Statistical Methods in Nuclear Physics

: Practical Approach

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Nuclear Physics School 2020
(Asia Pacific Center for Theoretical Physics)

Outlines

- Neutron Stars
- LDM
  - Root Mean Square Deviation, Lagrange Multiplier Method
- Energy density functional
- Tolmann Oppeheimer Volkov (T.O.V) equations
- Bayesian Statistics
  - Conditional probabilities
- Application to nuclear matter and neutron stars
  - Mass and radius according to the observational constraints

Neutron stars

Figure: Cassiopeia A is among the best-studied supernova remnants. This image blends data from NASA's Spitzer (red), Hubble (yellow), and Chandra (green and blue) observatories. NASA/JPL-CALTECH/STSCI/CXC/SAO

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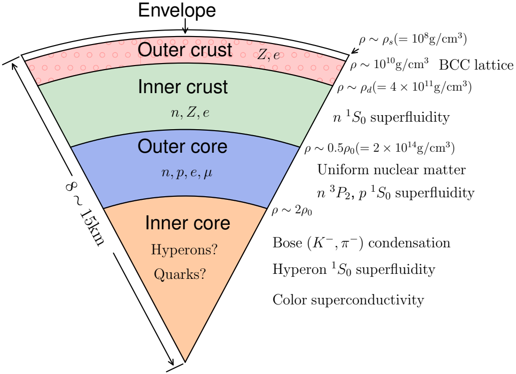
- Inner structure of neutron stars



- Neutron Stars:
  - Dense nuclear matter physics

- TOV equations for macroscopic structure
  (spherically symmetric non-rotating NS)

$$\frac{dp}{dr} = -\frac{G(M(r) + 4\pi r^3 p/c^2)(\epsilon + p)}{r(r - 2GM(r)/c^2)c^2},$$
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  r; distance from the center
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- Microscopic quantities (Nuclear physics)
  p ; pressure
  ϵ; energy density

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Truth

You have to know that 'if someone can do it, you can do it'.

Nuclear Binding energy

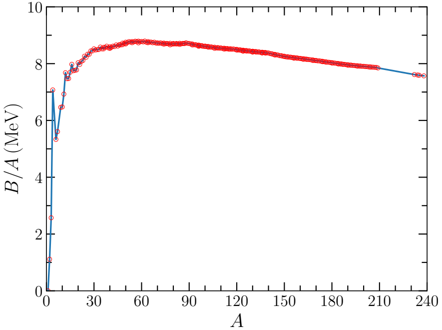

Figure: Binding energy per nucleon for stable nuclei

Nuclear Binding energy

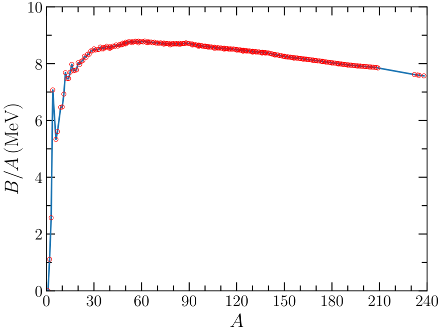

Figure: Binding energy per nucleon for stable nuclei

- There many methods to calculate binding energy of finite nuclei.
  - Liquid drop model, Thomas Fermi, Skyrme Hartee-Fock, Relativistic Mean field model, No-core shell, Quantum Monte Carlo, . . .

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- Fitting function(?)

$$f(A, Z) = a_0 A + a_1 A^{2/3} + a_2 A^{1/3} + \cdots + b_1 Z + b_2 Z^2 + \cdots \tag{3}$$

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- Origin of B, E_S, E_C
  - B : binding energy of bulk matter
  - E_S : Surface energy
  - E_C : Coulomb

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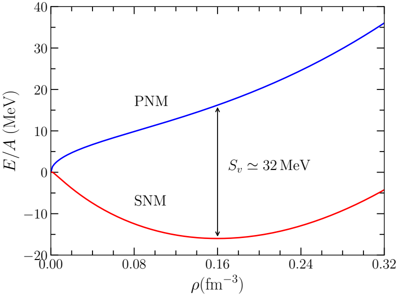
  Pure neutron matter and Symmetric nuclear matter show the different E/A.



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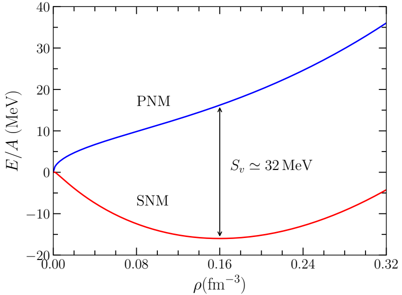

$$
\begin{aligned}
E(n,x) &= E_{SNM} + (1-2x)^2 S(n) \\
E_{SNM} &= E(n, x = \frac{1}{2}) \\
E_{PNM} &= E(n, x = 0) \\
S(n) &= E_{PNM} - E_{SNM}
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\tag{4}
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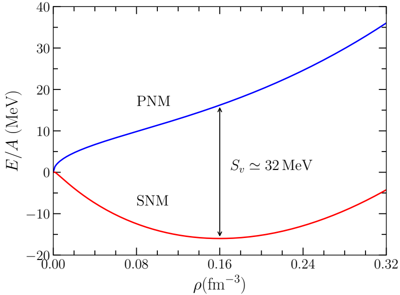

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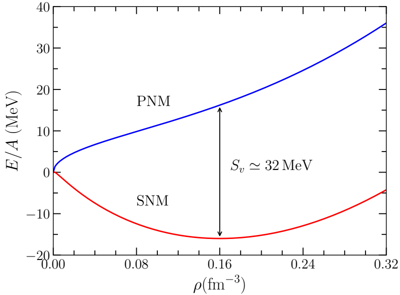

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$$-BA + E_{SYM} \frac{(N-Z)^2}{A} = \left[-B + E_{SYM}(1-2x)^2\right] A \tag{6}$$

- Correction term : Pairing Energy
  Even-odd Staggering $S_n = B(N, Z) - B(N-1, Z)$

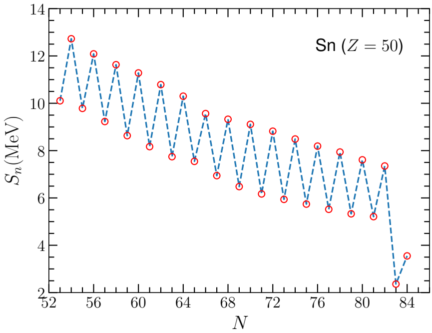

Figure: One neutron separation energy of Sn isotopes

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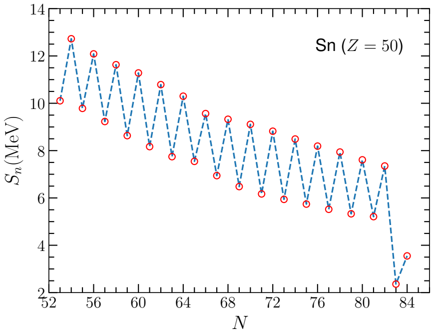

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$$E = -BA + E_S A^{2/3} + E_C \frac{Z^2}{A^{1/3}} + E_{SYM} \frac{(N-Z)^2}{A} + A_p \frac{\Delta}{\sqrt{A}} \quad (7)$$

$A_p = -1$ for even-even, $A_p = 0$ for even-odd, and $A_p = 1$ for odd-odd nuclei.

