Wrapped M5-branes
: From topological phases to blackholes

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APCTP

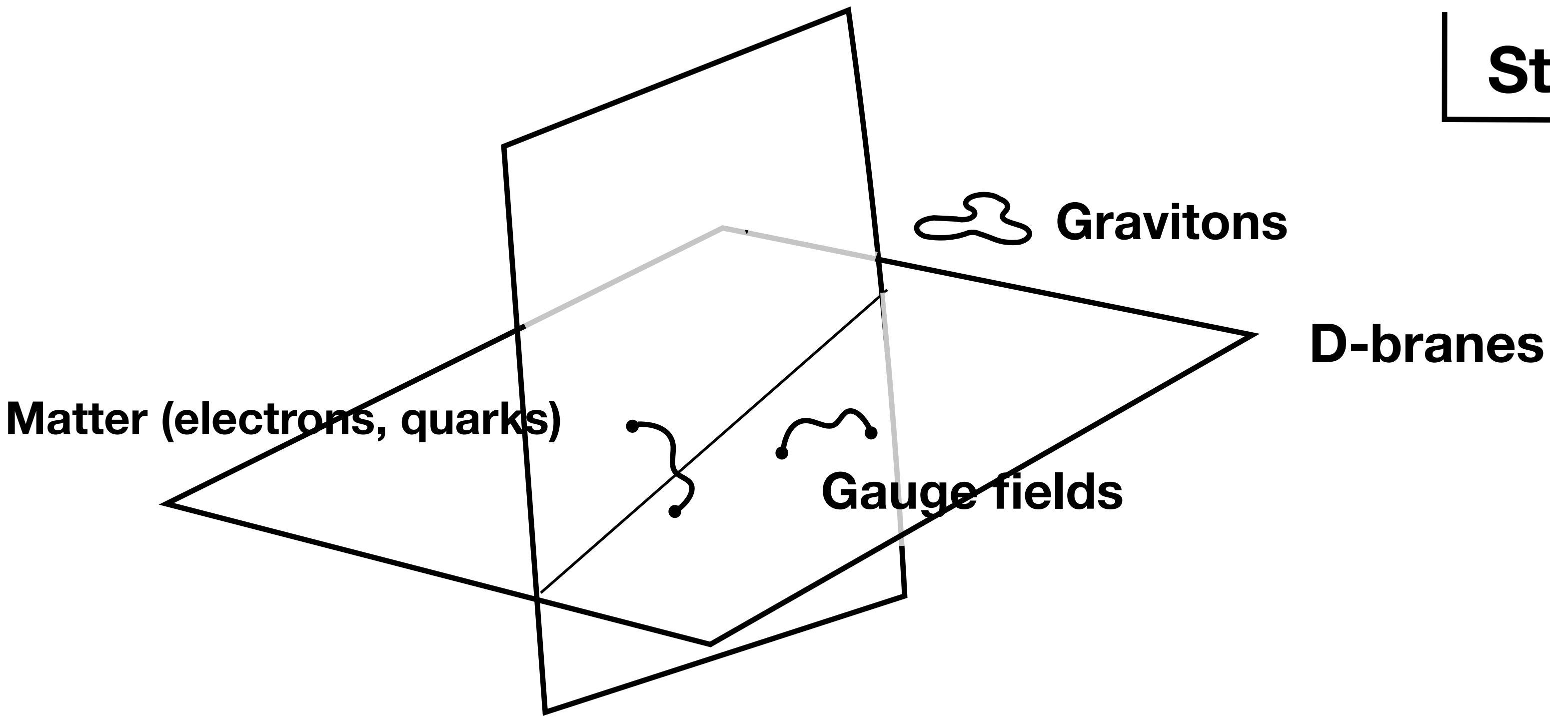
Arxiv : 1808.02797, with Nakwoo Kim (KyungHee U)

1905.01559, with Nakwoo Kim , Leopoldo A. Pando Zayas (Michigan U)

2007.01532, with Gil Young Cho and Hee-Cheol Kim (POSTECH)

Introduction

- **String/M-theory provide a unified consistent framework for (QFT)+(QG)**



String Theory

- **Various Quantum Systems can be geometrically engineered**

  – Particle physics : Standard model in string theory?

  – Condensed matter systems : Topological phases

  – Quantum Gravity Systems : Early universe, Blackholes …

"Unity of Theoretical Physics"

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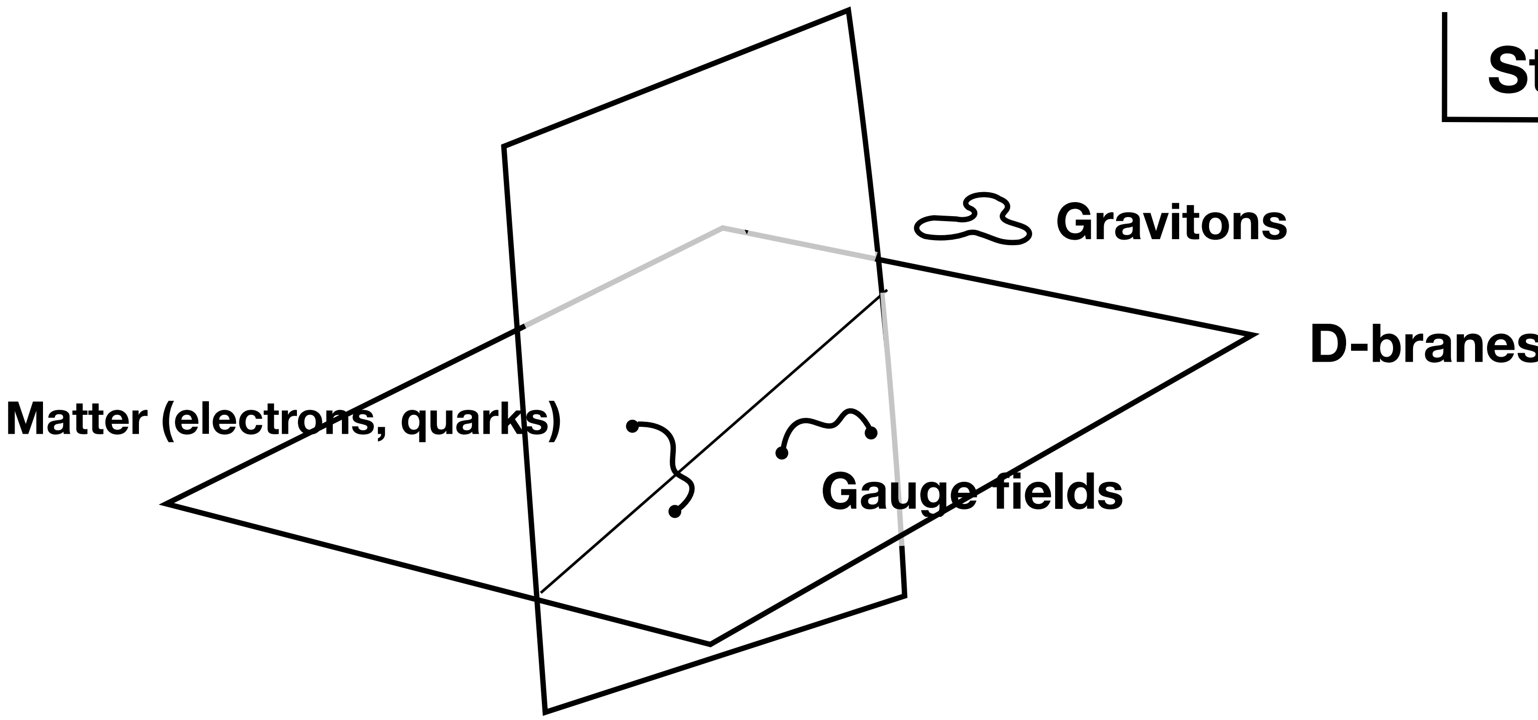

