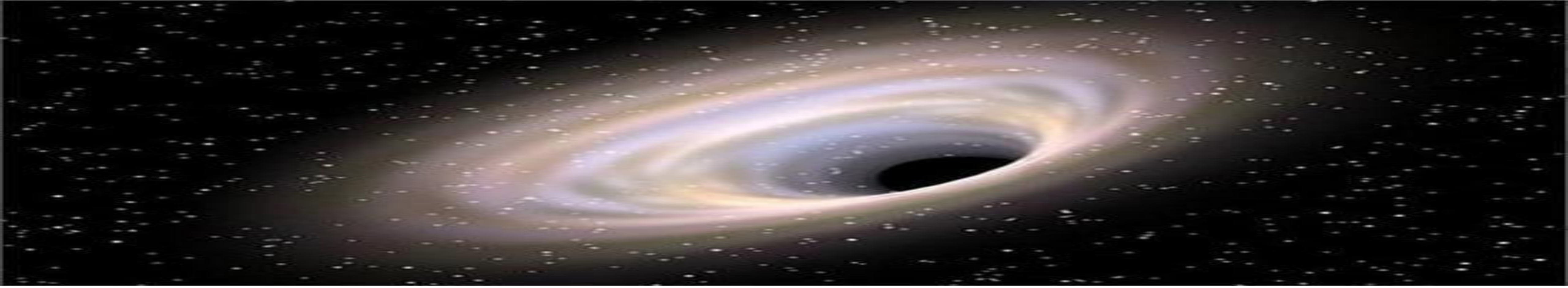

Upper bound about cross-sections inside black holes

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Based on arXiv:1911.12561

Content

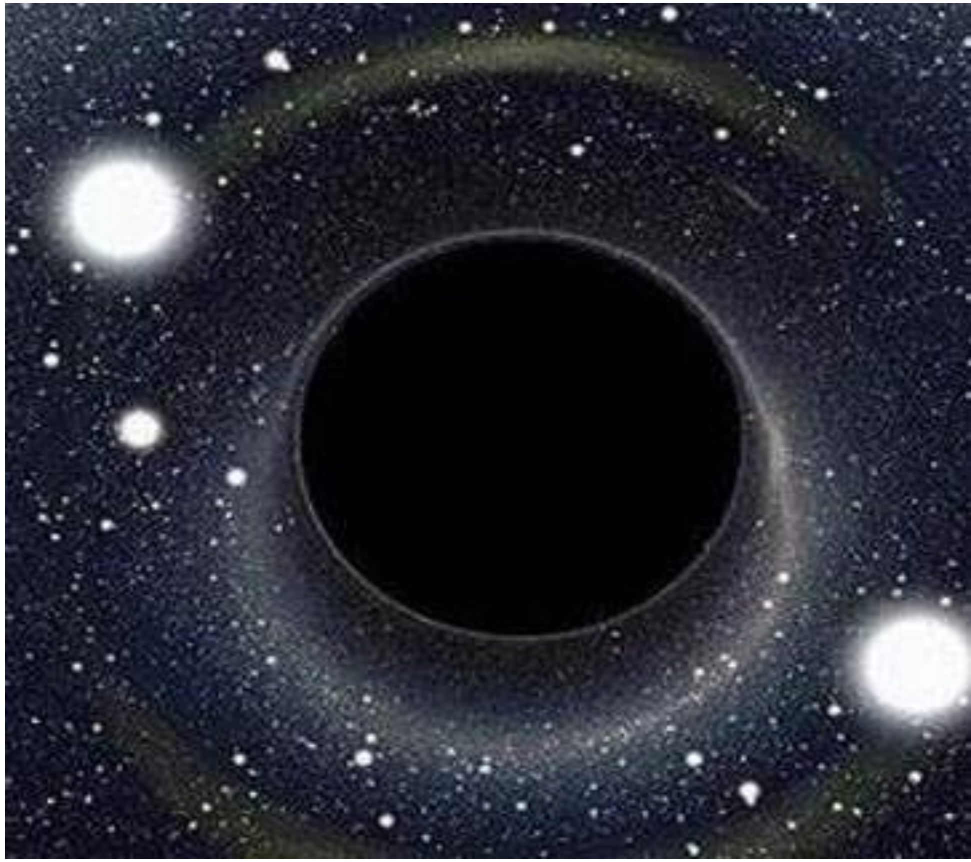
- Background and motivation;
- Examples in RN black hole and BTZ black hole;
- Relationship to the complexity growth rate;
- Overview on the proof;
- Discussion and summary.

Background and motivation

- Black holes, as ultra dense objects in universe, exhibit many fascinating physical and mathematical properties.

