Hamilton-Jacobi Formulation of Holographic Renormalization and Supersymmetry

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Based on 2006.16727 (NK, Se-Jin Kim),
and also 1902.00418, 1904.02038, 1904.05344, 2001.06789

Introduction

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- In holography (AdS/CFT), for many examples we are required to solve coupled nonlinear (ordinary) differential equations derived from Einstein gravity with matter fields.
- Then the ODEs can be understood as classical mechanics system, where the holographic coordinate is interpreted as "time".
- Evaluating the action is important because it gives partition function/generating function etc.
- This talk is concerned about how to solve the Hamilton-Jacobi equation perturbatively to get the on-shell action in AdS/CFT problems.

Hamilton-Jacobi equation

- HJ equation is obtained when one replaces $p_a \to \frac{\partial W}{\partial x^a}$.

$$H(x^a, p_a) = E \to H\left(x^a, \frac{\partial W}{\partial x^a}\right) = E$$

- W here is called Hamilton's characteristic function. It is part of the principal function $S = W - Et$ which generates canonical transformation to trivial Hamiltonian.
- W gives the action (Lagrangian integrated)

W as superpotential

- When the system is described by "superpotential" $\mathscr{W}$, it immediately gives the characteristic function W.

$$L = \frac{1}{2} g_{ab} \dot{x}^a \dot{x}^b + \frac{1}{2} g^{ab} \frac{\partial \mathscr{W}}{\partial x^a} \frac{\partial \mathscr{W}}{\partial x^b}$$
$$= \frac{1}{2} g_{ab} \left(\dot{x}^a \mp \frac{\partial \mathscr{W}}{\partial x^c} g^{ac} \right) \left(\dot{x}^b \mp \frac{\partial \mathscr{W}}{\partial x^d} g^{bd} \right) \pm \frac{d\mathscr{W}}{dt}$$

- BPS equation is $\dot{x}^a = \pm g^{ab}(\partial \mathscr{W} / \partial x^b)$, and on-shell action is $S = \int dt L = \pm \mathscr{W}$.
- It is also obvious that HJ eq is satisfied by $W = \mathscr{W}$.
- But this nice property (W being the superpotential) does not always hold, and there are many non-trivial examples in various contexts, including holography.

HJ in practical holography

- HJ approach has been applied to AdS/CFT from early days (see Skenderis, Papadimitriou etc)
- But for practical problems, like AdS/CMT or precision matching with susy localization, the field equations themselves are usually solved numerically.
- In this talk we propose "perturbative" approach where the holographic solutions are small deformation of AdS vacuum, as a method to extract semi-analytic results.

Plan

HJ approach to holography of ABJM with real mass

Setup for ABJM on S^3

- For concreteness and simplicity let us consider the ABJM.
- It is the theory on M2-branes and dual to M-theory in $AdS_4 \times S^7$.
- Chern-Simons-Matter theory with $U(N) \times U(N)$, CS levels $(k, -k)$ and quartic superpotential for 4 bi-fundamental chiral multiplets.
- $k \neq 1$ leads to orbifolding: $S^7/\mathbb{Z}_k$ and susy from $\mathcal{N} = 8$ to $\mathcal{N} = 6$.
- S^3 localization for partition function and Wilson loops developed by Kapustin, Willett, Yaakov (2009) and later generalized to less susy or different backgrounds such as squashed sphere Jafferis (2010) Hama, Hosomichi, Lee (2011), general $U(1)$ fibration over Riemann surface etc Closset, Kim, Willett (2017).

ABJM partition function

- Z as a function of N_1, N_2, k and an ordinary integral over eigenvalues.

$$Z_{ABJM} = \frac{1}{N_1!N_2!}\int \prod_i^{N_1} \frac{d\mu_i}{2\pi} \prod_j^{N_2} \frac{d\nu_j}{2\pi} e^{\frac{ki}{4\pi}(\sum \mu_i^2 - \sum \nu_j^2)} \frac{\prod_{i<j}(2\sinh(\mu_i-\mu_j))^2 \prod_{i<j}(2\sinh(\nu_i-\nu_j))^2}{\prod_{i,j}(2\cosh(\mu_i-\nu_j)/2)^2}$$

