TIME-DEPENDENT ENTANGLEMENT ENTROPY IN FLRW COSMOLOGIES

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Outline

1. Brief summary on the standard cosmology

2. Review on the Braneworld (Randall-Sundrum) model

3. Time-dependent entanglement entropy in expanding universes

5. Discussion

1. Brief summary on the standard cosmology

- Before discussing the braneworld cosmology, we first summarize the standard cosmology in four-dimensional flat space

Friedmann equation $$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\kappa^2}{3}\rho,$$

Continuity equation $$0 = \dot{\rho} + 3\left(\frac{\dot{a}}{a}\right)(\rho + p),$$

$a(t) \sim \tau^{\frac{2}{3(1+w)}}.$ in terms of the cosmological time

Standard cosmology relying on the matter content

1. Inflation with $w = -1,$: $a \sim e^{H\tau}$

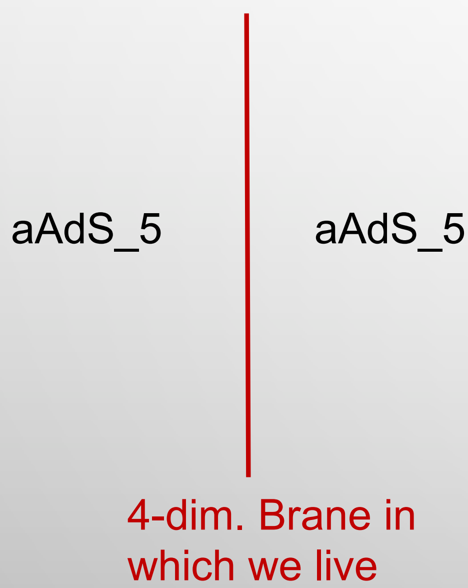
2. radiation-dominated era $(w = 1/3)$: $a \sim \tau^{1/2}$

3. matter-dominated era $(w = 0)$: $a \sim \tau^{2/3}$

2. Braneworld model

We take into account another model called the braneworld or RS model, which can describe the expanding universe of the dual QFT.

Let us first assume that M± are two five-dimensional bulk spaces with each own well-defined metric, g_(±)MN, and that they are bordered through a four-dimensional brane ∂M.



$$S = S_{\mathcal{M}_+} + S_{\mathcal{M}_-} + S_{\partial\mathcal{M}},$$

$$S_\pm = \frac{1}{2\kappa^2}\int_{\mathcal{M}_\pm} d^5x\sqrt{-g}\,\left(\mathcal{R} - 2\Lambda^{(\pm)} + \mathcal{L}_m^{(\pm)}\right) - \frac{1}{\kappa^2}\int_{\partial\mathcal{M}} d^4x\sqrt{-\gamma}K^{(\pm)},$$

$\mathcal{L}_m^{(\pm)}$ denotes the Lagrangian of bulk matter fields.

we hereafter focus on the case with the Z_2 symmetry

Radial motion of a brane in the braneworld model

Assume that the bulk has the following general metric

$$ds^2 = g_{MN} dx^M dx^N = -A(r)dt^2 + B(r)dr^2 + C(r)\ \delta_{ij} dx^i dx^j,$$

The Israel junction equation governs the motion of the brane

$$K_{\mu\nu} = -\frac{\sigma}{d-1}\gamma_{\mu\nu}.$$

σ : tension of the brane

$\gamma_{\mu\nu}$: induced metric on the brane

Introducing a cosmological time on the brane

$$-d\tau^2 = -Adt^2 + Bdr^2.$$

the induced metric on the brane becomes the FRLW metric form

$$ds_\Sigma^2 = -d\tau^2 + a(\tau)^2\ \delta_{ij} dx^i dx^j. \qquad C(r) = a(\tau)^2$$

The scale factor is determined by the Israel junction equation

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{\sigma^2}{3^2}\frac{C^2}{C'^2} - \frac{1}{B}.$$

How can we realize the standard cosmologies on the brane?