- Correction term : Shell corrections

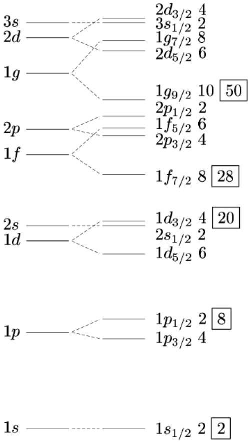

Figure: Single particle energy level (wikipedia)

Low-lying energy levels in a single-particle shell model with an oscillator potential without spin-orbit (left) and with spin-orbit (right) interaction.

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + U(\mathbf{x})\right]\psi(\mathbf{x}) = E\psi(\mathbf{x}) \tag{8}$$

$$U(\mathbf{x}) = \begin{cases} \frac{1}{2}k\mathbf{x}^2 \\ -\frac{U_0}{1+e^{(r-R)/a}} \end{cases} \tag{9}$$

$U(\mathbf{x}) \to U(\mathbf{x}) + W_{LS}(\mathbf{x})$,

$W_{LS}(r) = f(r)\vec{L} \cdot \vec{S}$, $\mathbf{J} = \mathbf{L} + \mathbf{S}$, $\vec{L} \cdot \vec{S} = \frac{1}{2}(\mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2)$.

$$W_{LS} = \frac{\partial}{\partial r} f(r) \left[J(J+1) - L(L+1) - S(S+1)\right] \quad (10)$$

Solve Schrödinger equation, sum up all wave functions and obtain density profile.

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DFT

Roughly speaking, Density Functional Theory for nuclei is to replace $U(\mathbf{r})$ with $U(\rho_n, \rho_p)$ and obtain wave functions or densities until it reaches self-consistency.

- We employ algebraic function depends on magic number and valence number of nucleus (Duflo and Zuker Phys. Rev. C 52, R23(R)).

$$E_{shell} = a_1 S^2 + a_2 (S_2)^2 + a_3 S_3 + a_{np} S_{np}, \tag{12}$$

where

$$\begin{aligned} S_2 &= \frac{n_v \bar{n}_v}{D_n} + \frac{p_v \bar{p}_v}{D_p} \\ S_3 &= \frac{n_v \bar{n}_v (n_v - \bar{n}_v)}{D_n} + \frac{p_v \bar{p}_v (p_v - \bar{p}_v)}{D_p} \\ S_{np} &= \frac{n_v \bar{n}_v p_v \bar{p}_v}{D_n D_p} \end{aligned} \tag{13}$$

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^{56}Fe

We obtain $n_v = |30 - 28| = 2$, $p_v = |26 - 20| = 6$. $D_n(D_p)$ is the degeneracy number, $D_n = 50 - 28 = 22$, $D_p = 28 - 20 = 8$, $\bar{n}_v = 50 - 30 = 20$, $\bar{p}_v = 28 - 26 = 2$

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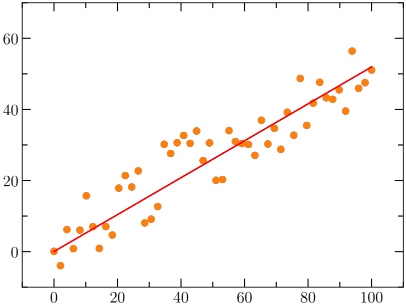

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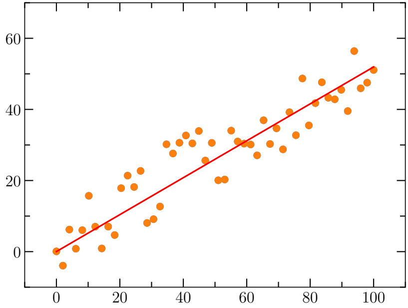

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$$y = ax + b \quad \rightarrow \quad \chi^2 = \frac{1}{N} \sum_i (ax_i + b - y_i)^2 \qquad (14)$$

- To minimize χ^2, we take derivatives w.r.t. a and b

$$\begin{aligned} f_1 &= \frac{\partial \chi^2}{\partial a} = \frac{2}{N}\sum_i (a x_i + b - y_i) x_i = 0\,, \\ f_2 &= \frac{\partial \chi^2}{\partial b} = \frac{2}{N}\sum_i (a x_i + b - y_i) = 0 \end{aligned} \tag{15}$$

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Good news !

In this type of LDM, we can linearize by changing variables $A_i = x_i$, $A^{2/3} = s_i$, ..., $A_{pi}/\sqrt{A} = u_i$. Follow the ways as in the linear regression.

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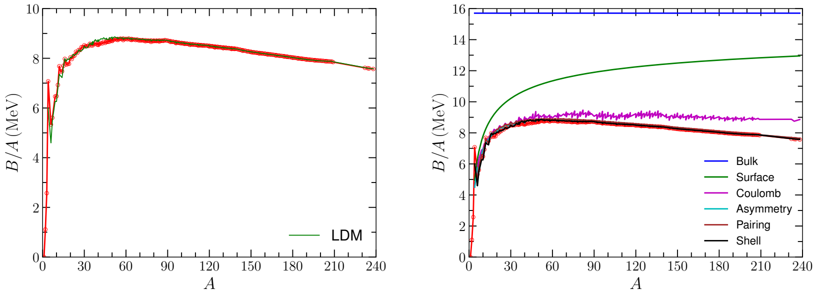

Figure: Left :Experimental binding energy (red circles) and LDM calculations (green line), Right: Binding energy contributions

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Bound nuclei

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Coding:

Except theoretical parts, Computer coding : Do Loop & If and else

- Exercise ! DIY

  We have the equations to get the optimized linear equation or correlation line, i.e., Eq. (15). Can you get the analytic solution for a and b?