- Drukker, Marino, Putrov (2010,2011): The integral at hand is related to Lens space matrix model whose exact solution is already known: Shown that $F \equiv -\log|Z| \sim k^{1/2}N^{3/2}$. (free energy)
- For more general cases (e.g. with less susy), one can employ the matrix model technique developed by Herzog, Klebanov, Pufu, Tesileanu (2010) and others including Martelli, Sparks, Cheon, Kim, Kim, Jafferis, Klebanov, Pufu, Safdi (2011)

F-maximization

- More precisely, the action depends on the R-charge assignments of chiral multiplets which then affects the free energy.
- The correct free energy is obtained via F-maximization, as a function of R-charge Δ.
- For ABJM, $F = \frac{4\sqrt{2}\pi N^{3/2}}{3}\sqrt{\Delta_1\Delta_2\Delta_3\Delta_4}$ with constraint $\sum \Delta_i = 2$ (whose Im part gives real mass of fermions)
- Question: Can we extend the correspondence for general Δ?

Supergravity dual of ABJM on S^3

- Euclidean action with 3 complex scalars and BPS equations obtained from $D = 4, \mathcal{N} = 8$ sugra. Freedman, Pufu (2013)

$$S = \frac{1}{8\pi G_4} \int d^4x \sqrt{g} \left[-\frac{1}{2} R + \sum_{\alpha=1}^{3} \frac{\partial_\mu z^\alpha \partial^\mu \tilde{z}^\alpha}{(1 - z^\alpha \tilde{z}^\alpha)^2} + \frac{1}{L^2} \left(3 - \sum_{\alpha=1}^{3} \frac{2}{1 - z^\alpha \tilde{z}^\alpha} \right) \right]$$

- Metric ansatz in conformal gauge $ds^2 = e^{2A}(dr^2/r^2 + ds^2(S^3))$

$$\begin{aligned} r(1 + \tilde{z}^1\tilde{z}^2\tilde{z}^3) z^{\alpha\prime} &= (\pm 1 - rA')(1 - z^\alpha \tilde{z}^\alpha) \left(z^\alpha + \frac{\tilde{z}^1\tilde{z}^2\tilde{z}^3}{\tilde{z}^\alpha} \right), \\ r(1 + z^1 z^2 z^3) \tilde{z}^{\alpha\prime} &= (\mp 1 - rA')(1 - z^\alpha \tilde{z}^\alpha) \left(\tilde{z}^\alpha + \frac{z^1 z^2 z^3}{z^\alpha} \right), \\ -1 &= -r^2 (A')^2 + e^{2A} \frac{(1+z^1z^2z^3)(1+\tilde{z}^1\tilde{z}^2\tilde{z}^3)}{\prod_{\beta=1}^{3}(1 - z^\beta \tilde{z}^\beta)}. \end{aligned}$$

Exact solutions and holographic free energy

- Scalars $z^\alpha(r) = c_\alpha f(r), \quad \tilde{z}^\alpha(r) = -\frac{c_1 c_2 c_3}{c_\alpha} f(r)$, with $f(r) = \frac{1-r^2}{1+c_1 c_2 c_3 r^2}$
- Metric $e^{2A} = \frac{4r^2(1+c_1c_2c_3)(1+c_1c_2c_3r^4)}{(1-r^2)^2(1+c_1c_2c_3r^2)^2}$
- 3 integration constants, eventually related to Δ_i.
- To evaluate holographic free energy, one evaluates on-shell action, after adding Gibbons-Hawking and counterterms, and through Legendre transformation w.r.t. UV asymptotics coefficients: the result matches with the field theory.
- Luckily, finding the exact solution was not very difficult here. However, we may ask if there is a general and systematic method.
- Idea: Treat c_α as small parameters and solve perturbatively!

Solving Perturbatively

- Introduce an expansion parameter ϵ and write

$$z^\alpha(r) = \sum_{k=1}^{\infty} \epsilon^k z_k^\alpha(r)\,, \quad \tilde{z}^\alpha(r) = \sum_{k=1}^{\infty} \epsilon^k \tilde{z}_k^\alpha(r)\,,$$
$$e^{2A(r)} = \frac{4r^2}{(1-r^2)^2}\left(1 + \sum_{k=1}^{\infty} \epsilon^k a_k(r)\right)\,.$$

- Then at leading order, a_1 should be zero (due to regularity) and

$$z_1^{\alpha\prime} + \frac{2r}{1-r^2} z_1^\alpha = 0, \qquad \tilde{z}_1^{\alpha\prime} + \frac{2}{r(1-r^2)} \tilde{z}_1^\alpha = 0.$$