- many such properties can be presented by some universal inequalities；

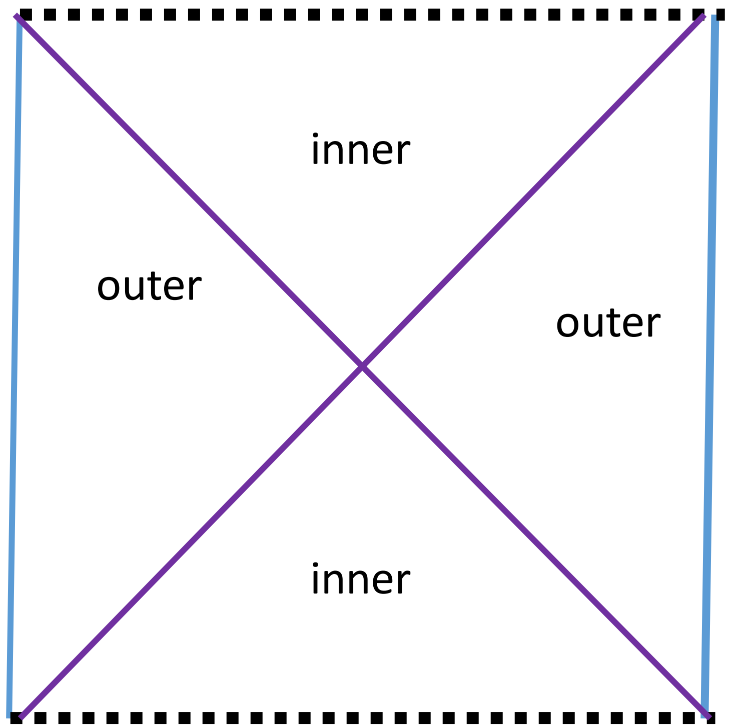
- Positive mass theorem M≥0;
- Second law of black holes $\delta A_H \geq 0$
- Penrose inequality $M \geq \sqrt{A_H/4\pi}$;



Deepen our understanding about foundations of gravity

Background and motivation

- Almost all such studies involve only the horizon and its outside;
- However, recent developments in AdS/CFT shows that the inner region also has boundary correspondence;
- Complexity growth rate in a Schwarzschild-AdS black hole is determined completely by the inner geometry



Is there any universality inside black holes?

Background and motivation

- The simplest object is the volume of interior.

- Consider a (d + 1)-dimensional Schwarzschild-AdS black hole with metric

$$\mathrm{d}s^2 = \frac{1}{z^2}\left[-f(z)\mathrm{d}t^2 + \frac{\mathrm{d}z^2}{f(z)} + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right]$$

- Inside the black hole, t is spatial coordinate, but z is time. The equal-z space has volume

$$V = \int dt V_{d-1} z^{-d}\sqrt{-f(z)}$$

- This is infinite as t is the orbit of a Killing vector field!

Background and motivation

- However, the infinity is trivial become t is a Killing vector;

$$V = \int dt V_{d-1} z^{-d} \sqrt{-f(z)}$$

Integration with respective to t
causes the divergence

- Let us define a size of cross-section

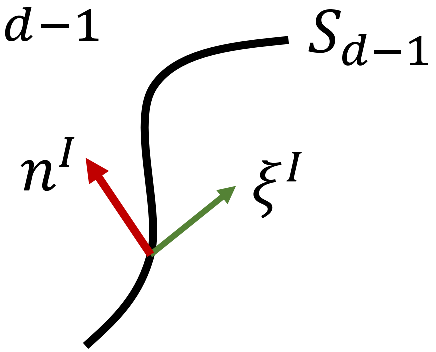
$$\Xi = V_{d-1} z^{-d} \sqrt{-f(z)}$$

- After a few of simple algebras, we can find that

$$0 \leq \Sigma \leq 8\pi E \ell_{\mathrm{AdS}}/(d-1)$$

Definition of the "size of cross-section'' inside black holes;

- Consider a (d+1) dimensional stationary black hole with a globally defined Killing vector ξ^I which is timelike outside Killing horizon. S_{d-1} is a spacelike co-dimensional 2 surface inside Killing horizon.



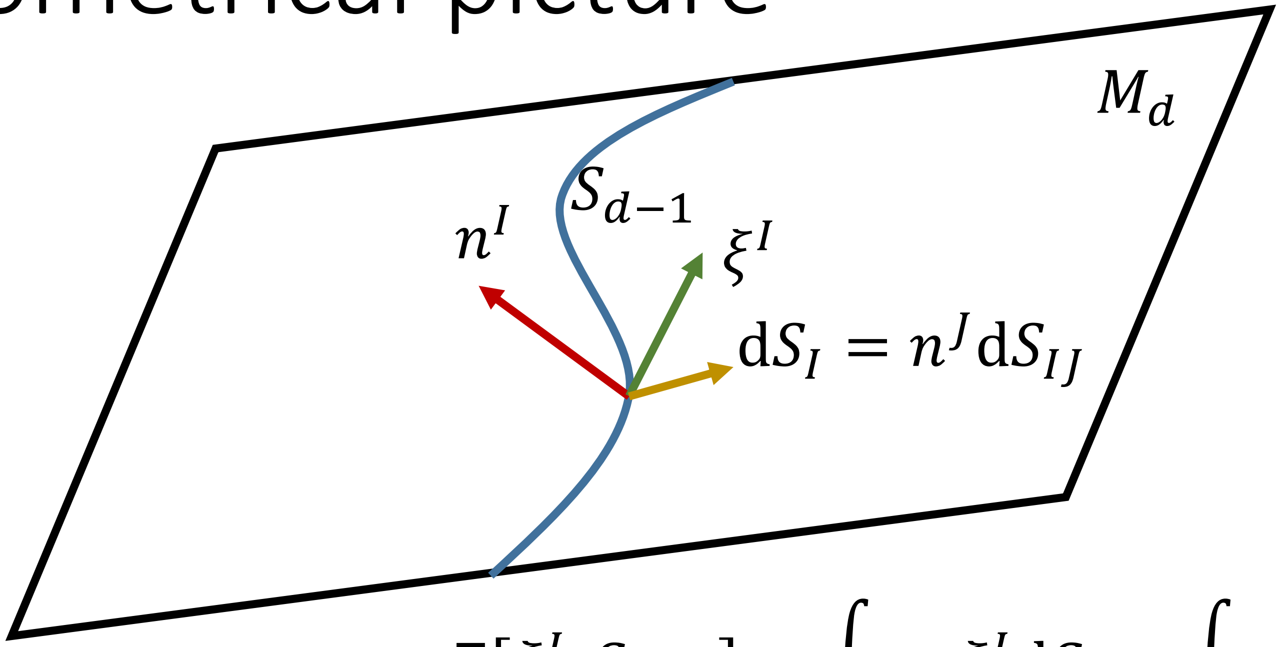
$$\Xi[\xi^I, S_{d-1}] = \int_{S_{d-1}} \xi^I n^J \mathrm{d}S_{IJ}$$

Here n^I is unit timelike vector which is normal to S_{d-1} and $n^I \xi_I = 0$.

