A model of quantum gravity with emergent spacetime

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Sung-Sik Lee

McMaster   Perimeter Institute

Dynamical geometry, topology and dimension

- In general relativity, geometry is dynamical
- There is in priori no reason why topology and dimension remain well defined in the presence of strong quantum fluctuations of geometry

- Goal :
  - Quantum gravity in which dimension, topology and geometry are dynamical

[Other related works : Quantum graphity, Konopka, Markopoulou, Smolin (06); Geometry from entanglement, Cao, Carroll, Michalakis(17), ..]

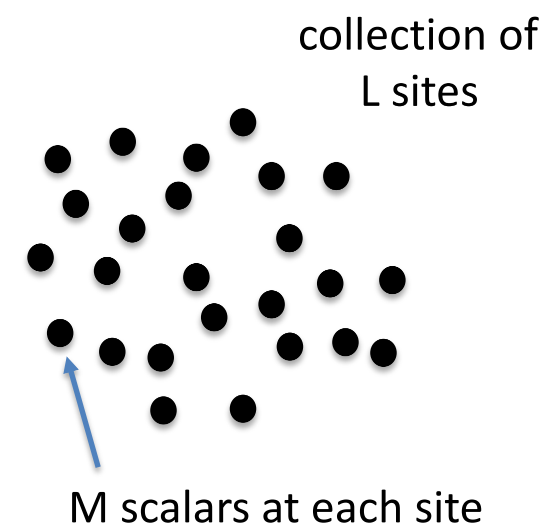
Model

- Fundamental degree of freedom : M × L real matrix

$$\Phi^A{}_i \qquad A = 1, 2, .., M, \quad i = 1, 2, .., L$$

(M≫L)

  - row index (A) : flavor
  - column index (i) : site



- Hilbert space : $\{ \ |\Phi\rangle \ \}$

$$\hat{\Phi}^A{}_i|\Phi\rangle = \Phi^A{}_i|\Phi\rangle$$

- Inner product : $\langle\Phi'|\Phi\rangle = \prod_{i,A} \delta\left(\Phi'^A_i - \Phi^A_i\right)$

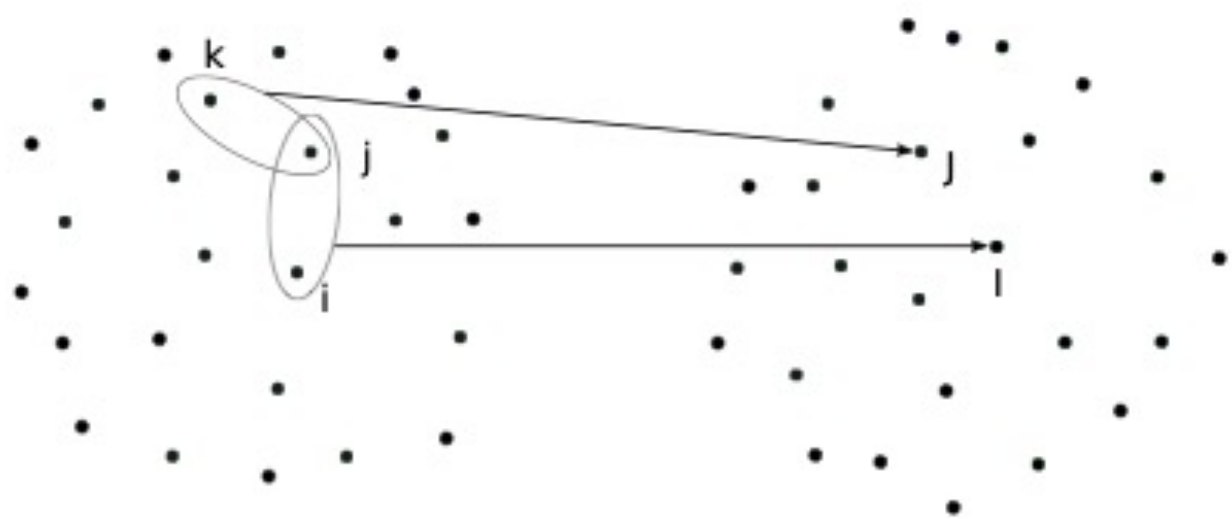
Frame

- A decomposition of the total Hilbert space as a direct product of local Hilbert spaces

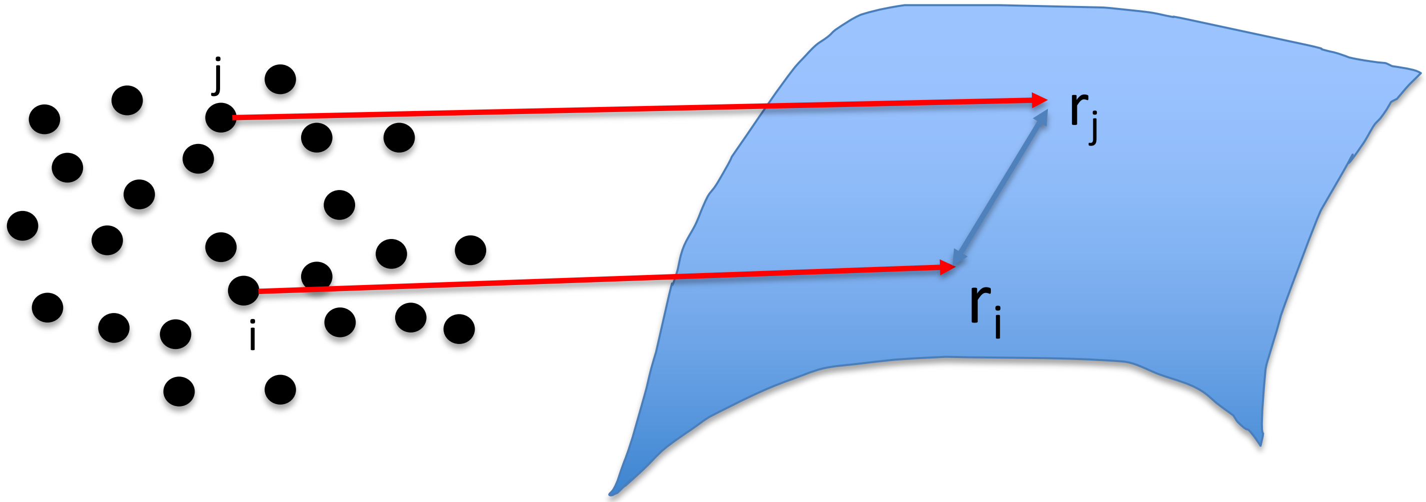
$$\mathbb{H} = \otimes_i \mathbb{H}_i$$

- Choice of frame is not unique $\mathbb{H} = \otimes_I \mathbb{H}'_I$

$$\tilde{\Phi}^A{}_i = \Phi^A{}_I \; g^I{}_i \qquad g \in SL(L, \mathbb{R})$$



Local structure



- A state is defined to have a local structure in a frame if
  - there exists a mapping from the set of sites into a Riemannian manifold
  - the mutual information between two points decay exponentially in the geodesic distance between the images of the points

$$I_{ij} = S_i + S_j - S_{i \cup j} \propto e^{-d(r_i, r_j)/\xi}$$

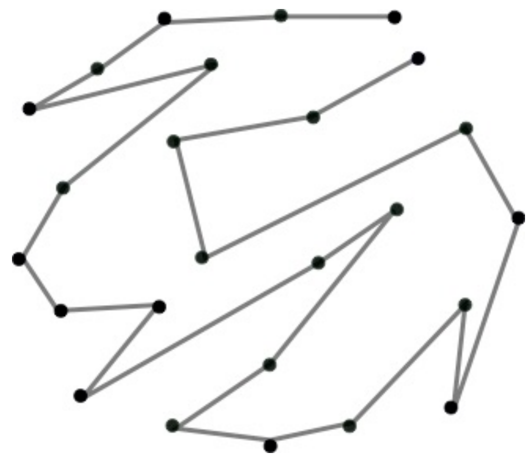
[earlier use of mutual information for distance measure :
Qi (2013); Cao, Carroll, Michalakis(17)]

