Updates on Precision Holography

Nakwoo Kim (KHU/KIAS)

Precision Holography

- Usually refers to the comparison of classical supergravity solutions and large-*N* QFT computations.
- In early days of AdS/CFT, anomaly coefficients and CFT quantities which are independent of coupling constants.
- Can account for more non-trivial quantities, now using supersymmetric localization results.

Free energy of ABJM using Localization

Saddle point approx. makes us treat the integration variables as particles on complex plane under 1-particle and 2-particle interactions

$$Z_{ABJM} = \frac{1}{N_1!N_2!}\int \prod_i^{N_1} \frac{d\mu_i}{2\pi} \prod_j^{N_2} \frac{d\nu_j}{2\pi} e^{\frac{ki}{4\pi}(\sum \mu_i^2 - \sum \nu_j^2)}$$

$$\times \quad \frac{\prod_{i<j}(2\sinh(\mu_i-\mu_j))^2 \prod_{i<j}(2\sinh(\nu_i-\nu_j))^2}{\prod_{i,j}(2\cosh(\mu_i-\nu_j)/2)^2}$$

1. A perturbative approach in holography

1902.00418 NK
1904.02038 NK, Se-Jin Kim
1904.09465 NK, Se-Jin Kim

Sphere Partition Functions with Mass terms

- Localization formula is independent of superpotential interactions, but does depend on mass.
- Exhibits interesting phase transition.
- For small mass, the free energy (ln Z) calculation is not too difficult and the results are available.

ABJM

$$F = \frac{4\sqrt{2}\pi N^{3/2}}{3}\sqrt{\Delta_1\Delta_2\Delta_3\Delta_4} \text{ with constraint } \sum \Delta_i = 2.$$

N=2* deformation of N=4 SYM

$$\frac{d^3 F_{S^4}}{d(ma)^3} = -2N^2 \frac{ma(m^2a^2+3)}{(m^2a^2+1)^2},$$

Brandhuber-Oz D4-D8, USp(2N) with massless hyper and massive fundamental

$$F(\mu) = \frac{\pi}{135}\left((N_f - 1)|\mu|^5 - \sqrt{\frac{2}{8-N_f}}\,(9+2\mu^2)^{5/2}\right) N^{5/2}$$

- For all the field theory side considerations in the last slide, we know of the holographic side BPS equations in certain Einstein-scalar systems.

Example: Dual of N=1*

$$\mathcal{L} = -\frac{1}{4}R + 3\frac{(\partial\eta_1)^2}{\eta_1^2} + \frac{(\partial\eta_2)^2}{\eta_2^2} + \frac{1}{2}\mathcal{K}_{a\bar{b}}\partial_\mu z^a \partial^\mu \bar{z}^{\bar{b}} - \mathcal{P},$$

$$\mathcal{P} = \frac{1}{8}e^{\mathcal{K}}\left(\frac{\eta_1^2}{6}\partial_{\eta_1}\mathcal{W}\partial_{\eta_1}\widetilde{\mathcal{W}} + \frac{\eta_2^2}{2}\partial_{\eta_2}\mathcal{W}\partial_{\eta_2}\widetilde{\mathcal{W}} + \mathcal{K}^{\bar{b}a}\nabla_a\mathcal{W}\nabla_{\bar{b}}\widetilde{\mathcal{W}} - \frac{8}{3}\mathcal{W}\widetilde{\mathcal{W}}\right),$$

$$\begin{aligned}\mathcal{W} = {} & \eta_1^{-2}\eta_2^{-2}\left(1 + z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4 + z_1z_2z_3z_4\right) \\ & + \eta_1^{-2}\eta_2^{2}\left(1 - z_1z_2 + z_1z_3 - z_1z_4 - z_2z_3 + z_2z_4 - z_3z_4 + z_1z_2z_3z_4\right) \\ & + \eta_1^{4}\left(1 + z_1z_2 - z_1z_3 - z_1z_4 - z_2z_3 - z_2z_4 + z_3z_4 + z_1z_2z_3z_4\right),\end{aligned}$$

10 real scalars and 1 warp factor
Regular solutions parametrized by
3 mass parameters