Determines the unit normal vector n^I uniquely up to an orientation

- We only consider the maximally extended cross-section, i.e., the cross-section which is not a real subset of any other cross-section.

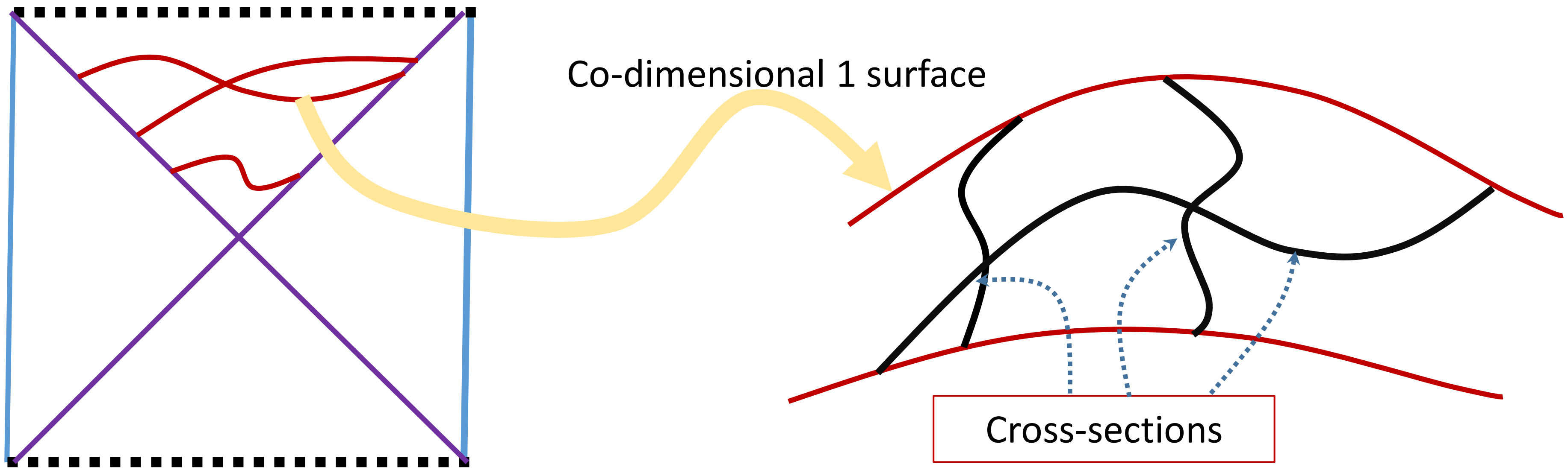
Geometrical picture



$$\Xi[\xi^I, S_{d-1}] = \int_{S_{d-1}} \xi^I \mathrm{d}S_I = \int_{S_{d-1}} \vec{\xi} \cdot \mathrm{d}\vec{S}$$

- Geometrically, Σ stands for the flux of vector field ξ^I on the surface S_{d-1}.

Bound on the size of cross-section



- For arbitrary co-dimensional 1 hypersurface inside a stationary black hole which is asymptotically spherical/planar symmetric, and for its arbitrary cross-section, we conjecture that if dominate energy condition is satisfied, the cross-section will satisfy

$$0 \leq \Sigma \leq \frac{8\pi E \ell_{AdS}}{d-1}$$

In general coordinates gauge

- In a general stationary (d + 1)-dimensional spacetime, the metric can locally be expressed as

$$\mathrm{d}s^2 = \frac{1}{z^2}[-f\mathrm{d}t^2 + \chi\mathrm{d}z^2 + 2v_i\mathrm{d}t\mathrm{d}x^i + h_{ij}\mathrm{d}x^i\mathrm{d}x^j]$$

- A general cross-section inside black hole then can be parameterized by

$$t = t_S(x^i), z = z_S(x^i)$$

- Then after some algebras, we can find

$$\Sigma[\xi^I, \mathcal{S}_{d-1}] = \int_{z=z_S(x^i)} z^{-d}\sqrt{-f-|v|^2}\sqrt{\tilde{h}}\mathrm{d}^{d-1}x\,.$$

$$\tilde{h}_{ij} := h_{ij} + \chi\partial_i z_S \partial_j z_S$$

Example 1: Spherical/planar Schwarzschild-AdS black hole

- In SAdS black hole, we have $\chi = f^{-1}, v^i = 0$.

$$f = kz^2 + \frac{1}{\ell_{\text{AdS}}^2} - f_0 z^d \qquad M = \frac{(d-1)V_{d-1}}{16\pi} f_0$$