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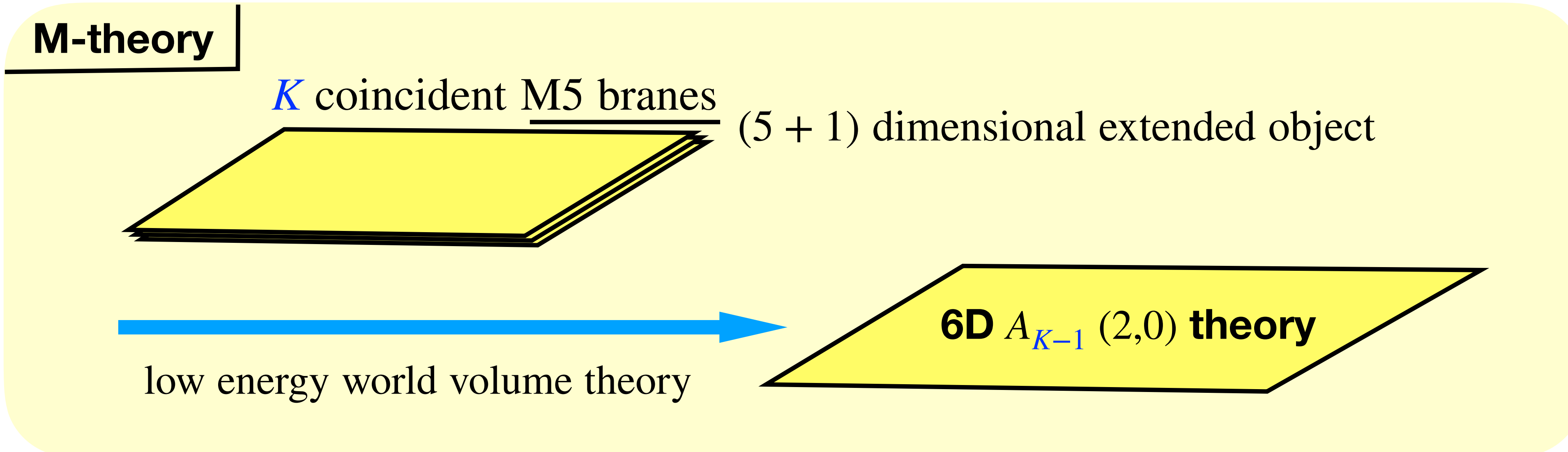
$\boxed{\text{M5 branes wrapped on } M_3}$

"Unity of Theoretical Physics"

Quantum Flatland from M5-branes

- **(5+1)D A_{K-1} (2,0) theory** : **world-volume theory for K M5-branes**



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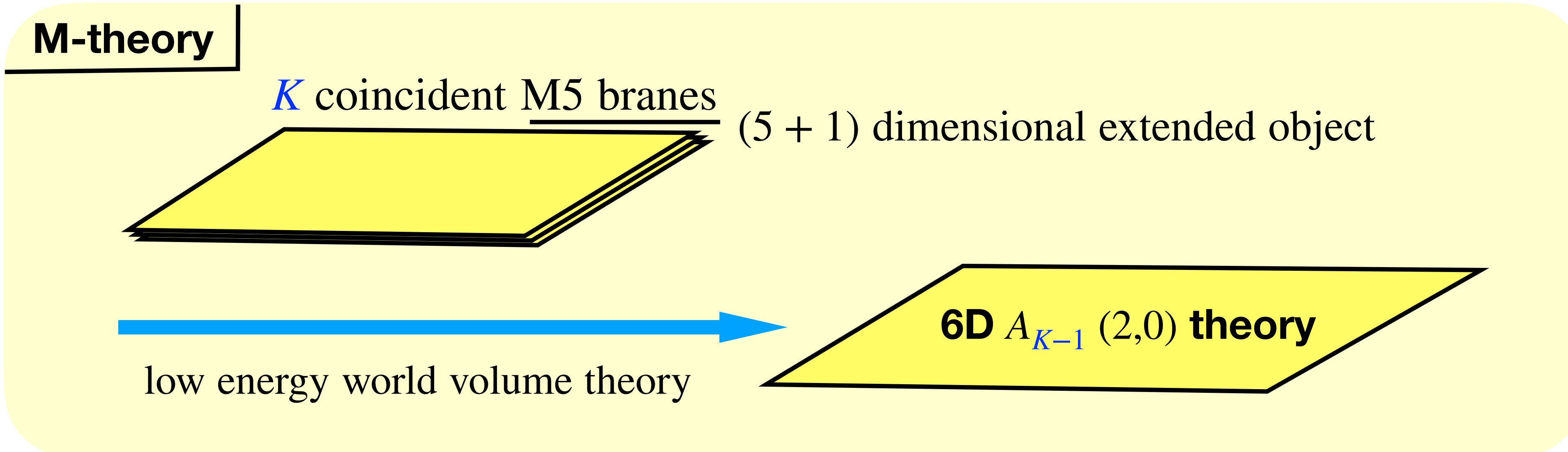
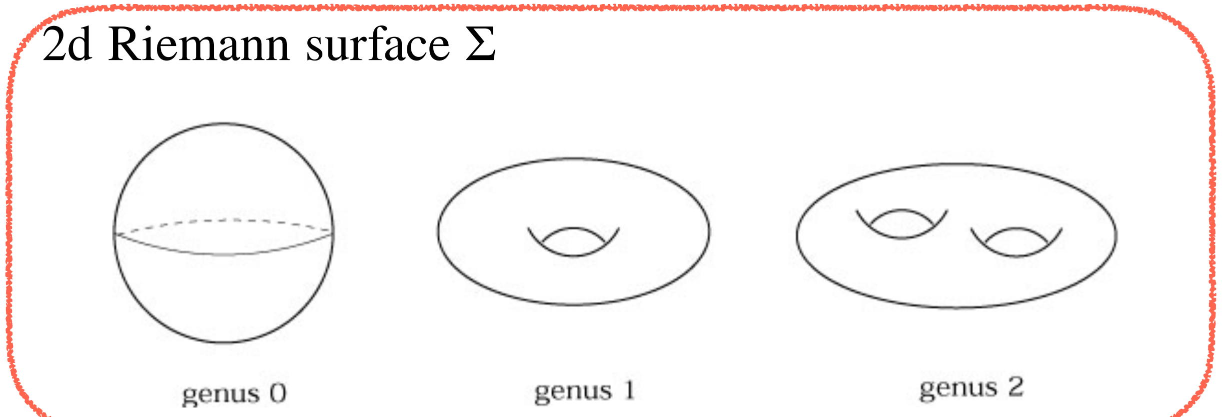