Examples of states with/without local structures

$$|t\rangle = \int d\Phi \, e^{i t^{ij} \Phi_i^A \Phi_j^A} |\Phi\rangle$$

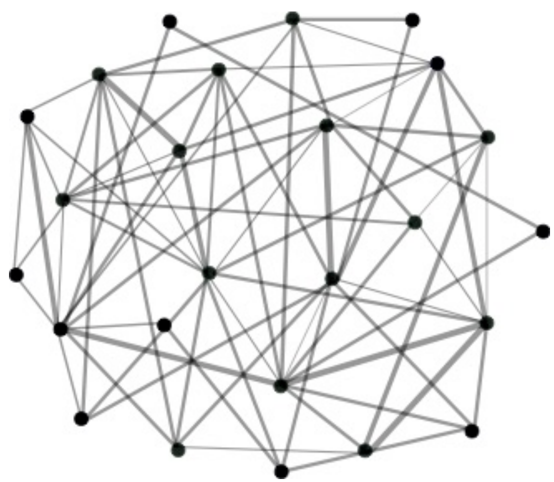
t^{ij} describes entanglement bonds

$$I_{ij} = 2M\left(-\ln\frac{|t^{ij}|^2}{4Imt^{ii}Imt^{jj}} + 1\right)\frac{|t^{ij}|^2}{4Imt^{ii}Imt^{jj}} + \ldots$$



1D local structure

No local structure



2D local structure

Generalized spatial diffeomorphism

- In GR, spatial diffeomorphism is generated by momentum constraint
$$\left\{P\left[\vec{\xi}_1\right], P\left[\vec{\xi}_2\right]\right\}_{PB} = P\left[\mathscr{L}_{\vec{\xi}_1}\vec{\xi}_2\right] \qquad \vec{\xi} \quad : \quad \text{shif}$$
- The dimension and topology of manifold is determined from the pattern of entanglement
- Generalized spatial diffeomorphism should include
  - smooth diffeomorphism in any dimension and topology
  - a map that takes any chosen point in the set to any other chosen point
- SL(L,R) frame rotation

L × M   M× L

$$\hat{G}_y = \frac{1}{2}\text{tr}\left(\hat{\Pi}\hat{\Phi}y\right) + h.c.$$

$$\Pi^i{}_A = -i\frac{\partial}{\partial \Phi^A{}_i}$$

y : L × L traceless matrix (shift tensor)

Hamiltonian constraint

- In GR, Hamiltonian density transforms as a scalar density under spatial diffeomorphism

$$\left\{P\left[\vec{\xi}\right], H\left[\theta\right]\right\}_{PB} = H\left[\mathscr{L}_{\vec{\xi}}\,\theta\right],$$
$$\left\{H\left[\theta_1\right], H\left[\theta_2\right]\right\}_{PB} = P\left[\vec{\xi}_{\theta_1,\theta_2}\right].$$

$$\theta \quad : \quad \text{lapse}$$

$$\xi^{\mu}_{\theta_1,\theta_2} = -\mathcal{S}\, g^{\mu\nu}\left(\theta_1 \nabla_\nu \theta_2 - \theta_2 \nabla_\nu \theta_1\right)$$

- A Hamiltonian that satisfies $[H, H] \sim G$ is

$$\hat{H}_v = \operatorname{tr}\left\{\left(-\hat{\Pi}\hat{\Pi}^T + \frac{\tilde{\alpha}}{M^2}\hat{\Pi}\hat{\Pi}^T\hat{\Phi}^T\hat{\Phi}\hat{\Pi}\hat{\Pi}^T\right) v\right\}$$

↑ lapse tensor (symmetric matrix)

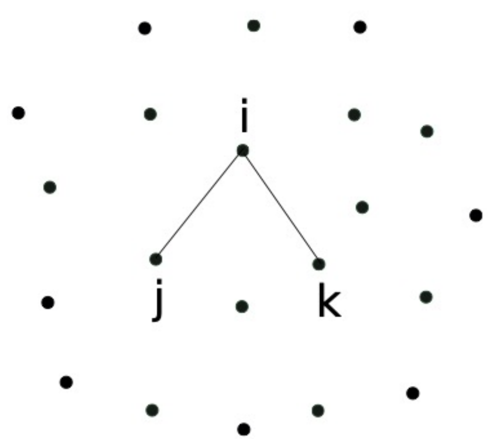
Physical meaning

In the frame in which the lapse is diagonal,

$$\hat{H}_v = n_v \sum_i S_i \left[-\hat{\Pi}'^{i}_{A}\hat{\Pi}'^{i}_{A} + \frac{\tilde{\alpha}}{M^2} \sum_{j,k} \hat{\Pi}'^{i}_{A}\hat{\Pi}'^{j}_{A}\hat{\Phi}'^{B}_{j}\hat{\Phi}'^{B}_{k}\hat{\Pi}'^{k}_{C}\hat{\Pi}'^{i}_{C} \right]$$

ultra-local kinetic term

Relatively local hopping term



$$\left\langle \hat{\Pi}'^{i}_{A}\hat{\Pi}'^{j}_{A} \right\rangle \hat{\Phi}'^{B}_{j}\hat{\Phi}'^{B}_{k} \left\langle \hat{\Pi}'^{k}_{C}\hat{\Pi}'^{i}_{C} \right\rangle$$

$$\frac{1}{M}\frac{\langle t|(\hat{\Pi}^{j}{}_{A}\hat{\Pi}^{i}{}_{A})|t\rangle}{\langle t|t\rangle} = -2i(t^{-1} - t^{*-1})^{-1}_{ij}$$

the strength of hopping between sites j and k is given by the strength of mutual information formed by a third site i

$$|t\rangle = \int d\Phi \; e^{it^{ij}\Phi^{A}_{i}\Phi^{A}_{j}} |\Phi\rangle$$

Constraint Algebra

$$\left[\hat{G}^{i}{}_{j}, \hat{G}^{k}{}_{l}\right] = A^{ikn}_{jlm}\ \hat{G}^{m}{}_{n}$$

$$\left[\hat{G}^{i}{}_{j}, \hat{H}^{kl}\right] = B^{ikl}_{jmn}\ \hat{H}^{mn}$$

sub-leading

$$\left[\hat{H}^{ij}, \hat{H}^{kl}\right] = C^{ijkln}_{m}\ \hat{G}^{m}{}_{n} + \frac{1}{M} D^{ijkl}_{mn} \hat{H}^{mn}$$

Algebra of GR

$$\left\{P\left[\vec{\xi}_1\right], P\left[\vec{\xi}_2\right]\right\}_{PB} = P\left[\mathscr{L}_{\vec{\xi}_1}\vec{\xi}_2\right],$$
$$\left\{P\left[\vec{\xi}\right], H\left[\theta\right]\right\}_{PB} = H\left[\mathscr{L}_{\vec{\xi}}\theta\right],$$
$$\left\{H\left[\theta_1\right], H\left[\theta_2\right]\right\}_{PB} = P\left[\vec{\xi}_{\theta_1,\theta_2}\right].$$

$$\xi^{\mu}_{\theta_1,\theta_2} = -\mathcal{S} g^{\mu\nu}\left(\theta_1\nabla_\nu\theta_2 - \theta_2\nabla_\nu\theta_1\right)$$