- The maximum size of cross-section is expressed in terms of

$$\max \Sigma = V_{d-1}\sqrt{\max P_d(z)}$$

$$P_d(z) := -[kz^2 + \frac{1}{\ell_{\text{AdS}}^2} - f_0 z^d] z^{-2d}$$

- After some algebras, we find

$$\max P_d(z) - \frac{\ell_{\text{AdS}}^2 f_0^2}{4} = P_d(z_m) - \frac{\ell_{\text{AdS}}^2 f_0^2}{4}$$
$$= -\frac{\ell_{\text{AdS}}^2 f_0^2}{4} \frac{[kz_m^2 \ell_{\text{AdS}}^2 (d-1)^2 + d^2] k z_m^2 \ell_{\text{AdS}}^2}{(dk\ell_{\text{AdS}}^2 z_m^2 + d - k\ell_{\text{AdS}}^2 z_m^2)^2}$$

$\Rightarrow$

If k≥0， we find

$$\max \Sigma \le \frac{\ell_{AdS} V_{d-1} f_0}{2} = 8\pi M \ell_{AdS}/(d-1)$$

Example 2: RN black hole

- In the second example, we consider planar or spherical RN black hole

$$f = kz^2 + 1/\ell_{\mathrm{AdS}}^2 - f_0 z^d + \tilde{q} z^{2d-2}$$

- We still have $\max \Sigma = V_{d-1}\sqrt{\max P}$, where $P = f z^{-2d}$
- Assume that $z_m(\tilde{q})$is the point which maximizes P. We then have

$$f'(z_m, \tilde{q}) z_m - 2df(z_m, \tilde{q}) = 0 \qquad \max P_d = \mathcal{P}(\tilde{q}) := -f(z_m(\tilde{q}), \tilde{q}) z_m^{-2d}(\tilde{q})$$

- After some simple algebras, we can find

$$\mathrm{d}\mathcal{P}(\tilde{q})/\mathrm{d}\tilde{q} = -z_m(\tilde{q})^{-2d} \partial_{\tilde{q}} f|_{z_m = z_m(\tilde{q})} = -z_m^{-2}(\tilde{q}) < 0$$

- This shows that the charge only decreases the maximum size of cross-section inside black hole!

$$\forall\ \tilde{q} \geq 0, \quad \mathcal{P}(\tilde{q}) \leq \mathcal{P}(0) \leq \frac{\ell_{\mathrm{AdS}}^2 f_0^2}{4} \quad \Longrightarrow \quad \max \Sigma \leq \frac{8\pi M \ell_{AdS}}{d-1}$$

Example 3: BTZ black hole

- In the third example, we consider a non-static stationary black hole. The simplest one is the BTZ black hole

$$\mathrm{d}s^2 = \frac{1}{z^2}\left[-\tilde{f}(z)\mathrm{d}t^2 + \frac{\mathrm{d}z^2}{\tilde{f}(z)} + (\mathrm{d}\phi - Jz^2\mathrm{d}t/2)^2\right] \qquad \tilde{f}(z) = 1/\ell_{\mathrm{AdS}}^2 - f_0 z^2 + J^2 z^4/4$$

- Then we see

$$\mathrm{d}s^2 = \frac{1}{z^2}[-f\mathrm{d}t^2 + \chi\mathrm{d}z^2 + 2v_i\mathrm{d}t\mathrm{d}x^i + h_{ij}\mathrm{d}x^i\mathrm{d}x^j]$$

$$v_i = v = Jz^2/2, \quad f = \tilde{f} - J^2z^4/4 = \tilde{f} - v^2$$

- The size of cross-section then reads

$$\Sigma = \int_{z=z_S(\phi)} z^{-2}\sqrt{-f - v^2}\mathrm{d}\phi = \int z_S^{-2}\sqrt{f_0 z_S^2 - 1/\ell_{\mathrm{AdS}}^2 - J^2 z_S^4/4}\mathrm{d}\phi$$

