# Statistical Methods in Nuclear Physics
## : Practical Approach

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  2. How can we find that?
     Write down the total binding energy as a function density and think the way to solve it !

$$E = -BA + E_S A^{2/3} + E_C \frac{Z^2}{A^{1/3}}; \quad \text{In compressible model} \qquad (1)$$

$$-B \rightarrow -B + S_v(1-2x)^2 + \frac{K}{18}\left(1 - \frac{n}{n_0}\right)^2$$

$A \rightarrow \frac{4\pi}{3} R^3 n,\ E_S \rightarrow \sigma(x);\ x \rightarrow$ proton fraction

※ $\sigma(x)$ is the surface tension which can be obtained semi-infinite nuclear matter density profile. This is an quantities related to two phase(dense matter, dilute matter) equilibrium. Thus it is a thermodynamic quantity.

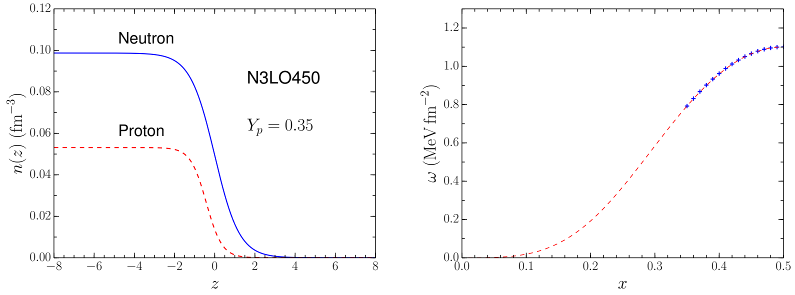


Figure: surface density profile (left) and surface tension (right)

$$\sigma(x) \simeq \sigma(x = 0.5) - \sigma_\delta (1 - 2x)^2 \qquad (2)$$

$$f(n,x) = \left[-B + S_v(1-2x)^2 + \frac{K}{18}\left(1-\frac{n}{n_0}\right)^2\right]A \\ + 4\pi R^2\sigma(x) + \frac{3}{5}\frac{Z^2e^2}{R}; \quad \text{Compressible model} \tag{3}$$

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No we forgot the constraints

$$A = \frac{4\pi}{3}R^3 n\,, \quad x = \frac{Z}{A} \tag{5}$$

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$$2x + \lambda = 0,\ 2y + \lambda = 0,\ x + y - 1 = 0 \tag{9}$$

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- Apply Lagrange Multiplier method to the compressible liquid drop model

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\begin{aligned}
g(n,x) = &\left[-B + S_v(1-2x)^2 + \frac{K}{18}\left(1-\frac{n}{n_0}\right)^2\right]A \\
&+ 4\pi R^2[\sigma_0 - (1-2x)^2\sigma_\delta] + \frac{3}{5}\frac{Z^2e^2}{R} \\
&+ \lambda_1\left(A - \frac{4\pi R^3}{3}n\right) + \lambda_2\left(\frac{Z}{A} - x\right)
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- What are unknowns ? $n$, $x$, $R$, $\lambda_1$. $\lambda_1$, $\lambda_2$

Using the notation $f_B(n,x) = -B + S_v(1-2x)^2 + \frac{K}{18}\left(1 - \frac{n}{n_0}\right)^2$

$$\frac{\partial g}{\partial n} = 0; \quad \frac{\partial f_B}{\partial n} A - \frac{4\pi R^3}{3}\lambda_1 = 0, \tag{15}$$

$$\frac{\partial g}{\partial x} = 0; \quad \frac{\partial f_B}{\partial x} A - 16\pi R^2 \delta^2 \sigma_\delta - \lambda_2 = 0, \tag{16}$$

$$\frac{\partial g}{\partial R} = 0; \quad 8\pi R[\sigma_0 - \delta^2 \sigma_\delta] + \frac{\partial E_C}{\partial R} - \lambda_1 4\pi R^2 n = 0, \tag{17}$$

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Next job is to reduce the equations as many as possible!

- $\lambda_1$ and $\lambda_2$ are related to chemical potential of neutrons and protons

$$\lambda_1 = n\frac{\partial f_B}{\partial n}, \quad \lambda_2 = \frac{\partial f_B}{\partial n}A - 16\pi R^2 \delta\sigma_\delta, \quad x = \frac{Z}{A} \tag{20}$$

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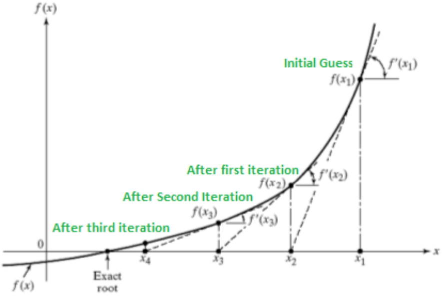
**Non-linear equations !**

It's time to learn how to solve non-linear solutions.

