Workshop:
Quantum Matter and Quantum Information With Holography

APCTP

Probing Inverted Harmonic Oscillator using Circuit Complexity

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Primarily Based on
Phys.Rev.D 101 (2020) 2, 026021, Phys.Rev.D 101 (2020) 10, 106020 , Phys. Rev. Research 2, 033273 (2020)

(Shajid Haque, Jeff Murugan, Nathan Moynihan (Cape Town U.), Eugene H. Kim (Windsor U.), Tibra Ali (Perimeter Institute), Bret Underwood (Pacific Lutheran U.) , Saurya Das (Lethbridge U.))

Outline

– **Introduction :**

(for more about complexity
Run-Qiu and Vijay's talk)

– **Circuit Complexity :**

Basic Setup & Assumptions (via example)

– **Application:**

(for more about quantum chaos please refer to
Yan, Keun-Young,Viktor, Mitsuhiro's talk)

Symptom of quantum chaos?

Model: Inverted Harmonic Oscillator

– **Summary:**

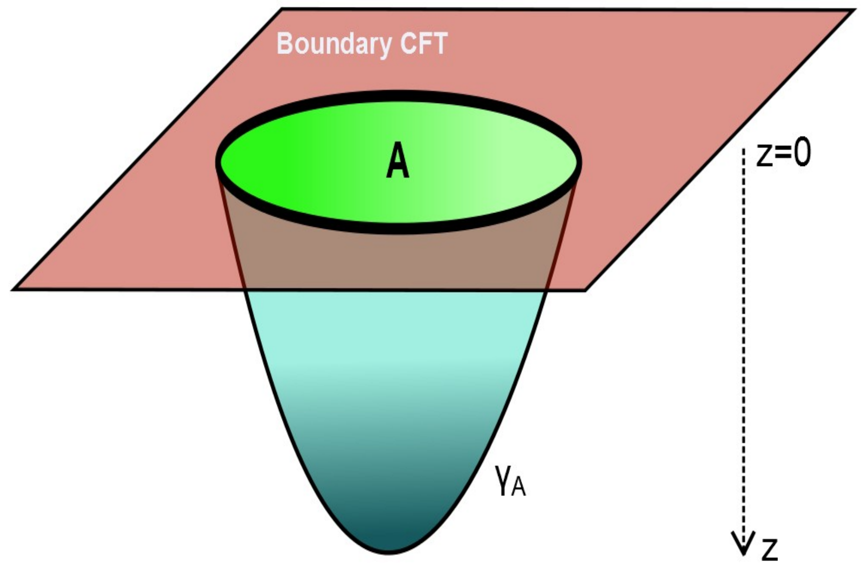
Introduction

In recent time tools from QI has played important role
to advance our understanding about the mechanism of AdS/CFT

For eg: Entanglement entropy

Ryu-Takayanagi prescription:

(Ryu -Takayanagi, Phys.Rev.Lett.96:181602,2006)

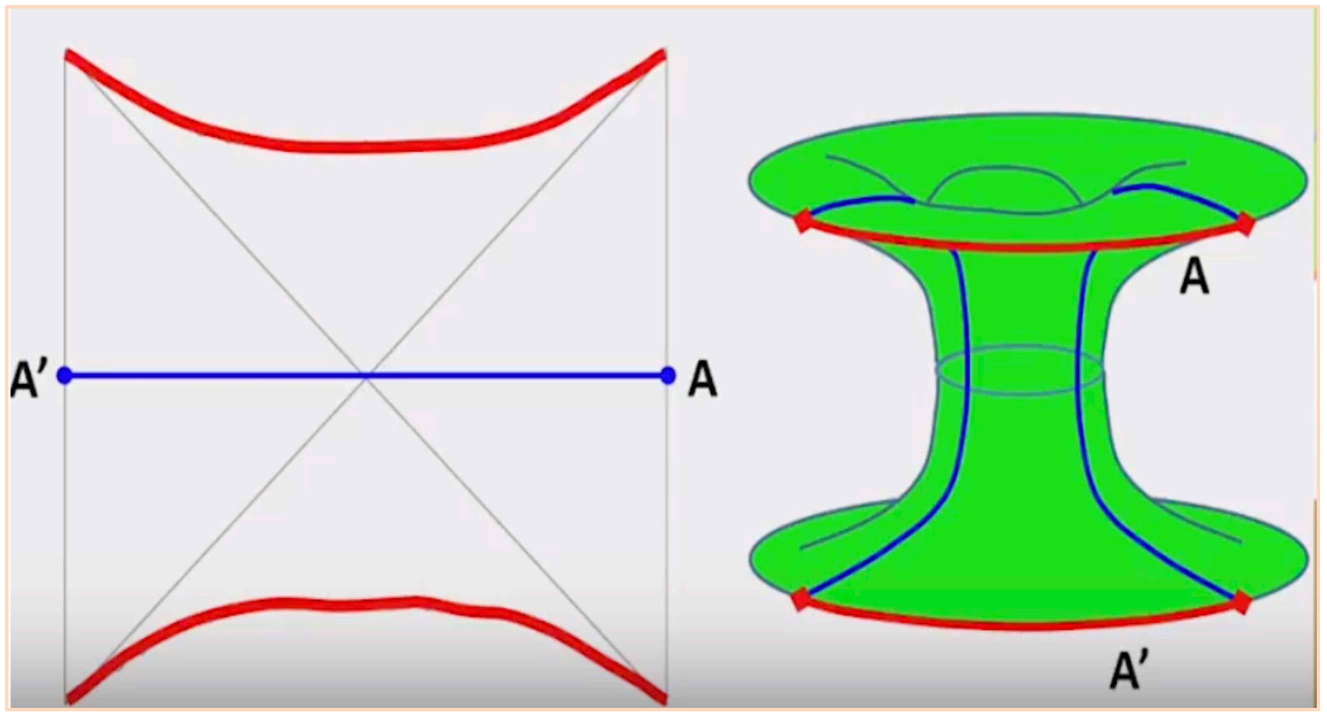


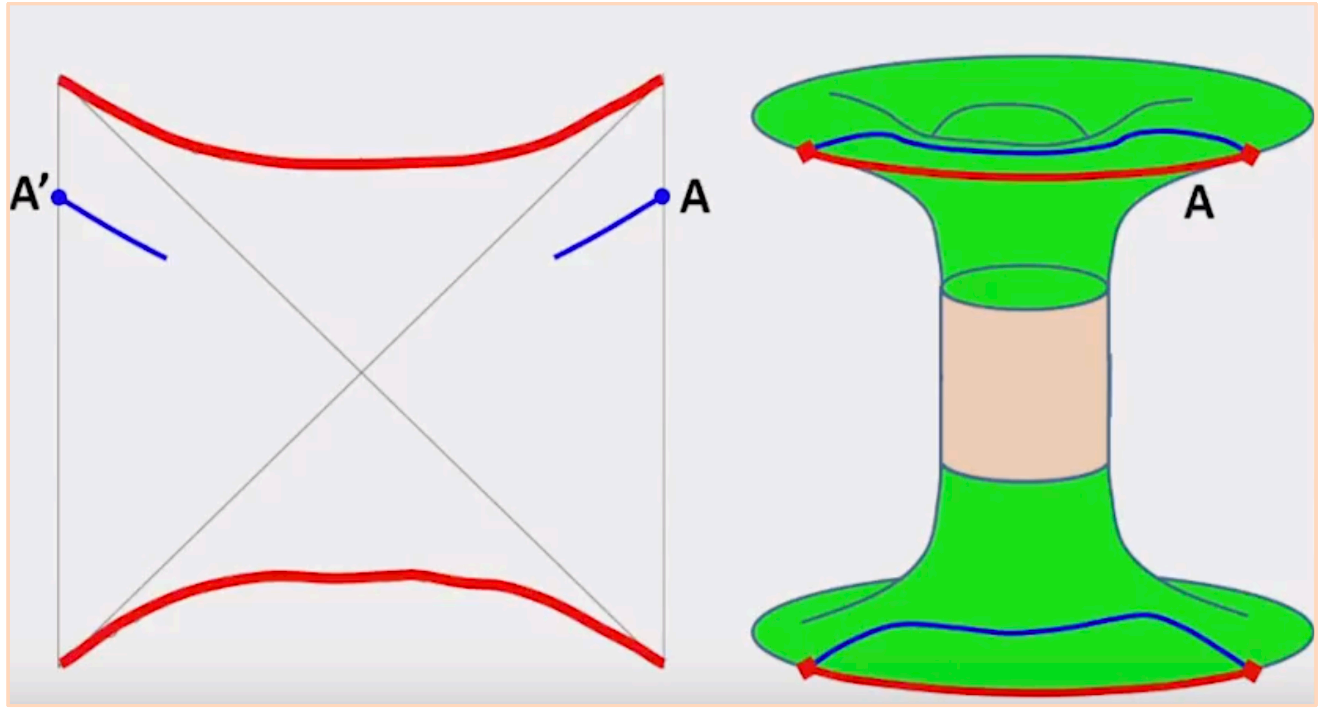
This duality becomes more stimulating in the context of Black hole