BPS equations

$$\partial_r z^a = -\frac{3}{2}(\partial_r A \pm 1/r)\mathcal{K}^{a\bar{b}}\frac{\partial}{\partial \bar{z}^b}\log(\mathcal{W}\widetilde{\mathcal{W}}e^{\mathcal{K}}),$$

$$\partial_r \bar{z}^{\bar{b}} = -\frac{3}{2}(\partial_r A \mp 1/r)\mathcal{K}^{a\bar{b}}\frac{\partial}{\partial z^a}\log(\mathcal{W}\widetilde{\mathcal{W}}e^{\mathcal{K}}),$$

$$\partial_r \eta_1 = -\frac{\eta_1^2}{72}\frac{e^{2A}}{r^2\partial_r A}\frac{\partial}{\partial \eta_1}(\mathcal{W}\widetilde{\mathcal{W}}e^{\mathcal{K}}),$$

$$\partial_r \eta_2 = -\frac{\eta_2^2}{24}\frac{e^{2A}}{r^2\partial_r A}\frac{\partial}{\partial \eta_2}(\mathcal{W}\widetilde{\mathcal{W}}e^{\mathcal{K}}),$$

$$(\partial_r A)^2 = \frac{1}{r^2} + \frac{1}{9}\frac{e^{2A}}{r^2}(\mathcal{W}\widetilde{\mathcal{W}}e^{\mathcal{K}}),$$

$$\frac{\partial}{\partial \eta_i}\mathcal{W} = \frac{\partial_r A \pm 1/r}{\partial_r A \mp 1/r}\frac{\mathcal{W}}{\widetilde{\mathcal{W}}}\frac{\partial}{\partial \eta_i}\widetilde{\mathcal{W}}.$$

Expansion near AdS boundary

$$
\begin{aligned}
(z_1+z_2+z_3+z_4+\bar{z}_1+\bar{z}_2+\bar{z}_3+\bar{z}_4)/4 &= \left(1-s^2\right)\left(2\mu_1\rho+v_1-s\mu_2\mu_3\right)e^{-2\rho}+\mathcal{O}(\rho^2 e^{-4\rho}),\\
(z_1-z_2+z_3-z_4+\bar{z}_1-\bar{z}_2+\bar{z}_3-\bar{z}_4)/4 &= \left(1-s^2\right)\left(2\mu_2\rho+v_2-s\mu_1\mu_3\right)e^{-2\rho}+\mathcal{O}(\rho^2 e^{-4\rho}),\\
(z_1+z_2-z_3-z_4+\bar{z}_1+\bar{z}_2-\bar{z}_3-\bar{z}_4)/4 &= \left(1-s^2\right)\left(2\mu_3\rho+v_3-s\mu_1\mu_2\right)e^{-2\rho}+\mathcal{O}(\rho^2 e^{-4\rho}),\\
(z_1-z_2-z_3+z_4+\bar{z}_1-\bar{z}_2-\bar{z}_3+\bar{z}_4)/4 &= 2s-\frac{s}{2}\left(1-s^2\right)\left(\mu_1^2+\mu_2^2+\mu_3^2\right)e^{-2\rho}+\mathcal{O}(\rho e^{-4\rho}),\\
(z_1-z_2-z_3+z_4-\bar{z}_1+\bar{z}_2+\bar{z}_3-\bar{z}_4)/4 &= -\frac{1}{2}\left(1-s^2\right)\Big[2w-\left(1-3s^2\right)\mu_1\mu_2\mu_3\\
&\quad -2s\left(\mu_1 v_1+\mu_2 v_2+\mu_3 v_3\right)-4s\left(\mu_1^2+\mu_2^2+\mu_3^2\right)\rho\Big]e^{-3\rho}+\mathcal{O}(\rho e^{-5\rho}).
\end{aligned}
$$

Holographic Renormalization
(Gravity action after regularization and including counterterm)

$$\frac{\partial^3 F}{\partial \mu_i^3} = -\frac{N^2}{2}\frac{\partial^2 v_i}{\partial \mu_i^2}.$$

Our result

Numerical method is too demanding because there are many DOF.
We can treat the integration constants as small perturbation and
Impose regularity at each order (in IR)

$$v_i = -2\mu_i + \left(\frac{16\pi^4}{525} - \frac{1}{5}\right)\mu_i^3 + \left(\frac{3}{5} - \frac{8\pi^4}{525}\right)\mu_i\left(\mu_1^2 + \mu_2^2 + \mu_3^2\right) + \mathcal{O}(\mu_i^5).$$

Prediction of AdS/CFT, to be verified in QFT

with Se-Jin Kim

Gaugino condensate

$$w = 2\mu_1\mu_2\mu_3 + \mathcal{O}(\mu_i^5),$$

Results of other models

- ABJM: exact solutions are known, which can be constructed using our method.
- D4-D8: the series form of F can be summed. Disagrees with the field theory prediction.
- Mass deformed ABJM (An N=2 Fixed point when one bifundamental in ABJM is given mass): Consistent with field theory and numerical results.