- $z_1^\alpha = c(1-r^2), \tilde{z}_1^\alpha = \tilde{c}(1-r^{-2})$. We want regular solutions, so should set $\tilde{c} = 0$.
- Continuing this, one can construct the solutions found by Freedman and Pufu. NK (1902.00418, JHEP)

Mass-deformed $\mathcal{N} = 4$ super Yang-Mills

- $\mathcal{N} = 4$ SYM with (susy-compatible) mass terms for adjoint chiral multiplets are called $\mathcal{N} = 2^*$ or $\mathcal{N} = 1^*$ theories.
- For $\mathcal{N} = 2^*$, the localization formula is available and the large-N limit gives Pestun (2007), Buchel, Russo, Zarembo (2013)

$$\frac{d^3 F_{S^4}}{d(ma)^3} = -2N^2 \frac{ma(m^2 a^2 + 3)}{(m^2 a^2 + 1)^2} \qquad (*)$$

- The supergravity dual of $\mathcal{N} = 2^*$ on S^4 was constructed by Bobev, Elvang, Freedman, Pufu (2013) considering a subsector of $\mathcal{N} = 8, D = 5$ gauged supergravity in Euclidean signature; Argued that the numerics of supergravity solution is compatible with $(*)$.
- We applied perturbative approach to $\mathcal{N} = 2^*$ and $\mathcal{N} = 1^*$, and obtained (partial) analytic results.

$\mathcal{N} = 1^*$ results

- We applied our perturbative prescription and established relation between UV expansion coefficients. NK, S-J Kim (1904.02038 JHEP)
- For the sphere partition function one can write in the following form

$$
\begin{aligned}
F_{S^4}/N^2 &= A_1(\mu_1^4+\mu_2^4+\mu_3^4) + A_2(\mu_1^2+\mu_2^2+\mu_3^2)^2 \\
&\quad + B_1(\mu_1^6+\mu_2^6+\mu_3^6) + B_2(\mu_1^2+\mu_2^2+\mu_3^2)^3 + B_3\mu_1^2\mu_2^2\mu_3^2 + \mathcal{O}(\mu^8). \\
A_1 &= (105-16\pi^4)/4200 \approx -0.346082 \\
A_2 &= (8\pi^4-315)/4200 \approx 0.110541
\end{aligned}
$$

- For the μ^6-order coeffs, numerics of Bobev et. al gives $B_1 \approx -0.146$ and $B_2 \approx 0.026$ but failed to calculate B_3. Our results give $B_1 \approx -0.146573, B_2 \approx 0.0262266, B_3 \approx -0.267706.$

Mass deformation of an AdS_6 example

- Although YM theory is not renormalizable in $D=5$, string theory implies there do exist superconformal field theories.
- Massive IIA allows AdS_6 from D4-D8-O8 system. Can be described using $D=6, F(4)$ gauged supergravity.
- Dual theory has $USp(2N)$ gauged group with N_f matter hypermultiplets in fundamental rep, and one in antisymmetric tensor rep. $N^{5/2}$ dof scaling matched using localization formula Brandhuber, Oz (1999) Jafferis, Pufu (2012): $F=-\frac{9\sqrt{2}\pi N^{5/2}}{5\sqrt{8-N_f}}$.
- One can consider adding mass to matter in fundamental rep. Action and BPS equations found by Gutperle, Kaidi, Raj (2018), who also presented numerical solutions.
- We obtained the holographic result in closed form (NK 1902.00418, JHEP)

Solving HJ perturbatively

HJ equation

- We use metric ansatz: $ds^2 = d\tau^2 + \kappa^{-2}\alpha^2(\tau)ds^2_{S^3}$

$$\mathscr{L} = \alpha\dot{\alpha}^2 + \kappa^2\alpha - \frac{\alpha^3}{3}\sum_{i=1}^{3}\left[\frac{\dot{z}_i\dot{\tilde{z}}_i}{(1-z_i\tilde{z}_i)^2} - g^2\frac{1+z_i\tilde{z}_i}{1-z_i\tilde{z}_i}\right]$$

- On-shell action cannot be a total derivative, even though it allows BPS equations: substituting the equation for α,

$$\mathscr{L}_{\mathrm{on-shell}} = \frac{2}{3}\frac{d}{d\tau}\left(\alpha^2\dot{\alpha}\right) + \frac{2}{3}\kappa^2\alpha$$