Maldacena '2001

Eternal Black Hole ⟷ Thermofield Double

AdS/CFT



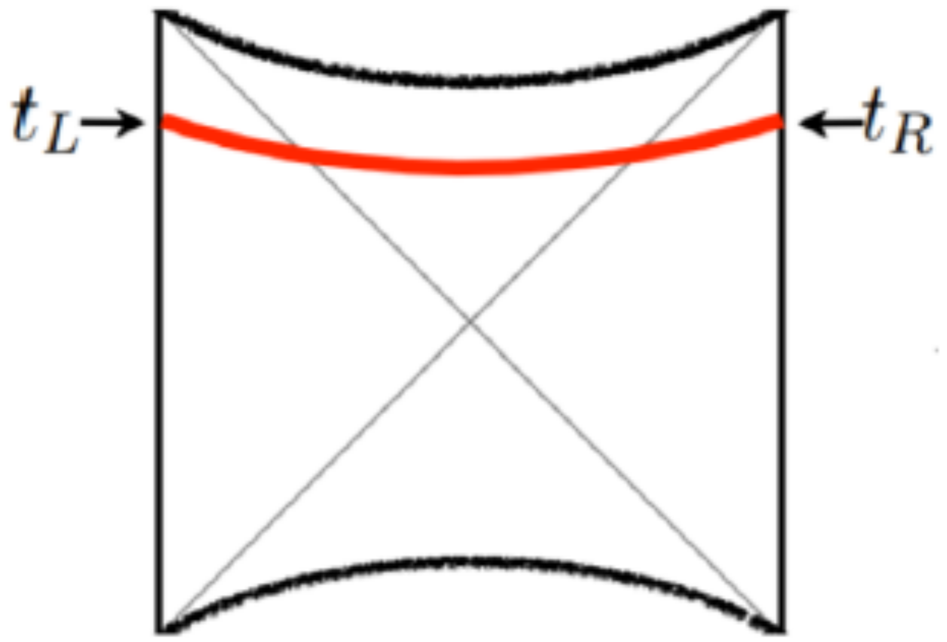


EE is not a good probe for physics behind horizon

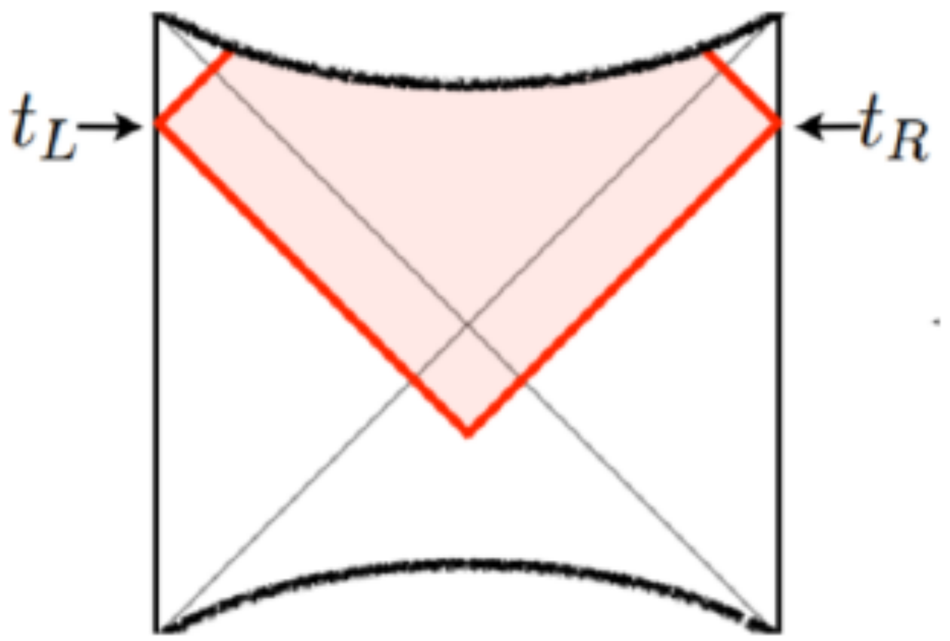
Two interesting objects probing the interior of black hole

(for more Run-Qiu talk)

Complexity = Volume



Complexity = Action



(Brown, Roberts, Swingle, Susskind & Zhao)

(Carmi, Chapman, Lehner, Myers, Marrochio, Poisson, Sorkin, Sugishita, Heller et al, Simon Ross et al, Run-Qiu Yang , Keun-Young Kim et al, Mohesn Alishahia, et al…)

$$\mathcal{C}_V(\Sigma) = \max\left[\frac{\mathcal{V}(\mathcal{B})}{G_N\, l}\right]$$

$$\mathcal{C}_A(\Sigma) = \frac{I_{WDW}}{\pi\hbar}$$

(picture courtesy Jefferson-Myers, 1707.08570 [hep-th])

Grows with time and keep growing even after the thermalization time

“Complexity” is dual to these two objects ?

Can we compute it field theory (or even in quantum mechanical systems) ?

Computational Complexity

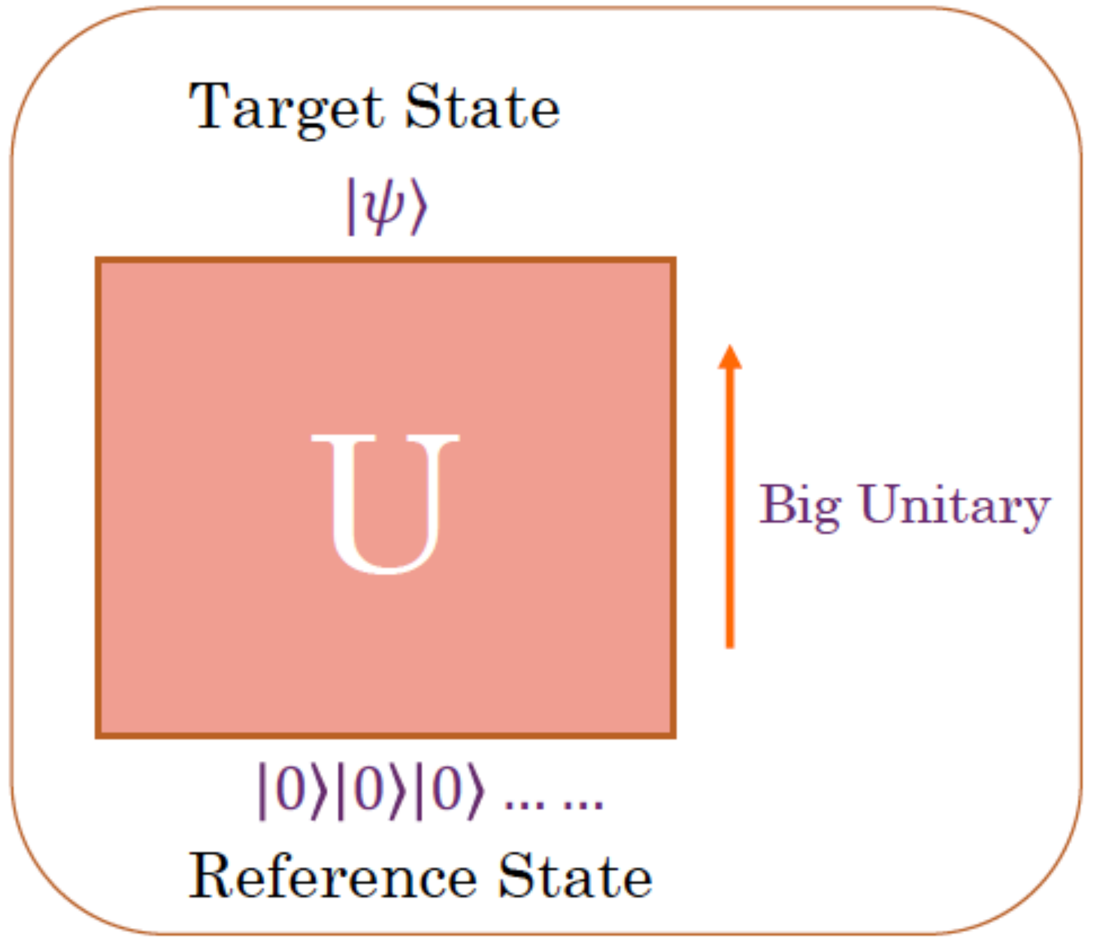
Generically: How difficult is to implement a task ?

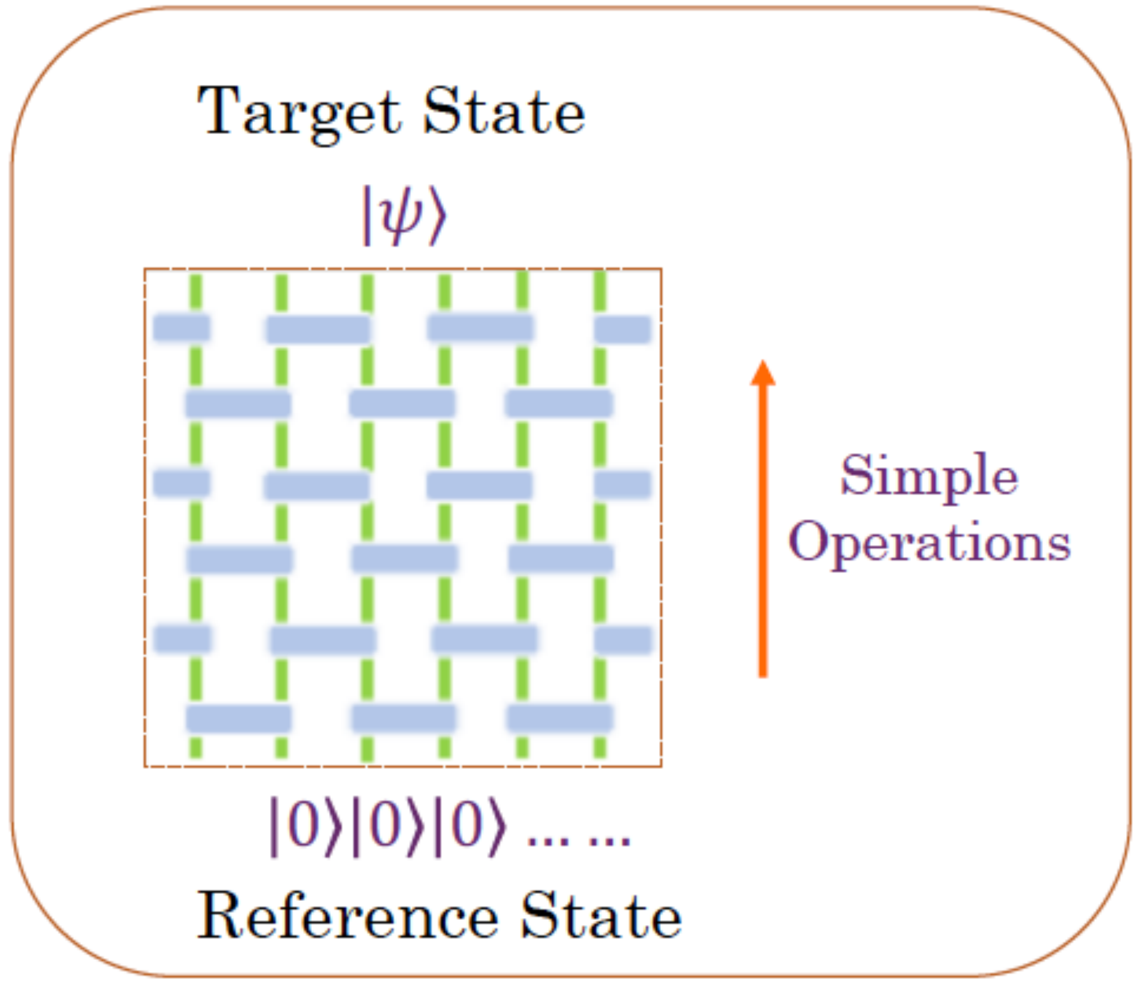
Important applications in QI and Quantum Many body physics

(Vidal '03, '04, F. Verstraete and I.Cirac '06,09
N. Schuch, I. Cirac, and F. Verstraete '08,
D. Aharonov, I. Arad, Z. Landau, and U. Vazirani '11)

Here we will use the notion of "Circuit complexity"

how difficult is to prepare a particular state ?





"minimize the number of operations"

will depend on the choice of the reference state

Basic Setup & Assumptions

Jefferson Myers '17

AB, A.Sinha, A Shekar `18

Lets us illustrate this via a simple example:

$$\mathcal{H} = \frac{1}{2}\Big[p_1^2 + p_2^2 + \omega^2(x_1^2 + x_2^2) + \Omega^2(x_1 - x_2)^2\Big]$$

Next we solve the ground state:

$$\tilde{x}_0 = \frac{1}{\sqrt{2}}(x_1 + x_2), \quad \tilde{x}_1 = \frac{1}{\sqrt{2}}(x_1 - x_2),$$
$$\tilde{p}_0 = \frac{1}{\sqrt{2}}(p_1 + p_2), \quad \tilde{p}_1 = \frac{1}{\sqrt{2}}(p_1 - p_2),$$
$$\tilde{\omega}_0^2 = \omega^2, \quad \tilde{\omega}_1^2 = \omega^2 + 2\Omega^2.$$

$$\psi_{0,0}(x_1, x_2) = \frac{(\omega_0\omega_1)^{1/4}}{\sqrt{\pi}} \exp\Big[-\frac{1}{2}\Big(\frac{a_1 + a_2}{2}(x_1^2 + x_2^2) + (a_1 - a_2)x_1x_2\Big)\Big],$$

We will refer to as "Target State"
$\psi^T(x_1, x_2)$

$$a_1 = \tilde{\omega}_0, a_2 = \tilde{\omega}_1$$

This is a Gaussian State

Lets compute the circuit complexity ("Circuit Depth") for the state

The reference state:

No entanglement in the original basis i.e in the position space

$$\psi^R(x_1,x_2) = \mathcal{N}\exp\Big[-\frac{\tilde{\omega}_{ref}}{2}(x_1^2+x_2^2)\Big]$$

Now lets build the circuit:

We choose the following natural set of gates:

$$Q_{11} = \exp\Big[\underbrace{\frac{i\,\epsilon}{2}(x_1\,p_1+p_1\,x_1)}_{\substack{\text{scaling}\\ O_{11}}}\Big],\, Q_{22} = \exp\Big[\underbrace{\frac{i\,\epsilon}{2}(x_2\,p_2+p_2\,x_2)}_{\substack{\text{scaling}\\ O_{22}}}\Big],\, Q_{12} = \exp\Big[\underbrace{i\,\epsilon\,x_1\,p_2}_{\substack{\text{entangling}\\ O_{12}}}\Big],\, Q_{21} = \exp\Big[\underbrace{i\,\epsilon\,x_2\,p_1}_{\substack{\text{entangling}\\ O_{21}}}\Big]$$

Note that the operators O's form a closed algebra.