Unlike A and B, C is a dynamical variable (function of $\hat{\Phi}, \ \hat{\Pi}$)

- The constraints obey a **first-class quantum** algebra
- In the continuum limit, the constraint algebra reduces to an algebra that includes general relativity once we identify the metric as

$$g^{\mu\nu}(r_m) = \frac{1}{2}\sum_{i,k,n} C^{iikkn}_{m}\left(r^{\mu}_{n} - r^{\mu}_{m}\right)\left(r^{\nu}_{k} - r^{\nu}_{i}\right)$$

- The metric identified in this way indeed encodes information on entanglement
- However, the metric alone does not fully specify entanglement : ER $\subsetneq$ EPR

Gauge Invariant State

$$\hat{G}_y \big| 0 \big\rangle = \hat{H}_v \big| 0 \big\rangle = 0$$

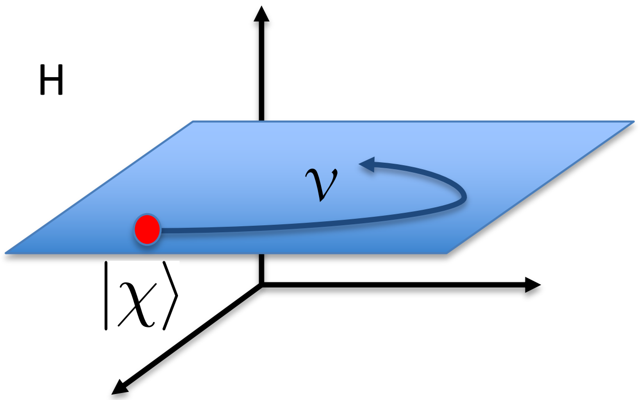
- All gauge invariant states have infinite norm
  - Gauge group is non-compact, and wavefunctions for gauge invariant states are extended in the space of ϕ
- States to which probabilities can be assigned break the gauge symmetry (spontaneously)
- A natural object is an overlap between gauge invariant state, $|0\rangle$ and a state with finite norm, $|\chi\rangle$
- $\langle 0 | \chi \rangle$: wavefunction of gauge invariant state written in the basis of states with finite norm

Projection

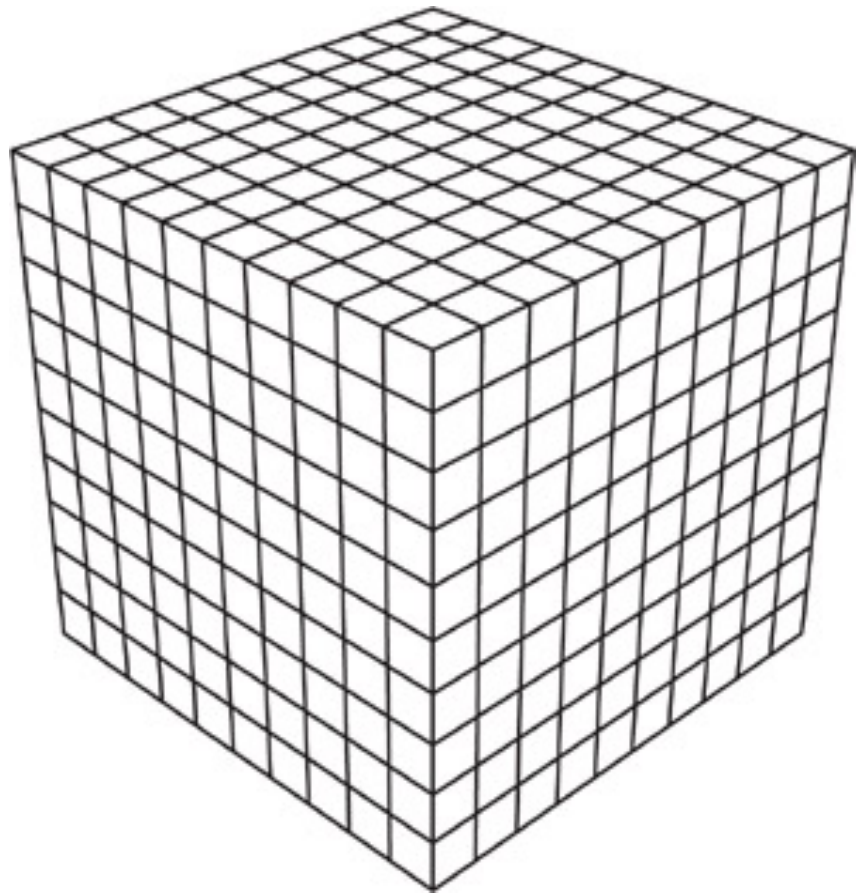
$$\langle 0|\chi\rangle = \langle 0|e^{-i\epsilon\left(\hat{H}_{v(k)}+\hat{G}_{y(k)}\right)}..e^{-i\epsilon\left(\hat{H}_{v(2)}+\hat{G}_{y(2)}\right)}e^{-i\epsilon\left(\hat{H}_{v(1)}+\hat{G}_{y(1)}\right)}|\chi\rangle$$

$$|\chi(\tau)\rangle = \mathcal{T}e^{-i\int_0^\tau d\tau'\left(\hat{H}_{v(\tau')}+\hat{G}_{y(\tau')}\right)}|\chi\rangle$$

- A series of successive gauge transformations generates an evolution of the state with finite norm
- The evolution describes paths along which the state with finite norm is projected to the gauge invariant state
- The sub-Hilbert space ($\mathcal{V}$) within which paths lie is determined by global symmetry of $|\chi\rangle$



Initial state with a classical 3d local structure

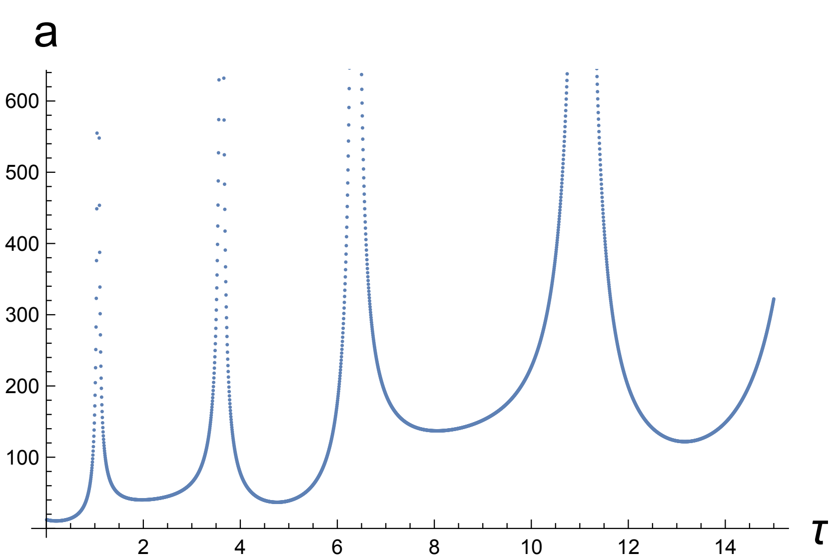


Three torus with nearest neighbor entanglement bonds in three-dimensional local structure