- Newton Raphson is a very powerful method to get solution for non-linear equations
  Find the solution $f(x) = 0$



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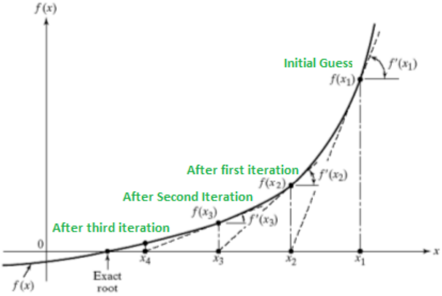


$$x_{n+1} = x_n - f(x_n)/f'(x_n) \quad (22)$$

- Numerical derivative

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Depending on the relative size of $x_j$, $\lambda_j$ (scale factor) is introuduced, $\Delta x_j = -\lambda_j D_{ij}^{-1} f_i$

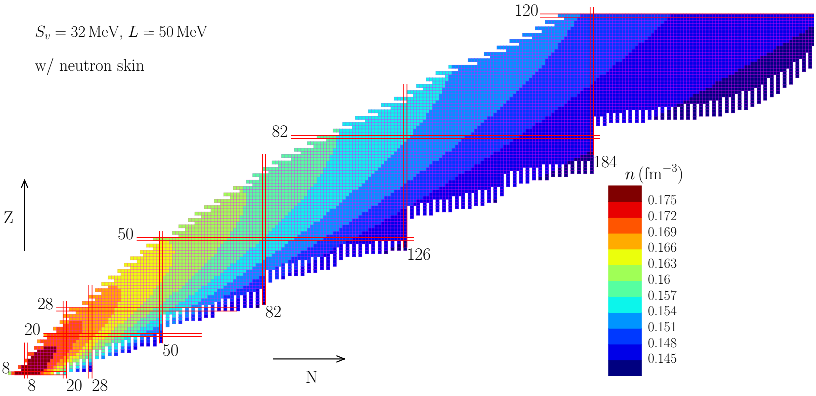


Figure: Bound nuclei with compressible model

# Take home problem III

- If you can write a code (python, fortran, c, c++,...), Try to solve Eq.(21) for $^{56}$Fe.
- You may use $B = -16\,\mathrm{MeV}$, $S_v = 32\,\mathrm{MeV}$, $K = 235\,\mathrm{MeV}$, $\sigma_0 = 1.12\,\mathrm{MeV\,fm^{-2}}$, $\sigma_\delta = 2.0\mathrm{MeV\,fm^{-2}}$.
- You can try with initial guess $n = 0.16\,\mathrm{fm^{-3}}$, $R = 1.12A^{1/3}\,\mathrm{fm}$.
- If you can solve for a given value of $B$, $S_v$, $K$, $\sigma_0$, $\sigma_\delta$, you can also find the optimized $\sigma_0$ and $\sigma_\delta$.
- In the same manner, you can also find the pairing gap $\Delta$ and Shell corrections in the compressible model.
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**Tips**

As you may know, if you meet some numerical problems, it always works if you can simplify the problems. In this case, you may start $f_B = -B$ or $\sigma_\delta = 0$ and increase the complexities to check your code.

# Take Home Problem IV

- Energy density functioal is a simple analytic function to decribe energy density as a function of baryon number density and proton fraction $(n, x)$.

$$\mathcal{E}(n,x) = \frac{\hbar}{2m}\tau_n + \frac{\hbar}{2m}\tau_p + (1-2x)^2 f_n(n) + \left[1-(1-2x)^2\right] f_s(n)\,,$$

- We can get the coefficients for Energy Density Functional utilizing the results from state of the art calculation for pure neutron matter EOS and symmetric nuclear matter properties.
First, the kinetic energy densities $(\tau_n, \tau_p)$ (in the uniform) are given as

$$\tau_n = \frac{3}{5}(3\pi^2)^{2/3}(n(1-x))^{5/3},\ \ \tau_p = \frac{3}{5}(3\pi^2)^{2/3}(nx)^{5/3}$$