  What if you have a vector $\vec{x}$ instead of scalar x? That is, there are data points $(x_i^1, x_i^2, x_i^3, \ldots, x_i^N, y_i)$ and we want to find, $(a^1, a^2, \ldots, a^N, b)$.

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  Nuclear matter saturates at a density of about 0.16 $\mathrm{baryons}/\mathrm{fm}^3$ where the energy per baryoon is about $-16\,\mathrm{MeV}$. The nuclear surface tension is about $1\,\mathrm{MeV}/\mathrm{fm}^2$. Estimate the mass number of the nucleus with the largest binding energy per baryon, assuming symmetric nuclear matter (equal number of neutrons and protons).

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  Hint:

$$E = -BA + 4\pi R^2\sigma + \frac{3}{5}\frac{Z^2e^2}{R}\,, \quad n_0\frac{4\pi R^3}{3} = A \tag{17}$$

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  1. The first thing we can do is to allow the variation of density of nuclei $n_0 \rightarrow n$; compressible model, $R \neq r_0 A^{1/3}$

Compressible Liquid Drop Model

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  2. How can we find that?
     Write down the total binding energy as a function density and think the way to solve it !

$$E = -BA + E_S A^{2/3} + E_C \frac{Z^2}{A^{1/3}}; \quad \text{In compressible model} \qquad (18)$$

$$-B \to -B + S_v(1-2x)^2 + \frac{K}{18}\left(1 - \frac{n}{n_0}\right)^2$$
$A \to \frac{4\pi}{3}R^3 n,\ E_S \to \sigma(x);\ x \to$ proton fraction

※ $\sigma(x)$ is the surface tension which can be obtained semi-infinite nuclear matter density profile. This is an quantities related to two phase(dense matter, dilute matter) equilibrium. Thus it is a thermodynamic quantity.

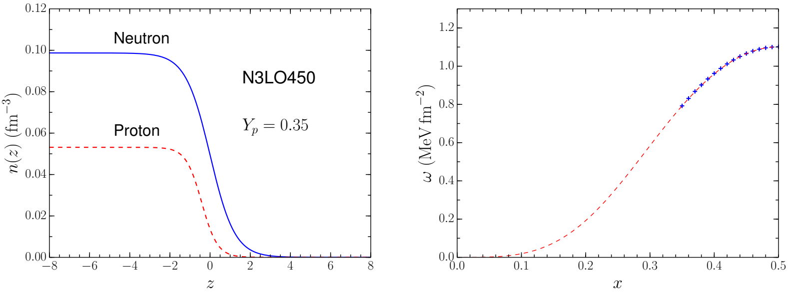

Figure: surface density profile (left) and surface tension (right)

$$\sigma(x) \simeq \sigma(x = 0.5) - \sigma_\delta(1 - 2x)^2 \qquad (19)$$

$$
\begin{aligned}
f(n,x) = &\left[-B + S_v(1-2x)^2 + \frac{K}{18}\left(1-\frac{n}{n_0}\right)^2\right]A \\
&+ 4\pi R^2\sigma(x) + \frac{3}{5}\frac{Z^2e^2}{R}; \quad \text{Compressible model}
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\tag{20}
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No we forgot the constraints

$$A = \frac{4\pi}{3}R^3 n\,, \quad x = \frac{Z}{A} \tag{22}$$

Lagrange Multiplier Method

- How can we minimize some quantities with certain constraints?

$$f(x, y) = x^2 + y^2 - 1; \quad x + y = 1 \tag{23}$$

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$$2x + \lambda = 0,\ 2y + \lambda = 0,\ x + y - 1 = 0 \tag{26}$$

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$$\boxed{\mu_n = \mu_p + \mu_e\,, \quad \mu_e = \mu_\mu,} \tag{30}$$

- Apply Lagrange Multiplier method to the compressible liquid drop model

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\begin{aligned}
g(n,x) = & \left[-B + S_v(1-2x)^2 + \frac{K}{18}\left(1 - \frac{n}{n_0}\right)^2 \right] A \\
& + 4\pi R^2[\sigma_0 - (1-2x)^2\sigma_\delta] + \frac{3}{5}\frac{Z^2 e^2}{R} \\
& + \lambda_1 \left(A - \frac{4\pi R^3}{3} n \right) + \lambda_2 \left(\frac{Z}{A} - x \right)
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- What are unknowns ? n, x, R, λ_1. λ_1, λ_2

Using the notation $f_B(n,x) = -B + S_v(1-2x)^2 + \frac{K}{18}\left(1 - \frac{n}{n_0}\right)^2$

$$\frac{\partial g}{\partial n} = 0; \quad \frac{\partial f_B}{\partial n} A - \frac{4\pi R^3}{3}\lambda_1 = 0, \tag{32}$$

$$\frac{\partial g}{\partial x} = 0; \quad \frac{\partial f_B}{\partial x} A - 16\pi R^2 \delta^2 \sigma_\delta - \lambda_2 = 0, \tag{33}$$

$$\frac{\partial g}{\partial R} = 0; \quad 8\pi R[\sigma_0 - \delta^2 \sigma_\delta] + \frac{\partial E_C}{\partial R} - \lambda_1 4\pi R^2 n = 0, \tag{34}$$

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Next job is to reduce the equations as many as possible!

- λ_1 and λ_2 are related to chemical potential of neutrons and protons

$$\lambda_1 = n\frac{\partial f_B}{\partial n}, \quad \lambda_2 = \frac{\partial f_B}{\partial n}A - 16\pi R^2 \delta\sigma_\delta, \quad x = \frac{Z}{A} \tag{37}$$

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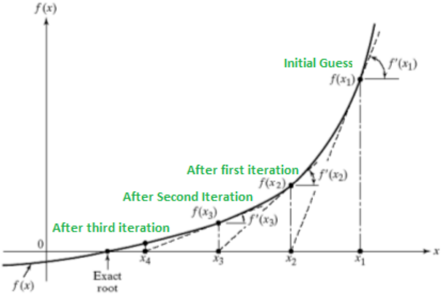
Non-linear equations !

It's time to learn how to solve non-linear solutions.