- **Geometrical construction of lower dimensional theory**

ex) **(5+1)D A_{K-1} (2,0) Theory on $\mathbb{R}^{1,3}\times\Sigma$** $\xrightarrow{\text{size}(\Sigma)\to 0}$ **(3+1)D $T[\Sigma, K]$ theory on $\mathbb{R}^{1,3}$**

[Gaiotto-Moore-Neitzke;2009]



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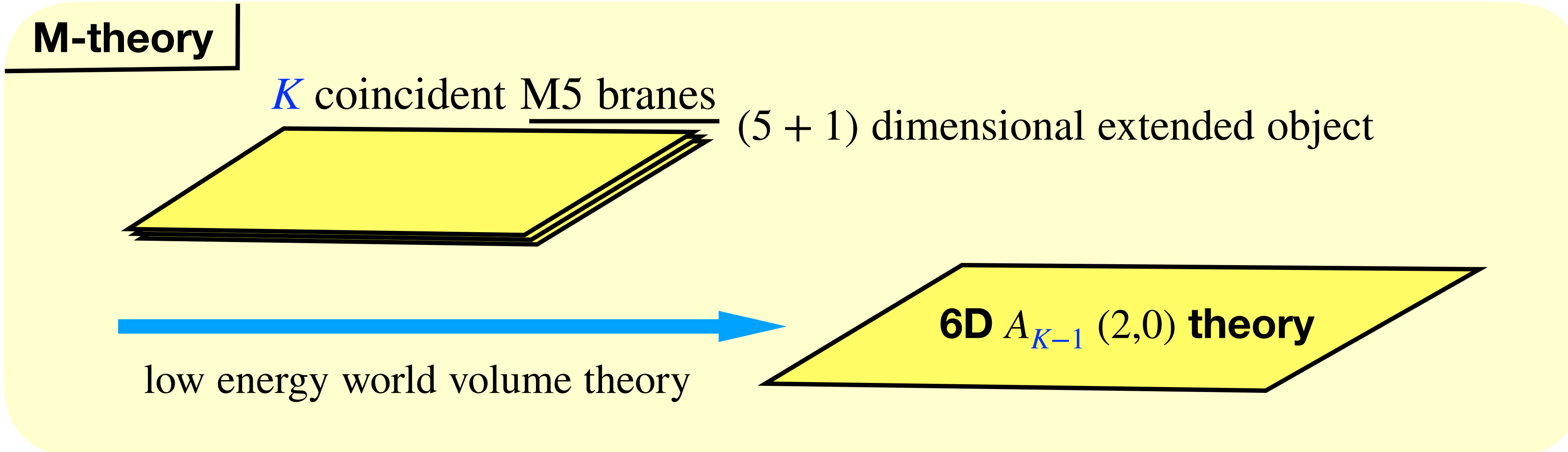

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(5+1)D A_{K-1} (2,0) theory on $\mathbb{R}^{1,2} \times M \xrightarrow{\text{size}(M)\to 0}$ **(2+1)D $T[M, K]$ theory on** $\mathbb{R}^{1,2}$

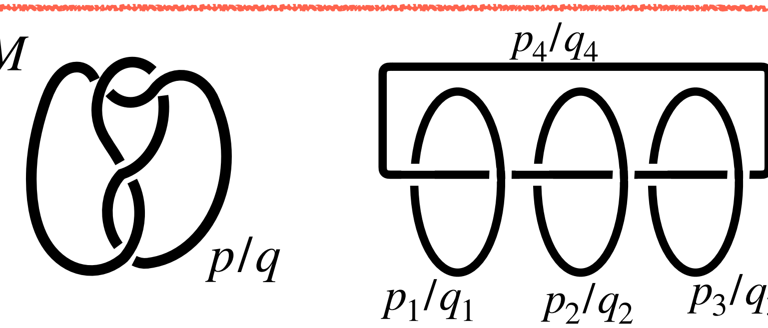
[Terashima-Yamazaki;2011]

"geometrically engineered quantum flatland"

[Dimofte-Gaiotto-Gukov;2011]

"Very rich"

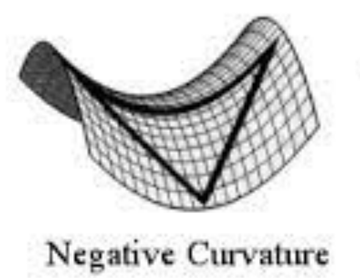
3-manifold M



Dehn surgery representation [Lickorish – Wallace theorem]