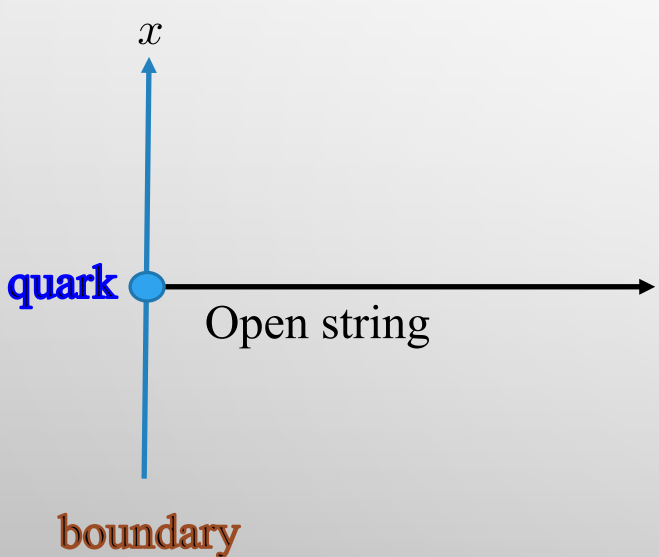
In order to realize the matter in the braneworld model, we consider uniformly distributed open strings which can be identified with fundamental matter.

$$S = \frac{1}{16\pi G}\int d^{d+1}x\sqrt{-g}\,(\mathcal{R}-2\Lambda) - \frac{d-1}{4}\sum_i \mathcal{T}_i \int d^2\xi\sqrt{-h}h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}$$

In the static gauge with $t = \xi^0$ and $z = \xi^1$

the density of the open string is given by

$\mathcal{E} = N\mathcal{T}/V$ is an energy density of open strings



Using the density of the open string, the Einstein equation reduces to

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

with the following open string's energy-momentum tensor

$$T^{\mu\nu} = -\frac{(d-1)\ \rho}{2} \frac{\sqrt{-h}}{\sqrt{-g}} \partial_\alpha X^\mu \partial_\beta X^\nu$$

Intriguingly, this Einstein equation allows the following exact solution

$$ds^2 = \frac{R^2}{z^2}\left(-f(z)dt^2 + \frac{1}{f(z)}dz^2 + \delta_{ij}dx^i dx^j\right)$$

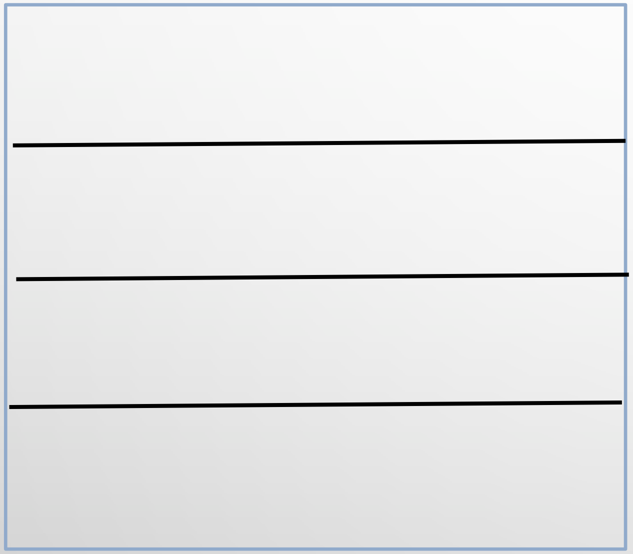
with $f(z) = 1 - \frac{\mathcal{E}}{r^3}$ (cf. $f(z) = 1 - m z^d$ for a Schwarzschild BH)

which is called the string cloud geometry. This string cloud geometry looks like a black hole geometry.

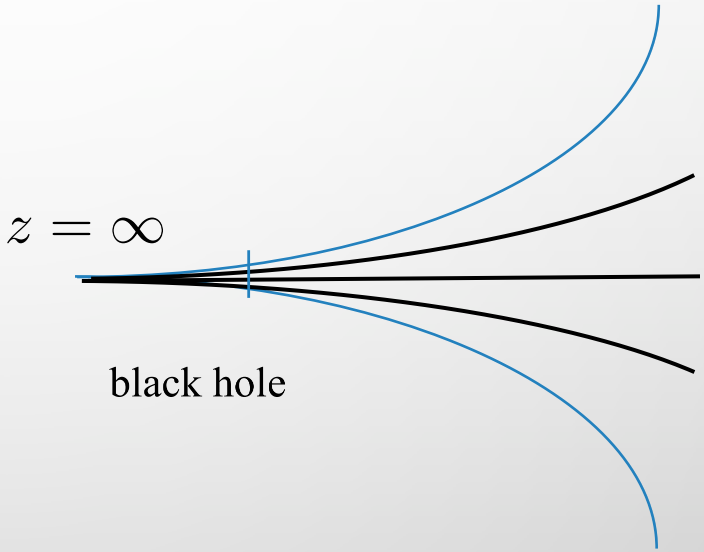
In the present model, we considered uniformly distributed strings.