Newton Raphson

- Newton Raphson is a very powerful method to get solution for non-linear equations
  Find the solution $f(x) = 0$



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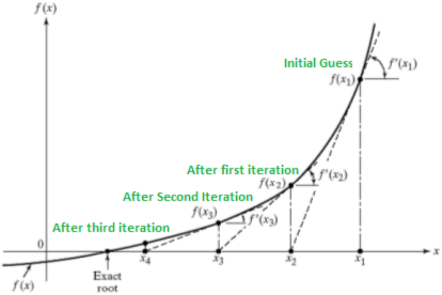

$$x_{n+1} = x_n - f(x_n)/f'(x_n) \tag{39}$$

- Numerical derivative

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \tag{40}$$

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$\partial x_j \Delta x_j$ means summation over j, D_{ij}^{-1} is the matrix inverse for $\partial f_i / \partial x_j$
Depending on the relative size of x_j, λ_j (scale factor) is introuduced, $\Delta x_j = -\lambda_j D_{ij}^{-1} f_i$

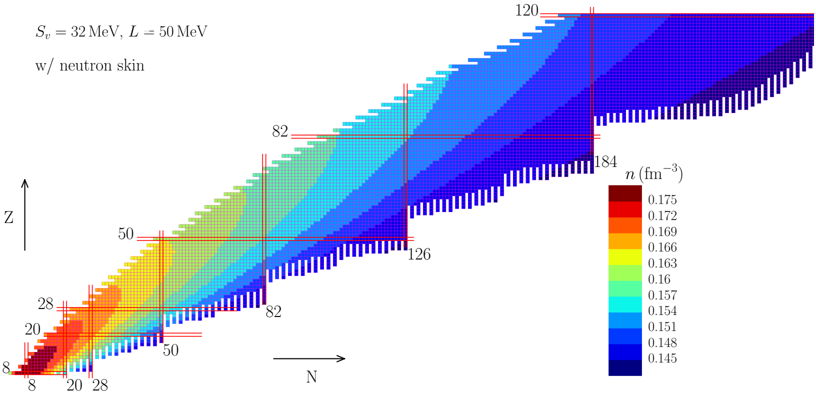

Figure: Bound nuclei with compressible model

Take home problem III

- If you can write a code (python, fortran, c, c++,...), Try to solve Eq.(38) for ^{56}Fe.
- You may use $B = -16\,\mathrm{MeV}$, $S_v = 32\,\mathrm{MeV}$, $K = 235\,\mathrm{MeV}$, $\sigma_0 = 1.12\,\mathrm{MeV\,fm}^{-2}$, $\sigma_\delta = 2.0\mathrm{MeV\,fm}^{-2}$.
- You can try with initial guess $n = 0.16\,\mathrm{fm}^{-3}$, $R = 1.12A^{1/3}\,\mathrm{fm}$.
- If you can solve for a given value of B, S_v, K, σ_0, σ_δ, you can also find the optimized σ_0 and σ_δ.
- In the same manner, you can also find the pairing gap Δ and Shell corrections in the compressible model.
- See how the energy difference changes for N, Z, especially magic number.

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Tips

As you may know, if you meet some numerical problems, it always works if you can simplify the problems. In this case, you may start $f_B = -B$ or $\sigma_\delta = 0$ and increase the complexities to check your code.

Take Home Problem IV

- Energy density functioal is a simple analytic function to decribe energy density as a function of baryon number density and proton fraction (n, x).

$$\mathcal{E}(n,x) = \frac{\hbar}{2m}\tau_n + \frac{\hbar}{2m}\tau_p + (1-2x)^2 f_n(n) + \left[1-(1-2x)^2\right] f_s(n)\,,$$

- We can get the coefficients for Energy Density Functional utilizing the results from state of the art calculation for pure neutron matter EOS and symmetric nuclear matter properties.
  First, the kinetic energy densities (τ_n, τ_p) (in the uniform) are given as

$$\tau_n = \frac{3}{5}(3\pi^2)^{2/3}(n(1-x))^{5/3},\ \ \tau_p = \frac{3}{5}(3\pi^2)^{2/3}(nx)^{5/3}$$

  The potential energy parts are assumed to be

$$f_s(n) = \sum_{i=0}^{3} a_i n^{(2+i/3)}\,, \quad f_n(n) = \sum_{i=0}^{3} b_i n^{(2+i/3)}$$

- The definition of symmetric nuclear matter properties are

$$-B = \frac{\mathcal{E}}{n}, P = n^2 \frac{\partial}{\partial n}\left(\frac{\mathcal{E}}{n}\right), K = 9n^2 \frac{\partial^2}{\partial n^2}\left(\frac{\mathcal{E}}{n}\right), Q = 27n^3 \frac{\partial^3}{\partial n^3}\left(\frac{\mathcal{E}}{n}\right).$$

where every quantity is evaluated at $n = n_0$, $x = 1/2$.
From the properties finite nuclei, we have $B = 16$MeV, $P = 0(n_0 = 0.16\,\mathrm{fm}^{-3})$, $K = 235$MeV, $Q = -300$MeV
- Find the coefficient a_i from the symmetric nuclear matter properties !
- Find the coefficient b_i from the neutron matter EOS !

Fitting

This problem is related with the last lecture and a problem in there.

This is the data file for pure neutron matter from Many body perturbation calculation using chiral potential and three body forces.

$k_F\,(\mathrm{fm}^{-1})$	$n\,(\mathrm{fm}^{-3})$	$E/A\,(\mathrm{MeV})$	$k_F\,(\mathrm{fm}^{-1})$	$n\,(\mathrm{fm}^{-3})$	$E/A\,(\mathrm{MeV})$
0.66651	0.01	2.87933	1.48231	0.11	11.73104
0.83975	0.02	4.27096	1.52593	0.12	12.56379
0.96127	0.03	5.33345	1.56719	0.13	13.42159
1.05802	0.04	6.23757	1.60639	0.14	14.29794
1.13972	0.05	7.05834	1.64376	0.15	15.19301
1.21113	0.06	7.83732	1.6795	0.16	16.10768
1.27499	0.07	8.59843	1.71379	0.17	17.03337
1.33302	0.08	9.35958	1.74675	0.18	17.97216
1.3864	0.09	10.13324	1.77852	0.19	18.91497
1.43595	0.1	10.92077	1.80919	0.20	19.86317

LDM for neutron star crust

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- There are several diffference between finite nuclei and nuclei in neutron star crust.
- First of all, we have to think the presence of electrons
- As density increases, neutrons drips out of neutron rich heavy nuclei Unbound neutron exists
- Write energy contribution and apply Lagrange Multiplier Method with constraints !