Then we construct the circuit U with these gates

$$|\psi^T(x_1,x_2) - U\psi^R(x_1,x_2)|^2 < \epsilon$$

We can tune $\epsilon \to 0$ to get a very precise match

Nielsen approach:

(Nielsen quant-ph/0502070,
Nielsen, Dowling, Gu, Doherty,quant-ph/0603161
M.~A. Nielsen and M.~R. Dowling,quant-ph/0701004)

To achieve the optimal circuit instead of working in discrete picture we work in continuous picture, the circuit is parametrized by continuous parameter "s" and consists of continuous functions

$$U(s) = \overleftarrow{\mathcal{P}} \exp(i \int_0^s ds\, Y^I(s)\, O_I(s)),$$

$$O_I = \{O_{11}, O_{22}, O_{12}, O_{21}\}$$

Path ordering

In the discrete picture

$$\Delta s = \epsilon$$

$Y^I(s)$ control functions

Boundary conditions:

$$\psi^T(x_1, x_2) = U(s=1)\psi^R(x_1, x_2),$$
$$U(s=0) = I$$

(note: there is a freedom in choosing end point for "s", we fix it to be at s=1)

Optimal Circuit: We need to find optimal $Y^I(s)$

This can be typically achieved by minimizing some kind of action "***Cost function***" $\mathcal{F}(U, \dot{U})$ for these $Y^I(s)$

Complexity: $\mathcal{D}(U) = \int_0^1 \mathcal{F}(U, \dot{U})\, ds.$

Some desirable properties of cost functions:

1. Continous
2. Positivity
3. Homogeneity
4. Satisfy triangle inequality

(Nielsen quant-ph/0502070,
Nielsen, Dowling, Gu, Doherty,quant-ph/0603161
M.~A. Nielsen and M.~R. Dowling,quant-ph/0701004,
Jefferson-Myers, 1707.08570 [hep-th])

These help us to identify these functions as distance function between two point on a given manifold

Several Choice:

Jefferson-Myers, 1707.08570 [hep-th],
Hackl-Myers, 1803.10638 [hep-th],
Guo-Hernandez-Myers-Ruan, 1807.07677[hep-th]

$$\mathcal{F}_2(U,Y) = \sqrt{\sum_I p_I (Y^I)^2}, \mathcal{F}_\kappa(U,Y) = \sum_I p_I \, |Y^I|^\kappa, \quad \kappa \quad \text{is an integer and}\ , \kappa \geq 1,$$

$$\mathcal{F}_p(U,Y) = (Tr(V^\dagger V)^{p/2}))^{1/p}, V^I = Y^I(s) M_I, \quad p \quad \text{is an integer}$$

$\mathcal{F}_{\kappa=1}(\text{for } p_I = 1, \forall I)$ counts number of gates

$\mathcal{F}_2$ is the distance on Riemannian manifold

(for this talk we will mainly consider these two)

Now the strategy is to minimize these cost functions.
For this we first solve for the geodesics

Remember : $Y^I(s)O_I = \partial_s U(s).U(s)^{-1}$

Next we define a metric (right invariant !)

$$ds^2 = G_{IJ} dY^I dY^J$$

Find the Geodesic and evaluate the the action on it

$$\mathcal{D}(U) = \int_0^1 ds \sqrt{\sum_{I,J} G_{IJ} Y^I Y^J}$$ (here we have used the $\mathcal{F}_2$)

G_{IJ} Penalty factor

To practically compute this metric we first note: *Jefferson-Myers, 1707.08570 [hep-th],*

Our wavefunction can be written in the following way:

$$\psi^s(x_1, x_2) = \mathcal{N}^s \exp\Big[-\frac{1}{2}\vec{v}.A(s).\vec{v}\Big] \qquad \vec{v} = \{x_1, x_0\}$$

And: $s = 0, \text{Reference State}$

$s = 1, \text{Target State}$

For our case:

$$A(s=0) = \begin{pmatrix} \tilde{\omega}_{\text{ref}} & 0 \\ 0 & \tilde{\omega}_{\text{ref}} \end{pmatrix}$$

$$A(s=1) = \begin{pmatrix} \frac{1}{2}(a_1 + a_2) & \frac{1}{2}(a_1 - a_2) \\ \frac{1}{2}(a_1 - a_2) & \frac{1}{2}(a_1 + a_2) \end{pmatrix}$$

Now given this basis: $\vec{v}$

We can find the representations of these operators O's

$$O_{ij}.\vec{v} = (M_{ij})_{ab} v_a$$

We then get,

$$O_{11} \to M_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$
$$O_{22} \to M_{22} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$O_{12} \to M_{12} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$
$$O_{21} \to M_{21} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

They are nothing but generators of GL(2, R)

$$Tr(M_I M_J^T) = 2\delta_{IJ}$$

then

$$Y^I(s) = \frac{1}{Tr(M_I.M_I^T)}\left(\partial_s U(s).U(s)^{-1}.M_I^T\right)$$

Now we have to find the geodesic on GL(2,R) group manifold

A parametrization : $GL(2,R) = R \times SL(2,R)$

$$U(s) = \exp(y(s)) \begin{pmatrix} \cosh(\rho(s))\cos(\tau(s)) - \sin(\theta(s))\sinh(\rho(s)) & \cos(\theta(s))\sinh(\rho(s)) - \cosh(\rho(s))\sin(\tau(s)) \\ \cos(\theta(s))\sinh(\rho(s)) + \cosh(\rho(s))\sin(\tau(s)) & \sin(\theta(s))\sinh(\rho(s)) + \cosh(\rho(s))\cos(\tau(s)) \end{pmatrix}$$

Choose the penalty factor: $G_{IJ} = \delta_{IJ}$

Line element:

$$\begin{aligned} ds^2 &= \delta_{IJ} Tr(dU.U^{-1}.M_I^T) Tr(dU.U^{-1}.M_J^T), \\ &= 2dy^2 + 2d\rho^2 + 2\cosh(2\rho)\cosh(\rho)^2 d\tau^2 + 2\cosh(2\rho)\sinh(\rho)^2 d\theta^2 - 2\sinh(2\rho)^2 d\tau d\theta \end{aligned}$$

We now solve for the geodesic on this background

Boundary conditions:

We observe that the unitary operator acts on the wavefunction in the following way

$$A(s) = U(s).A(s=0).U(s)^T$$

arbitrary, either θ or τ can be uniquely determined

$$s := 0, \{y(0) = 0, \rho(0) = 0, \theta(0) + \tau(0) = c_0\}$$

$$s := 1, \exp(2y(1)) = \sqrt{\frac{a_1\, a_2}{\tilde{\omega}_{ref}^2}}, \cosh(2\rho(1)) = \frac{a_1 + a_2}{2\sqrt{a_1 a_2}}, \tan(\theta(1) + \tau(1)) = 0.$$

Choosing: $c_0 = 0$ Geodesic became straight line

$$y(s) = y(1)\, s, \rho(s) = \rho(1)\, s$$

$$\tau(s) = 0, \theta(s) = \theta_0$$

We can show $\mathcal{D}(U) = \int_0^1 ds \sqrt{\sum_{I,J} G_{IJ} Y^I Y^J}$ gets minimized on this geodesic

In fact one can check that it is the global minimum

Complexity: $\mathcal{D}(U) = \sqrt{y(1)^2 + \rho(1)^2}$

In fact one can check $\mathcal{F}_{\kappa=1} \left(\sum_I |Y^I|\right)$ and the associated complexity

$$\mathcal{C}_{\kappa=1} = \int_0^1 ds \mathcal{F}_{\kappa=1}$$

also gets minimized when evaluated on this geodesic

In terms of normal mode frequency:

$$\mathcal{C}_{\kappa=1} = \frac{1}{2}\left[\left|\log \frac{\tilde{\omega}_0}{\tilde{\omega}_{ref}}\right| + \left|\log \frac{\tilde{\omega}_1}{\tilde{\omega}_{ref}}\right|\right]$$

This is the strategy we will follow in the remaining of the talk.

A point to note:

A Gaussian state can alternatively characterized by a "Covariance Matrix"

$$G_{ab} =< \psi(x_k, t)|\xi_a\xi_b + \xi_b\xi_a|\psi(x_k, t) >$$

$$\xi_a = \{x_1, p_1, x_2, p_2, \cdots\}$$

Hugo A. Camargo, Pawel Caputa,Diptarka Das, Michal P. Heller, and Ro Jefferson, Phys. Rev. Lett. 122, 081601 (2019)
AB, T.Ali, E.Kim, S.Haque, N.Moynihan JHEP 1904 (2019) 087 [arXiv: 1810.02734 [hep-th]]

We can compute the complexity in terms of this Covariance matrix

eg: Reference State ($\tau = 0$)

$$\psi^R = \mathcal{N}\exp(-\frac{\omega_r\, x^2}{2})$$

ω_r is real

$$G^{\tau=0} = \begin{pmatrix} \frac{1}{\omega_r} & 0 \\ 0 & \omega_r \end{pmatrix}$$

eg: Target State ($\tau = 1$)

$$\psi^R = \mathcal{N}\exp(-\frac{\omega(t)\, x^2}{2})$$

$\omega(t)$ is complex in general

$$G^{\tau=1} = \begin{pmatrix} \frac{1}{\mathrm{Re}(\omega(t))} & -\frac{\mathrm{Im}(\omega(t))}{\mathrm{Re}(\omega(t))} \\ -\frac{\mathrm{Im}(\omega(t))}{\mathrm{Re}(\omega(t))} & \frac{|\omega(t)|^2}{\mathrm{Re}(\omega(t))} \end{pmatrix}$$

We can compute the complexity in terms of this Covariance matrix

$$G^{T} = U(s=1).G^{R}.U^{\mathcal{T}}(s=1)$$

Given this: We can proceed in the same way as before and compute the complexity

$$\mathcal{C}_{\kappa=2} = \frac{1}{2}\Big(\cosh^{-1}\Big[\frac{\omega_r^2 + |\omega(t)|^2}{2\omega_r \Re(\omega(t))}\Big]\Big)$$

Sometimes we will work with Covariance matrix instead of wavefunction for eg: in the context of single inverted oscillator and for that case the general conclusions will not depend on this much.