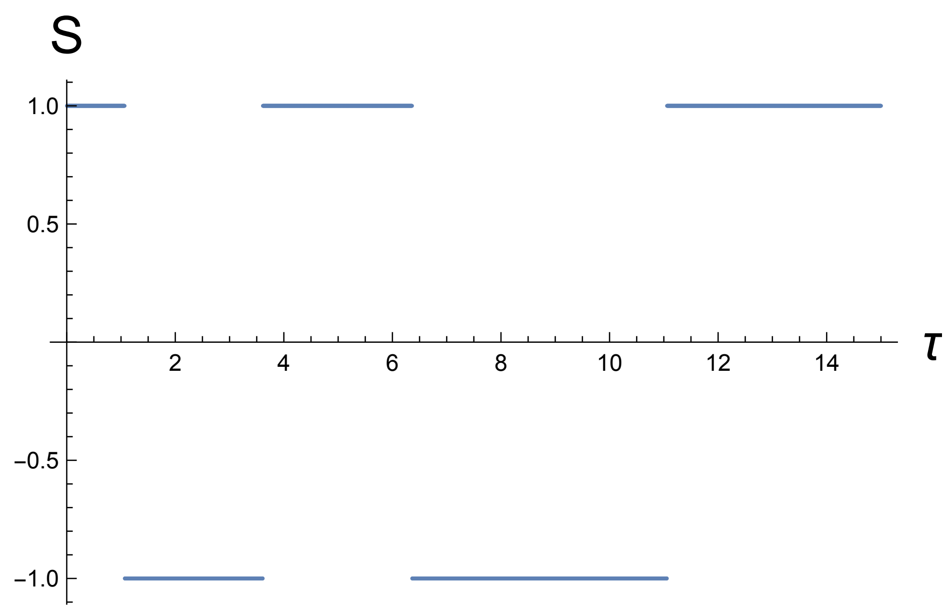
Time evolution

numerical solution for $L=10^6$

Scale factor of space ($g_{\mu\nu} = a\delta_{\mu\nu}$)

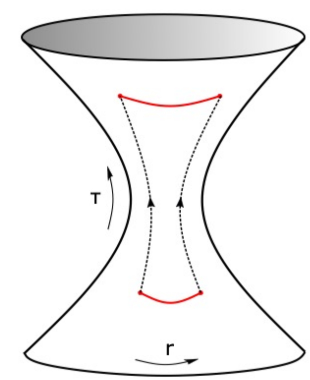


Signature of time



Euclidean spacetime

One epoch of de Sitter-like spacetime with Lorentzian signature



Conclusion

- A background independent quantum gravity in which dimension, topology and geometry are dynamical collective variables of underlying quantum matter
- Saddle-point solution that describes a series of de Sitter-like spacetimes

Open question

- Physical spectrum
- A background independent theory which has a small number of low-energy modes

SUPPLEMENTARY MATERIAL

Review of GR in Hamiltonian formalism

[Arnowitt-Deser-Misner]

$$S = \int d\tau d^3r \left[\pi^{\mu\nu}\partial_\tau g_{\mu\nu} - \xi^\mu(r)\mathscr{P}_\mu(r) - \theta(r)\mathscr{H}(r)\right]$$

momentum constraint $\quad P\left[\vec{\xi}\right] = \int d^3r\, \xi^\mu(r)\mathscr{P}_\mu(r)$

Hamiltonian constraint $\quad H[\theta] = \int d^3r\, \theta(r)\mathscr{H}(r)$

Hypersurface deformation algebra

$$\begin{aligned}
\left\{P\left[\vec{\xi}_1\right], P\left[\vec{\xi}_2\right]\right\}_{PB} &= P\left[\mathscr{L}_{\vec{\xi}_1}\vec{\xi}_2\right], \\
\left\{P\left[\vec{\xi}\right], H[\theta]\right\}_{PB} &= H\left[\mathscr{L}_{\vec{\xi}}\theta\right], \\
\left\{H[\theta_1], H[\theta_2]\right\}_{PB} &= P\left[\vec{\xi}_{\theta_1,\theta_2}\right].
\end{aligned}$$

$$\xi^\mu_{\theta_1,\theta_2} = -\mathcal{S} g^{\mu\nu}\left(\theta_1\nabla_\nu\theta_2 - \theta_2\nabla_\nu\theta_1\right)$$

$(\mathcal{S}, +, +, +)$ signature

spatial metric

SL(L,R) frame rotation : generalized spatial diffeomorphism

$$\hat{G}_y = \mathbf{tr}\left\{\hat{G}y\right\}$$

$$\hat{\mathbf{G}}^i{}_j = \frac{1}{2}\left(\hat{\Pi}^i{}_A\hat{\Phi}^A{}_j + \hat{\Phi}^A{}_j\hat{\Pi}^i{}_A\right)$$

y : L × L traceless matrix (shift tensor)

$e^{-i\hat{G}_y}\ \hat{\Phi}\ e^{i\hat{G}_y}\ =\ \hat{\Phi}\ g_y,$	covariant
$e^{-i\hat{G}_y}\ \hat{\Pi}\ e^{i\hat{G}_y}\ =\ g_y^{-1}\ \hat{\Pi}$	contravariant

$$g_y = e^{-y} \in\ SL(L,\mathbb{R})$$

Smooth diffeomorphism from SL(L,R)

$$e^{-i\hat{G}_{\epsilon y}} \; \hat{\Phi}^A{}_i \; e^{i\hat{G}_{\epsilon y}} \;=\; \left(\hat{\Phi}\; e^{-\epsilon y}\right)^A{}_i$$

$$= \hat{\Phi}^A{}_i - \epsilon\; \hat{\Phi}^A{}_j \; y^j{}_i$$

if $\phi^A{}_i$ varies slowly in a manifold

$$= \hat{\Phi}^A(r_i) - \epsilon \left[\hat{\Phi}^A(r_i) + \partial_\mu \hat{\Phi}^A(r_i)(r^\mu_j - r^\mu_i) + ..\right] y^j{}_i$$

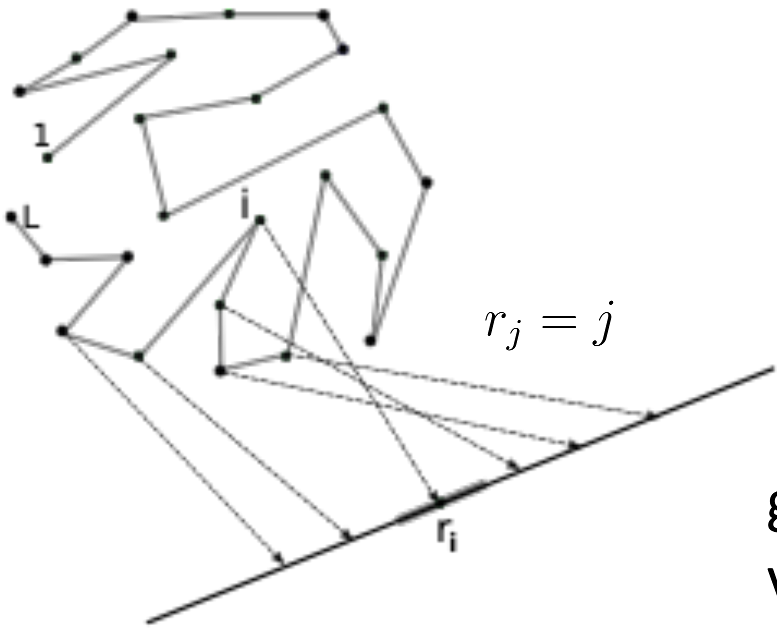
$$= \hat{\Phi}^A(r_i) - \epsilon\; \zeta_y(r_i)\hat{\Phi}^A(r_i) - \epsilon\; \xi^\mu_y(r_i)\partial_\mu \hat{\Phi}^A(r_i) + \ldots$$

$$\zeta_y(r_i) \;=\; \sum_j y^j{}_i, \qquad \text{Weyl scalar}$$

$$\xi^\mu_y(r_i) \;=\; \sum_j y^j{}_i (r^\mu_j - r^\mu_i) \qquad \text{shift vector}$$

