Scrambling and Pole-skipping in Hyperbolic Black Holes

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Based on arXiv:1907.08030 and arXiv:2006.00974 with **Keun-Young Kim**, **Mitsuhiro Nishida**, **Yongjun Ahn**, **Hyun-Sik Jeong** and **Kyung-Sun Lee**

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Overview

1. Introduction
2. Gravity setup
3. Scrambling in hyperbolic black holes
4. Pole-skipping in hyperbolic black holes
5. Final remarks

Introduction

- Pole-skipping is a phenomenon that takes place in **planar black holes** and it suggests a non-trivial connection between **scrambling** and **hydrodynamics** in the boundary theory;

- Does pole-skipping also takes place in black holes with non-planar horizons? Is it still related to scrambling?

- To answer that question, we study pole-skipping in hyperbolic black holes in AdS. In particular, we consider a Rindler-AdS geometry, which allows us to obtain some exact results in both sides of the AdS/CFT duality;

- AdS perspective $\rightarrow$ this talk!, CFT perspective $\rightarrow$ Mitsuhiro's talk

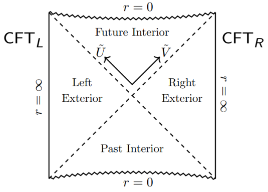
Gravity setup Emparan 1999, Czech, Karczmarek, Nogueira, Van Raamsdonk 2012

$(d+1)-$dimensional Einstein-Hilbert action:

$$S = \frac{1}{16\pi G_N}\int d^{d+1}x\sqrt{-g}\left(R + \frac{d(d-1)}{\ell^2}\right)$$

General asymptotically AdS **hyperbolic black hole**:

$$\boxed{ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 dH^2_{d-1}}\,, \quad f(r) = \frac{r^2}{\ell^2} - 1 - \frac{r_0^{d-2}}{r^{d-2}}\left(\frac{r_0^2}{\ell^2} - 1\right)$$

$|TFD\rangle =$ 

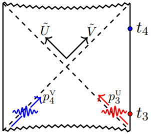
CFTs live in $\mathbb{R} \times H_{d-1}$

Figure 1: Penrose diagram for two-sided black holes with asymptotically AdS geometry.

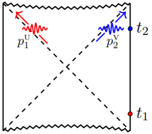
Special case: $r_0 = \ell \rightarrow$ Rindler-AdS$_{d+1}$ geometry $\rightarrow$ Exact solutions

$$F = \langle \text{TFD} | V_{\mathbf{x_1}}(t_1) W_{\mathbf{x_2}}(t_2) V_{\mathbf{x_3}}(t_3) W_{\mathbf{x_4}}(t_4) | \text{TFD} \rangle = \langle \text{out} | \text{in} \rangle$$

$$|\text{in}\rangle = V_{x_3}(t_3) W_{x_4}(t_4) |\text{TFD}\rangle \,, \quad |\text{out}\rangle = W_{x_2}(t_2)^\dagger V_{x_1}(t_1)^\dagger |\text{TFD}\rangle \,.$$



$V_{\mathbf{x_3}}(t_3) W_{\mathbf{x_4}}(t_4) |\text{TFD}\rangle$



$W_{\mathbf{x_2}}(t_2)^\dagger V_{\mathbf{x_1}}(t_1)^\dagger |\text{TFD}\rangle$

These two-particle states are described by a shock wave geometry:

$$ds^2_{\text{shock}} = ds_0^2 + h_{UU} dU^2 + h_{VV} dV^2$$

$$F = \int \overbrace{K_V K_W K_V K_W}^{\text{bulk-bdry propagators}} \; \underbrace{\overbrace{\langle \phi_V \phi_W \phi_V \phi_W \rangle}^{e^{i\delta(s,b)}}}_{\text{bulk 4pt-function}} , \quad b = \text{collision impact parameter} \,, \quad s = E_{CM}^2$$

$$\delta(s,b) = \frac{1}{4} \int d^{d+1}x \sqrt{-g} \left(h_{UU} T^{UU} + h_{VV} T^{VV} \right) \propto G_N \, s \, f(b) \sim G_N e^{\frac{2\pi}{\beta}\left(t - \frac{b}{v_B}\right)} ,$$