(For detailed comparison for various methods of computation of circuit complexity refer to AB, T.Ali, E.Kim, S.Haque, N.Moynihan JHEP 1904 (2019) 087 [arXiv: 1810.02734 [hep-th]]

Application

(Related papers by: Run-Qiu Yang, Keun-Young Kim et al *JHEP* 05 (2020) 045, 1906.02063, Vijay Balasubramanian et al Matthew DeCross, Arjun Kar, Onkar Parrikar, JHEP01(2020)134)

AB, T.Ali, E.Kim, S.Haque, J.Murugan,N.Moynihan, Phys.Rev.D 101 (2020) 2, 026021

Symptom of Quantum Chaos ?

- Classical chaotic systems are characterized by a hypersensitivity to perturbations in initial conditions under the Hamiltonian evolution.

- This hypersensitivity is usually diagnosed by studying individual orbits in phase space — The orbits diverges!

- For quantum system: we cannot specify both position and momentum " Uncertainty Principle" .

The volume occupied by a single quantum state in the classical phase space is ~

$$\left(\frac{h}{2\pi}\right)^N$$ for a system with N degrees of freedom.

We no longer have the luxury to follow individual orbits !!!

New diagnostics needed.

(for more details of quantum chaos Please refer to Yan, Keun-Young,Viktor, Mitsuhiro's talk and also Vijay's talk)

We can only talk about "**symptoms**" of quantum chaos !!!

(PhD thesis of Nicholas Hunter Jones "Chaos and Randomness in Strongly-Interacting Quantum Systems")

(M. Berry, "Quantum chaology, not quantum chaos," Phys. Scr. 40 (1989) 335)

There are several ones !

Wigner 1950: Statistical Properties of energy spectra

Distribution Eigenvalue spacing of energy eigenvalue of quantum chaotic Hamiltonian is similar to the one for Gaussian Random Matrix ensembles

But computation of eigen-spectrum is computationally quite taxing

So other diagnostics ("Symptoms" !!) are being developed

(Viktor Jahnke's review Arxiv: 1811.06949+ yesterday's talk)

A popular one is
(Out of Time ordered Correlator) :

$$OTOC(t) =< B^{\dagger}(0)A^{\dagger}(t)B(0)A(t) >_{\beta}, T = \frac{1}{\beta}$$

$$C_T(t) = - < [A(t), B(0)]^2 >_{\beta} = 2(1 - Re(OTOC(t))),$$

Quantum analog of classical expectation value:

$$< \left(\frac{\partial x(t)}{\partial x(0)}\right)^2 > \approx \sum_n c_n e^{\lambda_n t}\lambda_n = \text{ Lyapunov Exponent}$$

Early Time characteristic decay of OTOC:

$$OTOC(t) \approx 1 - e^{\lambda_L (t - t_*)}$$

Quantum Lyapunov Exponent

Recently well explored in Holography: "Chaos Bound" (conjectured)

$$\lambda_L \leq \frac{2\pi}{\beta}$$

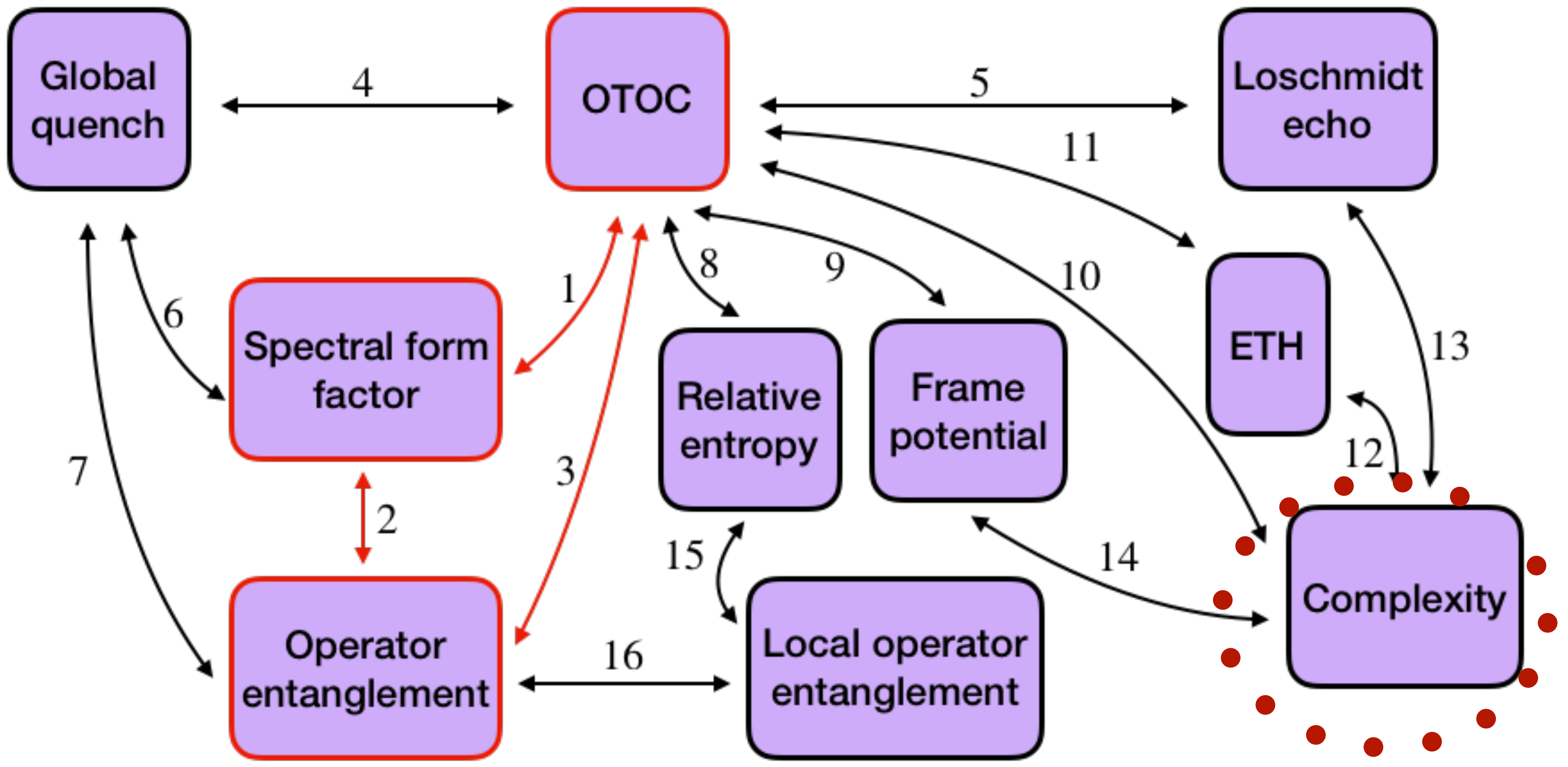
(Maldacena Shenker Stanford '15)

But it has its own issue: Reliability breaks down at late time and for some well studied single particle quantum chaotic system like stadium billiard ball model, some quantum spin chains it does not give the expected Lyapunov decay

(eg K.Hashimoto et al JHEP 10 138 (2017))

One need other measures.

In this context we ask can we use "complexity" as a diagnostics of quantum chaos ?



This our hero
for this talk

(picture courtesy: Jonah Kudler-Flam, Laimei Nie, Shinsei Ryu, JHEP01(2020)175)

The Model

We use “Inverted Harmonic” oscillator model.

Classically it has an unstable fixed point: $x = 0, p = 0$

Its a toy model but nonetheless a powerful and an interesting one

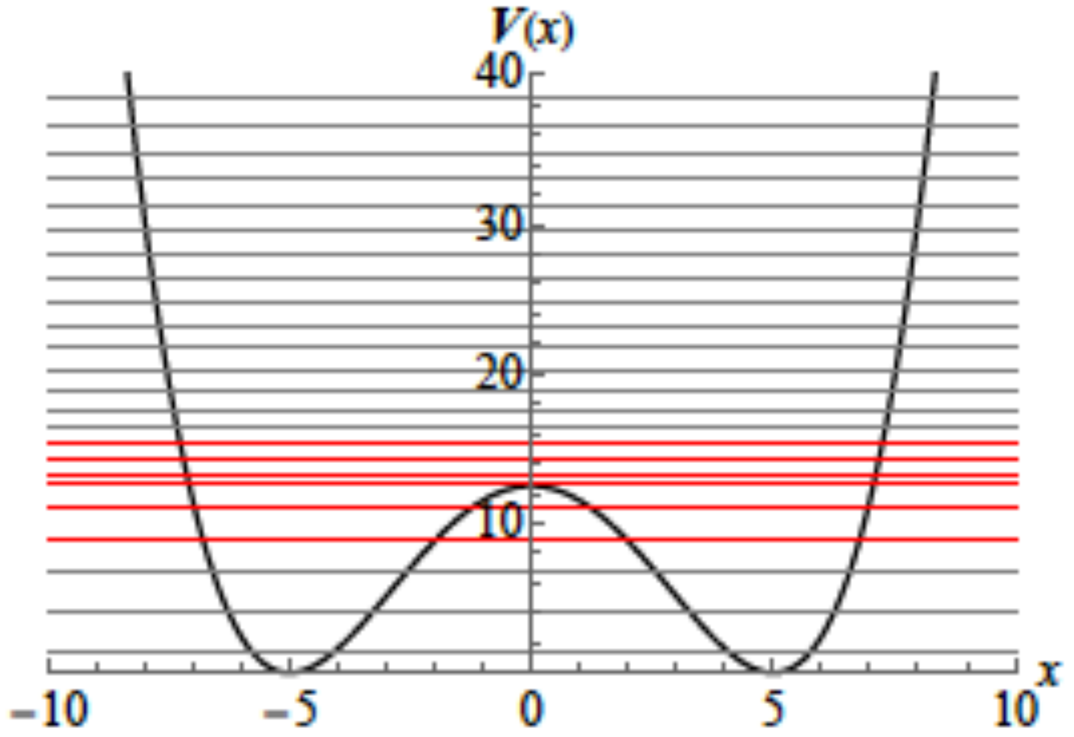
This model has also been demonstrated rich and fruitful in a wider context in the field of quantum chaos

(Physical Review A 68, 032104, Physical Review Letters 122, 101603 and many more, experimentally realized: Sci. Rep. 5, 15816 (2015))

It can appear as a local maxima inside various interesting potentials.

$$V = g(x^2 - \frac{\lambda^2}{8g})^2 = -\frac{1}{4}\lambda^2 x^2 + g x^4 + \frac{\lambda^4}{64g}$$

Picture courtesy Koji Hashimoto, Kyoung-Bum Huh, Keun-Young Kim, Ryota Watanabe
ArXiv: 2007.04746



The Setup

Hamiltonian: $H = \frac{1}{2}p^2 + \frac{\Omega^2}{2}x^2$ where $\Omega^2 = m^2 - \lambda$