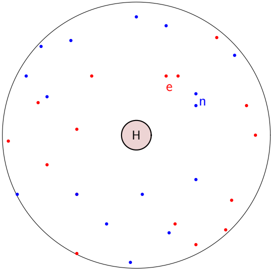
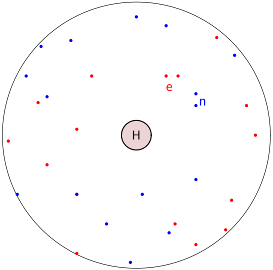


Figure: Wigner Seitz cell



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F_i : Energy density of a heavy nucleus

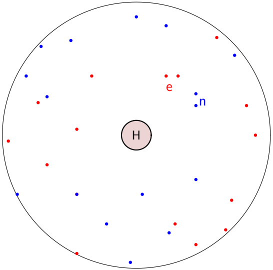


Figure: Wigner Seitz cell

F_i : Energy density of a heavy nucleus

F_s : Surface energy density

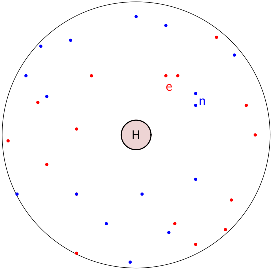
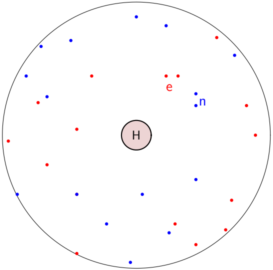


Figure: Wigner Seitz cell

F_i : Energy density of a heavy nucleus

F_s : Surface energy density

F_c : Coulomb energy density



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F_i : Energy density of a heavy nucleus

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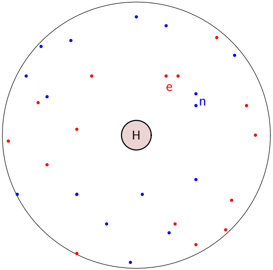
F_o : Energy density of outside nucleons



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F_i : Energy density of a heavy nucleus

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F_o : Energy density of outside nucleons

F_e : Electron energy density

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- $\partial F / \partial r_N \rightarrow F_S = 2F_C$ (the nuclear virial theorem)
- $\partial G / \partial n_e = 0$, the beta equilibrium is made.
- Finally, the unknowns will be (n_i, x_i, u, n_{no}) for a given (n, Y_p).

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Condition

However, the chance will depend on the age, level of cholesterol, blood pressure, family history, and so on.

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Example

A is the even that the first coin lands face up. B is the event that the the second coin lands face up, then $p(A) = p(B) = 0.5$,
$p(A \cap B) = p(A)p(B) = 0.25$.

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$P(B|A) = P(B)$ means they are independent!

Example

Suppose that A means that it rains today and B means that it rains tomorrow. If I know that it rained today, it is more likely that it will rain tomorrow, so $p(B|A) > p(B)$.

- The Cookie problems

Simple consideration

Suppose there are two bowls of cookies. Bowl 1 contains 30 vanilla cookies and 10 chocolate cookies. Bowl 2 contains 20 of each. Now suppose you choose one of the bowls at random and, without looking, select a cookie at random. The cookie is vanilla. What is the probability that it came from Bowl 1?

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It may not seem to be easy but we know that $P(Vanilla|Bowl1) = 1/4$

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- The observation that conjunction is commutative

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We can simply guess that $p(B_1|V) > p(B_2|V)$ because Bowl1 has more vanilla cookies.

We can mathematically or statistically confirm that $p(B_1|V) > p(B_2|V)$.

The diachronic interpretation

- "Diachronic" means that something is happening over time; in this case the probability of the hypotheses changes, over time, as we see new data.

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4. $p(D)$ is the probability of the data under any hypothesis, called the **normalizing constant**.

TOV equations

- Connect nuclear EOS to neutron star's mass and radius.

$$\frac{dp}{dr} = -\frac{G(M(r) + 4\pi r^3 p/c^2)(\epsilon + p)}{r(r - 2GM(r)/c^2)c^2},$$
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- Choose p as integration variables and chages $r^2 = z$, dr^2/dp , dM/dp

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- Integrate from the center to the surface with Runge-Kutta 4th oder method.

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- First : find the nearest data point for a given. **x**
- Choose your own interpolation scheme.

Legendre interpolation

For a given set (x_i, f_i), $(i = 1, 2, \ldots, k)$,

$$L(x) = \sum_{i=1}^{k} f_i l_i(x) \tag{54}$$

where

$$l_i(x) = \prod_{\substack{j=1,\cdots,k \\ j \neq i}} \frac{x - x_j}{x_i - x_j} \tag{55}$$

$l_i(x)$ is a $(k-1)$th order polynomial.

$$l_i(x_m) = \begin{cases} 1 & \text{if} \quad m = i \\ 0 & \text{if} \quad m \neq i \end{cases} \quad \rightarrow \quad L(x_i) = f_i \tag{56}$$

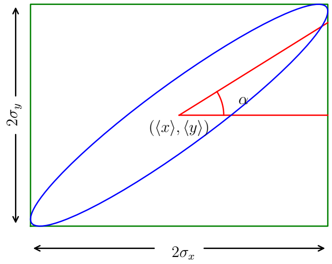
Ex. Four points interpolation,
$x_1 < x_2 < x < x_3 < x_4$,

$$\begin{aligned} f(x) = & \frac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} f_1 \\ & + \frac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} f_2 \\ & + \frac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} f_3 \\ & + \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} f_4 \end{aligned} \tag{57}$$

You may have your own favorite!

Correlation or confidence interval

- You may meet some statisitical quantities with correlation between x and y, with σ_x, σ_y, σ_{xy}, $\langle x \rangle$, $\langle y \rangle$.



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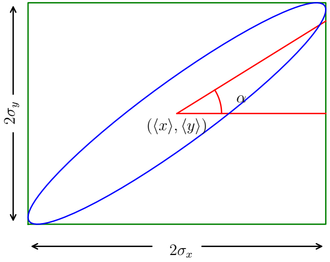

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Correlation or confidence interval

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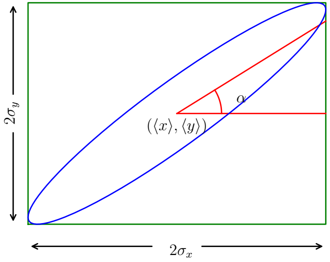

If there's an extreme case, $y = ax + b$, perfect correlation, then we don't expect that there are some data at two corners.
Thus, it is not statistally correct to draw a box with $2\sigma_x$ width and $2\sigma_y$ centered at $(\langle x \rangle, \langle y \rangle)$

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$$\sigma_x = \sqrt{b^2\sin^2\alpha + a^2\cos^2\alpha}, \quad \sigma_y = \sqrt{b^2\cos^2\alpha + a^2\sin^2\alpha}, \tag{58}$$

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$$\boxed{\tan(2\alpha) = \frac{2R_{xy}\sigma_x\sigma_y}{\sigma_x^2 - \sigma_y^2}} \tag{60}$$