Assuming $\Delta_W >> \Delta_V >> 1$, we obtain:

$$\frac{\langle V_{\mathbf{x_1}}(t_1) W_{\mathbf{x_2}}(t_2) V_{\mathbf{x_1}}(t_3) W_{\mathbf{x_2}}(t_4)\rangle}{\langle V_{\mathbf{x_1}}(i\epsilon_1) V_{\mathbf{x_1}}(i\epsilon_3)\rangle\langle W_{\mathbf{x_2}}(i\epsilon_2) W_{\mathbf{x_2}}(i\epsilon_4)\rangle} = \frac{1}{\left[1 - \frac{16\pi i G_N \Delta_W e^{t-(d-1)b}}{\epsilon_{13}\epsilon_{24}^*}\right]^{\Delta_V}}$$

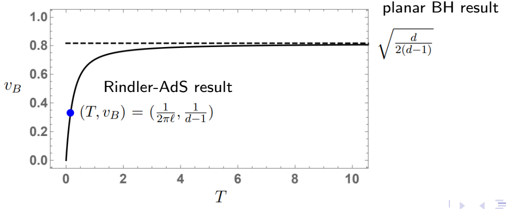
where $b = d(\mathbf{x_1}, \mathbf{x_2})$ is the geodesic distance between $\mathbf{x_1}$ and $\mathbf{x_2}$ in H_{d-1}

This implies

- Lyapunov exponent: $\lambda_L = \frac{2\pi}{\beta} = 1$
- Butterfly speed: $v_B = \frac{1}{d-1}$ $\rightarrow$ matches the CFT result obtained by Perlmutter 2016

In more general hyperbolic black holes, v_B is temperature dependent:

$\lambda_L = 2\pi T,$



Pole-skipping in planar black holes Grozdanov, Schalm, Scopelliti 2018

Energy-density retarded two-point function:

$$G^R_{T^{00}T^{00}} = \frac{b(\omega, k)}{a(\omega, k)} \leftrightarrow \text{gravitational fluctuations } \delta g_{\mu\nu} \text{ in the sound channel}$$

(ω_*, k_*) is a pole-skipping point if $\boxed{a(\omega_*, k_*) = b(\omega_*, k_*) = 0}$

The leading pole-skipping point of $G^R_{T^{00}T^{00}}$ satisfies:

$$\boxed{\omega_* = i2\pi T\,, \quad k_* = i\frac{2\pi T}{v_B}}$$

- This suggests a non-trivial connection between hydrodynamics and chaos;
- Is there a form of pole-skipping in non-planar black holes?

Pole-skipping in hyperbolic black holes in AdS

- In planar black holes:

  EOM involve $\Box_{\mathbb{R}^{d-1}} \to$ planar wave decomposition

$$\delta g_{\mu\nu}(r,t,x) = \delta g_{\mu\nu}(r;\omega,k)e^{-i\omega t + ik\cdot x} \to G^R(\omega,k)$$

- In hyperbolic black holes:

  EOM involve $\Box_{H_{d-1}} \to$ decomposition in terms of hyperspherical harmonics

$$\delta g_{\mu\nu}(r,t,\chi,\Omega_{d-2}) = \delta g_{\mu\nu}(r;\omega,L)e^{-i\omega t}\, Y_{LK}^{(d-1)}(i\chi,\theta_{d-2})$$

where $\boxed{\Box_{H_{d-1}} Y_{LK}^{(d-1)}(i\chi,\theta_{d-2}) = L(L+d-2)\,Y_{LK}^{(d-1)}(i\chi,\theta_{d-2})}$.

Here $K = (K_1, K_2, \ldots, K_{d-2})$ and $\theta \in S^{d-2}$, $dH_{d-1}^2 = d\chi^2 + \sinh^2\chi\, d\Omega_{d-2}^2$

In this case: $G^R(\omega,L) = \frac{B(\omega,L)}{A(\omega,L)}$

Pole-skipping points: (ω_*, L_*) such that $B(\omega_*, L_*) = A(\omega_*, L_*) = 0$

Pole-skipping in hyperbolic black holes in AdS

- gravitational perturbations: $\delta g_{\mu\nu} \to G^R_{T_{\mu\nu} T_{\mu\nu}}(\omega, L)$