- Generalized diffeomorphism includes Weyl transformations, smooth diffeomorphism and more
- This is an active transformation in a fixed coordinate system

Example



$$\hat{G}_y = \text{tr}\left\{\hat{G}y\right\}$$ with

$$y^j_{\ i} = \frac{\xi_i}{2}\left(\delta_{j,i+1} - \delta_{j,i-1}\right)$$

generates diffeomorphism with shift vector $\xi(r_i) = \xi_i$
in the continuum limit

$$|t\rangle = \int d\Phi \, e^{it^{ij}\Phi^A_{\ i}\Phi^A_{\ j}}|\Phi\rangle$$

$$t^{ij} = i\left(\delta_{ij} + \epsilon\delta_{|i-j|,1}\right)$$

Full Constraint Algebra

$$\left[\hat{G}_x, \hat{G}_y\right] = i\underline{\hat{G}_{(yx-xy)}} \qquad \left[\hat{G}_x, \hat{H}_v\right] = i\underline{\hat{H}_{vx+x^Tv}}$$

$$\begin{aligned}
\left[\hat{H}_u, \hat{H}_v\right] = &-i\frac{4\tilde{\alpha}}{M^2}\,\mathrm{tr}\left\{\left[(\hat{\Pi}\hat{\Pi}^T)u(\hat{\Pi}\hat{\Pi}^T)v - (\hat{\Pi}\hat{\Pi}^T)v(\hat{\Pi}\hat{\Pi}^T)u\right]\underline{\hat{G}}\right\} \\
&+i\frac{4\tilde{\alpha}^2}{M^4}u_{nk}v_{n'k'}\Big[-(\hat{\Pi}\hat{\Pi}^T)^{kl}(\hat{\Phi}^T\hat{\Phi})_{li}(\hat{\Pi}\hat{\Pi}^T)^{k'i'}(\hat{\Pi}\hat{\Pi}^T)^{j'n'}\delta^n_j \\
&\quad +(\hat{\Pi}\hat{\Pi}^T)^{ki'}(\hat{\Phi}^T\hat{\Phi})_{jl}(\hat{\Pi}\hat{\Pi}^T)^{ln'}(\hat{\Pi}\hat{\Pi}^T)^{j'n}\delta^{k'}_i \\
&\quad +(\hat{\Pi}\hat{\Pi}^T)^{ki'}(\hat{\Pi}\hat{\Pi}^T)^{k'l}(\hat{\Phi}^T\hat{\Phi})_{li}(\hat{\Pi}\hat{\Pi}^T)^{j'n}\delta^{n'}_j \\
&\quad -(\hat{\Pi}\hat{\Pi}^T)^{k'i'}(\hat{\Pi}\hat{\Pi}^T)^{j'n'}(\hat{\Phi}^T\hat{\Phi})_{jl}(\hat{\Pi}\hat{\Pi}^T)^{ln}\delta^k_i \\
&\quad +M(\hat{\Pi}\hat{\Pi}^T)^{ki'}(\hat{\Pi}\hat{\Pi}^T)^{j'n'}\delta^{k'}_i\delta^n_j + (M+2)(\hat{\Pi}\hat{\Pi}^T)^{ki'}(\hat{\Pi}\hat{\Pi}^T)^{k'n}\delta^{j'}_i\delta^{n'}_j \\
&\quad +2(\hat{\Pi}\hat{\Pi}^T)^{ki'}(\hat{\Pi}\hat{\Pi}^T)^{j'n}\delta^{k'}_i\delta^{n'}_j - 2(\hat{\Pi}\hat{\Pi}^T)^{kn'}(\hat{\Pi}\hat{\Pi}^T)^{k'i'}\delta^{j'}_i\delta^n_j \\
&\quad -2(\hat{\Pi}\hat{\Pi}^T)^{kn'}(\hat{\Pi}\hat{\Pi}^T)^{j'i'}\delta^{k'}_i\delta^n_j - 2(\hat{\Pi}\hat{\Pi}^T)^{nk}(\hat{\Pi}\hat{\Pi}^T)^{n'i'}\delta^{j'}_i\delta^{k'}_j\Big]\underline{\hat{G}^{[i}_{[i'}\delta^{j]}_{j']}} \\
&+\frac{\tilde{\alpha}}{M^2}\,\mathrm{tr}\left\{\left[(M-2)(v\hat{\Pi}\hat{\Pi}^Tu - u\hat{\Pi}\hat{\Pi}^Tv) + 4\hat{\Pi}\hat{\Pi}^T(vu-uv)\right]\underline{\hat{H}}\right\},
\end{aligned}$$

← Sub-leading in 1/M

Algebra of GR

$$\begin{aligned}
\left\{P\left[\vec{\xi}_1\right], P\left[\vec{\xi}_2\right]\right\}_{PB} &= P\left[\mathscr{L}_{\vec{\xi}_1}\vec{\xi}_2\right], \\
\left\{P\left[\vec{\xi}\right], H\left[\theta\right]\right\}_{PB} &= H\left[\mathscr{L}_{\vec{\xi}}\theta\right], \\
\left\{H\left[\theta_1\right], H\left[\theta_2\right]\right\}_{PB} &= P\left[\vec{\xi}_{\theta_1,\theta_2}\right].
\end{aligned}$$

Constraint algebra in the continuum I

$$\begin{aligned}\mathscr{G}_y &= \mathscr{G}^i{}_j y^j{}_i. \\ &= \left[\mathscr{G}^i{}_i + \left.\frac{\partial \mathscr{G}^i{}_j}{\partial r^\mu_j}\right|_{j=i} (r^\mu_j - r^\mu_i) + ..\right] y^j{}_i \\ &= \mathscr{G}^i{}_i \zeta_y(r_i) + \left.\frac{\partial \mathscr{G}^i{}_j}{\partial r^\mu_j}\right|_{j=i} \xi^\mu_y(r_i) + ..., \\ &= \int dr \Big(\mathscr{D}(r)\zeta_y(r) + \mathscr{P}_\mu(r)\xi^\mu_y(r) + ..\Big)\end{aligned}$$

$$\mathscr{D}(r_i) = V_i^{-1}\mathscr{G}^i{}_i,$$

$$\mathscr{P}_\mu(r_i) = V_i^{-1} \left.\frac{\partial \mathscr{G}^i{}_j}{\partial r^\mu_j}\right|_{j=i}$$

V_i : coordinate volume assigned to site i

$$\begin{aligned}&\left\{\int dr\Big(\mathscr{D}(r)\zeta_x(r) + \mathscr{P}_\mu(r)\xi^\mu_x(r) + ..\Big), \int dr'\Big(\mathscr{D}(r')\zeta_y(r') + \mathscr{P}_\nu(r')\xi^\nu_y(r') + ..\Big)\right\}_{PB} \\ &= \int dr\Big(\mathscr{D}(r)\zeta_{yx-xy}(r) + \mathscr{P}_\mu(r)\xi^\mu_{yx-xy}(r) + ..\Big),\end{aligned}$$

$$\zeta_{yx-xy}(r) = \mathscr{L}_{\xi_x}\zeta_y(r) + O(\partial^2)$$

$$\xi^\mu_{yx-xy}(r) = (\mathscr{L}_{\xi_x}\xi_y(r))^\mu + O(\partial^2)$$

Constraint algebra in the continuum II

$$\mathscr{H}_v \;=\; \int dr\, \theta_v(r)\mathscr{H}(r)$$

$$\mathscr{H}(r_i) \;=\; V_i^{-1}\mathscr{H}^{ii}, \qquad\qquad \theta_v(r_i) \;=\; v_{ii}$$

$$\left\{\int dr''\Big(\mathscr{D}(r'')\zeta_x(r'') + \mathscr{P}_\mu(r'')\xi_x^\mu(r'') + ..\Big), \int dr\theta_v(r)\mathscr{H}(r)\right\}_{PB} = \int dr\; \theta_{vx+x^Tv}(r)\mathscr{H}(r)$$

$$\theta_{vx+x^Tv}(r) \;=\; 2\zeta_x(r)\theta_v(r) + \mathscr{L}_{\xi_x}\theta_v(r) + O(\partial^2)$$

