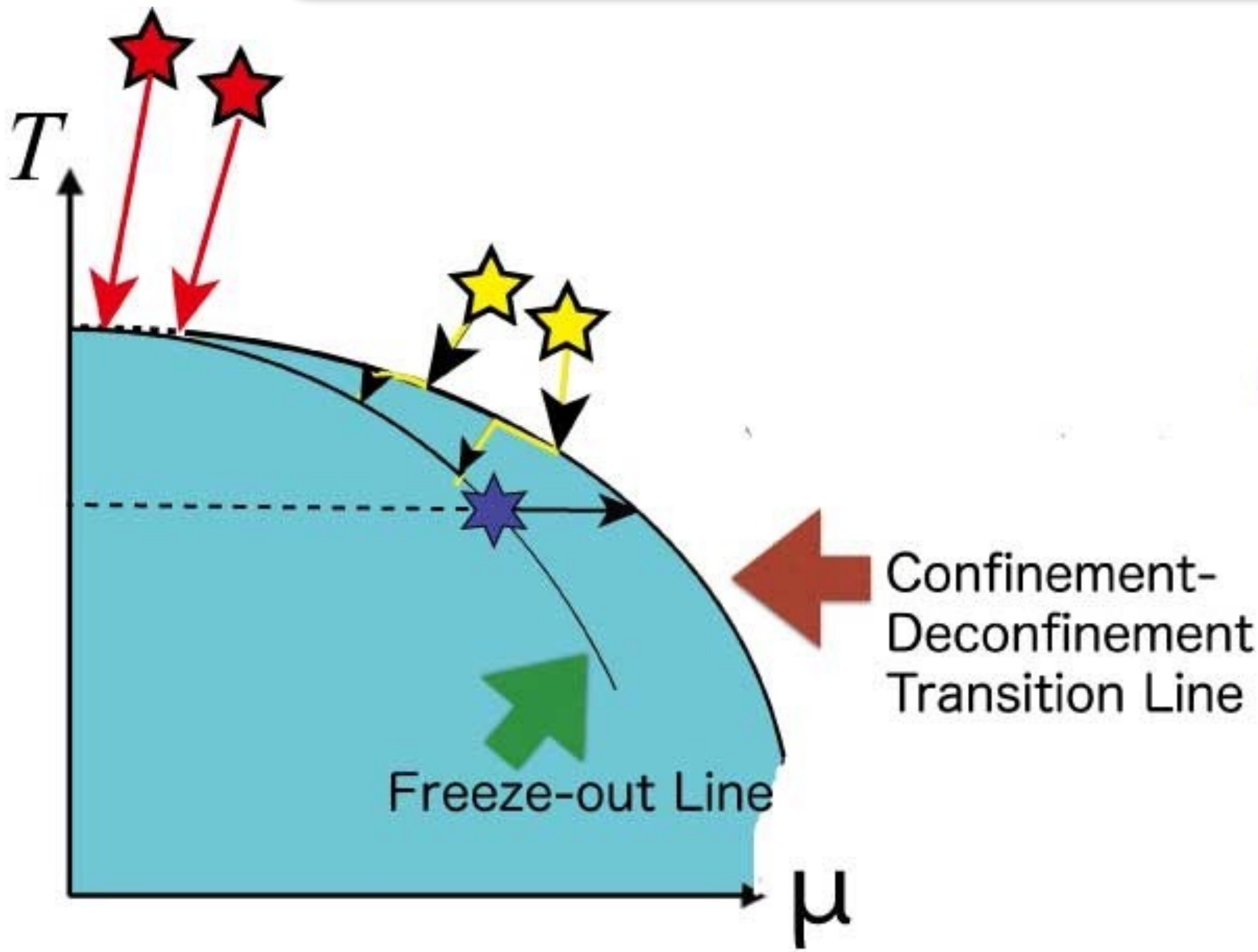
$$Z(\mu, T) = \sum_n z_n(T)(e^{\mu/T})^n$$

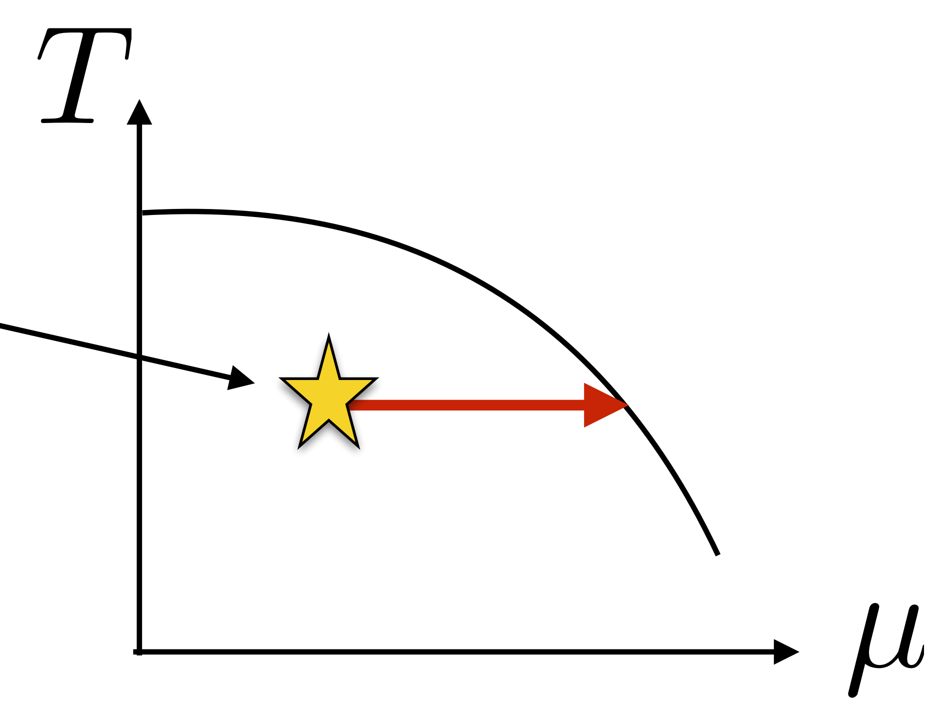


Information hidden in Fugacity Expansion ?

$$Z(\mu, T) = \sum_n z_n(T)(e^{\mu/T})^n$$

☆ <u>Experimentally</u>

Determine $Z_n(T)$ here.
Then see QCD Phase
at higher density !



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Sign Problem is now solved for T>0, and
it is time to analyze the finite density QCD.

But people do not know it. Why ?

It takes very
long time until
your idea is understood.

Because your
Approach is
different.

But I use
only Statistical
Mechanics !?

What should we do next ?

- Let the world to know that the Sign Problem was solved by Vladivostok group
  - Canonical approach + Multiple precision beat the sign problem
- Quark mass in the present lattice QCD calculation is very heavy, and we want go to more realistic quark masses.
  - Physical quak mass lattice simulations have been done by several groups at zero density. (Algorithm is known)

OK, we explore the new world, Hadronic matter at Finite Density, with our Tool !

Agenda

Future lattice topics, possible topics for

small groups, low-cost computations