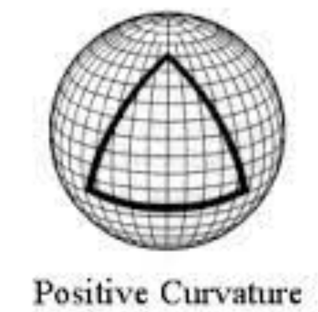
Summary

Geometrically egineered
Qauntum Flatland $T[M_3, K]$



Hyperbolic case

Non-Hyperbolic case

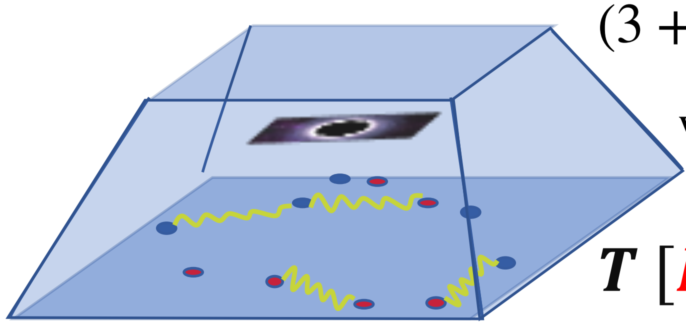


(gapless) **Conformal field theory**

$\updownarrow$

(3 + 1)D quantum gravity

with $L^2_{Ads4}/G_4 = \dfrac{2K^3 \text{vol}(M_3)}{3\pi^2}$



$\boldsymbol{T}\ [\boldsymbol{M_3}; \boldsymbol{K}]$

- Various blackholes in the quantum gravity
  - magnetically charged
  - rotating
- Quantum entropy computation

Topological phases (gapped) **[Cho-DG-Kim: 2020]**

- Geometry/(topological phase) correspondence

  $(M_3, K = 2) \longrightarrow \text{TFT}[M_3]$

  $\text{TFT}[M_3] := (\text{IR topological phase of } T[M_3, K = 2])$

- Modular data of TFT$[M]$ from topological data of M

Spectrum (topological spin, quantum dimension) of anyons Ground state degeneracy on genus g Riemann surfaces , …

- Geometrical construction/classification of
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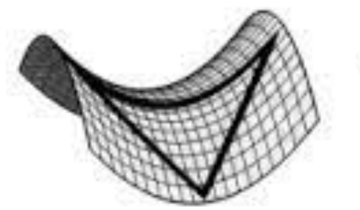

Negative Curvature

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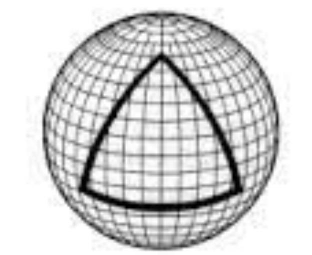

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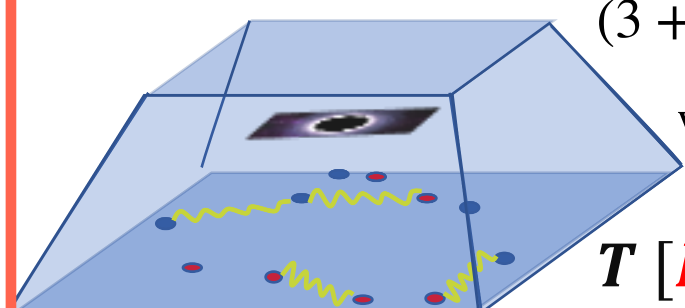

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Blackholes from wrapped wrapped M5 branes

- Gravity Dual of $T[M, K]$ for hyperbolic M

M theory on $AdS_4 \times M \times \tilde{S}^4 \xrightarrow{\text{consistent truncation}} AdS_4$ supergravity ($G_{\mu\nu}, A_\mu$ + fermionic fields)

[Pernici : 1985]
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with $L^2_{Ads4}/G_4 = \dfrac{2K^3 \mathrm{vol}(M_3)}{3\pi^2}$ and $L_{Ads4}/L_P \propto N^{1/3}$

hyperbolic volume : volume measured using the unique hyperbolic metric $R_{\mu\nu} = -2g_{\mu\nu}$
topological invariant

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$$\boxed{ds^2/L^2_{Ads4} = -\left(\rho - \frac{1}{2\rho}\right)^2 dt^2 + \left(\rho - \frac{1}{2\rho}\right)^{-2} d\rho^2 + \rho^2 ds^2(\Sigma_g)\,,\ F = \frac{\mathrm{vol}(\Sigma_g)}{L^2_{Ads4}}} \longleftarrow K \text{ M5 branes on } \mathbb{R} \times \Sigma_g \times M$$

Labelled by : $(M, K, g) \longrightarrow (G_4, L_{Ads4}, g)$

microscopic macroscopic

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Labelled by : $\underset{\text{microscopic}}{(M,K,g)} \longrightarrow \underset{\text{macroscopic}}{(G_4, L_{Ads4}, g)}$

- Quantum entropy using holography

$S_{\mathrm{BH}}(M,K;g) = \log(\mathrm{GSD}_g \text{ of } T[M,K])$

$$\xrightarrow{K\to\infty} \underline{\log\left(2\cos\left((g-1)\theta_{K,M}\right)\right)} + (g-1)\left(\underline{\frac{\mathrm{vol}(M)}{3\pi}(K^3-K)} + \underline{\frac{\mathrm{vol}(M)}{6\pi}(K-1)} + \underline{\log K} + \underline{\sum_\gamma \mathrm{PE}\left[\frac{q_\gamma^{N+1}-q_\gamma^2}{(1-q_\gamma)^2}\right]}\right)$$

relative phase factor in complex saddles
[WIP with S. Choi and N. Kim]

Bekenstein Hawking entropy
$S_{BH} = \dfrac{A_{\mathrm{horizon}}}{4G}$ [DG-Kim;2018]

higher derivative corrections
[Bobev, Charles, Hristov, Reys : 2020]

SUGRA zero modes on $AdS_4 \times M \times \tilde{S}^4$
[DG-Kim-Pando Zayas;2019]

??