Constraint algebra in the continuum III

$$\left\{\int dr\Big(\theta_u(r)\mathscr{H}(r)+..\Big),\int dr'\Big(\theta_v(r')\mathscr{H}(r')+..\Big)\right\}_{PB}$$
$$=\int dr\Big(F^{\nu}(r)\mathscr{D}(r)+G^{\mu\nu}(r)\mathscr{P}_{\mu}(r)+..\Big)\Big(\theta_u(r)\nabla_{\nu}\theta_v(r)-\theta_v(r)\nabla_{\nu}\theta_u(r)\Big)+O(\partial^2)$$

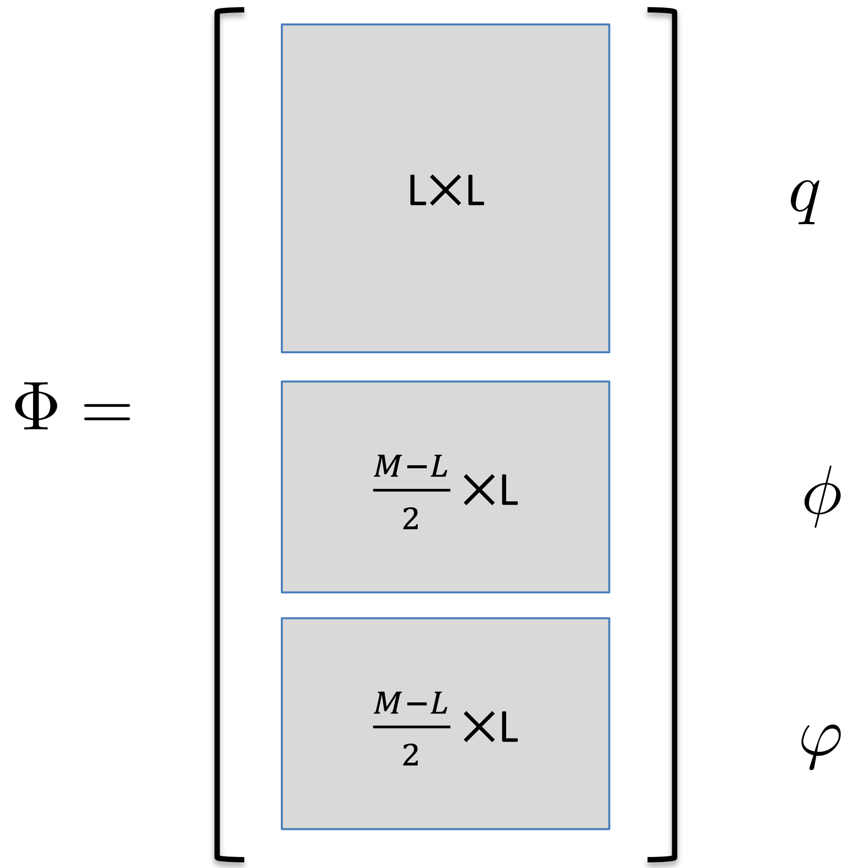
$$F^{\nu}(r_m) \;=\; \frac{1}{2}\sum_{i,k,n}\mathscr{C}_m^{iikkn}\left(r_k^{\nu}-r_i^{\nu}\right),$$
$$G^{\mu\nu}(r_m) \;=\; \frac{1}{2}\sum_{i,k,n}\mathscr{C}_m^{iikkn}\left(r_n^{\mu}-r_m^{\mu}\right)\left(r_k^{\nu}-r_i^{\nu}\right)$$

- Signature and metric are determined from the collective variables
- There exists additional fields such as anti-symmetric rank 2 tensor and higher spin fields

$$-\mathscr{S}g^{\mu\nu} \;=\; \frac{G^{\mu\nu}+G^{\nu\mu}}{2},$$
$$b^{\mu\nu} \;=\; \frac{G^{\mu\nu}-G^{\nu\mu}}{2}.$$

Sub-Hilbert space ($\mathcal{V}$)

- We consider a sub-Hilbert space with unbroken flavor symmetry : $S_L \times O\left(\frac{M-L}{2}\right) \times O\left(\frac{M-L}{2}\right) \subset O(M)$



Basis states of the $\mathcal{V}$

- Basis states of the sub-Hilbert space are labeled by collective variables : s, t_1, t_2

$$\left|s, t_1, t_2\right\rangle = \int DqD\phi D\varphi \; e^{i \,\mathbf{tr}\left\{Nsq + t_1(\phi^T\phi) + t_2(\varphi^T\varphi)\right\}} \left|q, \phi, \varphi\right\rangle'$$

$$\left|q, \phi, \varphi\right\rangle' = \sum_{P^f \in S_L^f} \left|P^f q, \phi, \varphi\right\rangle$$

Collective variables :

general L×L matrix → $s :$ conjugate to q

L×L symmetric matrix { $t_1 :$ conjugate to $\phi^T\phi$; $t_2 :$ conjugate to $\varphi^T\varphi$

States in $\mathcal{V}$

- General states in the sub-Hilbert space can be written as linear superpositions of the basis states

$$|\chi\rangle = \int DsDt \, |s, t_1, t_2\rangle \chi(s, t_1, t_2)$$

Wavefunction defined in the space of collective variables

Constraints for the collective variables

- Gauge constraints, being O(M) invariant, maps $\mathcal{V}$ into $\mathcal{V}$
- Constraints can be written in terms of the collective variables (s,t_1,t_2) and their conjugate variables (q,p_1,p_2)

$$\mathscr{H}[q,s,p_1,t_1,p_2,t_2] = -\left(ss^T + \sum_c \left[4t_c p_c t_c - it_c\right]\right)$$
$$+\tilde{\alpha}\left(ss^T + \sum_c \left[4t_c p_c t_c - it_c\right]\right)\left(q^T q + p_1 + p_2\right)\left(ss^T + \sum_{c'} \left[4t_{c'} p_{c'} t_{c'} - it_{c'}\right]\right) + O\left(\frac{1}{N}\right)$$

$$\mathscr{G}[q,s,p_1,t_1,p_2,t_2] = \left(sq + 2\sum_c t_c p_c - i\frac{M}{2N} I\right)$$

# of physical phase space variables :

$$2\left(L^2 + L(L+1)\right) - 2\left(\left(L^2-1\right) + \frac{L(L+1)}{2}\right) = L(L+1) + 2$$

Path integral representation of state projection

The projection can be written as a path integration of the collective variables

$$\langle 0|\chi\rangle = \int Ds^{(0)} Dt^{(0)} \int \mathscr{D}s\mathscr{D}t\mathscr{D}q\mathscr{D}p\mathscr{D}v\mathscr{D}y \, \langle 0|s^{(\infty)}, t_1^{(\infty)}, t_2^{(\infty)}\rangle e^{iS} \, \chi(s^{(0)}, t_1^{(0)}, t_2^{(0)})$$

$$S = N\int_0^\infty d\tau \, \mathrm{tr}\Big\{ -q\partial_\tau s - p_c\partial_\tau t_c - v(\tau)\mathscr{H}[q(\tau), s(\tau), p_1(\tau), t_1(\tau), p_2(\tau), t_2(\tau)] \\ - y(\tau)\mathscr{G}[q(\tau), s(\tau), p_1(\tau), t_1(\tau), p_2(\tau), t_2(\tau)] \Big\}$$