  - near-horizon analysis gives for $G^R_{T^{00} T^{00}}$: $\boxed{\omega_* = i2\pi T\,, \; L_* = -\frac{2\pi T}{v_B(T)}}$

  - relation between **scrambling** and **pole-skipping** still holds in H_{d-1}

- vector field perturbations: $\delta A_\mu \to G^R_{J_\mu J_\nu}(\omega, L)$

- scalar field perturbations: $\delta \phi \to G^R_{\mathcal{O}\mathcal{O}}(\omega, L)$

- Rindler-AdS$_{d+1}$ allows us to obtain exact results for $G^R_{J_\mu J_\nu}(\omega, L)$ and $G^R_{\mathcal{O}\mathcal{O}}(\omega, L)$ and compute the full set of pole-skipping points.

- For more general hyperbolic black holes, we can obtain the leading pole-skipping points using a near-horizon analysis.

Scalar Field in Rindler-AdS$_{d+1}$

We consider a massive scalar field

$$S_{\text{scalar}} = -\frac{1}{2}\int d^{d+1}x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right)$$

propagating in the background $ds^2_{\text{Rindler-AdS}} = -\sinh^2 r\, dt^2 + dr^2 + \cosh^2 r\, dH^2_{d-1}$.

EoM:

$$\partial_r^2\phi - \frac{\partial_t^2\phi}{\sinh^2 r} + \frac{\Box_{H_{d-1}}\phi}{\cosh^2 r} + \left[\coth r + (d-1)\tanh r\right]\partial_r\phi - m^2\phi = 0$$

Ansatz: $\phi(t, r, \chi, \theta_i) = F_{\omega,L}(r)e^{-i\omega t}Y^{(d-1)}_{LK}(i\chi, \theta_i)$,

$$\Box_{H_{d-1}}Y^{(d-1)}_{LK}(i\chi, \theta_{d-2}) = L(L+d-2)Y^{(d-1)}_{LK}(i\chi, \theta_{d-2})$$

$$F''(r) + \left[\coth r + (d-1)\tanh r\right]F'(r) + \left[\frac{\omega^2}{\sinh^2 r} + \frac{L(L+d-2)}{\cosh^2 r} - m^2\right]F(r) = 0$$

Scalar Field in Rindler-AdS$_{d+1}$

Solution:

$$F(z) = (1-z)^{\Delta} z^{\pm i\omega} {}_2F_1(a, b, a+b+n; z)\,, \quad z = \tanh^2 r\,.$$

where $\Delta = d/2 + \sqrt{(d/2^2 + m^2)}$, and

$$a = -\frac{i\omega + L + (d-2-\Delta)}{2}\,, \quad b = \frac{-i\omega + L + \Delta}{2} \quad n = \frac{d}{2} - \Delta$$

Horizon: $z = 0$, Boundary: $z = 1$

Near-boundary:

$$F(z) = (1-z)^{d-\Delta} A(\omega, L) + (1-z)^{\Delta}\Big[B(\omega, L) + C(\omega, L)\log(1-z)\Big]$$

Holographic dictionary:

$$G^R_{\mathcal{O}\mathcal{O}}(\omega, L) \propto \frac{B(\omega, L)}{A(\omega, L)} \propto \frac{\Gamma\left(-\frac{i\omega + L + (d-2-\Delta)}{2}\right)\Gamma\left(\frac{-i\omega + L + \Delta}{2}\right)}{\Gamma\left(-\frac{i\omega + L + \Delta - 2)}{2}\right)\Gamma\left(\frac{-i\omega + L + d - \Delta}{2}\right)}\,, \text{ non-integer } d/2 - \Delta$$

Scalar Field in Rindler-AdS$_{d+1}$ Kim, Nishida, Ahn, Jeong, Lee, VJ 2020

Retarded Green's function:

$$G^R_{\mathcal{O}\mathcal{O}}(\omega, L) \propto \frac{B(\omega, L)}{A(\omega, L)} \propto \frac{\Gamma\left(-\frac{i\omega+L+(d-2-\Delta)}{2}\right)\Gamma\left(\frac{-i\omega+L+\Delta}{2}\right)}{\Gamma\left(-\frac{i\omega+L+\Delta-2)}{2}\right)\Gamma\left(\frac{-i\omega+L+d-\Delta}{2}\right)}, \text{ non-integer } d/2-\Delta$$

Pole-skipping points occur at special values of (ω, L) such that the poles of the Gamma functions in the denominator and numerator coincide:

$$\omega_*^{(n)} = -i\, n\,, \;\; L_*^{(n,q)} = \frac{d-2}{2} \pm \left(-n + 2q + \Delta - \frac{d+2}{2}\right)$$

where $n = 1, 2, ...$ and $q = 1, 2, .., n$. Here $2\pi T = 1$.