$\gamma \in$ conjugacy classes of π_1M , $q_\gamma := e^{-\underline{\ell(\gamma)}}$
length of γ

+ (exponenetially smaller terms)

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microscopic

3D-3D relation + Mathematical fact

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3D/3D relations

$$Z\left[T[M, K] \text{ on } B\right] = \left(\textbf{invariant}_B \textbf{ of } SL(K, \mathbb{C}) \textbf{ Chern-Simons theory on } M\right)$$

[Terashima-Yamazaki;2011]

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curved Euclidean background

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$\mathscr{A}_\alpha$: irreducible $SL(K,\mathbb{C})$ flat connections on M

- **Perturbative invariants of** $SL(K,\mathbb{C})$ **CS theory**

$$\int \frac{[D(\delta\mathscr{A})]}{(\textbf{gauge})} \exp\left(-\frac{4\pi^2}{\hbar} CS[\mathscr{A}_\alpha + \delta\mathscr{A}; M]\right) \xrightarrow{\hbar\to 0} \frac{1}{\textbf{vol}(\textbf{Stab}(\mathscr{A}_\alpha))} \exp\left(\frac{1}{\hbar}F_0^\alpha + F_1^\alpha + F_2^\alpha \hbar + \dots\right)$$

with $CS[\mathscr{A};M] := \frac{1}{8\pi^2}\int_M \mathrm{Tr}(\mathscr{A}\wedge d\mathscr{A} + \frac{2}{3}\mathscr{A}^3)$

$F_0^\alpha = -4\pi^2 \boxed{\textbf{CS}[\mathscr{A}_\alpha]}$ (**classical part**)

$e^{-2F_1^\alpha} = \frac{(det'\Delta_1^\alpha)^{1/2}}{(det'\Delta_0^\alpha)^{3/2}} = \boxed{\textbf{Tor}(\mathscr{A}_\alpha)}$ (**1-loop part**)

Ray – Singer – Reidemeister torsion

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Mathematically, it is known that

$$\log\left|\mathbf{Tor}(\underline{\mathscr{A}_{\text{hyp}}})\right| = \frac{\mathrm{vol}(M)}{3\pi}(K^3 - K) + \frac{\mathrm{vol}(M)}{6\pi}(K-1) + \sum_\gamma \log \mathrm{PE}\left[\frac{q_\gamma^{N+1} - q_\gamma^2}{(1-q_\gamma)^2}\right]$$

$SL(K,\mathbb{C})$ flat connection from the hyperbolic metric on M

[Muller;2014] [Park;2017] using Selberg's trace formula

Blackholes from wrapped wrapped M5 branes

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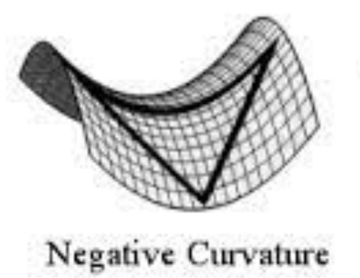

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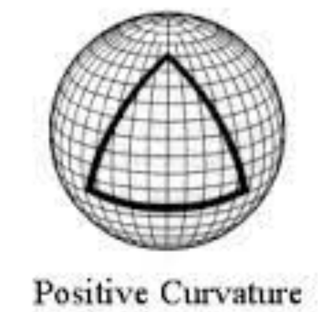

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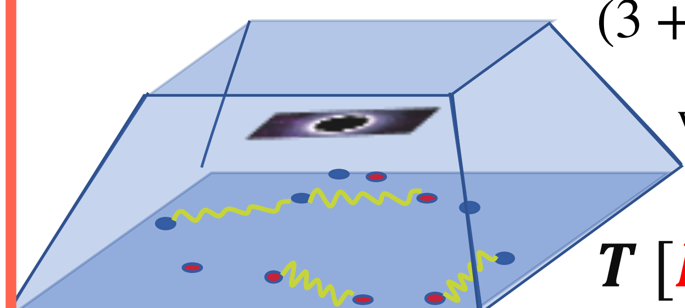

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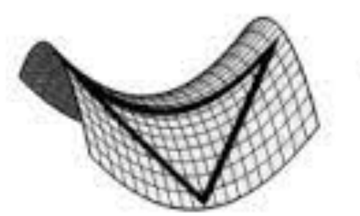

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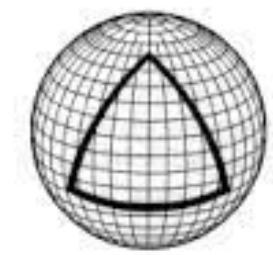

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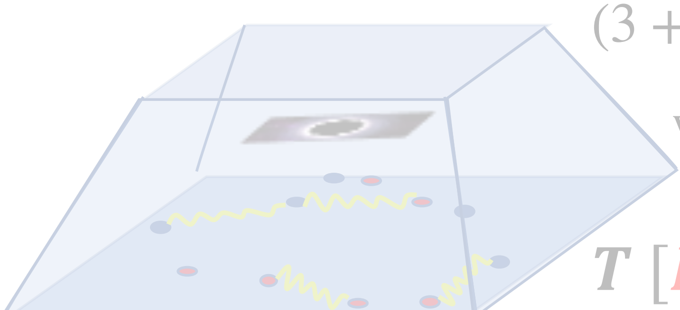

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Topological phases from wrapped wrapped M5 branes

[Cho-DG-Kim: 2020]

- Topological phases from certain class of non hyperbolic 3 manifold

If all irreducible $SL(2,\mathbb{C})$ flat connections on M are real (conjugate to $SU(2)$ or $SL(2,\mathbb{R})$) ,

$$T[M, K=2] \longrightarrow \underline{\mathrm{TFT}[M]}$$

A topological phase

More universal (non-SUSY/non-relativistic)

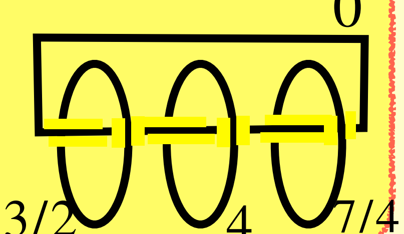
Gapped phases { in geometrical engineering / in non-SUSY field theory / in lattice model } $\Longrightarrow$ **Topological Field theories (TFTs)** **(anyon theory)**

ex)

A continuum field theory

$U(1)$ coupled to a complex scalar ϕ of charge $+2$ with $V(\phi) = (|\phi|^2 - 1)^2$

$\mathbf{T}[M, K=2]$ with

$M =$ 

Toric code model

$$\hat{H} = -\sum_v \prod_{e\in v} \sigma_x - \sum_f \prod_{e\in f} \sigma_z$$

$$4_0^B$$

(using nomenclature in [Wen: 2015])

Topological phases from wrapped wrapped M5 branes

[Cho-DG-Kim: 2020]