Different choices of lapse and shift tensors give rise to multi-fingered time

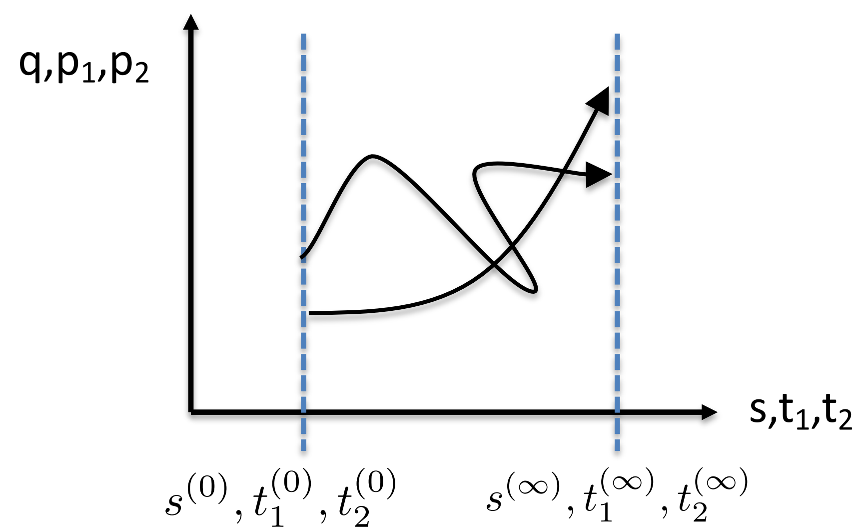
Path integral representation of state projection

The projection can be written as a path integration of the collective variables

$$\langle 0|\chi\rangle = \int Ds^{(0)}Dt^{(0)} \int \mathscr{D}s\mathscr{D}t\mathscr{D}q\mathscr{D}p\mathscr{D}v\mathscr{D}y \,\langle 0|s^{(\infty)}, t_1^{(\infty)}, t_2^{(\infty)}\rangle e^{iS}\,\chi(s^{(0)}, t_1^{(0)}, t_2^{(0)})$$

$$S = N\int_0^\infty d\tau\,\mathrm{tr}\Big\{-q\partial_\tau s - p_c\partial_\tau t_c - v(\tau)\mathscr{H}[q(\tau), s(\tau), p_1(\tau), t_1(\tau), p_2(\tau), t_2(\tau)] \\ -y(\tau)\mathscr{G}[q(\tau), s(\tau), p_1(\tau), t_1(\tau), p_2(\tau), t_2(\tau)]\Big\}$$

Different choices of lapse and shift tensors give rise to multi-fingered time



Constraint Algebra in the classical limit

Poisson bracket :

$$\{A,B\}_{PB} = \left(\frac{\partial A}{\partial q^{\alpha}_{\ i}}\frac{\partial B}{\partial s^{i}_{\ \alpha}} - \frac{\partial A}{\partial s^{i}_{\ \alpha}}\frac{\partial B}{\partial q^{\alpha}_{\ i}}\right) + \delta^{kl}_{ij}\left(\frac{\partial A}{\partial p_{c,ij}}\frac{\partial B}{\partial t^{kl}_{c}} - \frac{\partial A}{\partial t^{kl}_{c}}\frac{\partial B}{\partial p_{c,ij}}\right)$$

Constraint algebra :

$$\{\mathscr{G}^{i}_{\ j}, \mathscr{G}^{k}_{\ l}\}_{PB} = \mathscr{A}^{ikn}_{jlm}\,\mathscr{G}^{m}_{\ n},$$

$$\{\mathscr{G}^{i}_{\ j}, \mathscr{H}^{kl}\}_{PB} = \mathscr{B}^{ikl}_{jmn}\,\mathscr{H}^{mn},$$

$$\{\mathscr{H}^{ij}, \mathscr{H}^{kl}\}_{PB} = \mathscr{C}^{ijkln}_{m}\,\mathscr{G}^{m}_{\ n},$$

The constraint algebra is reduced to the algebra of an extended general relativity in the continuum limit

Constraint Algebra in the classical limit

Poisson bracket :

$$\{A,B\}_{PB} \;=\; \left(\frac{\partial A}{\partial q^{\alpha}_{\ i}}\frac{\partial B}{\partial s^{i}_{\ \alpha}} - \frac{\partial A}{\partial s^{i}_{\ \alpha}}\frac{\partial B}{\partial q^{\alpha}_{\ i}}\right) + \delta^{kl}_{ij}\left(\frac{\partial A}{\partial p_{c,ij}}\frac{\partial B}{\partial t^{kl}_{c}} - \frac{\partial A}{\partial t^{kl}_{c}}\frac{\partial B}{\partial p_{c,ij}}\right)$$

Constraint algebra :

$$\{\mathscr{G}^{i}_{\ j},\mathscr{G}^{k}_{\ l}\}_{PB} \;=\; \mathscr{A}^{ikn}_{jlm}\,\mathscr{G}^{m}_{\ n},$$

$$\{\mathscr{G}^{i}_{\ j},\mathscr{H}^{kl}\}_{PB} \;=\; \mathscr{B}^{ikl}_{jmn}\,\mathscr{H}^{mn},$$

$$\{\mathscr{H}^{ij},\mathscr{H}^{kl}\}_{PB} \;=\; \mathscr{C}^{ijkln}_{m}\,\mathscr{G}^{m}_{\ n},$$

$$\begin{aligned}
\mathscr{A}^{ikn}_{jlm} &= \delta^{k}_{j}\delta^{i}_{m}\delta^{n}_{l} - \delta^{i}_{l}\delta^{k}_{m}\delta^{n}_{j}\\
\mathscr{B}^{ikl}_{jmn} &= \delta^{k}_{j}\delta^{il}_{mn} + \delta^{l}_{j}\delta^{ki}_{mn},\\
\mathscr{C}^{ijkln}_{m} &= -4\tilde{\alpha}\Big[U^{n[j}U^{i][l}\delta^{k]}_{m} - U^{n[l}U^{k][j}\delta^{i]}_{m}\Big]\\
&\quad +4\tilde{\alpha}^{2}\Big[U^{n[j}U^{i]m'}Q_{m'n'}U^{n'[l}\delta^{k]}_{m} + U^{n[j}U^{i][l}U^{k]m'}Q_{m'n'}\delta^{n'}_{m}\\
&\quad -U^{n[l}U^{k]m'}Q_{m'n'}U^{n'[j}\delta^{i]}_{m} - U^{n[l}U^{k][j}U^{i]m'}Q_{m'n'}\delta^{n'}_{m}\Big]
\end{aligned}$$

$$U^{ij} \;=\; \left(ss^{T} + \sum_{c}\,[4t_c p_c t_c - i t_c]\right)^{ij},$$

$$Q_{ij} \;=\; \left(q^{T}q + \sum_{c} p_c\right)_{ij}.$$

Semi-classical state (wavepacket)