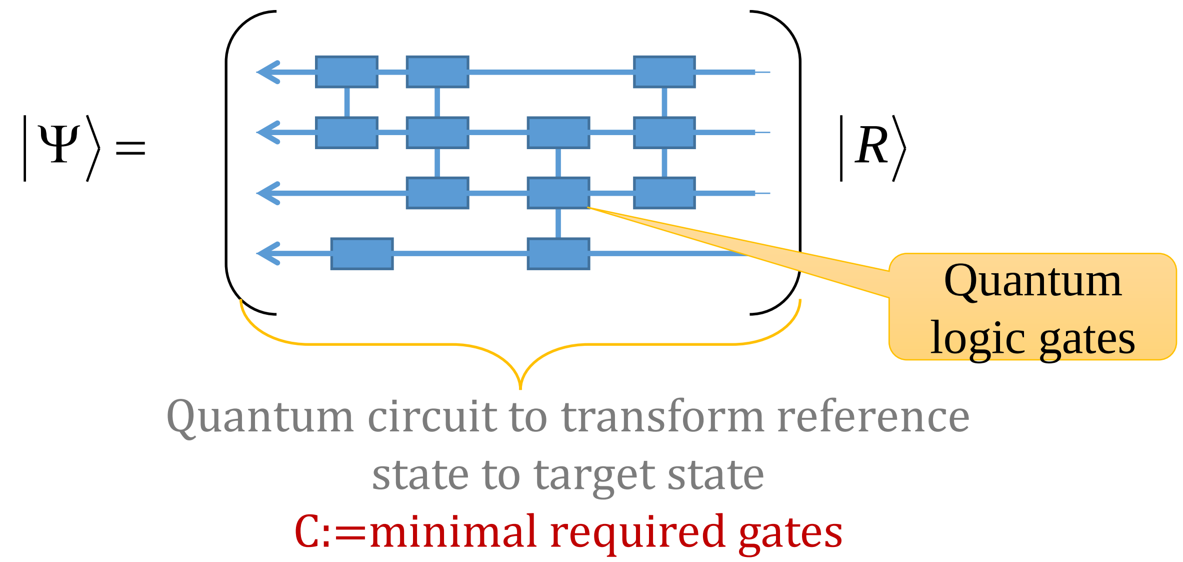
$$\Rightarrow$$

$$\max\Sigma = \int\sqrt{\ell_{\mathrm{AdS}}^2 f_0^2/4 - J^2/4}\mathrm{d}\phi = V_1\sqrt{\ell_{\mathrm{AdS}}^2 f_0^2/4 - J^2/4} \leq 8\pi\ell_{\mathrm{AdS}}M$$

Angular momentum only decrease the maximal size of cross-section!

Relationship to holographic complexity growth rate

- A quantum state can be converted into an other state by a quantum circuit.



- Complexity between two states is defined as the minimal required gates when we finish this transformation by quantum circuits.

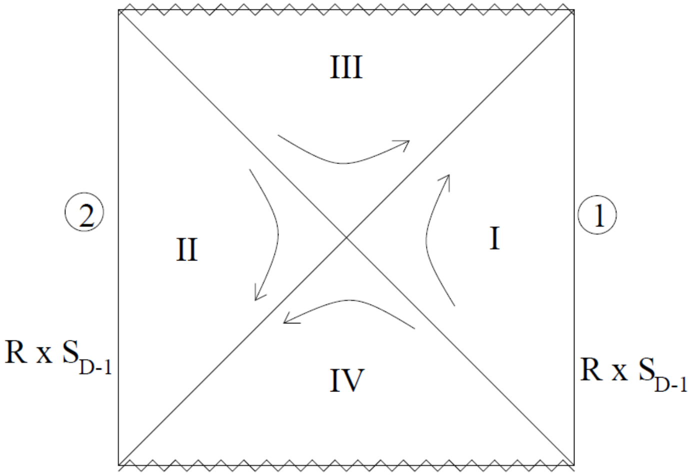
- Complexity growth rate stands for the speed of quantum computation

Background and main results

- The thermofield double state is two copies of the CFT;

$$|\Psi\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta E_n/2} |E_n\rangle_1 \times |E_n\rangle_2$$

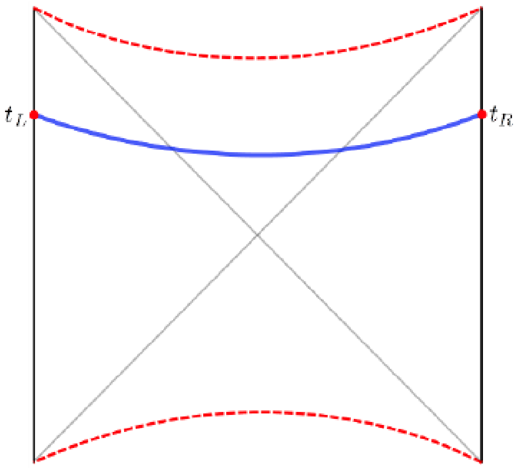
- It is approximately described by gravity on the eternal AdS Schwarzschild spacetime



Background and main results

CV (complexity-volume)

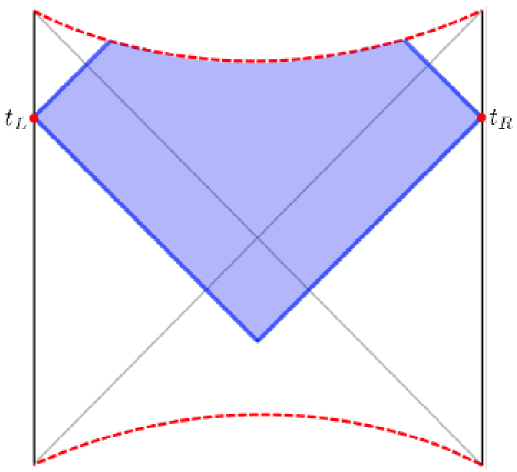
[Susskind: 1402.5674
Stanford and Susskind: 1406.2678]



$$\mathcal{C}_V = \max_{\partial\Sigma = t_L \cup t_R} \left[\frac{V(\Sigma)}{G_N \ell}\right]$$

CA (complexity-action)

[Brown, Roberts, Susskind Swingle and Zhao:
1509.07876, 1512.04993]