2. GK geometry

0511029 NK
0607093 NK, Jong-Dae Park
0612253 J. Gauntlett, NK, D. Waldram
0710.2590 J. Gauntlett, NK
0807.4375 A. Donos, J. Gauntlett, NK
1206.1536 NK
1904.05344 Hyojoong Kim, NK

Wrapped D3 and M2

- In 2005, I found the general form of supergravity solutions for D3 wrapped on 2-cycles. (M2 case essentially the same in higher dimensions)

$$
\begin{aligned}
ds^2 &= e^{-B/2}[ds^2(AdS_3) + ds_7^2] \\
F_5 &= -\frac{1}{4}[Vol(AdS_3) \wedge F - *_7 F],
\end{aligned}
$$

$$
ds^2_{2n+1} = c^2(dz+P)^2 + e^B ds^2_{2n}.
$$

$$
\begin{aligned}
e^B &= c^2 \left(\frac{R}{2}\right) \\
F &= -\frac{1}{c} J_T + cd[e^{-B}(dz+P)].
\end{aligned}
$$

$$
\Box R + R_{ij}R^{ij} - \frac{1}{2}R^2 = 0.
$$

vs. Sasaki-Einstein

- The construction is similar to the relation between CY with conical singularity, whose base space is Sasaki-Einstein, which is in turn a twisted U(1) fibration over Kahler-Einstein
- In this case of wrapped branes, the metric cone is not Kahler, although the base of the U(1) fibration is Kahler

Example Solutions

$$ds^2(Y_7) = \frac{y^2 - 2y + a}{4y^2} Dz^2 + \frac{9dy^2}{4q(y)} + \frac{q(y)D\psi^2}{16y^2(y^2 - 2y + a)} + \frac{9}{4y} ds^2(KE_4^+)$$

where $D\psi = d\psi + 3B,\ Dz = dz - g(y)D\psi$ and

$$q(y) = 4y^3 - 9y^2 + 6ay - a^2$$
$$g(y) = \frac{a - y}{2(y^2 - 2y + a)}$$

$$a = \frac{m^2 q^2 (3p + mq)^2 (3p + 2mq)^2}{4(3p^2 + 3mpq + m^2 q^2)^3} ,$$

J. Gauntlett et al
0606221

And many more.
Interpreted as
Lower-dim SCFTs and their central charge
or Near horizon geometry of AdS BHs

Volume Minimization

- Recently, a "master-volume" formula was devised whose minimization, given toric data and choosing magnetic flux, gives the central charge/BH entropy. Couzens, Gauntlett, Martelli, Sparks 2018-2019

- Since it is a generalization of "volume minimization" for toric Sasaki-Einstein, let me sketch Martelli, Sparks, Yau 2005 instead.

Toric geometry

- Definition: Symplectic manifold ($M, 2n$-dim) with $U(1)^n$ Killing isometry

- Symplectic 2-form (ω) is closed ($\mathrm{d}\omega = 0$) and its n-th power is Volume form ($1/n!\omega^n = \mathrm{Vol}(M)$).

- Hamiltonian mechanics is described in terms of symplectic geometry: when X is time-evolution vector field, $\iota_X \omega = \mathrm{d}H$

- Thus having n Killing vector means we have n "Hamiltonians", and their range gives toric data.

$$\begin{aligned} \mathrm{d}H(q,p) &= \frac{\partial H}{\partial q}\mathrm{d}q + \frac{\partial H}{\partial p}\mathrm{d}p \\ &= -\dot{p}\mathrm{d}q + \dot{q}\mathrm{d}p \\ &= X^q\omega_{qp}\mathrm{d}p + X^p\omega_{pq}\mathrm{d}q \end{aligned}$$

$$X = \frac{\partial}{\partial t} = \dot{q}\frac{\partial}{\partial q} + \dot{p}\frac{\partial}{\partial p}$$

$$\omega = \mathrm{d}q \wedge \mathrm{d}p$$

Moment map

- S^2 (CP1) is symplectic (Kahler in fact) and the moment map y (Hamiltonian) is easily obtained

- Toric diagram for S^2 is an interval; For $S^2 \times S^2$ etc, one has the square. In general, polygons or polytopes (Δ), and its area is related to the volume of toric manifold

$$\omega = -\, \mathrm{d}\phi \wedge \sin\theta \mathrm{d}\theta$$

$$X = \frac{\partial}{\partial \phi}$$

$$\begin{aligned} \iota_X \omega &= -\sin\theta \mathrm{d}\theta \\ &= \mathrm{d}(\cos\theta) \end{aligned}$$

$$\begin{aligned} \mathrm{Vol}(M) &= \int \mathrm{d}y_1 \mathrm{d}\phi_1 \ldots \\ &= (2\pi)^n \mathrm{Vol}(\Delta) \end{aligned}$$

Toric Cone and Sasakian manifold

- One application is for vacuum moduli space of supersymmetric gauge field theory
    - The description is modified: (1) toric manifold is non-compact, and (2) we use the normal vector to the faces of the polytope
- D-term conditions and identification of gauge orbits lead to Kahler quotient: GLSM (gauged linear sigma model)
- One considers theories with $U(1)$ vector multiplets and charged chiral multiplets
- For each $U(1)$, D-term condition and gauge redundancy reduces the dimensionality by 2.