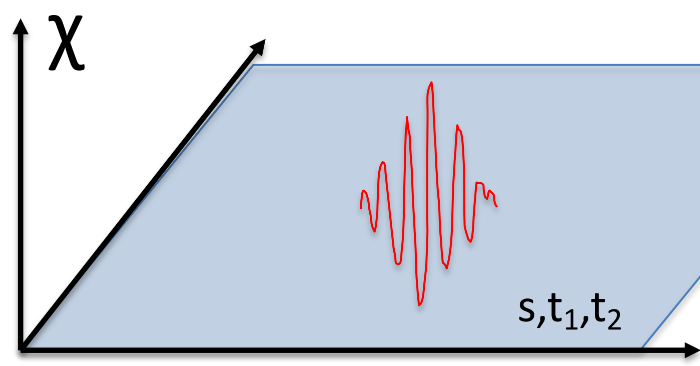
$$\chi_{\mathcal{q},\mathcal{s},\mathcal{p}_c,\mathcal{t}_c}(s,t_1,t_2) =$$

$$\exp\left(-iN\,\mathrm{tr}\left\{\mathcal{q}s + \sum_c \mathcal{p}_c t_c\right\} - \frac{\sum_{i,\alpha}\left[(s)^i{}_\alpha - \mathcal{s}^i{}_\alpha\right]^2 + \sum_c \sum_{ij}\left[t_c^{ij} - \mathcal{t}_c^{ij}\right]^2}{\Delta^2}\right)$$

In order for $\langle 0|\chi\rangle$ to be non-zero, the classical variables should satisfy the classical constraints

$$\mathcal{G}^i{}_j(\mathcal{s},\mathcal{q},\mathcal{t}_c,\mathcal{p}_c) = 0$$

$$\mathcal{H}^{ij}(\mathcal{s},\mathcal{q},\mathcal{t}_c,\mathcal{p}_c) = 0$$

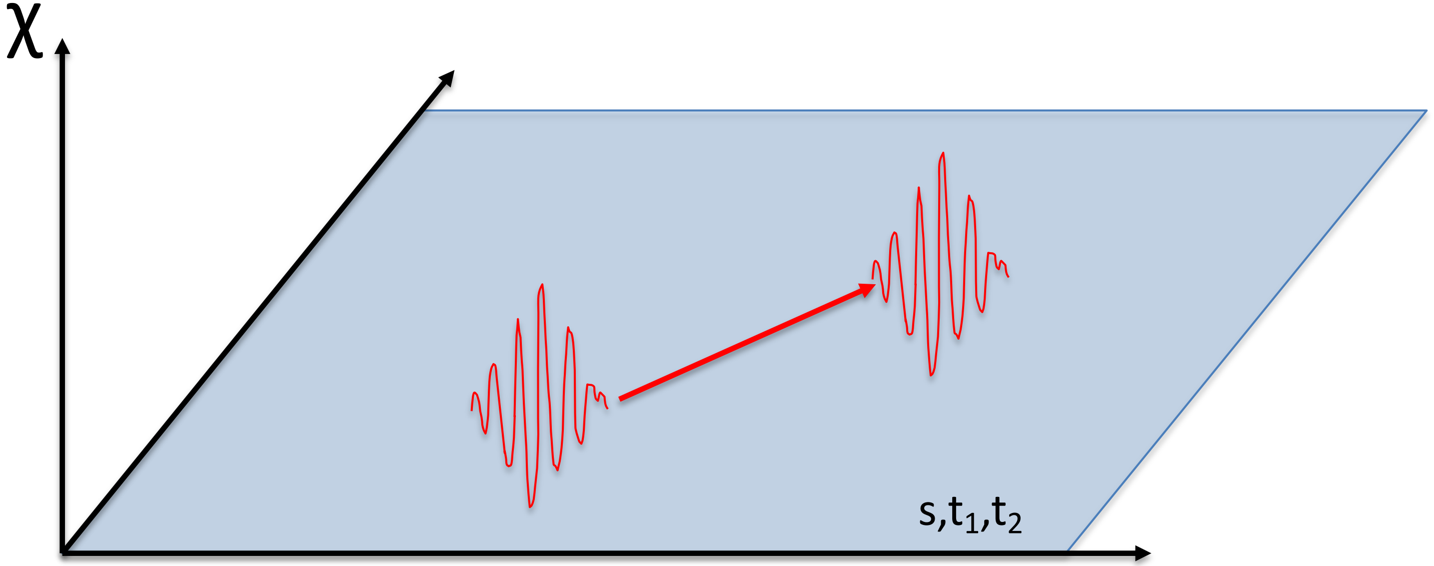


Saddle-Point EOM

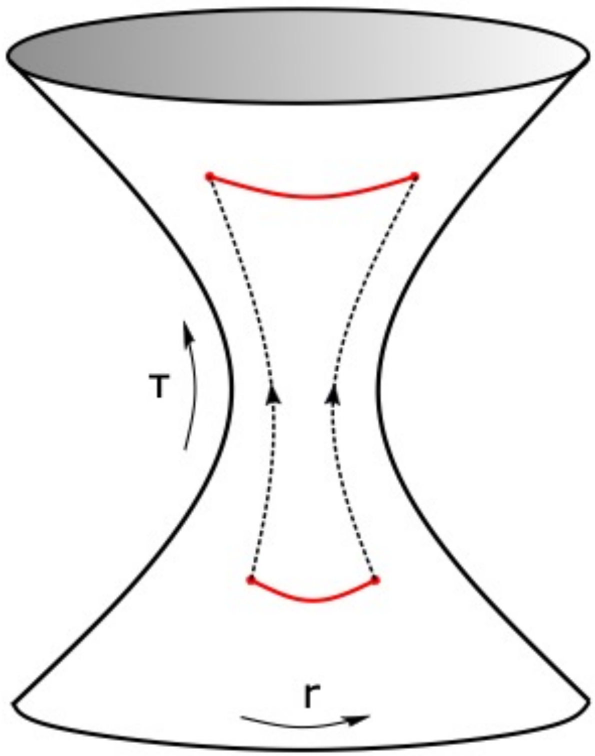
$$\begin{aligned}
\partial_\tau \bar{t}_c &= 4\bar{t}_c v \bar{t}_c - 4\tilde{\alpha}\left(\bar{t}_c \bar{Q}\bar{U} v \bar{t}_c + \bar{t}_c v \bar{U}\bar{Q}\bar{t}_c\right) - \tilde{\alpha}\bar{U} v \bar{U} - y\bar{t}_c - \bar{t}_c y^T, \\
\partial_\tau \bar{p}_c &= -\Big[4\bar{p}_c \bar{t}_c v + 4v\bar{t}_c \bar{p}_c - iv - 4\tilde{\alpha}\left(\bar{p}_c \bar{t}_c \bar{Q}\bar{U} v + v\bar{U}\bar{Q}\bar{t}_c \bar{p}_c\right) \\
&\quad - 4\tilde{\alpha}\left(\bar{Q}\bar{U} v \bar{t}_c \bar{p}_c + \bar{p}_c \bar{t}_c v \bar{U}\bar{Q}\right) + i\tilde{\alpha}\left(\bar{Q}\bar{U} v + v\bar{U}\bar{Q}\right)\Big] + \bar{p}_c y + y^T \bar{p}_c, \\
\partial_\tau \bar{s} &= -2\tilde{\alpha}\bar{U} v \bar{U}\bar{q}^T - y\bar{s}, \\
\partial_\tau \bar{q} &= -2\bar{s}^T v + 2\tilde{\alpha}\left(\bar{s}^T \bar{Q}\bar{U} v + \bar{s}^T v \bar{U}\bar{Q}\right) + \bar{q} y
\end{aligned}$$

$\bar{t}_c(0) = \mathscr{t}_c$, $\bar{p}_c(0) = \mathscr{p}_c$, $\bar{s}(0) = \mathscr{s}$ and $\bar{q}(0) = \mathscr{q}$

Time evolution of wavepacket



Low-energy effective theory



- Bi-local fields propagate (obeying local dynamics) in the background spacetime formed by the saddle-point configuration
- The end points of the bi-local fields freely propagate to the leading order in 1/M
- Only 1/M corrections can create `bound states'