Leading pole-skipping points (the ones with the biggest value of $\mathrm{Im}(\omega)$):

$$\boxed{\omega_* = -i\,, \;\; L_*^+ = 1 - \Delta\,, \;\; L_*^- = \Delta - d + 1}$$

- Scalar Field

$$\omega_* = -i\,,\;\; L_*^+ = 1-\Delta\,,\;\; L_*^- = \Delta - d + 1$$

- Vector Field (longitudinal channel)

$$\omega_* = 0\,,\;\; L_*^+ = 1-\Delta\,,\;\; L_*^- = \Delta - d + 1$$

- Spin-2 Field (sound channel, $\Delta = d$)

$$\omega_* = +i\,,\;\; L_*^+ = 1-d\,,\;\; L_*^- = 1$$

For a field of spin J and conformal dimension Δ, the above result can be summarized as follows

$$\boxed{\omega_* = i\,(J-1)\,,\;\; L_*^+ = 1-\Delta\,,\;\; L_*^- = \Delta - d + 1}$$

The above result can be derived/explained by a CFT analysis → **Mitsuhiro's talk**

Final remarks

- Rindler-AdS/CFT is a useful framework for studying pole-skipping since it allows us to obtain exact analytic results in both sides of the AdS/CFT duality;
- The relation between scrambling and pole-skipping still holds in hyperbolic space;
- Pole-skipping suggests a non-trivial connection between **chaos** and **hydrodynamics**. However, pole-skipping also takes place in hyperbolic space, where (as far as I know) there is no obvious definition of hydrodynamics. This indicates that pole-skipping is a very general phenomenon and maybe it is not necessarily related to hydrodynamics;

Open questions:

- Is it possible to define some form of hydrodynamics in hyperbolic space? $v_B^2 \leftrightarrow D_{\text{thermal diffusion}}$
- Does pole-skipping also happens in black holes with spherical horizon? If yes, that would imply that pole-skipping happens even in the absence of hydrodynamic modes

THANK YOU

The eikonal approximation 't Hooft 87 Verlinde^2 92 Kabat, Ortiz 92

Schematically

$$F = \langle V\, W\, V\, W \rangle = \int \overbrace{K_V K_W K_V K_W}^{\text{bulk-bdry propagators}} \underbrace{\langle \phi_V \phi_W \phi_V \phi_W \rangle}_{\text{bulk 4pt-function}}$$

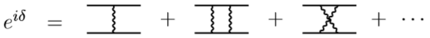
Using the eikonal approximation, we can write

$$\langle \phi_V \phi_W \phi_V \phi_W \rangle = e^{i\delta(s,b)}$$

where $s = -(p_1 + p_2)^2$ and b is the collision impact parameter.

Assumptions:

- Linearized gravity: $G_N << 1$
- Regge limit: $s >> 1$ and fixed b



The gravitational interaction dominates $\rightarrow$ Universality of OTOCs

Generalized hyperspherical harmonics

$$Y_{\mu_1,\mu_2,\dots} \sim e^{im\phi} \prod_{j=1}^{d-2} C_{\mu_j-\mu_{j+1}}^{\left(\frac{d-j-2}{2}+\mu_{j+1}\right)}(\cos\theta_j)(\sin\theta_j)^{\mu_j+1} ,$$

Gegenbauer functions

$$C_{\nu}^{(\alpha)}(z) = \frac{2^{1-2\alpha}\sqrt{\pi}\Gamma(\nu+2\alpha)}{\nu!\Gamma(\alpha)} {}_2F_1\left(-n, 2\alpha+n; \alpha+\frac{1}{2}; \frac{1-z}{2}\right) .$$