Statistical Methods in Nuclear Physics
: Practical Approach

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(Asia Pacific Center for Theoretical Physics)

Outlines

- Neutron Stars
- LDM
  - Root Mean Square Deviation, Lagrange Multiplier Method
- Energy density functional
- Tolmann Oppeheimer Volkov (T.O.V) equations
- Bayesian Statistics
  - Conditional probabilities
- Application to nuclear matter and neutron stars
  - Mass and radius according to the observational constraints

Neutron stars

Figure: Cassiopeia A is among the best-studied supernova remnants. This image blends data from NASA's Spitzer (red), Hubble (yellow), and Chandra (green and blue) observatories. NASA/JPL-CALTECH/STSCI/CXC/SAO

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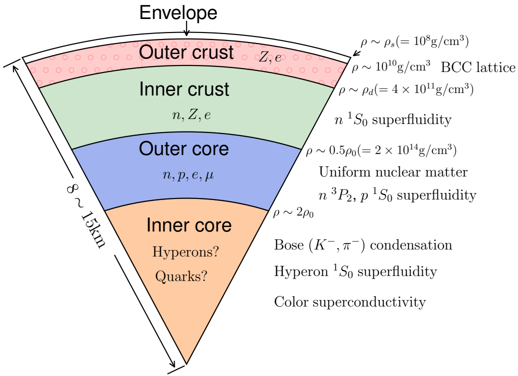
- Inner structure of neutron stars



- Neutron Stars:
  - Dense nuclear matter physics

- TOV equations for macroscopic structure
  (spherically symmetric non-rotating NS)

$$
\begin{aligned}
\frac{dp}{dr} &= -\frac{G(M(r) + 4\pi r^3 p/c^2)(\epsilon + p)}{r(r - 2GM(r)/c^2)c^2}, \\
\frac{dM}{dr} &= 4\pi \frac{\epsilon}{c^2} r^2,
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- Macroscopic quantities
  r; distance from the center
  $M(r)$; enclosed mass from the center
- Microscopic quantities (Nuclear physics)
  p ; pressure
  ϵ; energy density

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Truth

You have to know that 'if someone can do it, you can do it'.

Nuclear Binding energy

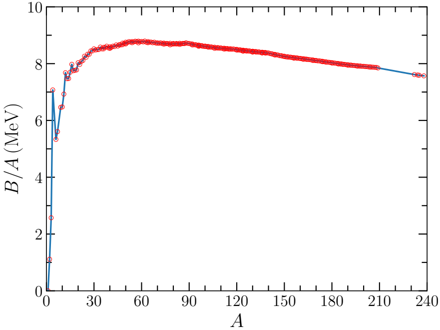

Figure: Binding energy per nucleon for stable nuclei

Nuclear Binding energy

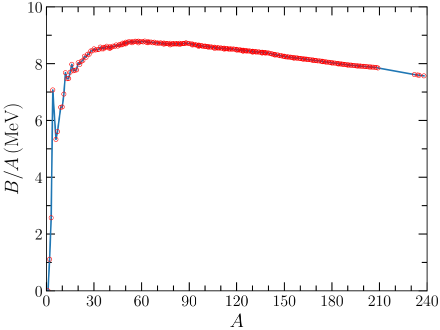

Figure: Binding energy per nucleon for stable nuclei

- There many methods to calculate binding energy of finite nuclei.
  - Liquid drop model, Thomas Fermi, Skyrme Hartee-Fock, Relativistic Mean field model, No-core shell, Quantum Monte Carlo, . . .

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- Fitting function(?)

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- Origin of B, E_S, E_C
  - B : binding energy of bulk matter
  - E_S : Surface energy
  - E_C : Coulomb

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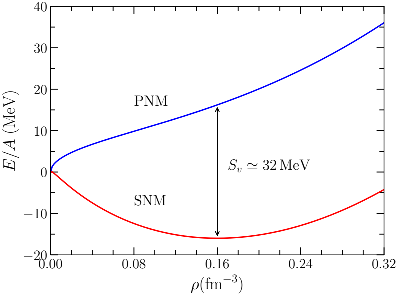
  Pure neutron matter and Symmetric nuclear matter show the different E/A.