The potential energy parts are assumed to be

$$f_s(n) = \sum_{i=0}^{3} a_i n^{(2+i/3)}\,, \quad f_n(n) = \sum_{i=0}^{3} b_i n^{(2+i/3)}$$

- The definition of symmetric nuclear matter properties are

$$-B = \frac{\mathcal{E}}{n}, P = n^2 \frac{\partial}{\partial n}\left(\frac{\mathcal{E}}{n}\right), K = 9n^2 \frac{\partial^2}{\partial n^2}\left(\frac{\mathcal{E}}{n}\right), Q = 27n^3 \frac{\partial^3}{\partial n^3}\left(\frac{\mathcal{E}}{n}\right).$$

  where every quantity is evaluated at $n = n_0$, $x = 1/2$.
  From the properties finite nuclei, we have $B = 16\text{MeV}$, $P = 0(n_0 = 0.16\,\mathrm{fm}^{-3})$, $K = 235\text{MeV}$, $Q = -300\text{MeV}$
- Find the coefficient $a_i$ from the symmetric nuclear matter properties !
- Find the coefficient $b_i$ from the neutron matter EOS !

**Fitting**

This problem is related with the last lecture and a problem in there.

This is the data file for pure neutron matter from Many body perturbation calculation using chiral potential and three body forces.

| $k_F\,(\mathrm{fm}^{-1})$ | $n\,(\mathrm{fm}^{-3})$ | $E/A\,(\mathrm{MeV})$ | $k_F\,(\mathrm{fm}^{-1})$ | $n\,(\mathrm{fm}^{-3})$ | $E/A\,(\mathrm{MeV})$ |
|---|---|---|---|---|---|
| 0.66651 | 0.01 | 2.87933 | 1.48231 | 0.11 | 11.73104 |
| 0.83975 | 0.02 | 4.27096 | 1.52593 | 0.12 | 12.56379 |
| 0.96127 | 0.03 | 5.33345 | 1.56719 | 0.13 | 13.42159 |
| 1.05802 | 0.04 | 6.23757 | 1.60639 | 0.14 | 14.29794 |
| 1.13972 | 0.05 | 7.05834 | 1.64376 | 0.15 | 15.19301 |
| 1.21113 | 0.06 | 7.83732 | 1.6795 | 0.16 | 16.10768 |
| 1.27499 | 0.07 | 8.59843 | 1.71379 | 0.17 | 17.03337 |
| 1.33302 | 0.08 | 9.35958 | 1.74675 | 0.18 | 17.97216 |
| 1.3864 | 0.09 | 10.13324 | 1.77852 | 0.19 | 18.91497 |
| 1.43595 | 0.1 | 10.92077 | 1.80919 | 0.20 | 19.86317 |

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- There are several diffference between finite nuclei and nuclei in neutron star crust.
- First of all, we have to think the presence of electrons
- As density increases, neutrons drips out of neutron rich heavy nuclei
  Unbound neutron exists
- Write energy contribution and apply Lagrange Multiplier Method with constraints !

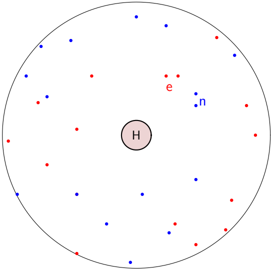
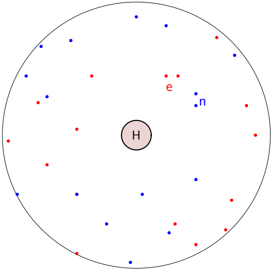



Figure: Wigner Seitz cell

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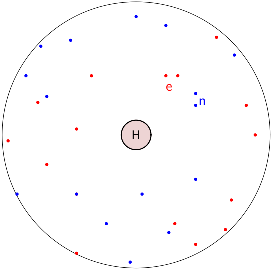
$F_i$ : Energy density of a heavy nucleus

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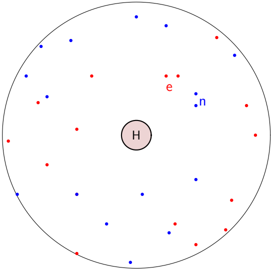
$F_i$ : Energy density of a heavy nucleus

$F_s$ : Surface energy density

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$F_i$ : Energy density of a heavy nucleus

$F_s$ : Surface energy density

$F_c$ : Coulomb energy density

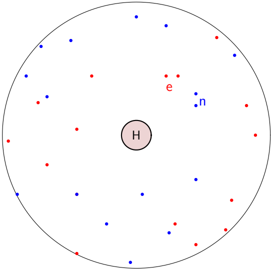


Figure: Wigner Seitz cell

$F_i$ : Energy density of a heavy nucleus

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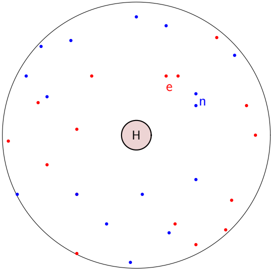


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$F_e$ : Electron energy density

# Nuclear pasta phase

- The ground state of nuclear shape is determined by the competition between Coulomb interaction and surface tension.