AB, T.Ali, E.Kim, S.Haque, J.Murugan,N.Moynihan, Phys.Rev.D 101 (2020) 2, 026021

$m^2 = \lambda$ Free particle

$m^2 > \lambda$ SHO

$m^2 < \lambda$ IHO

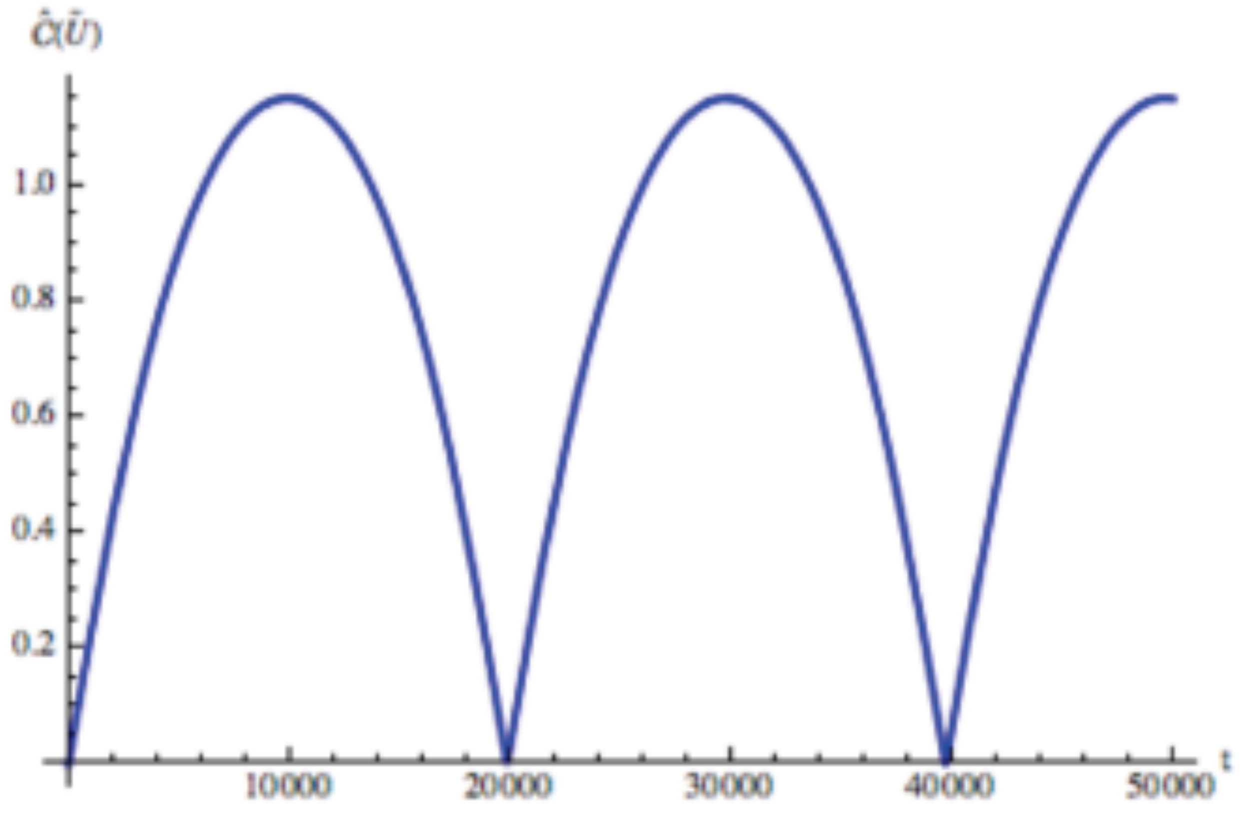
Reference State (at t=0): $\psi(x, t=0) = \mathcal{N}(t=0)\exp\left(-\frac{\omega_r x^2}{2}\right), \omega_r = m$

Target State (at t=0): $|\psi_2 >= e^{i(H+\delta H)t}e^{-iHt}|\psi_0 >$

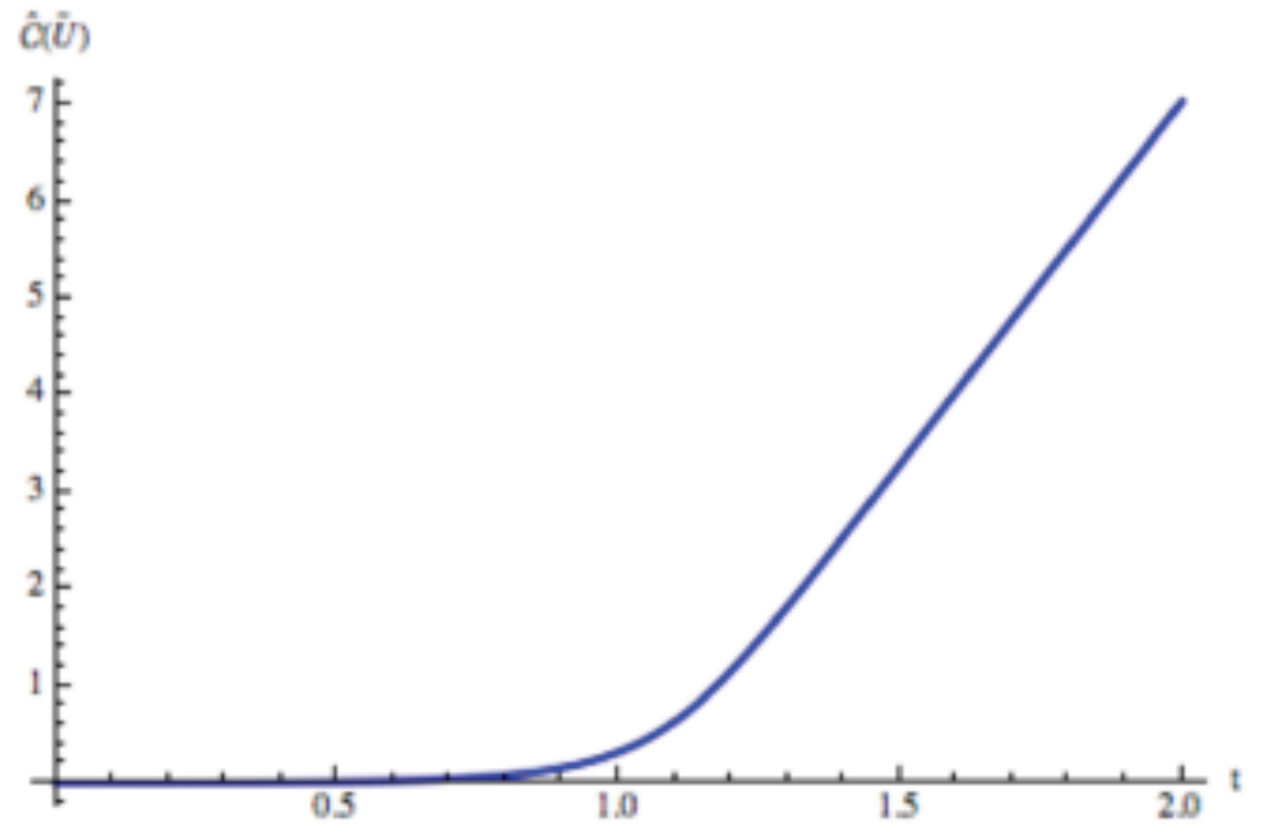
$$\psi_2(x,t) = \mathcal{N}(t)\exp\left[-\tfrac{1}{2}\hat{\omega}(t)x^2\right]$$

Then following previous method we compute complexity between them

$$\hat{C}(\tilde{U}) = \frac{1}{2}\left(\cosh^{-1}\left[\frac{\omega_r^2 + |\hat{\omega}|^2}{2\omega_r \Re(\hat{\omega}(t))}\right]\right), \quad \hat{\omega}(t) = i\,\Omega'\cot(\Omega' t) + \frac{\Omega'^2}{\sin^2(\Omega' t)(\omega(t) + i\,\Omega'\cot(\Omega' t))}$$



Regular Oscillator
(m=1, λ = 0.2)



Inverted Oscillator
(m=1, λ = 15)

Different behavior altogether

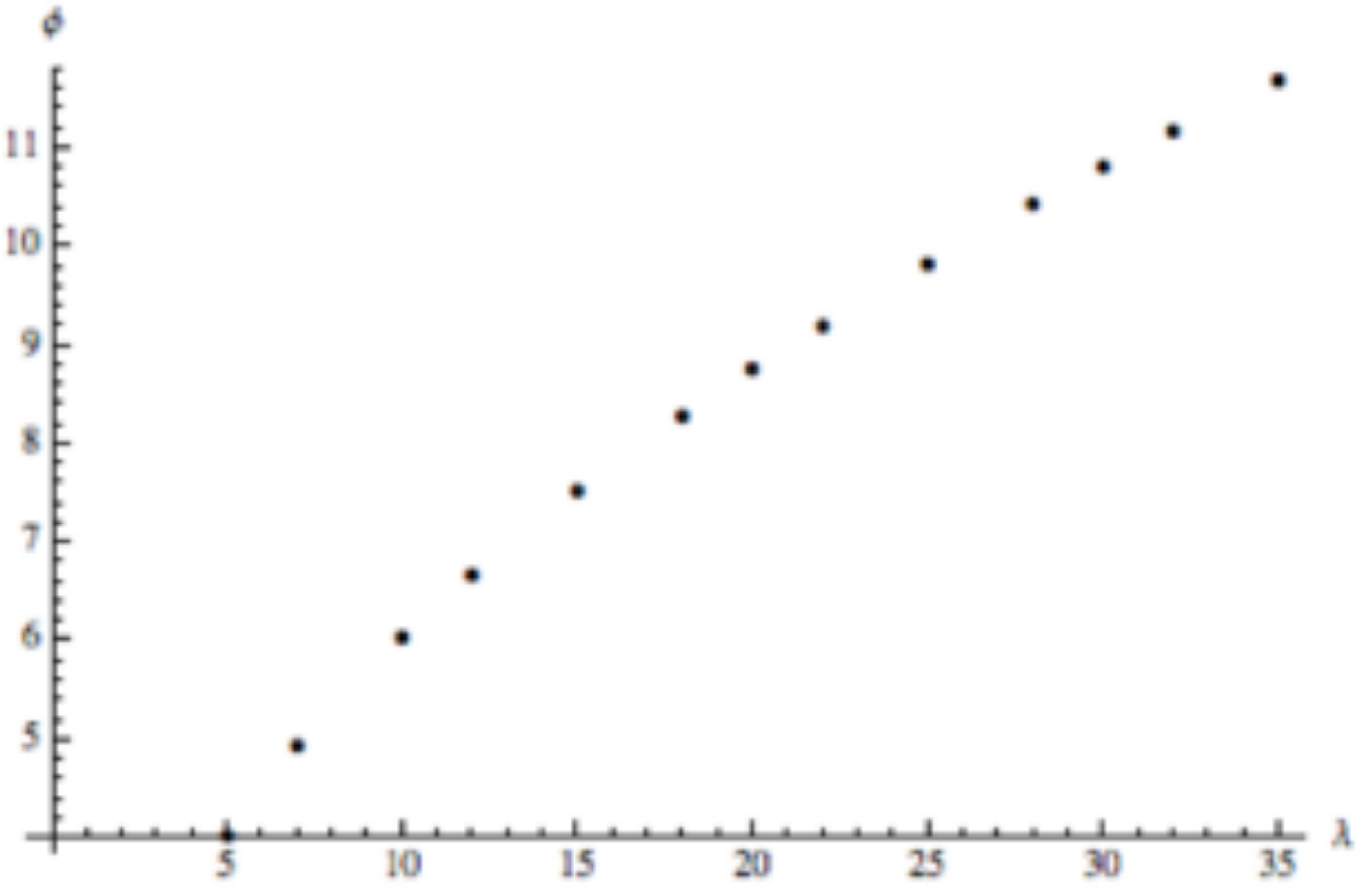
For inverted oscillator the complexity start to increase considerable after a certain time t_s