EOS and Uncertainties

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- EOS can be fitted by energy density functional

$$\mathcal{E}(n,x) = \frac{1}{2m}\tau_n + \frac{1}{2m}\tau_p + (1-2x)^2 f_n(n) + \left[1-(1-2x)^2\right] f_s(n)\,, \tag{61}$$

where n is the nucleon number density, τ_n and τ_p are the neutron and proton kinetic energy densities, x is the proton fraction,

$$f_s(n) = \sum_{i=0}^{3} a_i n^{(2+i/3)}\,, \quad f_n(n) = \sum_{i=0}^{3} b_i n^{(2+i/3)} \tag{62}$$

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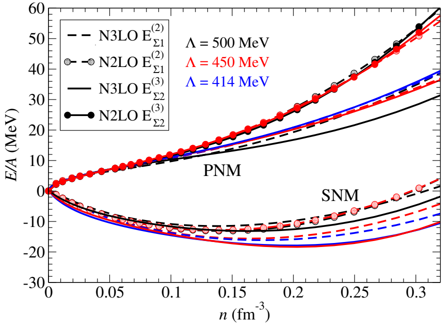
3. D can be symmetric nuclear matter properties and symmetry energy parameters.

$$\boxed{B, n_0, K, Q, J, L, K_{sym}, Q_{sym}}$$

$$\mathcal{E}(n, \frac{1}{2}) = -B + \frac{1}{2}K\left(\frac{n-n_0}{3n_0}\right)^2 + \frac{1}{6}Q\left(\frac{n-n_0}{3n_0}\right)^2 + \cdots$$

$$\mathcal{E}(n, 0) = J + L\left(\frac{n-n_0}{3n_0}\right) + \frac{1}{2}K_{sym}\left(\frac{n-n_0}{3n_0}\right)^2 + \frac{1}{6}Q_{sym}\left(\frac{n-n_0}{3n_0}\right)^2 +$$



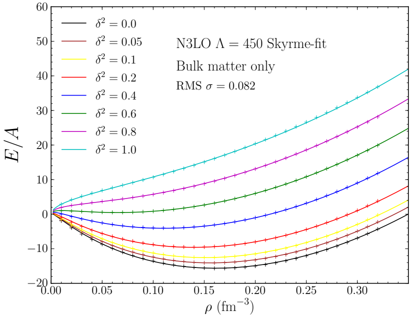


Figure: Theoretical calculation for PNM and SNM (left) and their fittig function results (right)

- Where can we find B, n_0, K, Q (symmetric matter) ?

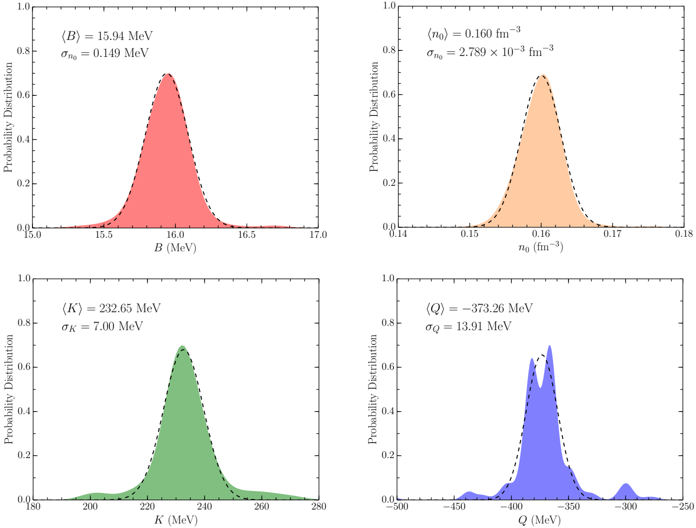


Figure: SNM probability distribution from Dutra et al. (2012)

- Where can we find J, L, K_{sym}, Q_{sym} (pure neutron matter) ?

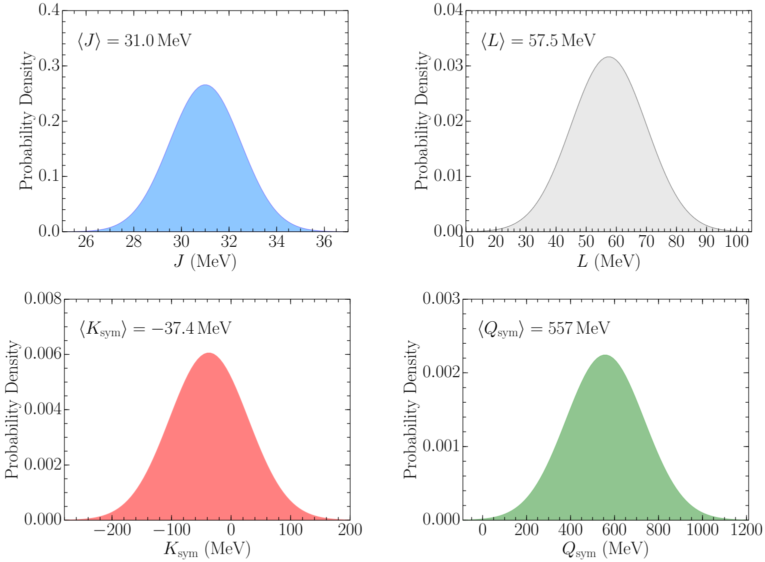


Figure: PNM probability distribution from Lim & Holt. (2019)

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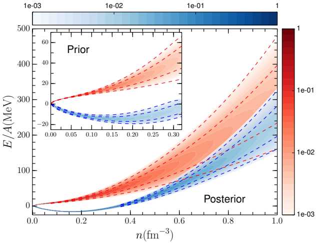
- Generate lots of $(\vec{a}, \vec{b})$ set from Σ_C, see the results

Neutron Star Phenomenology

- Statistical uncertainties originated from EOSs (Lim&Holt PRL 2018)



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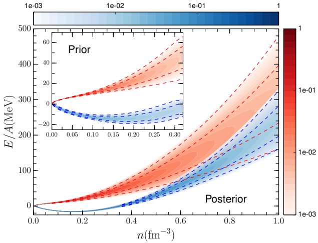