- Topological phases from certain class of non hyperbolic 3 manifold

If all irreducible $SL(2,\mathbb{C})$ flat connections on M are real (conjugate to $SU(2)$ or $SL(2,\mathbb{R})$) ,

$$T[M, K=2] \longrightarrow \underline{\text{TFT}[M]}$$

A topological phase

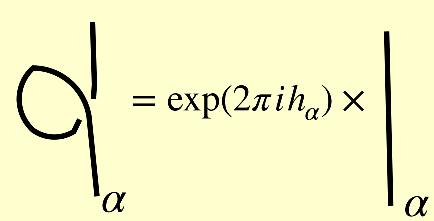
- Modular data of TFT$[M]$

$$S_{\alpha\beta}\,,T_{\alpha\beta} \quad (\alpha,\beta = 0,\ldots,N-1)$$

TFT$[M]$	**invariants on** M
Anyon $\alpha = 0,\ldots,N-1$	Irreducible $SL(2,\mathbb{C})$ flat connection $\mathscr{A}_\alpha$
$S_{0\alpha}^2$ ($d_\alpha := \frac{S_{0\alpha}}{S_{00}}$: quantum dimension)	$\frac{1}{2\mathbf{Tor}(\mathscr{A}_\alpha)}$
$T^{\alpha\beta} = \delta_{\alpha\beta}e^{2\pi i h_\alpha}$ (h_α : top'l spin)	$h_\alpha = (\mathbf{CS}[\mathscr{A}_\alpha] - \mathbf{CS}[\mathscr{A}_{\alpha=0}])$

$$\mathbf{GSD}_g = \sum_\alpha (S_{0\alpha})^{2g-2}$$: **Ground state degeneracy of genus g Riemman surface**

$$T_{\alpha\beta} = \delta_{\alpha\beta}\exp(2\pi i h_\alpha)$$

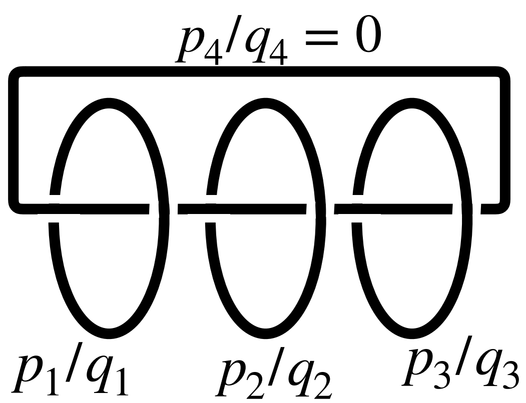


h_α : **topological spin (defined mod 1)**

$$N_\gamma^{\alpha\beta} = \sum_\delta \frac{S_{\delta\alpha}S_{\delta\beta}(S_{\delta\gamma})^*}{S_{0\delta}} \qquad [\alpha]\times[\beta] = \sum_\gamma N_\gamma^{\alpha\beta}[\gamma]$$

Fusion coefficients

Examples



$S^2\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}\right)$: **Sefert-fibered manifold**

$$\pi_1 M = \langle x_1, x_2, x_3, h \ : \ x_1^{p_1} h^{q_1} = x_2^{p_2} h^{q_2} = x_3^{p_3} h^{q_3} = 1 \ , x_1 x_2 x_3 = 1 \rangle$$

All irreducible flat connections, $\rho \in$ **Hom**$(\pi_1 M \to SL(2,\mathbb{C}))$ **, are real**

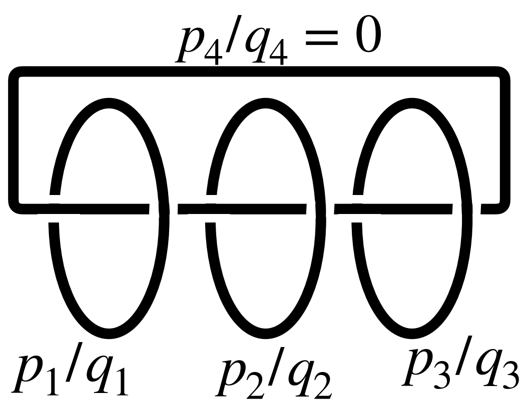
i.e. for all irreducible solutions to $\{X_1^{p_1} H^{q_1} = X_2^{p_2} H^{q_2} = X_3^{p_3} H^{q_3} = X_1 X_2 X_3 = 1\}$,

Tr$(X_1^{n_1} X_2^{n_2} X_3^{n_3} H^{n_4} \ldots) \in \mathbb{R}$ $\qquad X_i := \rho(x_i)$, $H := \rho(h) \ \in SL(2,\mathbb{C})$

Flat connections :

- $d\mathscr{A}_\alpha + \mathscr{A}_\alpha \wedge \mathscr{A}_\alpha = 0$
- $\mathscr{A}_\alpha \leftrightarrow \rho_\alpha \in \dfrac{\textbf{Hom}\left(\pi_1 M \to SL(2,\mathbb{C})\right)}{\textbf{conjugation}}$, $\quad \rho_\alpha(a) = P \exp\left(\oint_{a \in \pi_1 M} \mathscr{A}_\alpha\right)$.
- ρ_α is $\begin{cases} \textbf{reducible} & \textbf{if } [\rho_\alpha(a), \rho_\alpha(b)] = 0 \textbf{ for all } a, b \in \pi_1 M \\ \textbf{irreducible,} & \textbf{otherwise} \end{cases}$
- ρ_α is real if $\mathrm{Tr}[\rho_\alpha(a)] \in \mathbb{R} \ \forall a \in \pi_1 M$

Examples



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CS$[\rho]$ **and Tor**(ρ) **are known** [D. Freed,1992] [D. Auckly,1994]

$$CS[\rho] = \sum_{i=1}^{3}\left(\frac{r_i}{p_i}n_i^2 - q_is_i\lambda^2\right) \quad \textbf{(mod 1)}$$

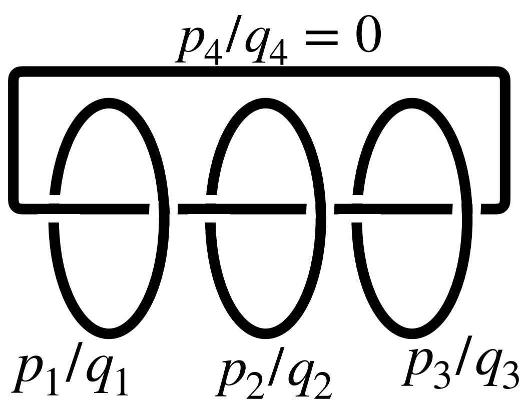
eigenvalues of $X_i = \left\{\exp\left(\pm 2\pi i\frac{n_i}{p_i}\right)\right\}$

$$\textbf{Tor}[\rho] = \prod_{i=1}^{3}\frac{p_i}{4\sin^2\left(2\pi\left(\frac{r_i}{p_i}n_i + s_i\lambda\right)\right)}$$

eigenvalues of $H = \left\{\exp\left(\pm 2\pi i\lambda\right)\right\}$

$(r_i, s_i) \in \mathbb{Z}$ **is chosen s. that** $p_is_i - q_ir_i = 1$

Examples



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On $S^2(3,3,-4/3)$ **there are 3 irreducible flat connections**