An example: the conifold

- In terms of gauge theory, in abelian version one has $U(1) \times U(1), A_1, A_2$ with charge (1,-1) and B_1, B_2 with charge (-1,1).
- D-term condition $|A_1|^2 + |A_2|^2 - |B_1|^2 - |B_2|^2 = 0$
- Gauge redundancy: $(A_1, A_2, B_1, B_2) \simeq (e^{i\alpha} A_1, e^{i\alpha} A_2, e^{-i\alpha} B_1, e^{-i\alpha} B_2)$
- The result is conifold: $\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}s^2_{T^{1,1}}$

Moment map of the conifold (0411238)

$$ds^2 = \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) + \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2 \ .$$

- Metric (Sasaki-Einstein, with $SU(2) \times SU(2) \times U(1)$ isometry)

- 3 commuting Killing vectors ($\psi = 2\nu$)

$$\begin{aligned} e_1 &= \frac{\partial}{\partial\phi_1} + \frac{1}{2}\frac{\partial}{\partial\nu} \\ e_2 &= \frac{\partial}{\partial\phi_2} + \frac{1}{2}\frac{\partial}{\partial\nu} \\ e_3 &= \frac{\partial}{\partial\nu} \ . \end{aligned}$$

- Moment maps, giving a "convex rational polyhedral cone", generated by 4 edge vectors

$$\vec{\mu} = \left(\frac{1}{6}r^2(\cos\theta_1 + 1), \frac{1}{6}r^2(\cos\theta_2 + 1), \frac{1}{3}r^2\right) \ .$$

- Alternatively defined by "outward pointing primitive normal vectors"

Toric Data

$$v_1 = [1, 0, -1], \quad v_2 = [0, 1, -1], \quad v_3 = [0, -1, 0], \quad v_4 = [-1, 0, 0] \ .$$

How to relate toric data to gauge theory

- The kernel of a matrix made of toric data gives charge assignment for Kahler quotient (GLSM). Delzant construction
- For the 3×4 matrix for conifold, the kernel is (1,1,-1,-1) and matches the matter contents of gauge theory dual.
- Calculating the volume of the base space requires choosing $r = \text{const}$ slice, or equivalently the Reeb vector
- Shown to equal to the extremized value of Ricci scalar. Martelli, Sparks, Yau 0503183

(Master) Volume formulas for 5d Sasaki-Einstein

$$\mathcal{V}(\vec{b};\{\lambda_a\}) = \frac{(2\pi)^3}{2}\sum_{a=1}^{d}\lambda_a \frac{\lambda_{a-1}(\vec{v}_a,\vec{v}_{a+1},\vec{b}) - \lambda_a(\vec{v}_{a-1},\vec{v}_{a+1},\vec{b}) + \lambda_{a+1}(\vec{v}_{a-1},\vec{v}_a,\vec{b})}{(\vec{v}_{a-1},\vec{v}_a,\vec{b})(\vec{v}_a,\vec{v}_{a+1},\vec{b})}$$

$$0 \;=\; A\sum_{a,b=1}^{d}\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial\lambda_b} - 2\pi n_1\sum_{a=1}^{d}\frac{\partial\mathcal{V}}{\partial\lambda_a} + 2\pi b_1\sum_{a=1}^{d}\sum_{i=1}^{3} n_i\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial b_i}\ ,$$

$$\mathrm{Vol}(Y_5) \;=\; \frac{\pi^3}{b_1}\sum_a \frac{(\vec{v}_{a-1},\vec{v}_a,\vec{v}_{a+1})}{(\vec{v}_{a-1},\vec{v}_a,\vec{b})(\vec{v}_a,\vec{v}_{a+1},\vec{b})}\ ,$$

$$\mathrm{Vol}(S_a) \;=\; 2\pi^2\frac{(\vec{v}_{a-1},\vec{v}_a,\vec{v}_{a+1})}{(\vec{v}_{a-1},\vec{v}_a,\vec{b})(\vec{v}_a,\vec{v}_{a+1},\vec{b})}\ ,$$

volume of SE5 and cycles

$$\frac{2(2\pi\ell_s)^4 g_s}{L^4}N \;=\; -\sum_{a=1}^{d}\frac{\partial\mathcal{V}}{\partial\lambda_a}\ , \qquad \textbf{\#D3}$$

$$\frac{2(2\pi\ell_s)^4 g_s}{L^4}M_a \;=\; \frac{A}{2\pi}\sum_{b=1}^{d}\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial\lambda_b} + b_1\sum_{i=1}^{3} n_i\frac{\partial^2\mathcal{V}}{\partial\lambda_a\partial b_i}\ \cdot \qquad \textbf{Flux}$$