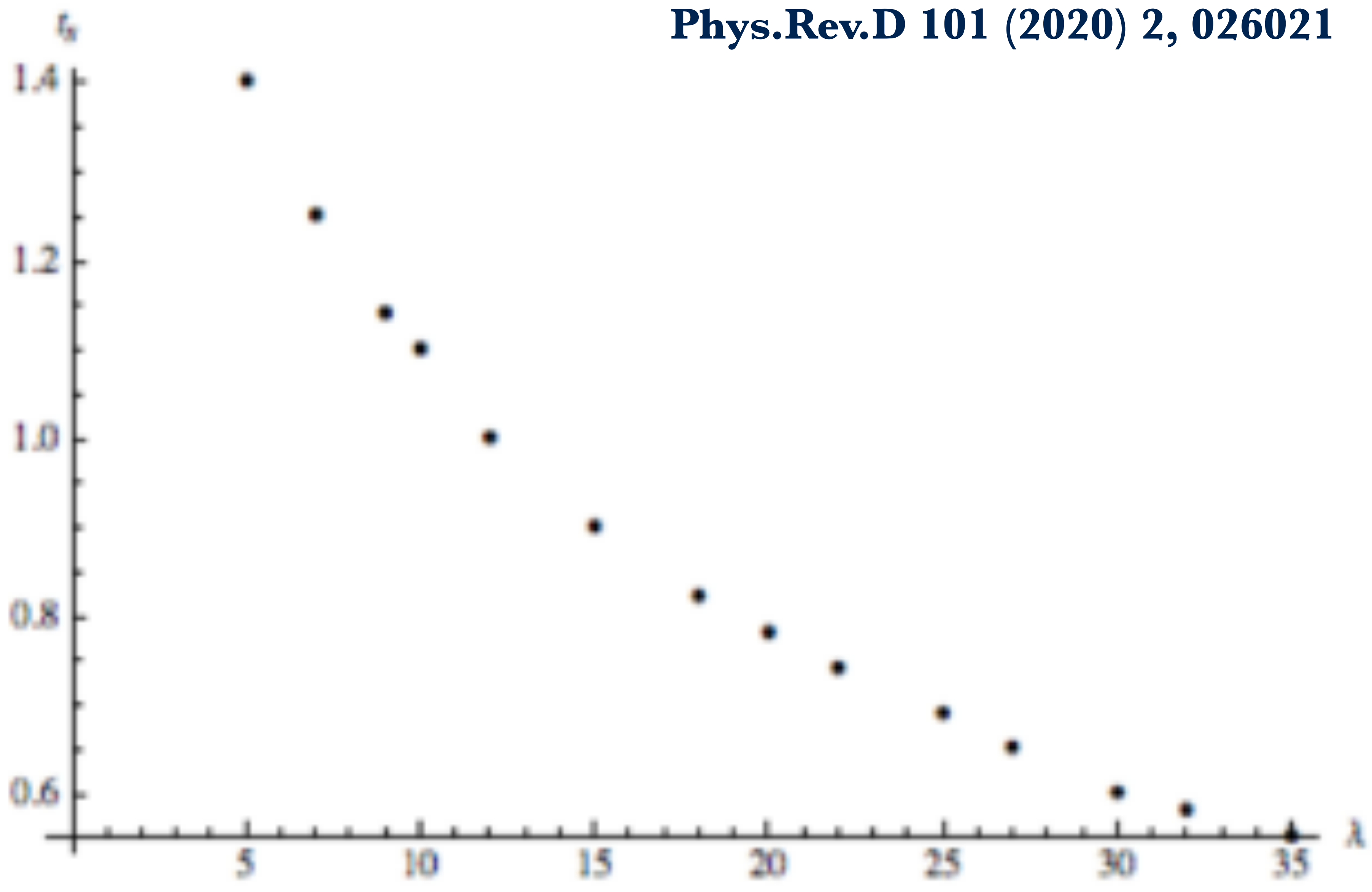
AB, T.Ali, E.Kim, S.Haque, J.Murugan,N.Moynihan, Phys.Rev.D 101 (2020) 2, 026021

Then grows linearly. We denote the slope of this portion by ϕ

We plot both of them w.r.t coupling λ

AB, T.Ali, E.Kim, S.Haque,
J.Murugan,N.Moynihan,
Phys.Rev.D 101 (2020) 2, 026021





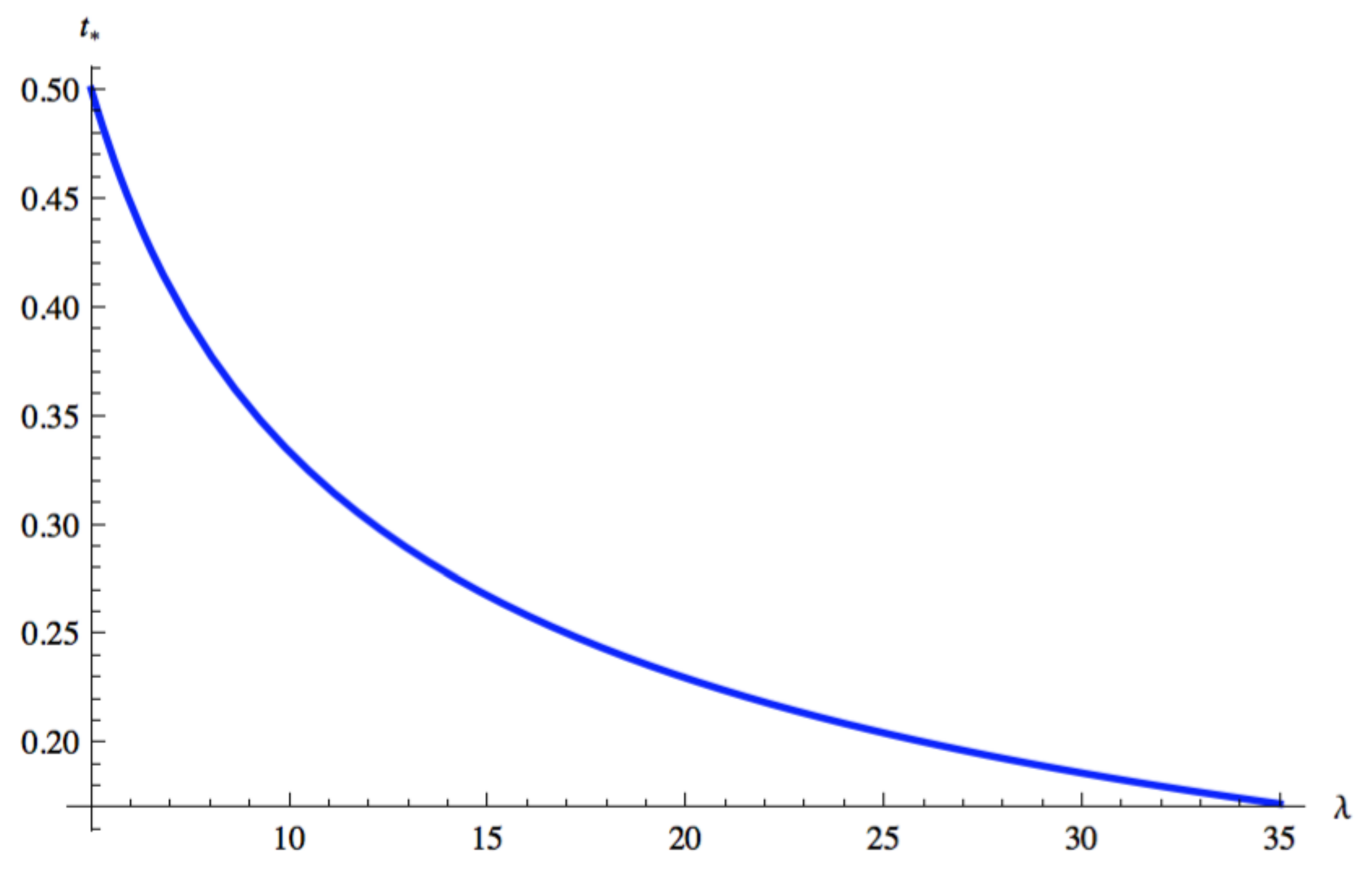
Further we compute for IHO,

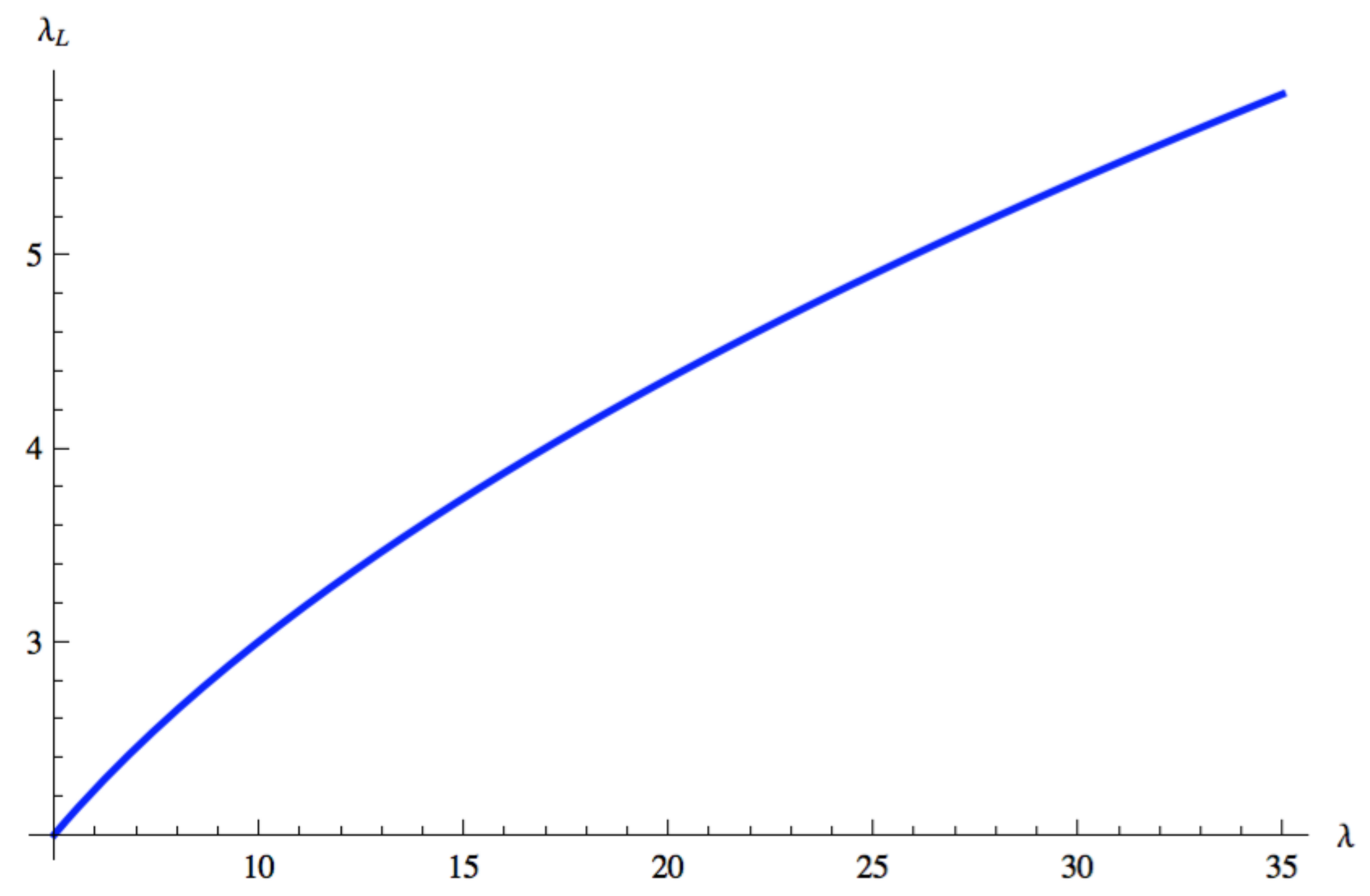
$$C_T(t) =< [x(t), p]^2 >\approx e^{2\lambda_L(t-t_*)}$$

$$\lambda_L = |\Omega|, t_0 = \frac{1}{\lambda_L}\log\frac{2\pi}{h}$$

(Koji Hashimoto, Keiju Murata,
Ryosuke Yoshii, JHEP10(2017)138)