$\alpha = 0:$

$$\rho_\alpha(x_1) = \begin{pmatrix} e^{\frac{\pi i}{3}} & 0 \\ 0 & e^{-\frac{\pi i}{3}} \end{pmatrix}, \; \rho_{\alpha=0}(h) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\rho_\alpha(x_2) = \begin{pmatrix} \frac{1}{2} + \frac{i}{6}\left(\sqrt{3} - 2\sqrt{6}\right) & -\frac{\sqrt{2}}{3} \\ 1 & \frac{1}{2} - \frac{i}{6}\left(\sqrt{3} - 2\sqrt{6}\right) \end{pmatrix}$$

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$$\rho_\alpha(x_3) = \begin{pmatrix} -\frac{i}{\sqrt{3}} & \frac{2}{3} e^{\frac{4i\pi}{3}} \\ e^{-\frac{i\pi}{3}} & \frac{i}{\sqrt{3}} \end{pmatrix}, \; \rho_\alpha(h) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$\alpha = 2:$

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$$CS[\rho_{\alpha=0}] = \frac{25}{48}, \; CS[\rho_{\alpha=1}] = \frac{1}{12}, \; CS[\rho_{\alpha=2}] = \frac{1}{48}, \quad \mathbf{Tor}[\rho_{\alpha=0}] = \mathbf{Tor}[\rho_{\alpha=2}] = 2, \; \mathbf{Tor}[\rho_{\alpha=1}] = 1$$

$$h_{\alpha=0} = 0, \; h_{\alpha=1} = \frac{9}{16}, \; h_{\alpha=2} = \frac{1}{2}$$

$$S_{00}^2 = \frac{1}{4}, \; S_{01}^2 = \frac{1}{2}, \; S_{02}^2 = \frac{1}{4}$$

TFT$[M]$	**invariants on** M
Anyon $\alpha = 0, \ldots, N-1$	Irreducible $SL(2,\mathbb{C})$ flat connection $\mathscr{A}_\alpha$
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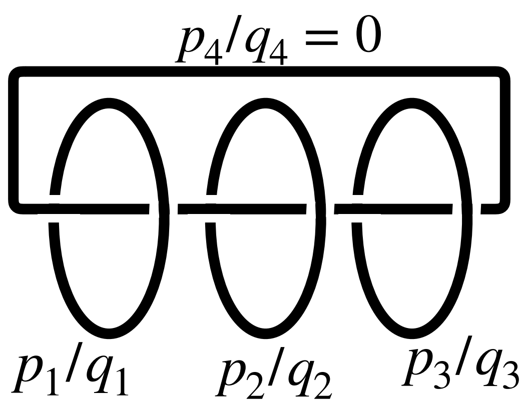
Examples



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Periodic table of bosonic topological phases

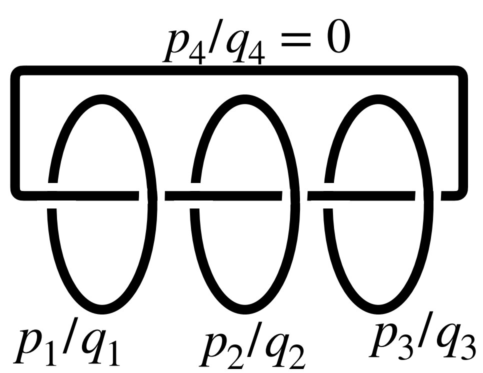
[Rowell-Stong-Wang : 2007]

(using nomenclature in [Wen: 2015])

N_c^B	$h_1, h_2, \cdots, h_N$	comment
1_1^B	0	
$2_{\pm 14/5}^B$	$0, \pm\frac{2}{5}$	Fibonacci
$2_{\pm 1}^B$	$0, \pm\frac{1}{4}$	Semion
$3_{\pm 8/7}^B$	$0, \mp\frac{1}{7}, \pm\frac{2}{7}$	$(A_1, 5)_{\frac{1}{2}}$
$3_{\pm 1/2}^B$	$0, \frac{1}{2}, \pm\frac{1}{16}$	Ising
$3_{\pm 7/2}^B$	$0, \frac{1}{2}, \pm\frac{7}{16}$	$SO(7)_1$
$3_{\pm 3/2}^B$	$0, \frac{1}{2}, \pm\frac{3}{16}$	$(A_1, 2)$
$3_{\pm 5/2}^B$	$0, \frac{1}{2}, \pm\frac{5}{16}$	$SO(5)_1$
$3_{\pm 2}^B$	$0, \pm\frac{1}{3}, \pm\frac{1}{3}$	$(A_2, 1)$
$4_{\pm 10/3}^B$	$0, \pm\frac{1}{3}, \pm\frac{2}{9}, \mp\frac{1}{3}$	$(A_1, 7)_{\frac{1}{2}}$
$4_{\pm 19/5}^B$	$0, \pm\frac{1}{4}, \mp\frac{7}{20}, \pm\frac{2}{5}$	
4^B [illegible]	$0, \mp$ [illegible]	

Geometrical construction up to rank 4

[Cho-DG-Kim: 2020]



$$S^2\left(\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}\right)$$

Sefert-fibered manifold

3-manifold
$S^2(2,3,3)$
$S^2(2,3,5)$
$S^2(3,3,3)$
$S^2(2,3,7)$
$S^2(3,\frac{3}{2},4)_{\text{I}}$
$S^2(3,\frac{3}{2},4)_{\text{II}}$
$S^2(3,3,4)_{\text{I}}$
$S^2(3,3,4)_{\text{II}}$
$S^2(2,3,6)$
$S^2(2,3,9)$
$S^2(3,3,5)_{\text{I}}$
$S^2(3,3,5)_{\text{II}}$
$S^2(2,5,5)$
$S^2(2,4,\frac{5}{4})$
$S^2(4,\frac{4}{7},\frac{3}{2})_{\text{I}}$
$S^2(4,\frac{4}{7},\frac{3}{2})_{\text{II}}$
$S^2(4,4,\frac{3}{2})_{\text{I}}$
$S^2(4,4,\frac{3}{2})_{\text{II}}$
$S^2(4,\frac{4}{3},\frac{3}{2})_{\text{I}}$
$S^2(4,\frac{4}{3},\frac{3}{2})_{\text{II}}$