- We consider for simplicity the single-scalar truncation. Then HJ equation is

$$\frac{1}{4\alpha}\left(\frac{\partial W}{\partial\alpha}\right)^2 - \frac{(1-z\tilde{z})^2}{\alpha^3}\frac{\partial W}{\partial z}\frac{\partial W}{\partial\tilde{z}} - \kappa^2\alpha - g^2\alpha^3\frac{1+z\tilde{z}}{1-z\tilde{z}} = 0$$

Solving HJ perturbatively

- Set $\kappa = g$ and re-define $x = z\tilde{z}, \quad y = z/\tilde{z}$. We obtain

$$\frac{1}{4\alpha}\left(\frac{\partial W}{\partial \alpha}\right)^2 - \frac{(1-x)^2}{\alpha^3 x}\left[x^2\left(\frac{\partial W}{\partial x}\right)^2 - y^2\left(\frac{\partial W}{\partial y}\right)^2\right] - g^2\alpha - g^2\alpha^3\frac{1+x}{1-x} = 0$$

- Still not solvable through separation of variables. Let us assume W is independent of $y = z/\tilde{z}$. (Compatible with BPS solution)

$$W(\alpha, x) = -\frac{2g}{3} + W_0(\alpha)\left(1 + \sum_{n=1}^{\infty}\frac{w_n(\alpha)}{(1+\alpha^2)^n}x^n\right)$$

- One can iteratively solve for w_n, which satisfy nonlinear ODE. With regularity condition, one obtains BPS solutions!

$$3\alpha^3\left(\alpha^2+1\right)w_1' - 2(1+\alpha^2)w_1^2 + 3\alpha^4 w_1 = 9\alpha^6 \Longrightarrow w_1 = \frac{3\alpha^2}{2}\left(1 - \frac{1}{\sqrt{1+\alpha^2}}\right)$$

Series in α and comparison with explicit solutions

- One can continue to higher orders, w_2, w_3, w_4 etc.

$$w_2 = \frac{3}{8}\alpha^2\left(\sqrt{\alpha^2+1}-1\right)\left(4\sqrt{\alpha^2+1}-\alpha^2\right).$$

$$w_3 = \frac{1}{16}\alpha^2\left(\sqrt{\alpha^2+1}-1\right)\left[(\alpha^4+24\alpha^2+24)\sqrt{\alpha^2+1}+\alpha^6-10\alpha^4-12\alpha^2\right]$$

$$w_4 = -\frac{3}{128}\alpha^2\left(\sqrt{\alpha^2+1}-1\right)\left[\left(\alpha^8-6\alpha^6-72\alpha^4-128\alpha^2-64\right)\sqrt{\alpha^2+1} + \alpha^{10}-5\alpha^8+27\alpha^6+80\alpha^4+48\alpha^2\right]$$

- One can compare with explicit solutions, noting $p_\alpha = 2\alpha\dot{\alpha} = \frac{\partial W}{\partial \alpha}$ etc.
- In the UV, $\alpha \to \infty$ and $x \to 0$, while $\alpha^2 x$ is finite.

$$W = \frac{2g}{3}\left(\alpha^3 + \frac{3}{2}\alpha + \frac{3}{2}\alpha^3 x\right) + \text{finite terms}$$

- One can check the 1st and 3rd terms are cancelled by supersymmetric counterterm, and the 2nd term cancelled by boundary curvature term.

BPS in HJ-friendly form and its trouble

$$\frac{\partial W}{\partial \alpha} \overset{?}{=} 2\alpha\dot{\alpha} = 2\alpha\sqrt{\kappa^2 + g^2\alpha^2\frac{(1+z^3)(1+\tilde{z}^3)}{(1-z\tilde{z})^3}} = \sqrt{4\kappa^2\alpha^2 + \left(\frac{\partial \mathcal{W}_0}{\partial \alpha}\right)^2},$$

$$\frac{\partial W}{\partial \tilde{z}} \overset{?}{=} -\frac{\alpha^3\dot{z}}{(1-z\tilde{z})^2} = \left(\sqrt{\kappa^2 + g^2\alpha^2\frac{(1+z^3)(1+\tilde{z}^3)}{(1-z\tilde{z})^3}} \mp \kappa\right)\frac{\alpha^2\left(z+\tilde{z}^2\right)}{(1-z\tilde{z})(1+\tilde{z}^3)}$$