We also plot them next





(m=1)

We fit the previous data with this and find that:

$$\phi = 2|\Omega| = 2\lambda_L, t_s = \frac{4\log(2)}{|\Omega|} = 4\log(2)t_*$$

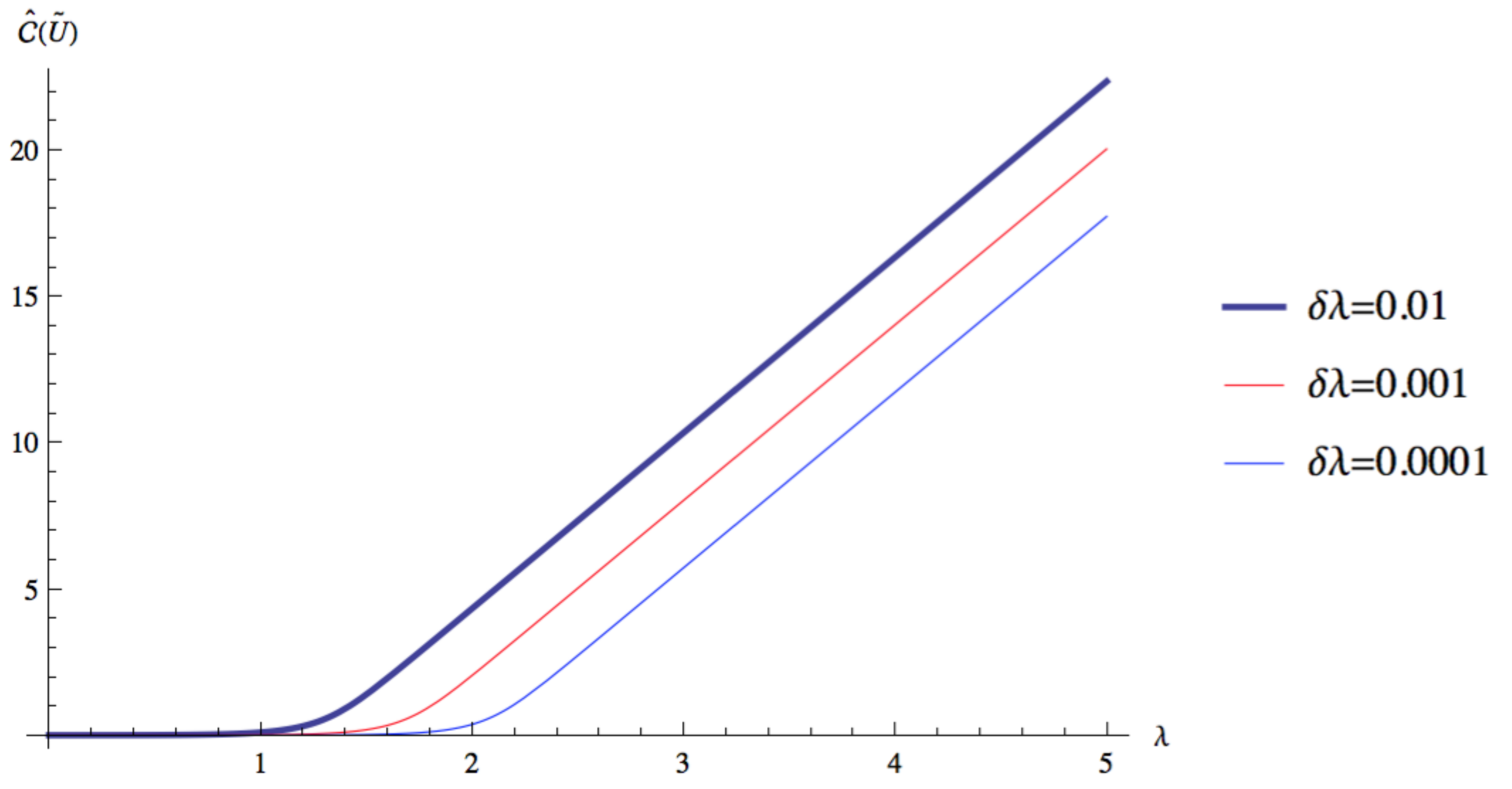
AB, T.Ali, E.Kim, S.Haque, J.Murugan,N.Moynihan, Phys.Rev.D 101 (2020) 2, 026021

We can extract Lyapunov Exponent and Scrambling time from the circuit complexity

One more check:

$$|\psi_2 >= e^{i(H+\delta H)t} e^{-iHt} |\psi_0 >$$

We plot complexity for various $\delta\lambda$



Slope remains same but the pick up time seems to be sensitive to the value of $\delta\lambda$

Towards a field theory analysis:

We consider now many coupled oscillator

$$H = H_0 + H_I = \frac{1}{2}\int dx\Big[\Pi_1^2 + (\partial_x\phi_1)^2 + \Pi_2^2 + (\partial_x\phi_2)^2 + m^2(\phi_1^2 + \phi_2^2)\Big] + \lambda\int dx(\partial_x\phi_1)(\partial_x\phi_2)$$

Discretize on lattice

$$x(\vec{n}) = \delta\phi(\vec{n}),\ p(\vec{n}) = \Pi(\vec{n})/\delta,\ \omega = m,\ \Omega = \frac{1}{\delta^2},\ \hat{\lambda} = \lambda\,\delta^{-4}\ \text{and}\ \hat{m} = \frac{m}{\delta}$$

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$$H = \frac{\delta}{2}\sum_n\Big[p_{1,n}^2 + p_{2,n}^2 + \Big(\Omega^2\,(x_{1,n+1} - x_{1,n})^2 + \Omega^2\,(x_{2,n+1} - x_{2,n})^2 +$$

$$\big(\hat{m}^2(x_{1,n}^2 + x_{2,n}^2) + \hat{\lambda}\,(x_{1,n+1} - x_{1,n})(x_{2,n+1} - x_{2,n})\Big)\Big].$$

Not Diagonal

Do a series of transformation

$$
\begin{aligned}
x_{1,a} &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp\left(\frac{2\pi\, i\, k}{N}\, a\right) \tilde{x}_{1,k}, \\
p_{1,a} &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp\left(-\frac{2\,\pi\, i\, k}{N} a\right) \tilde{p}_{1,k}, \\
x_{2,a} &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp\left(\frac{2\pi\, i\, k}{N}\, a\right) \tilde{x}_{2,k}, \\
p_{2,a} &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \exp\left(-\frac{2\,\pi\, i\, k}{N} a\right) \tilde{p}_{2,k}, \\
\tilde{p}_{1,k} &= \frac{p_{s,k} + p_{a,k}}{\sqrt{2}}, \;\; \tilde{p}_{2,k} = \frac{p_{s,k} - p_{a,k}}{\sqrt{2}}, \\
\tilde{x}_{1,k} &= \frac{x_{s,k} + x_{a,k}}{\sqrt{2}}, \;\; \tilde{x}_{2,k} = \frac{x_{s,k} - p_{a,k}}{\sqrt{2}},
\end{aligned}
$$

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We finally get,

$$H = \frac{\delta}{2} \sum_{k=0}^{N-1} \left[p_{s,k}^2 + \bar{\Omega}_k^2 x_{s,k}^2 + p_{a,k}^2 + \Omega_k^2 x_{a,k}^2 \right]$$

$$\bar{\Omega}_k^2 = \left(\hat{m}^2 + 4\,(\Omega^2 + \hat{\lambda})\, \sin^2\left(\frac{\pi\, k}{N}\right)\right), \; \Omega_k^2 = \left(\hat{m}^2 + 4\,(\Omega^2 - \hat{\lambda})\, \sin^2\left(\frac{\pi\, k}{N}\right)\right)$$

Gives rise to inverted oscillator,
We only focus on this part

Coupled Inverted Oscillator

$$\tilde{H}(m,\Omega,\hat{\lambda}) = \frac{\delta}{2}\sum_{k=0}^{N-1}\left[p_k^2 + \left(\hat{m}^2 + 4\,(\Omega^2 - \hat{\lambda})\,\sin^2\left(\frac{\pi\,k}{N}\right)\right)x_k^2\right]$$

Circuit Complexity:

$$\hat{\mathcal{C}}(\tilde{U}) = \frac{1}{2}\sqrt{\sum_{k=0}^{N-1}\left(\cosh^{-1}\left[\frac{\omega_{r,k}^2 + |\hat{\omega}_k(t)|^2}{2\,\omega_{r,k}\,\mathrm{Re}(\hat{\omega}_k(t))}\right]\right)^2}$$