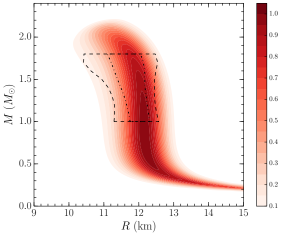

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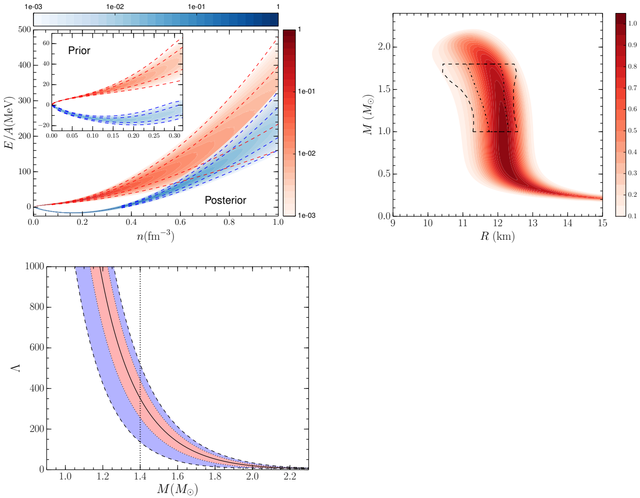

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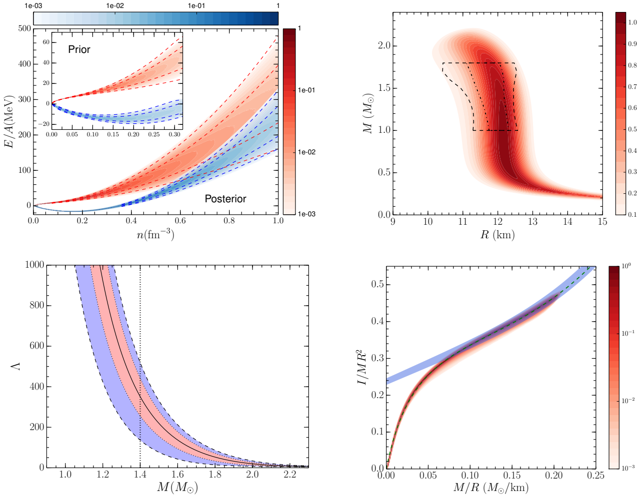

Probability distribution of central density I

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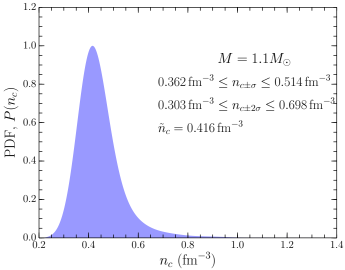

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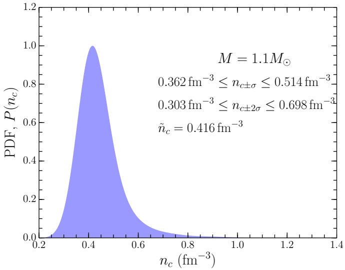

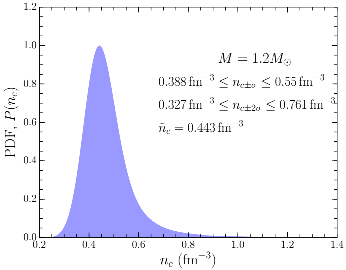

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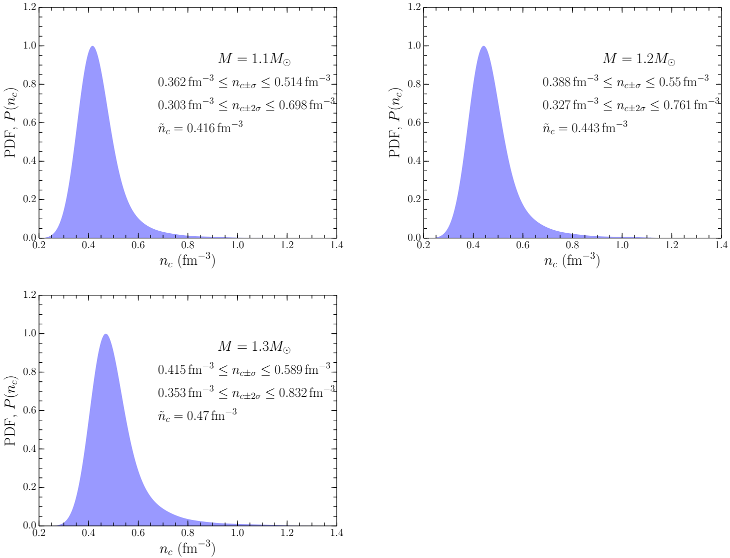

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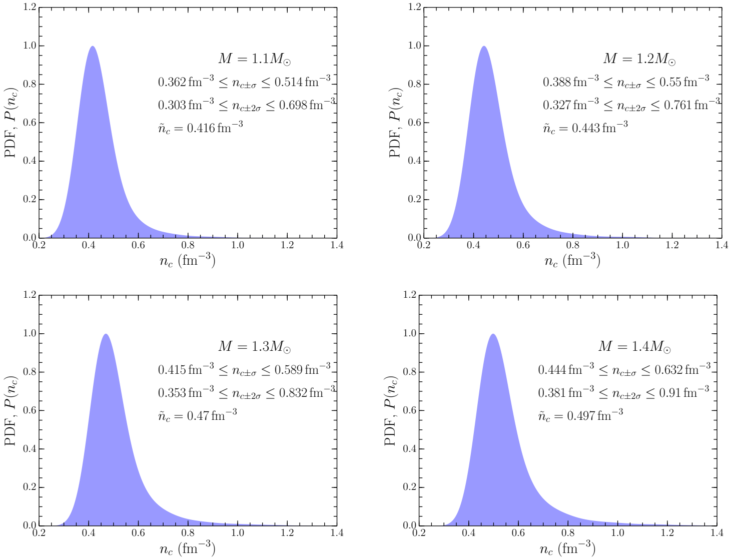

Probability distribution of central density II

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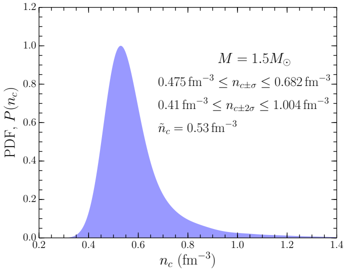

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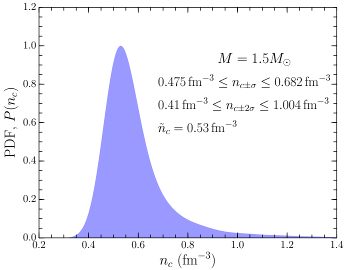

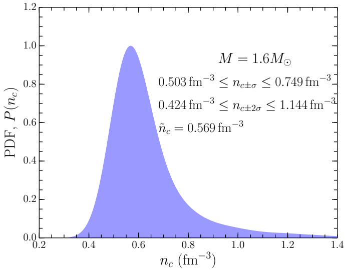

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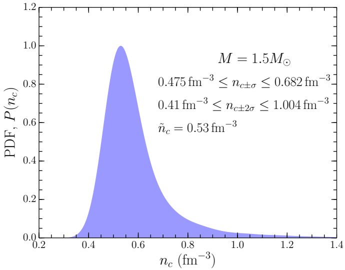