In a flat geometry, the uniformly distributed matter cannot make a black hole because it requires a well localized matter.

However, this is not true for the AdS sapce due to the nontrivial warping factor.



open strings in a flat space



open strings in an AdS space

We can also introduce a localized matter corresponding to a black hole mass m

$$f(z) = 1 - \frac{\mathcal{E}}{r^3} - \frac{m}{r^4}$$

Then, the dual geometry represents a quark-gluon plasma

- open strings : fundamental matter (quark, massive matter)
- black hole mass : adjoint matter (radiation, gluon)

A generalized string cloud geometry

$$ds^2 = \frac{r^2}{R^2}\left(-f(r)dt^2 + \delta_{ij}dx^i dx^j\right) + \frac{R^2}{r^2 f(r)}dr^2,$$

with the following blackening factor

$$f(z) = 1 - \frac{\mathcal{E}}{r^3} - \frac{m}{r^4},$$

where $\mathcal{E} = N\mathcal{T}/V$ is an energy density of open strings

In a generalized string cloud geometry, the junction equation reduces to

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{\sigma^2}{36} - \frac{1}{R^2}\right) r^2 + \frac{\mathcal{E}}{R^2}\frac{1}{r} + \frac{m}{R^2}\frac{1}{r^2}$$

The induced metric on the brane becomes

$$ds_\Sigma^2 = -d\tau^2 + \frac{r(\tau)^2}{R^2}\,\delta_{ij}dx^i dx^j.$$

Possible cosmologies in the braneworld model

1. Time-independent universe

For an AdS space with $\mathcal{E} = m = 0$, if the brane has a critical tension

$$\sigma_c = \frac{6}{R},$$

the brane does not move and the scale factor of the brane world becomes time-independent

2. Eternal inflationary era

For an AdS space with $\mathcal{E} = m = 0$, if the brane has a non-critical tension

$$\frac{dr}{d\tau} = \sqrt{\frac{\sigma^2}{36} - \frac{1}{R^2}}\, r.$$

the radial motion of the brane determines the cosmology on the brane

$$r(\tau) = r_i\, e^{H\tau},$$

with a Hubble constant

$$H = \frac{\sqrt{\sigma^2 - \sigma_c^2}}{6},$$

where r_i is the position of the brane at $\tau = 0$.

3. Matter-dominated era

Taking $\sigma = \sigma_c$ and $m = 0$, the open string's density determines the radial motion of the brane

$$\frac{dr}{d\tau} = \frac{\sqrt{\mathcal{E}}}{R} \frac{1}{r^{1/2}}$$

In this case, the scale factor results in

$$r(\tau) = \left(\frac{3}{2}\right)^{2/3} \frac{\mathcal{E}^{1/3}}{R^{2/3}}\, \tau^{2/3} + r_i,$$

4. Radiation-dominated era

In the string cloud geometry with $\sigma = \sigma_c$ and $\mathcal{E} = 0$, the brane's radial motion is determined by

$$\frac{dr}{d\tau} = \frac{\sqrt{m}}{R}\frac{1}{r}$$

The solution of this junction equation is

$$r(\tau) = \frac{\sqrt{2} m^{1/4}}{\sqrt{R}} \tau^{1/2} + r_i .$$

which corresponds to the radiation-dominated era, as expected.

Now, let us consider the time-dependent entanglement entropy in the above expanding universes

- To do so, we introduce a new coordinate $z = R^2/r$
- It is worth noting that the RT formula is sufficient to calculated the time-dependent entanglement entropy in the braneworld model, because the bulk geometry is time-independent.
- Time-dependence of the entanglement entropy comes from the time-dependent boundary condition.

Time-independent entanglement entropy in a static geometry

For a five-dimensional AdS space

$$ds^2 = \frac{R^2}{z^2}\left(dz^2 - dt^2 + du^2 + u^2 d\Omega_2^2\right)$$

- Divide the system into two parts, a subsystem and its complement, and parameterize the subsystem size as $0 \leq u \leq l$

- The entanglement entropy is determined by

$$S_E = \frac{R^{d-1}\Omega_2}{4G}\int_0^l du \, \frac{u^2\sqrt{1+z'^2}}{z^3},$$

- The equation governing the minimal surface configuration is

$$0 = 1 + z'^2 + zz''$$

- The minimal surface configuration satisfies

$$z(u) = \sqrt{l^2 + \bar{z}^2 - u^2}$$

where $\bar{z}$ indicates the position of the boundary.