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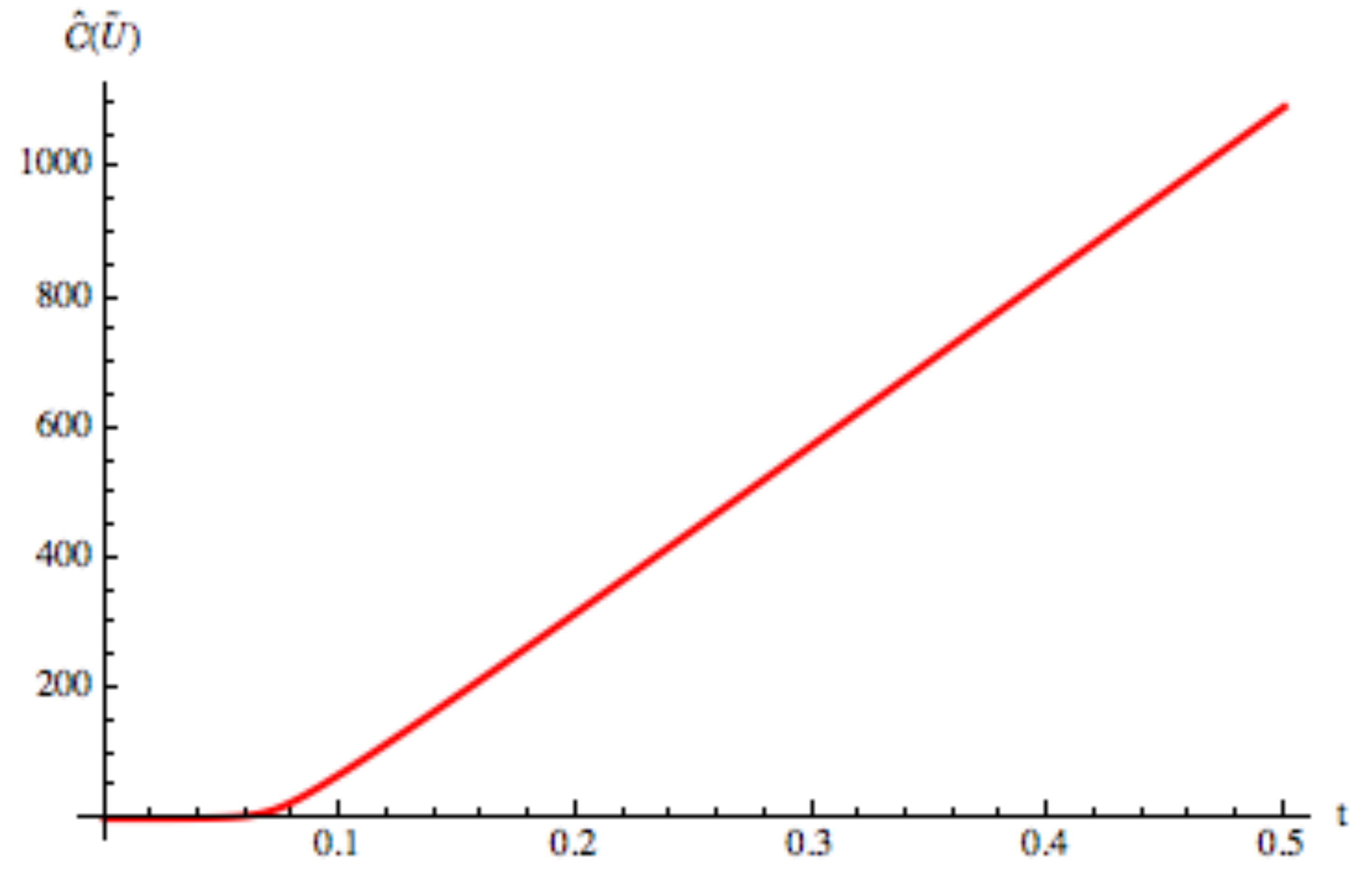
$$\hat{\omega}_k(t) = i\;\Omega'_k\cot(\Omega'_k t) + \frac{\Omega'^2_k}{\sin^2(\Omega'_k t)\,\left(\omega_k(t) + i\,\Omega'_k\cot(\Omega'_k t)\right)},\quad \Omega'^2_k = \hat{m}^2 + 4\,(\Omega^2 - \hat{\lambda} - \delta\hat{\lambda})\,\sin^2\left(\frac{\pi\,k}{N}\right)$$



$\hat{\mathcal{C}}(\tilde{U})$ vs time for the Inverted Oscillators ($\delta = 0.1, m = 1, N = 1000, \hat{\lambda}\delta^2 = 10, \delta\lambda = 0.01$)

Similar behaviour and
again we can extract the Lyapunov exponent
and Scrambling time

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Last but not the least:

We apply these concepts to that of scalar cosmological perturbation

We will consider a spatially flat
Friedmann-Lemaitre-Robertson-Walker (FLRW) metric

$$ds^2 = -dt^2 + a(t)d\vec{x}^2 = a(\eta)^2(-d\eta^2 + d\vec{x}^2)$$

Perturbation: Background scalar : $\phi(x) = \phi_0(t) + \delta\phi(x),$

metric: $ds^2 = a(\eta)^2(-(1+2\psi(x,\eta))d\eta^2 + (1-2\psi(x,\eta))d\vec{x}^2)$

The action (Einstein action + minimally coupled scar field)
upto quadratic order in fluctuation (scalar) can be written as :

The action becomes after some variable change

$$S = \frac{1}{2}\int d\eta d^3x \Big[v'^2 - (\partial_i v)^2 + \left(\frac{z'}{z}\right)^2 v^2 - 2\frac{z'}{z}v'v\Big]$$

(prime denotes derivative w.r.t conformal time)

Fourier Transform:

$$\hat{v}(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^2}\hat{v}_{\vec{k}}(\eta)e^{i\vec{k}.\vec{x}}$$

A. Albrecht, P. Ferreira, M. Joyce,T. Prokopec, Phys. Rev. D50 (1994) 4807–4820 [astro-ph/9303001].

J. Martin, Lect. Notes Phys. 738 (2008) 193–241 [0704.3540].

Creation and annihilation:

$$\hat{v}_{\vec{k}} = \frac{1}{\sqrt{2k}}(\hat{c}_{\vec{k}} + \hat{c}^{\dagger}_{-\vec{k}})$$

The Hamiltonian becomes:

$$\hat{H} = \int d^3x \hat{H}_{\vec{k}} = \int d^3k \Big[k(\hat{c}_{\vec{k}}\hat{c}^{\dagger}_{\vec{k}} + \hat{c}^{\dagger}_{-\vec{k}}\hat{c}_{-\vec{k}}) - i\frac{z'}{z}(\hat{c}_{\vec{k}}\hat{c}_{-\vec{k}} - c^{\dagger}_{\vec{k}}c^{\dagger}_{-\vec{k}})\Big]$$

$$\frac{z'}{z} >> k, \text{IHO dominates}$$

Free particle Hamiltonian

IHO

Reference State: $c_{\vec{k}}|0>_{\vec{k},-\vec{k}} = 0$

Target State: $|\psi>_{\vec{k},-\vec{k}} = e^{iHt}|0>_{\vec{k},-\vec{k}}$

We compute the complexity for single mode “k” given this two state:

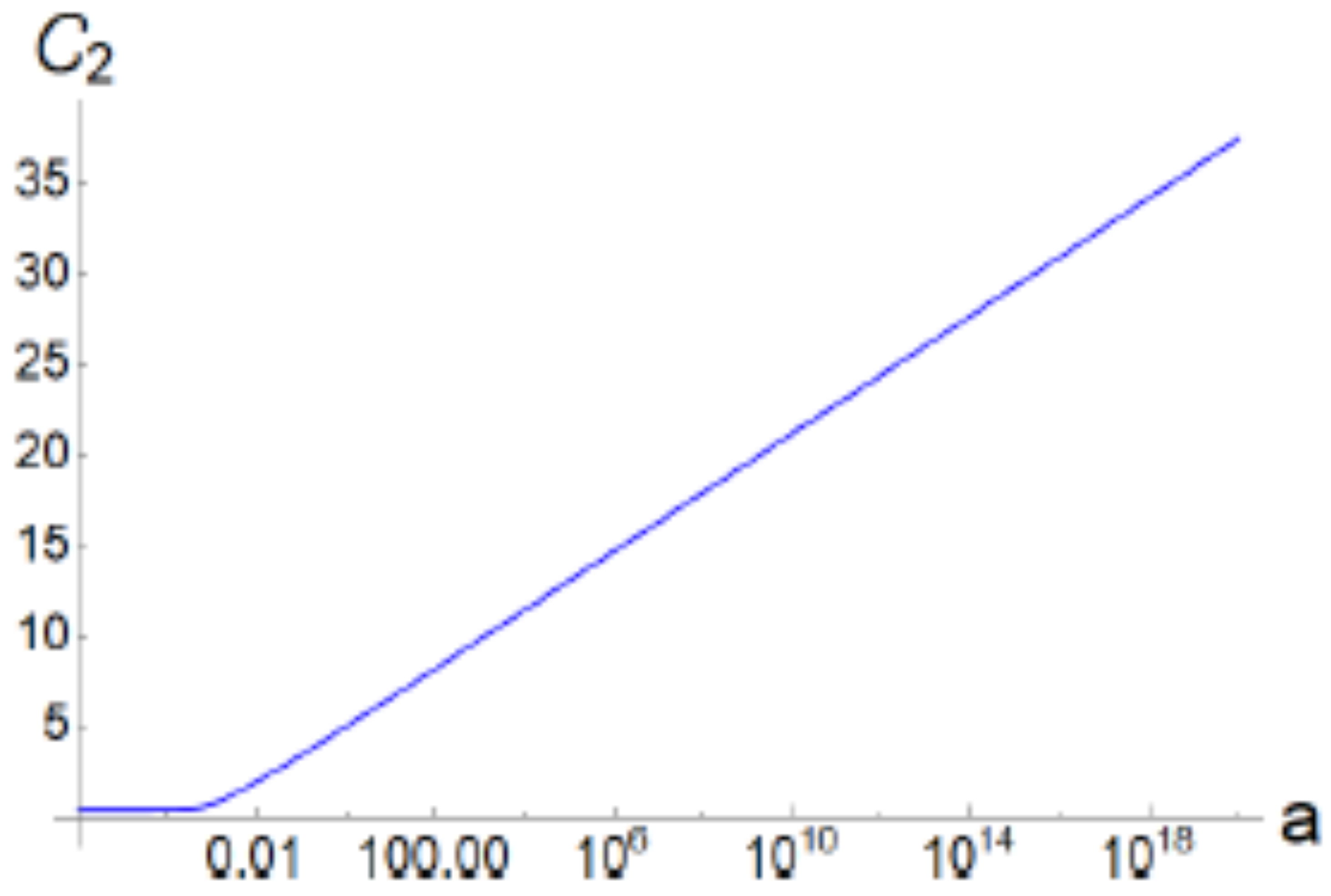
Off course we need to know the background : $a(\eta) \sim \left(\frac{\eta}{\eta_0}\right)^{\beta}$

$$a(\eta) = -\frac{1}{H\eta}, \ \beta = -1, \text{de Sitter}$$

$$a(\eta) = \frac{\eta}{\eta_0}, \beta = 1, \ \text{Radiation}$$

$$a(\eta) = \left(\frac{\eta}{\eta_0}\right)^2, \beta = 2, \ \text{Matter}$$

We focus on this case for now. Other cases can be similarly handled



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$$\text{At late time: } \frac{d\mathcal{C}}{dt} = \frac{H}{\sqrt{2}}$$

Slope is bounded by Hubble.

$$\text{At late time: } \frac{d\mathcal{C}}{dt} = \frac{H}{\sqrt{2}},$$

$$\frac{d\mathcal{C}}{dt} \sim \lambda \leq 2\pi T \sim H \quad (T \sim \frac{H}{2\pi})$$

Similar to the Chaos Bound. Recently verified by direct OTOC computation

(L. Aalsma and G. Shiu, 2002.01326)

Summary

- To give a proof of principle argument for circuit complexity as a symptoms of quantum chaos we have used the toy inverted harmonic oscillator model

- We can explicitly extract the information about Lyapunov Exponent and Scrambling time from the compelxity

- We discussed few interesting setup where inverted Harmonic oscillator can appear for eg. cosmological model

- Pleasingly for cosmological we can infer some information of Lyapunov exponent from circuit complexity

- Inverted Harmonic Oscillator is just a toy model we need to extend this for more realistic models.

    - eg. Kicked rotor, Kicked Top - Work in progress (hope to report soon)

- Certainly interesting to expand this line of study for non trivial chaotic field theories

Lots to explore !!!!

Quantum Information Theory

Complexity for Chaotic System (RMT)

Sub-System Complexity

(Work in progress)

Complexity Interacting QFT

(Gauge theory ?)

Complexity Experiment

Complexity for Cosmology

(go beyond linear fluctuation ? Direct computation of OTOC, various equation of state)

Black Hole and Holography

Open System ? (hopefully to appear soon)

Complexity for Tensor Network

Thank you