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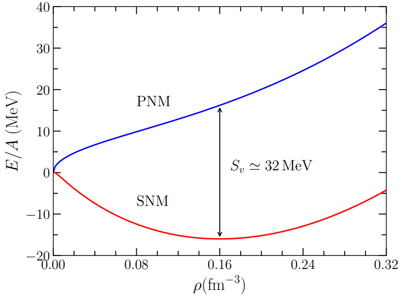

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E(n,x) &= E_{SNM} + (1-2x)^2 S(n) \\
E_{SNM} &= E(n, x = \frac{1}{2}) \\
E_{PNM} &= E(n, x = 0) \\
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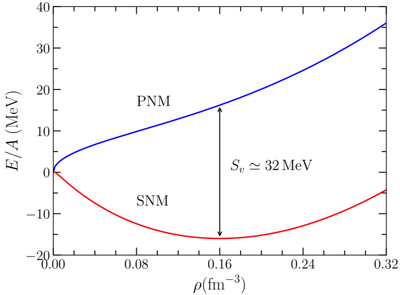

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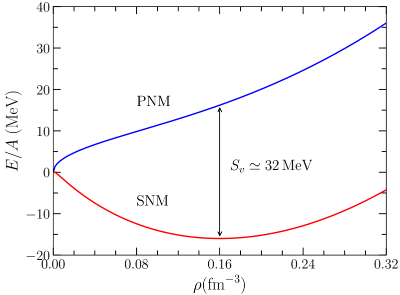

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$$-BA + E_{SYM} \frac{(N-Z)^2}{A} = \left[-B + E_{SYM}(1-2x)^2\right] A \quad (6)$$

- Correction term : Pairing Energy
  Even-odd Staggering $S_n = B(N, Z) - B(N-1, Z)$

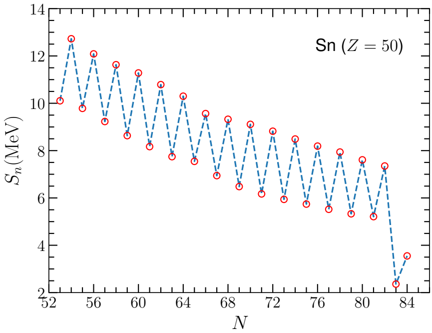

Figure: One neutron separation energy of Sn isotopes

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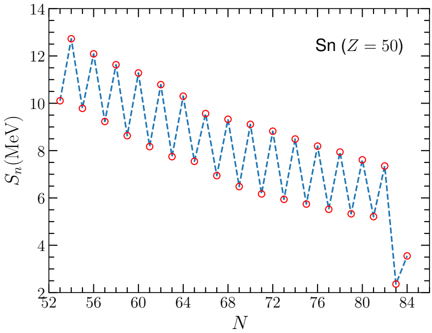

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$$E = -BA + E_S A^{2/3} + E_C \frac{Z^2}{A^{1/3}} + E_{SYM} \frac{(N-Z)^2}{A} + A_p \frac{\Delta}{\sqrt{A}} \qquad (7)$$

$A_p = -1$ for even-even, $A_p = 0$ for even-odd, and $A_p = 1$ for odd-odd nuclei.

- Correction term : Shell corrections

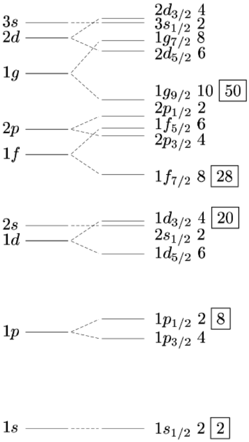

Figure: Single particle energy level (wikipedia)

Low-lying energy levels in a single-particle shell model with an oscillator potential without spin-orbit (left) and with spin-orbit (right) interaction.

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right]\psi(\mathbf{x}) = E\psi(\mathbf{x}) \qquad (8)$$

$$U(\mathbf{x}) = \begin{cases} \frac{1}{2}k\mathbf{x}^2 \\ -\frac{U_0}{1+e^{(r-R)/a}} \end{cases} \qquad (9)$$

$U(\mathbf{x}) \to U(\mathbf{x}) + W_{LS}(\mathbf{x})$,

$W_{LS}(r) = f(r)\vec{L} \cdot \vec{S}$, $\mathbf{J} = \mathbf{L} + \mathbf{S}$, $\vec{L} \cdot \vec{S} = \frac{1}{2}(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2)$.

$$W_{LS} = \frac{\partial}{\partial r} f(r) \left[J(J+1) - L(L+1) - S(S+1) \right] \tag{10}$$

Solve Schrödinger equation, sum up all wave functions and obtain density profile.

$$\rho(r) = \sum_{n,l,s} |\psi_{nls}(r)|^2 \tag{11}$$

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DFT

Roughly speaking, Density Functional Theory for nuclei is to replace $U(\mathbf{r})$ with $U(\rho_n, \rho_p)$ and obtain wave functions or densities until it reaches self-consistency.