- Finally, we obtain a time-independent entanglement entropy

$$S_E = \frac{R^3}{12G}\frac{l^3\Omega_2}{\bar{z}^2\sqrt{l^2+\bar{z}^2}}\, {}_2F_1\left(\frac{1}{2}, 1; \frac{5}{2}; \frac{l^2}{l^2+\bar{z}^2}\right)$$

3. **Time-dependent entanglement entropy in expanding universes**

1. Time-dependent entanglement entropy in the inflating universe

For an eternally inflating universe in the braneworld model, the bulk geometry is given by the same AdS geometry. Thus, the previous time-independent solution is still applicable. However, the brane moves in the braneworld model, we impose a time-dependent boundary condition

$$z(\tau, u) = \sqrt{l^2 + \bar{z}(\tau)^2 - u^2}.$$

- The brane, which plays a role of the boundary for the minimal surface, moves in the braneworld model

$$\bar{z}(\tau) = \frac{R^2}{\bar{r}(\tau)} = \frac{R^2}{r_0}\, e^{-H\tau}.$$

- In the late time era of the eternal inflation, the time-dependent entanglement entropy finally reduces to

$$S_E^{(inf)} = \frac{l^2 r_0^2 \Omega_2 e^{2H\tau}}{8GR} + \frac{R^4 \Omega_2 \left(1 + 2\log(2R^2/l r_0)\right)}{16GR} - \frac{H R^3 \Omega_2 \tau}{8G} + \mathcal{O}\left(e^{-2H\tau}\right).$$

- This result is consistent with the one obtained in the previous dS boundary model.

2. Entanglement entropy in the radiation- and matter-dominated eras

The entanglement entropy is given by

$$S_E = \frac{R^3\Omega_2}{4G}\int_0^l du\, \frac{u^2\sqrt{f+z'^2}}{z^3\sqrt{f}}$$

where

$$f(z) = 1 - \tilde{\mathcal{E}} z^3 - \tilde{m} z^4 \quad \text{with} \quad \tilde{\mathcal{E}} \equiv \frac{\mathcal{E}}{R^6}, \text{ and } \tilde{m} \equiv \frac{m}{R^8}.$$

2-1. In the radiation-dominated era

- The brane position is determined as

$$\bar{z} = \frac{z_h z_i}{z_h + \sqrt{2} z_i \sqrt{\tau}},$$

- In the late time era, the entanglement entropy increases linearly with time

$$S_E = \frac{cl^2\Omega_2}{6z_h^2}\tau + \frac{cl^2\Omega_2}{3\sqrt{2}z_h z_i}\sqrt{\tau} - \frac{c\Omega_2}{24}\log\tau + \frac{1}{120}c\Omega_2\left[2l^4\tilde{m} + \frac{10l^2}{z_i^2} + 10\log\left(\frac{z_h}{2\sqrt{2}l}\right) + 5\right] + \mathcal{O}\left(\frac{1}{\sqrt{\tau}}\right)$$

2-2. In the matter-dominated era

- The brane position is determined to be

$$\bar{z} = \frac{z_i}{1 + (3/2)^{2/3} z_i \, \tilde{\mathcal{E}}^{1/3} \, \tau^{2/3}}.$$

- In the late time era, the entanglement entropy increases by $\tau^{4/3}$

$$S_E = \left(\frac{3}{2}\right)^{1/3} \frac{c\Omega_2 l^2}{8} \tilde{\mathcal{E}}^{2/3} \tau^{4/3} + \frac{c\Omega_2 l^2 \tilde{\mathcal{E}}^{1/3}}{2^{5/3} \times 3^{1/3} z_i} \tau^{2/3}$$
$$- \frac{c\Omega_2}{18} \log \tau + \frac{c\Omega_2}{24} + \frac{c\Omega_2 l^2}{12 z_i^2} - \frac{c\Omega_2 \log\left(18 \tilde{\mathcal{E}} l^3\right)}{36} + \frac{7\pi c\Omega_2 \tilde{\mathcal{E}} l^3}{256} + \mathcal{O}\left(\tau^{-2/3}\right).$$

4. Discussion

- We have studied the time-dependent entanglement entropy in expanding universes by applying two different holographic models

- Despite the fact that they are casually disconnected, the quantum correlation leads to non-vanishing entanglement entropy, which increases with time in expanding universes:

  - For eternal inflation, $S_E \sim e^{2H\tau}$

  - In the radiation-dominated era, $S_E \sim \tau$

  - In the matter-dominated era, $S_E \sim \tau^{4/3}$

THANK YOU !