N_c^B	$h_1, h_2, \cdots, h_N$	comment
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$3_{\pm 7/2}^B$	$0, \frac{1}{2}, \pm\frac{7}{16}$	$SO(7)_1$
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$3_{\pm 5/2}^B$	$0, \frac{1}{2}, \pm\frac{5}{16}$	$SO(5)_1$
$3_{\pm 2}^B$	$0, \pm\frac{1}{3}, \pm\frac{1}{3}$	$(A_2, 1)$
$4_{\pm 10/3}^B$	$0, \pm\frac{1}{3}, \pm\frac{2}{9}, \mp\frac{1}{3}$	$(A_1, 7)_{\frac{1}{2}}$
$4_{\pm 19/5}^B$	$0, \pm\frac{1}{4}, \mp\frac{7}{20}, \pm\frac{2}{5}$	
$4_{\pm 9/5}^B$	$0, \mp\frac{1}{4}, \pm\frac{3}{20}, \pm\frac{2}{5}$	
$4_{\pm 12/5}^B$	$0, \mp\frac{2}{5}, \mp\frac{2}{5}, \pm\frac{1}{5}$	
$4_0^{B,c}$	$0, 0, \frac{2}{5}, -\frac{2}{5}$	
$4_0^{B,a}$	$0, 0, 0, \frac{1}{2}$	Toric code
4_4^B	$0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$	$(D_4, 1)$
$4_{\pm 1}^B$	$0, \pm\frac{1}{8}, \pm\frac{1}{8}, \frac{1}{2}$	$(A_3, 1)$
$4_{\pm 3}^B$	$0, \pm\frac{3}{8}, \pm\frac{3}{8}, \frac{1}{2}$	
$4_{\pm 2}^B$	$0, \pm\frac{1}{4}, \pm\frac{1}{4}, \frac{1}{2}$	
$4_0^{B,b}$	$0, 0, \frac{1}{4}, -\frac{1}{4}$	Double semion

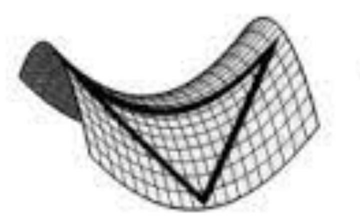
Bosonic topological phase up to rank 4

[Rowell-Stong-Wang : 2007]

(using nomenclature in [Wen: 2015])

Summary

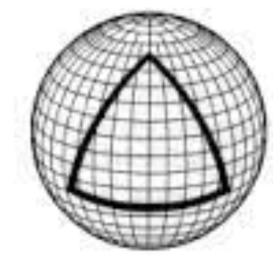
Geometrically egineered
Qauntum Flatland $T[M_3, K]$



Negative Curvature

Hyperbolic case

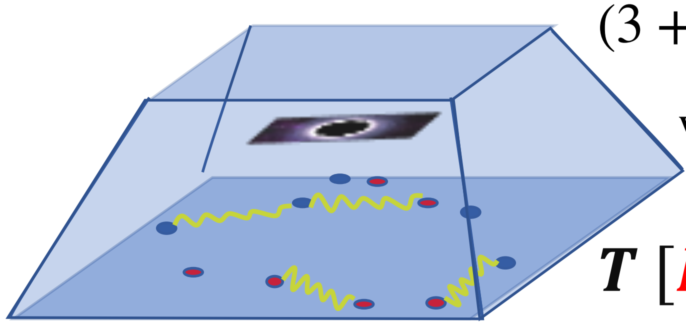
Non-Hyperbolic case



Positive Curvature

(gapless) **Conformal field theory**

$\updownarrow$



(3 + 1)D quantum gravity

with $L^2_{Ads4}/G_4 = \dfrac{2K^3 \text{vol}(M_3)}{3\pi^2}$

$\boldsymbol{T}\ [\boldsymbol{M_3}; \boldsymbol{K}]$

- Various blackholes in the quantum gravity
  - magnetically charged
  - rotating
- Quantum entropy computation

Topological phases (gapped) [Cho-DG-Kim: 2020]

- Geometry/(topological phase) correspondence

  $(M_3, K=2) \longrightarrow \text{TFT}[M_3]$

  $\text{TFT}[M_3] := (\text{IR topological phase of } T[M_3, K=2])$

- Modular data of TFT[M] from topological data of M

Spectrum (topological spin, quantum dimension) of anyons Ground state degeneracy on genus g Riemann surfaces , …

- Geometrical construction/classification of
  (2 + 1)D topological phases

Future Research Directions

- Quantum entropy from gravity side

SUGRA zero modes on $AdS_4 \times M \times \tilde{S}^4$

[DG-Kim-Pando Zayas;2019]

higher derivative corrections

[Bobev, Charles, Hristov, Reys : 2020]

??

$$S_{\rm BH}(M,K;g) = \log(\mathrm{GSD}_g \text{ of } T[M,K])$$

$$\xrightarrow{K\to\infty} \log\left(2\cos\left((g-1)\theta_{K,M}\right)\right) + (g-1)\left(\frac{\mathrm{vol}(M)}{3\pi}(K^3-K) + \frac{\mathrm{vol}(M)}{6\pi}(K-1) + \log K + \sum_\gamma \log \mathrm{PE}\left[\frac{q_\gamma^{N+1}-q_\gamma^2}{(1-q_\gamma)^2}\right]\right)$$

Bekenstein Hawking entropy

$$S_{BH} = \frac{A_{\rm horizon}}{4G}$$

[DG-Kim;2018]

relative phase factor in complex saddles

[WIP with S. Choi and N. Kim]

- Generalization of the "geometry/(topological phases) correspondence"

Geometrical Classification of $\begin{pmatrix} \text{fermionic} \\ \text{non unitary} \\ \text{Symmetry enriched} \end{pmatrix}$ $(2+1)$D topological phases

Geometrical Construction/Classification of $(3+1)$D topological phases

Search for exotic gapped phases (fractons?)

Thank you !!
(감사합니다.)