- We employ algebraic function depends on magic number and valence number of nucleus (Duflo and Zuker Phys. Rev. C 52, R23(R)).

$$E_{shell} = a_1 S^2 + a_2 (S_2)^2 + a_3 S_3 + a_{np} S_{np}, \tag{12}$$

where

$$\begin{aligned} S_2 &= \frac{n_v \bar{n}_v}{D_n} + \frac{p_v \bar{p}_v}{D_p} \\ S_3 &= \frac{n_v \bar{n}_v (n_v - \bar{n}_v)}{D_n} + \frac{p_v \bar{p}_v (p_v - \bar{p}_v)}{D_p} \\ S_{np} &= \frac{n_v \bar{n}_v p_v \bar{p}_v}{D_n D_p} \end{aligned} \tag{13}$$

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^{56}Fe

We obtain $n_v = |30 - 28| = 2$, $p_v = |26 - 20| = 6$. $D_n (D_p)$ is the degeneracy number, $D_n = 50 - 28 = 22$, $D_p = 28 - 20 = 8$, $\bar{n}_v = 50 - 30 = 20$, $\bar{p}_v = 28 - 26 = 2$

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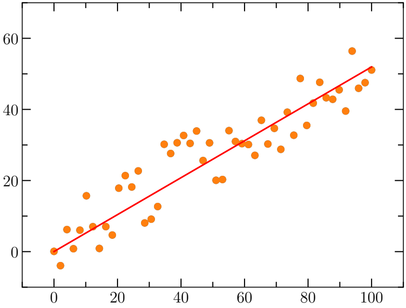

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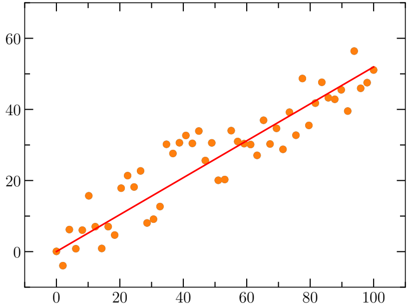

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$$y = ax + b \quad \rightarrow \quad \chi^2 = \frac{1}{N} \sum_i (ax_i + b - y_i)^2 \tag{14}$$

- To minimize χ^2, we take derivatives w.r.t. a and b

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\begin{aligned}
f_1 &= \frac{\partial \chi^2}{\partial a} = \frac{2}{N}\sum_i (a x_i + b - y_i) x_i = 0\,, \\
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Good news !

In this type of LDM, we can linearize by changing variables $A_i = x_i$, $A^{2/3} = s_i$, ..., $A_{pi}/\sqrt{A} = u_i$. Follow the ways as in the linear regression.

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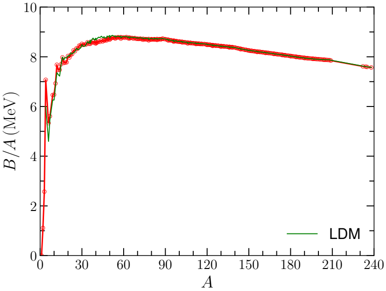

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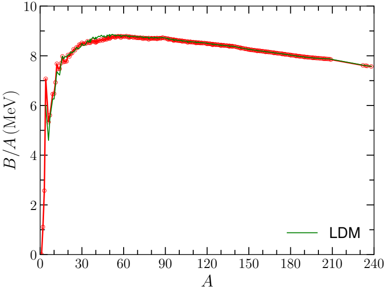

Figure: Left :Experimental binding energy (red circles) and LDM calculations (green line), Right: Binding energy contributions

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Coding:

Except theoretical parts, Computer coding : Do Loop & If and else

- Exercise ! DIY

  We have the equations to get the optimized linear equation or correlation line, i.e., Eq. (15). Can you get the analytic solution for a and b?

  What if you have a vector $\vec{x}$ instead of scalar x? That is, there are data points $(x_i^1, x_i^2, x_i^3, \ldots, x_i^M, y_i)$, $(i = 1, \ldots, N)$ and we want to find, $(a^1, a^2, \ldots, a^M, b)$.

- Exercise ! DIY
  Nuclear matter saturates at a density of about 0.16 $\mathrm{baryons/fm^3}$ where the energy per baryoon is about $-16\,\mathrm{MeV}$. The nuclear surface tension is about $1\,\mathrm{MeV/fm^2}$. Estimate the mass number of the nucleus with the largest binding energy per baryon, assuming symmetric nuclear matter (equal number of neutrons and protons).

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  Hint:

$$E = -BA + 4\pi R^2\sigma + \frac{3}{5}\frac{Z^2e^2}{R}, \quad n_0\frac{4\pi R^3}{3} = A \tag{17}$$