$$\mathcal{C}_A = \frac{I_{\mathrm{WDW}}}{\pi\hbar}$$

The CA conjecture was once favored due to

$$\dot{C}(\infty) \leq \frac{2E}{\pi}$$

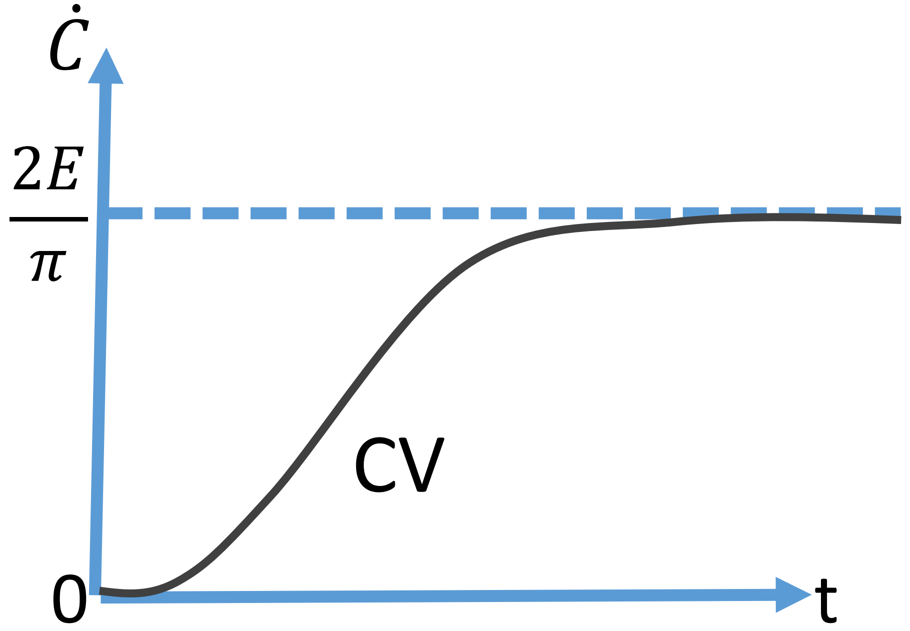
This is just the Lloyd bound.

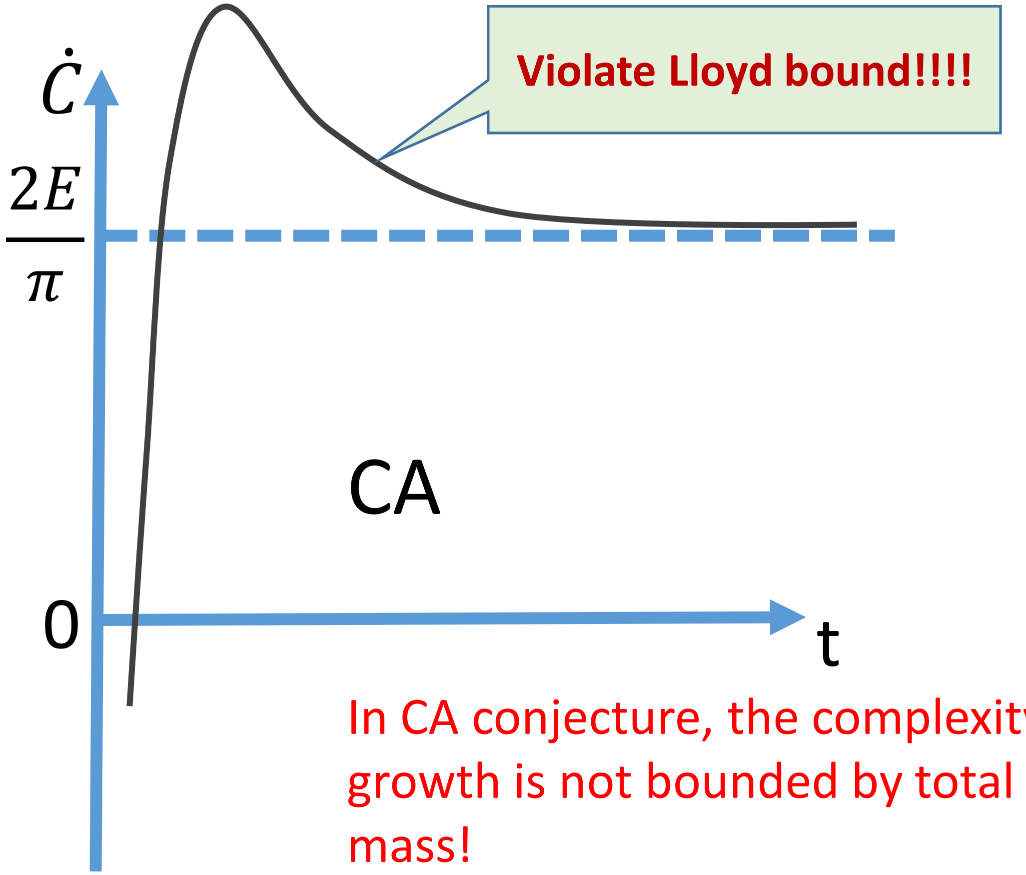
We take $\ell = \frac{4\pi^2 \ell_{AdS}}{d-1}$

Background and main results

- Complexity growth rate in CA and CV conjecture

arXiv:1709.10184, arXiv:1710.00600

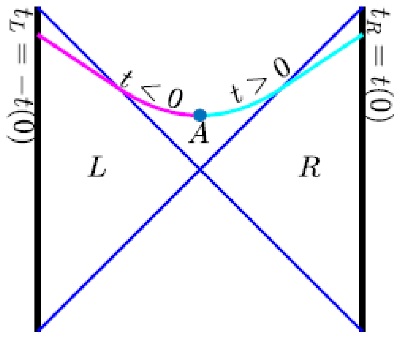




In CA conjecture, the complexity growth is not bounded by total mass!

- We can show that complexity growth rate in CV conjecture is given by the size of a special cross-section!