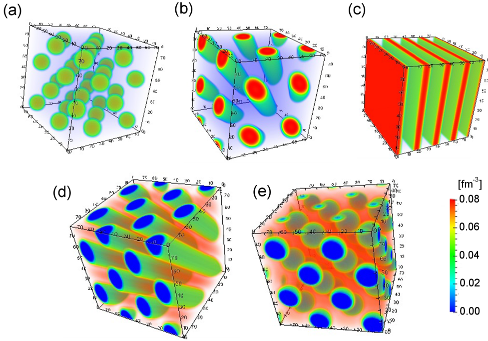


Figure: Numerical calculation of nuclear pasta phase. Figure from the work of Okamoto et al., Phys. Rev. C 88, 025801 (2013).

$$
\begin{aligned}
F =& u n_i f_i + \frac{\sigma(x_i) u d}{r_N} + 2\pi (n_i x_i e r_N)^2 u f_d(u) \\
&+ (1-u) n_{no} f_o + f_e \,,
\end{aligned}
\tag{26}
$$

with constraints

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\begin{aligned}
& n - u n_i - (1-u) n_{no} = 0 \,, \\
& n Y_p - u n_i x_i = 0 \,, \\
& n Y_p - n_e = 0 \,.
\end{aligned}
\tag{27}
$$

$$F = un_i f_i + \frac{\sigma(x_i) u d}{r_N} + 2\pi (n_i x_i e r_N)^2 u f_d(u) + (1-u) n_{no} f_o + f_e\,, \tag{26}$$

with constraints

$$\begin{aligned} n - un_i - (1-u) n_{no} &= 0\,, \\ nY_p - un_i x_i &= 0\,, \\ nY_p - n_e &= 0\,. \end{aligned} \tag{27}$$

- Unknows are $u$, $n_i$, $n_{no}$, $x_i$, $r_N$, $n_e$, $\lambda_1$, $\lambda_2$, and $\lambda_3$.

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with constraints

$$\begin{aligned} & n - un_i - (1-u) n_{no} = 0 \,, \\ & nY_p - un_i x_i = 0 \,, \\ & nY_p - n_e = 0 \,. \end{aligned} \quad (27)$$

- Unknows are $u$, $n_i$, $n_{no}$, $x_i$, $r_N$, $n_e$, $\lambda_1$, $\lambda_2$, and $\lambda_3$.
- $\partial F / \partial r_N \rightarrow F_S = 2F_C$ (the nuclear virial theorem)
- $\partial G / \partial n_e = 0$, the beta equilibrium is made.
- Finally, the unknowns will be $(n_i, x_i, u, n_{no})$ for a given $(n, Y_p)$.

# A little more : Hot dense matter

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- Thermodynamic quantities from free energy density and during the process of looking for solutions

# Take Home Problem V

- In this example, you may obtain the neutron star outer crust EOS using LDM.
- Since the outer crust does not have any unbound neutrons, Eq. (26) is simplified a lot.
- The valid dimension is $d = 3$. That is, spherically symmetric nuclei would be the ground state.
- The surface tension and dicrete dimensional Coulomb $f_d$ are given by

$$\sigma(x) = \sigma_0 \frac{2^{\alpha+1} + q}{(1-x)^{-\alpha} + q + x^{-\alpha}},$$
$$f_d(u) = \frac{1}{d+2}\left[\frac{2}{d-2}\left(1 - \frac{1}{2} d u^{1-2/d}\right) + u\right], \tag{28}$$

where $\sigma_0 = 1.12 \mathrm{MeV\,fm}^{-2}$, $q = 45$, and $\alpha = 3$.

- First, assume the electron energy but maintain $f_d$. Try to get Equations to solve in the outer crust, using the contraints for a given $(n, Y_p)$
- What happens if you add the electrons and minimize the energy density w.r.t. $Y_p$
- Estimate when $\mu_n > 0$ so that the neutron drip density is determined.