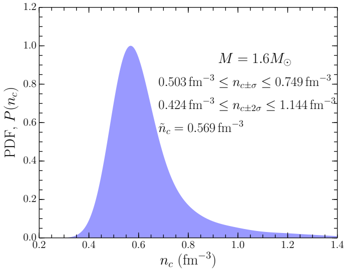

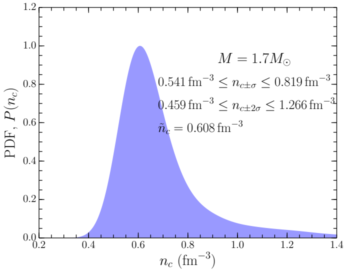

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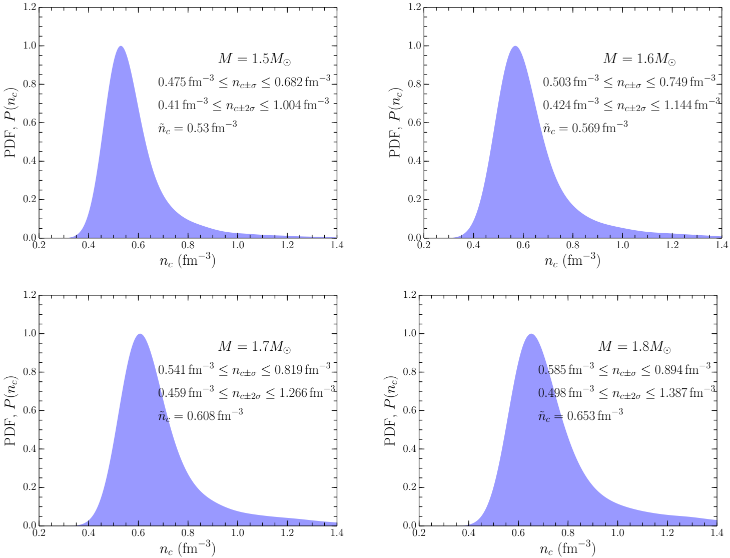


- Revised $S_v - L$ correlation from Bayesian modelling.

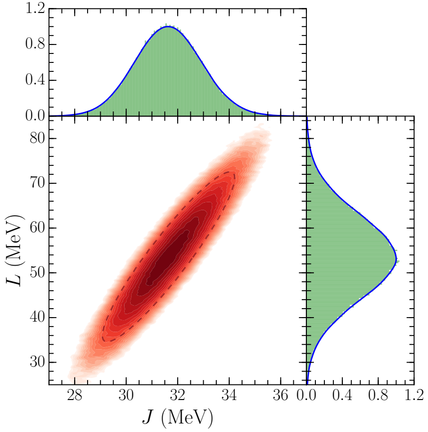
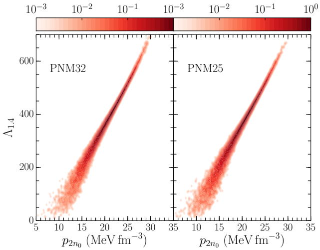
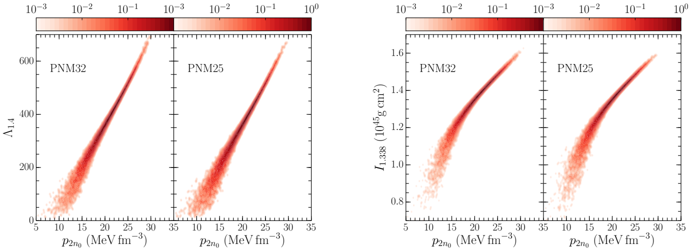


Figure: S_v and L constraints, Lim & Holt (2019).



10^{-3} 10^{-2} 10^{-1} 10^{-3} 10^{-2} 10^{-1} 10^{0}
PNM32
PNM25
$\Lambda_{1.4}$
p_{2n_0} (MeV fm^{-3})
p_{2n_0} (MeV fm^{-3})


PNM32
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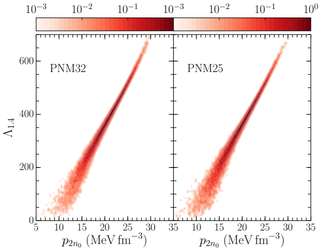
$I_{1.338}$ (10^{45}g cm^2)



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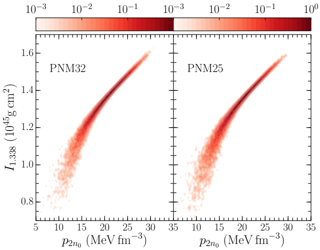

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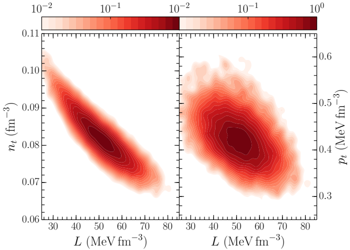

n_t (fm^{-3})
p_t (MeV fm^{-3})
L (MeV fm^{-3})
L (MeV fm^{-3})

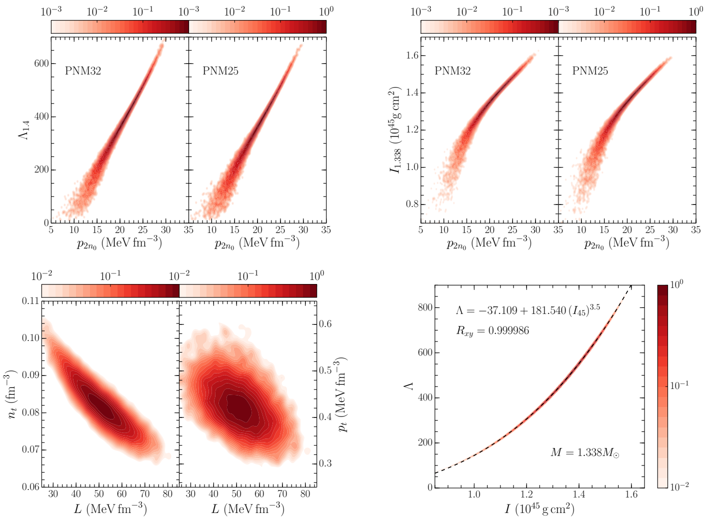

Figure: Correlations among nuclear properties and neutron stars, Lim & Holt (2019)