Relationship to the complexity growth rate



- The spacelike surface is parameterized by

$$t = t(s, x^i), \qquad z = z(s, x^i)$$

- Using Hamilton-Jacobi equation, we can show that the cc growth rate is given by canonical momentum of t;

$$\dot{C} = \frac{d-1}{4\pi^2 \ell_{AdS}} \left(\frac{\partial V_{on-shell}}{\partial t} \right)_{s=\infty} = \frac{d-1}{4\pi^2 \ell_{AdS}} P_t(\infty)$$

Hamilton-Jacobi equation

$$p = \frac{\partial S_{on-shell}}{\partial q}$$

- The volume functional does not depends on t explicitly; Momentum is conserved and so we can compute it at “time” s=0

$$\dot{C} = \frac{d-1}{4\pi^2 \ell_{AdS}} P_t(0)$$

Assume that W_d is the co-dimensional-one surface connecting two boundaries of an AdS black hole

The induced metric on W_d reads

Parameterize co-dim one surface W_d by
$t = t(x^i), z = z(x^i)$

$$\mathrm{d}s_W^2 = \frac{1}{z^2}[-f(t'\mathrm{d}s + \partial_i t \mathrm{d}x^i)^2 + \chi(z'\mathrm{d}s + \partial_i z \mathrm{d}x^i)^2 + 2v_i(t'\mathrm{d}s + \partial_j t \mathrm{d}x^j)\mathrm{d}x^i + h_{ij}\mathrm{d}x^i\mathrm{d}x^j] \,.$$

Here " $'$ " stand for the partial derivative with respect to s. Now we define

$$N = -ft'^2 + \chi z'^2, \ N_i = v_i t' + \chi z' \partial_i z - f t' \partial_i t \,,$$

and

$$\mathfrak{h}_{ij} = -f\partial_i t \partial_j t + \chi \partial_i z \partial_j z + 2v_{(i}\partial_{j)} t + h_{ij}$$

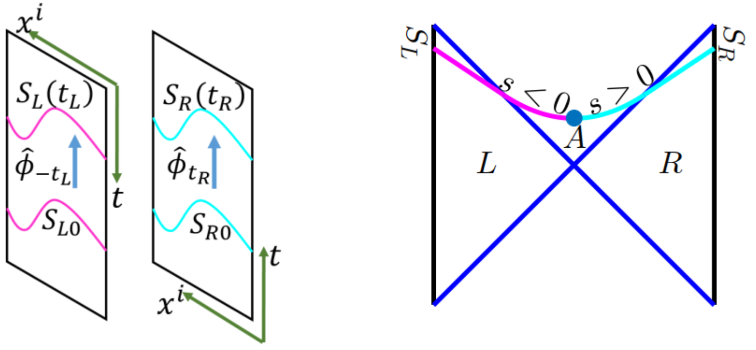
Then we have

$$\mathrm{d}s_W^2 = \frac{1}{z^2}[N\mathrm{d}s^2 + 2N_i \mathrm{d}s\mathrm{d}x^j + \mathfrak{h}_{ij}\mathrm{d}x^i\mathrm{d}x^j] \,.$$

The volume functional of W_d now reads

$$V[W_d] = \int z^{-d}\sqrt{N - \mathfrak{h}^{ij}N_i N_j}\sqrt{\mathfrak{h}}\mathrm{d}^{d-1}x\mathrm{d}s \,.$$

We take two boundary slices symmetrically



$$t(s,x^i)=-t(-s,x^i),\, z(s,x^i)=z(-s,x^i)$$

$$z'(0,x^i)=\partial_i t(0,x^i)=0$$

$$\mathfrak{h}_{ij}|_A=\tilde{h}_{ij},\quad N|_A=-ft'^2,\quad N_i|_A=v_i t'$$

Note that only N and N_i depend on the value of t'. The above volume functional is an analog of action functional. The two variables $t(s,x^i)$ and $z(s,x^i)$ are two "fields" and parameter s plays the role of "time". Thus the canonical momentum conjugate to "field" t reads

$$\begin{aligned}\mathcal{P}(s_0)&=\int_{s=s_0}\frac{\partial}{\partial t'}\left[\sqrt{N-\mathfrak{h}^{ij}N_iN_j}\frac{\sqrt{\mathfrak{h}}}{z^d}\right]\mathrm{d}^{d-1}x\\&=\int_{s=s_0}\frac{\frac{\partial N}{\partial t'}-2\mathfrak{h}^{ij}N_i\frac{\partial N_j}{\partial t'}}{2\sqrt{N-\mathfrak{h}^{ij}N_iN_j}}\frac{\sqrt{\mathfrak{h}}}{z^d}\mathrm{d}^{d-1}x\,.\end{aligned}$$

$$\frac{\partial N}{\partial t'}=-2ft',\quad \frac{\partial N_j}{\partial t'}=-f\partial_i t+v_j\,.$$

Thus, we have

$$\mathcal{P}(s_0)=\int_{s=s_0}\frac{-ft'-\mathfrak{h}^{ij}N_i(v_j-f\partial_i t)}{\sqrt{N-\mathfrak{h}^{ij}N_iN_j}}\frac{\sqrt{\mathfrak{h}}}{z^d}\mathrm{d}^{d-1}x$$

$$\ell\dot{\mathcal{C}}=\mathcal{P}(0)=\int_A z^{-d}\sqrt{-f-|v|^2}\sqrt{\tilde{h}}\mathrm{d}^{d-1}x=\Sigma[\xi^I,A]$$

The complexity growth rate is bounded by maximal size of cross-section!

$$\dot{C}\leq\frac{2E}{\pi}$$

$\Updownarrow$

$$\Sigma\leq\frac{8\pi M\ell_{AdS}}{d-1}$$

A general conclusion on the bound

For a stationary black hole, if

(i) outermost horizon is connected Killing horizon and has positive surface gravity,