$$= \left(\frac{\partial W}{\partial \alpha} \mp 2\kappa\alpha\right)\frac{\alpha}{3\mathcal{W}_0}\frac{\partial \mathcal{W}_0}{\partial \tilde{z}} = (A),$$

$$\frac{\partial W}{\partial z} \overset{?}{=} -\frac{\alpha^3\dot{\tilde{z}}}{(1-z\tilde{z})^2} = \left(\sqrt{\kappa^2 + g^2\alpha^2\frac{(1+z^3)(1+\tilde{z}^3)}{(1-z\tilde{z})^3}} \pm \kappa\right)\frac{\alpha^2\left(\tilde{z}+z^2\right)}{(1-z\tilde{z})(1+z^3)}$$

$$= \left(\frac{\partial W}{\partial \alpha} \pm 2\kappa\alpha\right)\frac{\alpha}{3\mathcal{W}_0}\frac{\partial \mathcal{W}_0}{\partial z} = (B)$$

$$\mathcal{W}_0 = \frac{2g\alpha^3}{3}\sqrt{\frac{(1+z^3)(1+\tilde{z}^3)}{(1-z\tilde{z})^3}}$$

$$\boxed{\frac{\partial}{\partial z}(A) - \frac{\partial}{\partial \tilde{z}}(B) = \mp\frac{4\kappa\alpha^2}{3}\frac{\partial^2 \ln \mathcal{W}_0}{\partial z \partial \tilde{z}}, \quad \text{Integrability breaks down}}$$

How to integrate the HJ-friendly BPS

- It is now obvious that the HJ version of BPS is satisfied only *on-shell*, in other words, only after substituting the explicit solutions in τ (holographic coordinate)
- We can nevertheless utilize the BPS equation: let us pretend we don't know the solution, and expand the RHS of the BPS equation for small $z, \tilde{z}$. For $W = W_0(\alpha) + W_1(\alpha) z\tilde{z} + \cdots$,

$$\frac{\partial W}{\partial \tilde{z}} = W_1 z + \cdots = g\alpha^2(\sqrt{1+\alpha^2} - 1)z + \cdots \,,$$
$$\frac{\partial W}{\partial z} = W_1 \tilde{z} + \cdots = g\alpha^2(\sqrt{1+\alpha^2} + 1)\tilde{z} + \cdots \,.$$

- Leads to inconsistency, as expected. At least one of them must be wrong: in fact for the solution we discussed, the latter one is wrong.
- Keep the first line and we get $W_1 = g\alpha^2(\sqrt{1+\alpha^2} - 1)$ ($\because$ regularity)

Solving BPS, higher-orders

- Naive expansion was wrong because $\tilde{z} \sim \mathcal{O}(z^2)$. Checking $\partial W/\partial \alpha$, we obtain an *on-shell* relation

$$W_1{}' z\tilde{z} = g\alpha^3 \frac{z^3 + 3z\tilde{z}}{\sqrt{1+\alpha^2}} \Longrightarrow \alpha^2 z^2 = 2(1 - \sqrt{1+\alpha^2})\tilde{z}$$

- Now we can make sense of the $\partial W/\partial z$ equation.

$$\begin{aligned}
\frac{\partial W}{\partial z} &= g\alpha^2(\sqrt{1+\alpha^2}+1)(\tilde{z}+z^2) + \cdots \\
&= g\alpha^2(\sqrt{1+\alpha^2}+1)\left(1 + \frac{2(1-\sqrt{1+\alpha^2})}{\alpha^2}\right)\tilde{z} + \cdots \\
&= g\alpha^2(\sqrt{1+\alpha^2}-1)\tilde{z} + \cdots \Longrightarrow W_1 \overset{!}{=} g\alpha^2(\sqrt{1+\alpha^2}-1)
\end{aligned}$$

- One can continue in this manner to higher orders. Note that this procedure does not involve integration!

Summary

Summary

- We have obtained the HJ solution of mass-deformed ABJM as a series form in scalar fields.
- The HJ equation (PDE) is reduced to ODEs which can be integrated explicitly.
- Using BPS equations, the solution of HJ is not obtained as superpotential, but one can solve it algebraically without having to do integration, order by order in scalar.
- Can also tackle other models of AdS/CFT, janus, black holes etc.

Thank You !