(ii) the spacetime is asymptotically spherical/planar Schwarzschild-AdS, and

(iii) dominate energy condition and Einstein equation are satisfied,

then following inequality is always true

$$\Sigma \leq 8\pi M \ell_{AdS}/(d-1)$$

As a corollary, the complexity growth rate in CV conjecture satisfies Lloyd bound

$$\dot{C} \leq 2E/\pi$$

The proofs for spherical/planar symmetric case in arbitrary dimension and general case in 4-dimensional case are shown in detailed in arXiv:1911.12561

A few of discussions

- If there is a next-outermost horizon, the cross-section will locate between two horizons;
- In the zero-temperature limit, i.e. two horizons coincident with each other, the size of cross-section is zero. For example, in RN black hole

$$\Xi = V_{d-1} z^{-d} \sqrt{-f(z)}, \qquad z \in (z_{h_1}, z_{h_2})$$
$$\Xi \to 0, \qquad if\ z_{h_1} \to z_{h_2}$$

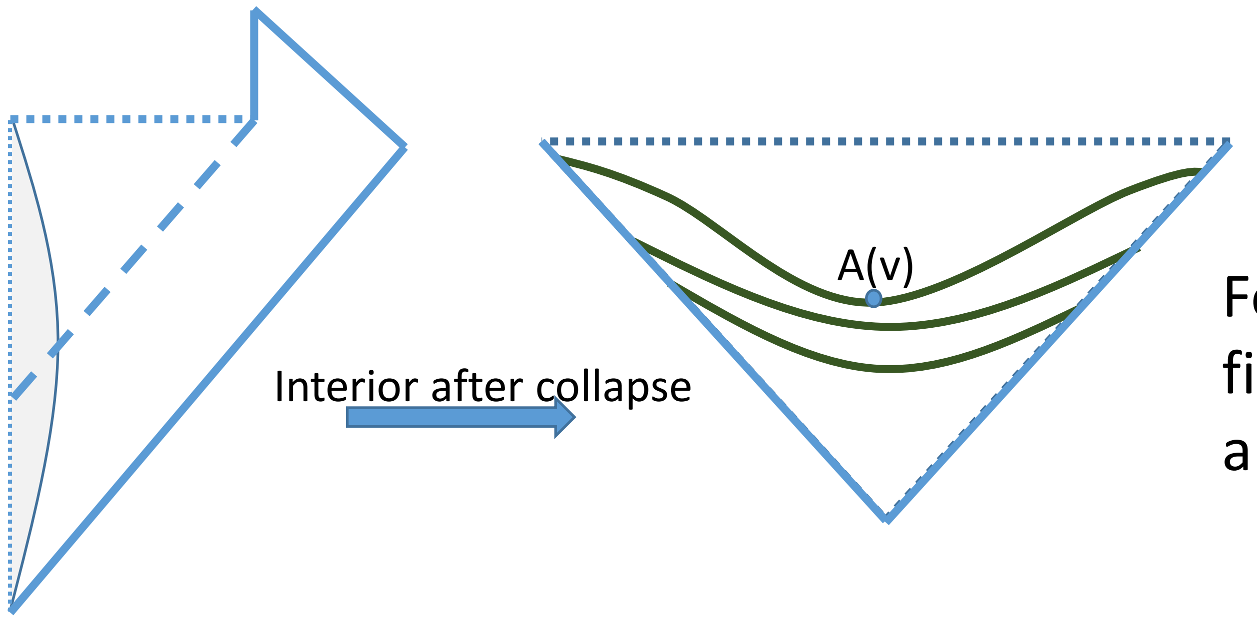
- But the mass be arbitrary large. This means in low temperature limit, there is tighter upper bound given by temperature.
- For BTZ black hole, direct computation shows $\Xi \leq 4\pi\, T_H S \ell_{AdS}$. What is in general cases? Complexity growth rate is also bounded by temperature?

Interior volume

- In asymptotic flat spacetime, we can also obtain a similar but different bound

$$\Sigma \leq 4a_d S \leq c_d V_{1,d-1} (E/V_{1,d-1})^{(d-1)/(d-2)}$$

- A black hole which is formed by collapse and disappears due to Hawking radiation, its maximal interior volume will be finite.



For large mass black hole, after the collapse finishes, spacetime is stationary approximately, it turns out to be

$$\dot{V}_{max} = \Sigma[A(v)] \leq \tilde{c}_d E^2$$

Interior volume

- Consider 4D asymptotically flat case:

Evaporating time order (order of stationary time)

$$\frac{t_{ve}}{t_p} \sim O\left(\frac{M^3}{M_P^3}\right)$$

t_p, M_p and V_p are Planck time, mass and volume.

Maximal interior volume growth rate

$$\dot{V} \sim \frac{V_p}{t_p} \times O\left(\frac{M^2}{M_P^2}\right)$$

- Thus, we find the maximal interior volume which appears in an evaporated black hole has order

$$V_{max} \sim O\left(\frac{M^5}{M_P^5}\right) \times V_p$$

For a black hole of mass$\sim 10^{28}$kg, it a small black hole in the sense horizon area≈9km^2. But its maximal interior volume≈10^{66}km^3 (huge!)

Area radius≈3km, Volume radius≈10^{22}km

What does it mean in physics?

Summary

- A universal inequality about the geometry inside black hole was proposed;
- This inequality has important application in holographic complexity and agrees with argument of quantum information;
- This makes a first step towards the holographic proof on the conjecture that vacuum black holes is the fastest “computers” in nature;
- It gives us an estimation on the largest interior volume of a large evaporating black hole.